Chapter 2
Fundamental Equations

The mathematical models, which simulate the physics involved, are the essential tools for the theoretical analysis of aerodynamical flows. These mathematical models are usually based on the equations which are nothing but the fundamental conservation laws of mechanics. The conservation equations are usually satisfied locally as differential equations; therefore, their unique solution requires initial and boundary conditions which are described with the farfield conditions and the time dependent motion of the body. Let us follow the historical development of the aerodynamics, and start our analysis with potential flow theory. The potential theory will help us to determine the aerodynamic lifting force which is in the direction normal to the flight and necessary to balance the weight of the body in flight. Since the viscous forces are neglected in potential theory, the drag force which is in the direction of flight cannot be calculated. On the other hand, the potential theory can determine the lift induced drag for three dimensional flows past finite wings. Now, in order to perform our aerodynamical analysis let us introduce further definitions and the simplification of the equations for first, (A) The Potential Theory with its assumptions and limitations, and then for the (B) Real Gas Flow which covers all sorts of viscous effects and the effect of composition changes in the gas because of high altitude flows with high speeds.

2.1 Potential Flow

2.1.1 Equation of Motion

Let us write the velocity vector $\mathbf{q}$ in Cartesian coordinates as $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Here, u,v and w denotes the velocity components in x,y,z directions, and **i,j,k** shows the corresponding unit vectors. At this stage it is useful to define the following vector operators.

Ü. Gülçat, *Fundamentals of Modern Unsteady Aerodynamics*,
DOI 10.1007/978-981-10-0018-8_2

The divergence of the velocity vector is given by

$$\operatorname{div}\mathbf{q} = \nabla.\mathbf{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

and the curl

$$\operatorname{curl}\mathbf{q} = \nabla \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The gradient of any function, on the other hand, reads as

$$\text{grad f} = \nabla \text{f} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

The material or the total derivative as an operator is shown with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Here, t denotes the time. Now, we can give the equations associated with the laws of classical mechanics.

Equation of Continuity:

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{q} = 0 \tag{2.1}$$

Momentum Equation:

$$\frac{D\mathbf{q}}{Dt} + \frac{1}{\rho}\nabla \text{p} = 0 \tag{2.2}$$

Energy Equation:

$$\frac{D}{Dt}\left(\frac{a^2}{\gamma - 1} + \frac{q^2}{2}\right) = \frac{1}{\rho}\frac{\partial p}{\partial t} \tag{2.3}$$

Equation of State:

$$\text{p} = \rho \text{RT} \tag{2.4}$$

Here, the pressure is denoted with p, density with ρ, temperature with T, speed of sound with a, specific heat ratio with γ and the gas constant with R.

In addition, the air is assumed to be a perfect gas and the body and frictional forces are neglected. It is also assumed that no chemical reaction takes place during the motion. The energy equation is given in Bisplinghoff et al. (1996).

Let us now see the useful results of Kelvin's theorem under the assumptions made above (Batchelor 1979). The following line integral on a closed path defines the

$$\text{Circulation}: \ \Gamma = \oint q.\,ds.$$

$$\text{The Kelvin's theorem}: \frac{D\Gamma}{Dt} = -\oint \frac{dp}{\rho}.$$

For incompressible flow or a barotropic flow where $p = p(\rho)$ the right hand side of Kelvin's theorem vanishes to yield

$$\frac{D\Gamma}{Dt} = 0.$$

This tells us that the circulation under these conditions remains the same with time. Now, let us analyze the flow with constant free stream which is the most referred flow case in aerodynamics. Since the free stream is constant then its circulation $\Gamma = 0$. The Stokes Theorem states that

$$\oint q.ds = \oiint \nabla \times \mathbf{q}\,.\,\mathbf{dA} = 0 \tag{2.5}$$

The integrand of the double integral must be zero in order to have Eq. 2.5 equal to zero for arbitrary differential area element. This gives $\nabla \times \mathbf{q} = 0$.

