Chapter 2
Formulation of the Problem

We will consider the system arising from the problem when a rigid body is translating with constant velocity and is rotating at constant angular velocity in an incompressible viscous fluid. The flow field around this body is usually described by a "modified" Navier–Stokes system, which can be written in a normalized form, and this system reads

$$\begin{aligned}
&\partial_t u + u\cdot\nabla u - \Delta_x u - \mathcal{R}e\Big((U+\omega\times x)\cdot\nabla_x u + \omega\times u\Big) + \nabla_x p = F, \qquad (2.1)\\
&\qquad \mathrm{div}_x u = 0\\
&\qquad \text{in } \ (\mathbb{R}^3\backslash\overline{\mathfrak{D}})\times(0,T),
\end{aligned}$$

where $\mathcal{R}e = d\,w\,l/\nu$ is the Reynolds number, w is a suitable scale velocity, l is a suitable scale length, d is a constant density and ν denotes the viscosity coefficient.

Here $\mathfrak{D}\subset\mathbb{R}^3$ is a bounded domain representing the rigid body. The function u denotes the dimensionless velocity of fluid with respect to a system of coordinates whose origin is located at the center of mass of the rigid body. The function p denotes the pressure in the fluid, the vector U corresponds to the translation of the body, the vector ω corresponds to the angular velocity of the body, and the function F stands for an exterior force exerted on the fluid.

Remark 3 We use same notation u, p for the normalized form of the Navier–Stokes equations.

In the work we also consider a stationary linearized variant of (2.1) given by

$$-\Delta u - (U+\omega\times x)\cdot\nabla u + \omega\times u + \nabla\pi = f, \quad \mathrm{div}\,u = 0 \quad \text{in } \ \mathbb{R}^3\backslash\overline{\mathfrak{D}} \qquad (2.2)$$

under the assumption that U and ω are parallel. We will consider that $\mathcal{R}e = 1$.

This condition does not imply loss of generality; see [33, Sect. 1]. Our aim is to derive a representation formula for the velocity part u of a solution (u,π) to (2.2).

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This formula is based on a fundamental solution to (2.2) proposed by Guenther and Thomann in the article [36] where they construct the fundamental solution to a linearized version of the time-dependent problem (2.1). In [36, p. 20], they indicate that by integrating this solution with respect to time on $(0, \infty)$, a fundamental solution to (2.2) could be obtained. They left the problem unsolved. It is this time integral we will use in our representation formula, see Theorem 5.3.

2.1 Notations, Definitions and Auxiliary Results

If $x, y \in \mathbb{R}^3$, we write $x \times y$ for the usual vector product of x and y. The open ball centered at $x \in \mathbb{R}^3$ and with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we will write B_r instead of $B_r(0)$. The symbol $|\cdot|$ will be used to denote the Euclidean norm in $\mathbb{R}^3$ and it will also stand for the length $\alpha_1 + \alpha_2 + \alpha_3$ of a multiindex $\alpha \in \mathbb{N}_0^3$.

We fix vectors $U, \omega \in \mathbb{R}^3\backslash\{0\}$ which are parallel. By another transformation of variables, we may suppose there is some $\tau > 0$ with $U = -\tau \cdot e_1 = -\tau \cdot (1, 0, 0)$, hence $\omega = \varrho \cdot (1, 0, 0)$ for some $\varrho \in \mathbb{R}\backslash\{0\}$. By the symbol $\mathfrak{C}$ we denote constants depending only on U and ω. We write $\mathfrak{C}(\gamma_1, \ldots, \gamma_n)$ for constants which additionally depend on quantities $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$ for some $n \in \mathbb{N}$. We further fix an open bounded set $\mathfrak{D}$ in $\mathbb{R}^3$ with Lipschitz boundary $\partial\mathfrak{D}$.

Set $p' := (1 - 1/p)^{-1}$ for $p \in (1, \infty)$.