$\nabla \times \mathbf{q} = 0$, on the other hand, implies that the velocity vector $\mathbf{q}$ can be obtained from the gradient of a scalar potential ϕ, i.e.

$$\mathbf{q} = \nabla \phi \tag{2.6}$$

At this stage if we expand the first term of the momentum equation into its local and convective derivative terms, and express the convective terms with its vector equivalent we obtain

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla \mathrm{p} \quad \text{and} \quad (\mathbf{q}.\nabla)\mathbf{q} = \nabla\frac{q^2}{2} - \mathbf{q} \times (\nabla \times \mathbf{q}).$$

From Eq. 2.5 we obtained ∇ **x q = 0.** Utilizing this fact the momentum equation reads as

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\frac{q^2}{2} + \frac{1}{\rho}\nabla \mathrm{p} = 0 \tag{2.7}$$

Now, we can use the scalar potential ϕ in the momentum equation in terms of Eq. 2.6.

For a baratropic flow we have the 3rd term of Eq. 2.7 as

$$\frac{1}{\rho}\nabla \mathrm{p} = \nabla \int \frac{dp}{\rho}.$$

Then collecting all the terms of Eq. 2.7 together

$$\nabla.\left(\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \int \frac{dp}{\rho}\right) = 0$$

we see that the scalar term under gradient operator is in general only depends on time, e.i,

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \int \frac{dp}{\rho} = \mathrm{F(t)} \tag{2.8}$$

According to Eq. 2.8, F(t) is arbitrarily chosen, and if we set it to be zero we obtain the classical Kelvin's equation

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \int \frac{dp}{\rho} = 0 \tag{2.9}$$

Let us try to write the continuity equation, Eq. 2.1, in terms of ϕ only,

$$\frac{\partial \rho}{\partial t} + (\mathbf{q}.\nabla)\rho + \rho\nabla.\mathbf{q} = 0 \tag{2.10}$$

The gradient of the velocity vector now reads as

$$\nabla.\mathbf{q} = \nabla^2 \phi.$$

Dividing Eq. 2.10 by ρ we obtain

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + (\frac{\mathbf{q}}{\rho}.\nabla)\,\rho + \nabla^2\phi = 0 \tag{2.11}$$

Note that Eq. 2.11 becomes the Laplace equation for incompressible flow

$$\nabla^2 \phi = 0 \tag{2.12}$$

We know that Laplace equation by itself is independent of time. The time dependent boundary conditions make us seek the time dependent solutions of Eq. 2.12.

Now, we can obtain the simplified version of Eq. 2.11 for the compressible flows. Let us rearrange Kelvin's equation, Eq. 2.9 in following form

$$\frac{\partial}{\partial t}\int\frac{dp}{\rho} = -\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}+\frac{q^2}{2}\right)$$

and the integral on the left hand side can be differentiated to give

$$\frac{\partial}{\partial t}\int\frac{dp}{\rho} = \frac{1}{\rho}\frac{\partial p}{\partial\rho}\frac{\partial\rho}{\partial t} \tag{2.13}$$

In Eq. 2.13, the speed of sound is related to the pressure and density changes: $\frac{\partial p}{\partial\rho} = a^2$

Hence, we obtain the following for the first term of the Eq. 2.11

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} = -\frac{1}{a^2}\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}+\frac{q^2}{2}\right) \tag{2.14}$$

Now, let us write Eq. 2.7 in terms of ϕ and the pressure gradient. Furthermore, expressing the pressure gradient in terms of the density gradient and the local speed of sound we obtain

$$\frac{1}{\rho}\nabla \mathrm{p} = \frac{a^2}{\rho}\nabla \mathrm{p} = -\nabla\left(\frac{\partial\phi}{\partial t}+\frac{q^2}{2}\right)$$

and with the aid of 2.14 and the multiplying term $\mathbf{q}/a^2$, the final form of Eq. 2.11 reads as

$$\nabla^2\phi - \frac{1}{a^2}\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial q^2}{\partial t}+\mathbf{q}.\nabla\frac{q^2}{2}\right) = 0 \tag{2.15}$$

In Eq. 2.15, we express the velocity vector in terms of the velocity potential. This way, the scalar non linear equation has the scalar function as the only unknown except the speed of sound. The equation itself models many kinds of aerodynamic problems. We need to impose, however, the boundary conditions in order to model a specific problem.

2.1.2 Boundary Conditions

Equation 2.15 as a fundamental equation is solved with the proper boundary conditions. In general the external flow problems will be studied. Therefore, we need to impose the boundary conditions accordingly as follows.

(i) At infinity, all disturbances must die out and only free stream conditions prevail.

ii) The time dependent boundary conditions at the body surface must be given as the time dependent motion of the body.

The equation of a surface for a 3-D moving body in Cartesian coordinate system is given as follows

$$\mathrm{B}(\mathrm{x},\ \mathrm{y},\ \mathrm{z},\ \mathrm{t}) = 0 \tag{2.16}$$

Let us take the material derivative of this surface in the flow field $\mathbf{q} = \mathrm{u}\mathbf{i} + \mathrm{v}\mathbf{j} + \mathrm{w}\mathbf{k}$.

$$\frac{DB}{Dt} = \frac{\partial B}{\partial t} + u\frac{\partial B}{\partial x} + v\frac{\partial B}{\partial y} + w\frac{\partial B}{\partial z} = 0 \tag{2.17}$$

For the steady flow it simplifies to

$$u\frac{\partial B}{\partial x} + v\frac{\partial B}{\partial y} + w\frac{\partial B}{\partial z} = 0$$

The external flows studied here require to find the pressure distribution at the lower and upper surfaces of the body immersed in a free stream. For this purpose, we need to know the upper and lower surface equations of a body in a free stream in x direction. If we show the direction normal to the flow with z, then the single valued surface equation, with the aid of Eq. 2.16, reads as

$$\mathrm{B}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}) = \mathrm{z} - \mathrm{z}_a(\mathrm{x},\mathrm{y},\mathrm{t}) = 0 \tag{2.18}$$

Now, we can take the material derivative of Eq. 2.18 with the aid of Eq. 2.17

$$w = \frac{\partial z_a}{\partial t} + u\frac{\partial z_a}{\partial x} + v\frac{\partial z_a}{\partial y} \tag{2.19}$$

Note that, $\frac{\partial B}{\partial z} = 1$ is used for the convective term in z direction. Here, the explicit expression of vertical velocity component w is named 'downwash' in aerodynamics. This downwash at the near wake is the indicative of the lifting force on the body. The direction of the force and the downwash are the same but their senses are opposite. Accordingly, for the downward downwash the force is then upward. In other words, downward velocity component at the wake region creates a clockwise circulation which in turn generates the lifting force together with the free stream.

Equations 2.15 and 2.19 are not linear. In order to solve those equations together, linearization is necessary. Once the equations are linearized we can also employ the superpositioning technique for solving them.

2.1.3 Linearization

Let us begin the linearization process with the boundary conditions. The small perturbations approach will be used here. Accordingly, let U be the free stream speed in positive x direction, Fig. 2.1.

Let u′ be the perturbation velocity component in x direction which makes the total velocity component in x direction: u = U + u′. In addition, defining function ϕ′ as the perturbation potential gives us the relation between the two potentials as follows: ϕ = ϕ′ + U x. As a result, we can write the relation between the perturbation potential and the velocity components in following form

$$\frac{\partial \phi^{/}}{\partial x} = \mathrm{u}', \quad \frac{\partial \phi^{/}}{\partial y} = \mathrm{v} \quad \text{and} \quad \frac{\partial \phi^{/}}{\partial z} = \mathrm{w}.$$