We fix parameters $\tau \in (0, \infty)$, $\varrho \in \mathbb{R}\backslash\{0\}$, and we set $\omega := \varrho e_1$ and

$$s_\tau(x) := 1 + \tau(|x| - x_1) \quad \text{for} \quad x \in \mathbb{R}^3.$$

Define the matrix by

$$\Omega := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

such that $\omega \times x = \Omega \cdot x$ for $x \in \mathbb{R}^3$. For open sets $V \subset \mathbb{R}^3$, sufficiently smooth functions $w : V \mapsto \mathbb{R}^3$, and for $z \in V$, we set

$$\mathcal{L}(w)(z) := -\Delta w(z) - (U + \omega \times z) \cdot \nabla w(z) + \omega \times w(z). \tag{2.3}$$

Let $\mathfrak{D}$ be an open bounded set in $\mathbb{R}^3$ with C^2-boundary $\partial\mathfrak{D}$. This set will be kept fixed throughout. We denote its outward unit normal by $n^{(\mathfrak{D})}$. For $T \in (0, \infty)$, put $\mathfrak{D}_T := B_T\backslash\overline{\mathfrak{D}}$ "truncated exterior domain").

For $p \in [1, \infty)$, $k \in \mathbb{N}$, and for open sets $A \subset \mathbb{R}^3$, we write $W^{k,p}(A)$ for the usual Sobolev spaces of order k and exponent p. Its standard norm will be denoted by $\|\cdot\|_{k,p}$. If $B \subset \mathbb{R}^3$ is open, define $W^{k,p}_{\text{loc}}(B)$ as the set of all functions $g : B \mapsto \mathbb{R}$ such that $g|U \in W^{k,p}(U)$ for any open bounded set $U \subset \mathbb{R}^3$ with $\overline{U} \subset B$. Also we will need the fractional order Sobolev space $W^{2-1/p,p}(\partial\mathfrak{D})$ equipped with its intrinsic norm, which we denote by $\|\cdot\|_{2-1/p,\,p}$ $\big(p \in (1,\infty)\big)$; see [51] for the corresponding definitions. If $\mathfrak{H}$ is a normed space whose norm is denoted by $\|\cdot\|_{\mathfrak{H}}$, and if $n \in \mathbb{N}$, we equip the product space $\mathfrak{H}^n$ with a norm $\|\cdot\|_{\mathfrak{H}}^{(n)}$ defined by $\|v\|_{\mathfrak{H}}^{(n)} := \Big(\sum_{j=1}^{n} \|v_j\|_{\mathfrak{H}}^2\Big)^{1/2}$ for $v \in \mathfrak{H}^n$. But for simplicity, we will write $\|\cdot\|_{\mathfrak{H}}$ instead of $\|\cdot\|_{\mathfrak{H}}^{(n)}$. We denote by $\mathcal{S}(\mathbb{R}^3)$ the usual Schwartz class of test functions.

Let $z \in \mathbb{R}^3\backslash\{0\}$, We define $\mathcal{N}$ as the fundamental solution of the Poisson equation,

$$\mathcal{N}(z) = (4\pi|z|)^{-1}\,,$$

i.e. as the kernel of the Newton potential.

For $z \in \mathbb{R}^3\backslash\{0\}$, $r \in (0,\infty)$, $\tau \in (0,\infty)$, $j,k \in \{1,\ 2,\ 3\}$, we define

$$\Psi(r) = \int_0^r (1-e^t)\ t^{-1}\,dt, \quad \Phi(z,\tau) := (4\pi\tau)^{-1}\cdot\Psi\big(\tau\cdot(|z|-z_1)/2\big),$$

$$E_{jk}(z,\tau) = \big(\delta_{jk}\ \Delta_z\ -\ \partial/\partial z_j\partial/\partial z_k\big)\Phi(z,\tau), \tag{2.4}$$

$$E_{4k}(z) = (4\cdot\pi)^{-1} z_k\ |z|^{-3}. \tag{2.5}$$

The matrix-valued function $(E_{jk})_{1\le j\le 4,\ 1\le k\le 3}$ is the fundamental solution of the Oseen system $-\Delta u + \tau\,\partial_1 u + \nabla\pi = f,\ \operatorname{div} u = 0$ in $\mathbb{R}^3$.

By $\mathcal{R}_i$ we denote the Riesz transforms.

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