The small perturbation method is based on the assumption that the perturbation speeds are quite small compared to the freestream speed, i.e. u′, v, w ≪ U. In addition, because of thin wing theory the slopes of the body surface are small therefore we can write

$$\frac{\partial z_a}{\partial x} \ll 1 \quad \text{and} \quad \frac{\partial z_a}{\partial y} \ll 1$$

Then the boundary condition 2.19 become

$$w = \frac{\partial z_a}{\partial t} + U\frac{\partial z_a}{\partial x} + u'\frac{\partial z_a}{\partial x} + v\frac{\partial z_a}{\partial y} \quad \text{where } u'\frac{\partial z_a}{\partial x}, v\frac{\partial z_a}{\partial y} \ll U\frac{\partial z_a}{\partial x}$$

which gives the approximate expression for the boundary condition

$$w = \frac{\partial z_a}{\partial t} + U\frac{\partial z_a}{\partial x} \tag{2.20}$$

Equation 2.20 is valid at angles of attack less than 12^0 for thin airfoils whose thickness ratio is less than 12 %. For the upper and lower surfaces, the linearized downwash expression will be denoted as follows.

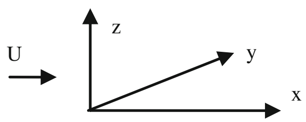

Fig. 2.1 Coordinate system and the free stream U

$$\text{Upper surface } (u): w = \frac{\partial z_u}{\partial t} + U\frac{\partial z_u}{\partial x}; \quad z = 0^+$$
$$\text{Lower surface } (l): w = \frac{\partial z_l}{\partial t} + U\frac{\partial z_l}{\partial x}; \quad z = 0^-.$$

Now, let us obtain an expression for the linearized surface pressure coefficient. For this purpose we are going to utilize the linearized version of Eq. 2.8. The second term of the equation is linearized as follows

$$\frac{q^2}{2} \cong \frac{U^2}{2} + 2Uu'$$

For the right hand side of Eq. 2.8 if we arbitrarily choose F(t) = U^2 /2 then the term with the integral reads as

$$\int \frac{dp}{\rho} = -\frac{\partial \phi}{\partial t} - 2Uu'.$$

The relation between the velocity potential and the perturbation potential gives: $\frac{\partial \phi}{\partial t} = \frac{\partial \phi^{/}}{\partial t}$. If we now evaluate the integral from the free stream pressure value p_∞ to any value p and omit the small perturbations in pressure and in density we obtain

$$\int_{p_\infty}^{p} \frac{dp}{\rho} \cong \frac{p - p_\infty}{\rho_\infty} = -\left(\frac{\partial \phi^{/}}{\partial t} + \mathrm{U}\frac{\partial \phi^{/}}{\partial x}\right)$$

Using the definition of pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2} = -\frac{2}{U^2}\left(\frac{\partial \phi^{/}}{\partial t} + \mathrm{U}\frac{\partial \phi^{/}}{\partial x}\right) \tag{2.21}$$

Here, the pressure coefficient is expressed in terms of the perturbation potential only.

Example Let the equation of the surface of a body immersed in a free stream U be

$$z_{u,l} = \pm a\sqrt{\frac{x}{l}} \quad (0 \le x \le l)$$

If this body pitches about its nose simple harmonically with a small amplitude, find the downwash at the upper and the lower surfaces of the body in terms of a, l and the amplitude and the frequency of the oscillatory motion.

Answer Let $\alpha = \bar{\alpha}\sin\omega t$ ($\bar{\alpha}$: small amplitude and ω: angular frequency) be the pitching motion, let x,z be the stationary coordinate and x', z' be the moving

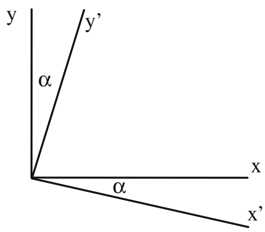

Fig. 2.2 α pitch angle and the coordinate systems

coordinate system attached to the body. The relation between the fixed and the moving coordinate system is given by Fig. 2.2 in terms of α.

The coordinate transformation gives

$$x' = x\cos\alpha - z\sin\alpha$$
$$z' = x\sin\alpha + z\cos\alpha$$

In body fixed coordinates the surface equations $z'_{u,l} = \pm a\sqrt{\frac{x'}{l}} \quad (0 \le x \le l)$

In terms of the stationary coordinate system B(x,z,t) = z′ $- z'_{u,l}(x') = x\sin\alpha + z\cos\alpha \mp a\left(\frac{x\cos\alpha - z\sin\alpha}{l}\right)^{1/2}$ for small $\alpha \sin\alpha \cong \alpha\, and\, \cos\alpha \cong 1$. Then B(x,z,t) $= x\alpha + z \mp a\left(\frac{x - z\alpha}{l}\right)^{1/2}$.

Equation 2.17 gives

$$w_{u,l} = -\left\{x\dot{\alpha} \pm \frac{a\dot{\alpha}z}{2l}\left(\frac{x - z\alpha}{l}\right)^{-1/2} + U\left[\alpha \mp \frac{a}{2l}\left(\frac{x - z\alpha}{l}\right)^{-1/2}\right]\right\} \Big/ \left[1 \pm \frac{a\alpha}{2l}\left(\frac{x - z\alpha}{l}\right)^{-1/2}\right]$$

Here $\dot{\alpha} = \omega\bar{\alpha}\cos\omega t$. Now, let us express the downwash for t = 0 $w_{u,l} = -\left[\bar{\alpha}\omega x \pm a\bar{\alpha}\omega\frac{z}{l}\left(\frac{x}{l}\right)^{-1/2} \mp \frac{Ua}{2l}\left(\frac{x}{l}\right)^{-1/2}\right]$. If we divide both sides with U and divide x and z with l the non dimensional form of the downwash expression becomes

$$\frac{w_{u,l}}{U} = -\left[\bar{\alpha}l\omega\frac{x}{Ul} \pm \frac{a}{Ul}\bar{\alpha}l\omega\frac{z}{l}\left(\frac{x}{l}\right)^{-1/2} \mp \frac{a}{2l}\left(\frac{x}{l}\right)^{-1/2}\right].$$

If we write the reduced frequency: $k = k = \frac{\omega l}{U}$, and the nondimensional coordinates $a^* = \frac{a}{l} : x^* = \frac{x}{l}\, ve\, z^* = \frac{z}{l}$, new form of the downwash becomes

$$\frac{w_{u,l}}{U} = -[\bar{\alpha}kx^* \pm a^*\bar{\alpha}kz^*(x^*)^{-1/2} \mp \frac{a^*}{2}(x^*)^{-1/2}].$$

In the last expression, the first two terms are time dependent and the last term is the term due to the steady flow.

Now, we can linearize Eq. 2.15 for the scalar potential with small perturbation approach. The nonlinear terms are the second and third terms in parentheses. The velocity vector in the second term is $\mathbf{q} = \mathrm{U}\mathbf{i} + \nabla\phi' = \mathrm{U}\mathbf{i} + u'\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

$$\frac{\partial q^2}{\partial t} = 2\mathbf{q}.\frac{\partial \mathbf{q}}{\partial t} = 2(U\mathbf{i} + \nabla\phi').\frac{\partial}{\partial t}(U\mathbf{i} + \nabla\phi')$$

If we include the time dependent derivative under the gradient operator we obtain

$$\begin{aligned} 2\mathbf{q}.\frac{\partial \mathbf{q}}{\partial t} &= 2(U\mathbf{i} + u'\mathbf{i} + v\mathbf{j} + w\mathbf{k}).\left(\frac{\partial^2 \phi\prime}{\partial t \partial x}\mathbf{i} + \frac{\partial^2 \phi'}{\partial t \partial y}\mathbf{j} + \frac{\partial^2 \phi'}{\partial t \partial z}\mathbf{k}\right) \\ &= 2(\mathrm{U} + u')\frac{\partial u'}{\partial t} + 2v\frac{\partial v}{\partial t} + 2w\frac{\partial w}{\partial t} \end{aligned}$$

Ignoring the second order perturbation terms, the approximate but linear form of the time derivative of the velocity reads

$$\frac{\partial q^2}{\partial t} \cong 2U\frac{\partial u'}{\partial t} = 2U\frac{\partial^2 \phi'}{\partial t \partial x} \tag{2.22}$$

Now, let us linearize the third term in parentheses

$$\begin{aligned} \mathbf{q}.\nabla\frac{q^2}{2} &= (U\mathbf{i} + \nabla\phi').\nabla\left(\frac{U^2}{2} + U\mathbf{i}.\nabla\phi' + \nabla\frac{\phi'}{2}\right) \\ &= (U + u')\left(U\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{}\right) + v\left(U\frac{\partial u'}{\partial y} + u'\frac{\partial u'}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y}\right) \\ &\quad + w\left(U\frac{\partial u'}{\partial z} + u'\frac{\partial u'}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z}\right) \end{aligned}$$

Neglecting the second and third order terms, the approximate convective term reads

$$\mathbf{q}.\nabla\frac{q^2}{2} \cong U^2\frac{\partial u'}{\partial x} = U^2\frac{\partial^2 \phi'}{\partial x^2} \tag{2.23}$$

Remembering $\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi'}{\partial t^2}$ with the aid of Eqs. 2.22 and 2.23 Eq. 2.15 becomes d

$$\nabla^2\phi' - \frac{1}{a^2}\left(\frac{\partial^2 \phi'}{\partial t^2} + 2U\frac{\partial^2 \phi'}{\partial t \partial x} + U^2\frac{\partial^2 \phi'}{\partial x^2}\right) = 0$$

If we write second term in the form of an operator square we obtain

$$\nabla^2\phi' - \frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \phi' = 0 \tag{2.24}$$

In Eqs. 2.15 and 2.24, one of the non linear quantities is the square of the local speed of sound a^2, which will be linearized next, to give us totally linear potential.

Let us start the linearization with the energy equation, Eq. 2.3 given in (Liepmann and Roshko 1963). The energy equation:

$$\frac{D}{Dt}\left(\frac{a^2}{\gamma - 1} + \frac{q^2}{2}\right) = \frac{1}{\rho}\frac{\partial p}{\partial t}$$

Writing the material derivative at the left hand side of the equation in its approximate form reads

$$\frac{D}{Dt}\left(\frac{a^2}{\gamma - 1} + \frac{q^2}{2}\right) = \left(\frac{\partial}{\partial t} + U\frac{\partial}{}\right)\left(\frac{a^2}{\gamma - 1} + \frac{q^2}{2}\right)$$

If we take the time derivative of the Kelvin's equation, Eq. 2.9, for the integral term we get

$$\frac{\partial}{\partial t}\int \frac{dp}{\rho} = \frac{\partial}{\partial t}\int f(p)\,dp = \frac{\partial F(p)}{\partial t} = \frac{dF(p)}{dp}\frac{\partial p}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial t} = -\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2}\frac{\partial q^2}{\partial t}$$

With the last line the energy equation reads

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{}\right)\left(\frac{a^2}{\gamma - 1} + \frac{q^2}{2}\right) = -\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2}\frac{\partial q^2}{\partial t}$$

Rearranging the equation gives

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{}\right)\left(\frac{a^2}{\gamma - 1}\right) + \frac{\partial^2\phi}{\partial t^2} = -\frac{2}{\partial t} - \frac{U}{2}\frac{\partial q^2}{}$$

If we take the derivative of the right hand side of the last equation we obtain

$$\begin{aligned} -2\mathbf{q}.\frac{\partial \mathbf{q}}{\partial t} - U\mathbf{q}.\frac{\partial \mathbf{q}}{\partial x} &= -2\ (\mathrm{U} + u')\frac{\partial u'}{\partial t} - 2v\frac{\partial v}{\partial t} - 2w\frac{\partial w}{\partial t} \\ &\quad - U(U + u')\frac{\partial u'}{\partial x} - Uv\frac{\partial v}{\partial x} - Uw\frac{\partial w}{\partial x} \\ &\cong -2U\frac{\partial u'}{\partial t} - U^2\frac{\partial u'}{\partial x} \end{aligned}$$

Now, the energy equation reads as

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\frac{a^2}{\gamma - 1}) = (-\frac{\partial}{\partial t} - U\frac{\partial}{\partial x})^2\phi' \qquad (2.24a)$$

Let us denote the perturbation of the local speed of sound as $a = a_\infty + a'$, and multiply the energy equation with $(\gamma - 1)/a_\infty^2$

$$-\frac{\gamma - 1}{a_\infty^2}(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})^2\phi' = (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(1 + \frac{a'}{a_\infty})^2 = (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})[2\frac{a'}{a_\infty} + (\frac{a'}{a_\infty})^2]$$
$$\cong (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})[2\frac{a'}{a_\infty}]$$

Here, $(\frac{a'}{a_\infty})^2 \ll 1$ is assumed. The linearization process has then given

$$\frac{a'}{a_\infty} = -\frac{\gamma - 1}{2a_\infty^2}(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\phi'$$

The presence of speed of sound at the denominator of the right hand side of the last line implies that the perturbation speed of sound is very small compared to the free stream speed of sound. Therefore, it can be neglected near the free stream speed of sound to give approximate value of the local speed of sound as the free stream speed of sound. Hence, the final form of the linearized potential flow equation reads as

$$\nabla^2\phi' - \frac{1}{a_\infty^2}(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})^2\phi' = 0 \qquad (2.24b)$$

2.1.4 Acceleration Potential

Another useful potential function which is used in aerodynamics is the acceleration potential. If we recall the momentum equation for barotropic flows:

$$\frac{D\mathbf{q}}{\mathrm{Dt}} = -\nabla\int\frac{dp}{\rho}$$

As seen in the left hand side of the equation, the material derivative of the velocity vector is obtained from the gradient of a function of pressure and density only. Hence, we can define the acceleration potential as follows

$$\frac{D\mathbf{q}}{\mathrm{Dt}} = \nabla\psi.$$

As a result of last line the momentum equation reads as,

$$\nabla\psi + \nabla\int\frac{dp}{\rho} = 0$$

The integral form of the last equation becomes

$$\psi = -\int\frac{dp}{\rho} + \mathrm{F(t)}$$

The pressure term integrated at the right hand side of the equation from free stream to the point under consideration gives,

$$\psi = \frac{p_\infty - p}{\rho}$$

Because of the direct relation between the pressure and the acceleration potential, this potential is also called the pressure integral. Let us rewrite the Kelvin's equation in gradient form

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{q^2}{2} + \int\frac{dp}{\rho}\right] = 0.$$

We can now find the relation between the velocity potential an the acceleration potential as follows

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{q^2}{2} - \psi\right] = 0.$$

The integral of the last equation

$$\frac{\partial\phi}{\partial t} + \frac{q^2}{2} - \psi = \mathrm{F(t)}$$

Once again if we choose $\mathrm{F(t)} = \mathrm{U}^2/2$ we can satisfy the flow conditions at infinity. Hence, the acceleration potential becomes,

$$\psi = \frac{\partial\phi}{\partial t} + \frac{q^2}{2} - \frac{U^2}{2}$$

With small perturbation approach, the linear form of the last line reads

$$\psi = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi' \tag{2.25}$$

If the linear operator

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})$$

operates on Eq. 2.24b to give

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla^2\phi' - \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi'\right] = 0,$$

Interchanging the operators and utilizing Eq. 2.25 gives us the final form of the equation for the acceleration potential

$$\left[\nabla^2\psi - \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\psi\right] = 0 \tag{2.26}$$

2.1.5 Moving Coordinate System

The linearized equations which are obtained previously enable us to analyze aerodynamical problems more conveniently. Let us now elaborate on the coordinate systems which will further simplify the equations. The type of external flows we study usually considers a constant free stream velocity U at the far field. The reference frame used for this type analysis is a body fixed coordinate system which moves in the negative x direction with velocity U. Another type of analysis is based on the moving reference system which moves with the free stream. With this type analysis, the form of the equations looks simpler to handle. Let us write Eq. 2.24b in the moving coordinate system which moves with the free stream. Let x, y, z be the body fixed coordinate system and, x', y', z' be the flow fixed coordinate system. As seen from Fig. 2.3, since the free stream velocity is U, after the time interval t the flow fixed coordinate system translates in x direction by an amount Ut.

The relation between the two coordinate system reads as

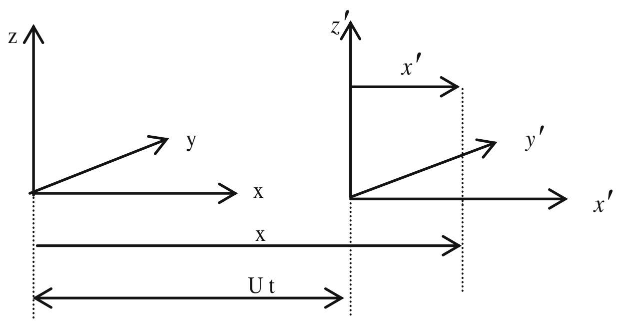

Fig. 2.3 Body fixed x,y,z and the flow fixed x',y',z' coordinate systems

$$x' = x - U\,t, \quad y' = y, \quad z' = z, \quad t' = t.$$

The derivative with respect to t' becomes

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x'}(-U).$$

Here, $\frac{\partial x'}{\partial t} = -U$.

The partial derivatives with respect to body fixed coordinates in terms of the flow fixed coordinates then become:

$$\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} = \frac{\partial}{\partial t'} \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

Equation 2.24b in the flow fixed coordinate system reads as

$$\nabla^2\phi' - \frac{1}{a_\infty^2}\frac{\partial^2\phi'}{\partial t'^2} = 0$$

The last equation is in the form of the classical wave equation whose solutions are well known in mathematical physics. The boundary conditions and the pressure coefficient expressions, Eqs. 2.20 and 2.21, become:

$$\text{Boundary condition:}\, w = \frac{\partial z_a}{\partial t'}$$

$$\text{Pressure coefficient:}\, \mathrm{C_p} = -\frac{2}{U^2}\frac{\partial\phi'}{\partial t'}.$$

2.1.5.1 Summary

Hitherto, we have given the linearized form of the potential equations which are applicable to various problems of classical aerodynamics. In order for these equations to be valid in our modeling, the following assumptions must be true:

1. The air is considered as a perfect gas.
2. Mass, momentum and the energy conservations are used.
3. Body forces, viscous forces and the chemical reactions are ignored.
4. The flowfield is assumed to be either incompressible or barotropic.
5. The slopes of the body surfaces and all the flowfield perturbations are assumed to be small.
6. The time rate of change of the flow parameters are assumed to be small.

In addition, the linearized form of the compressible flow is only valid for subsonic and supersonic flows. The nonlinear approaches for the transonic and the hypersonic flows will be seen separately in relevant chapters.

2.2 Real Gas Flow

The real gas flow equations are free of all the restrictions given above. Therefore, they are first introduced in their weak form, integral form, in terms of the system and control volume approaches.

2.2.1 System and Control Volume Approaches

Let **V**(x,y,z,t) be the velocity vector field given in a stationary space coordinate system x,y,z and time coordinate t. Shown in Fig. 2.4 is the closed system composed of air coalescing with a control volume at time t. The control volume remains the same at time t + Δt the system, however, as the collection of same particles, moves and deforms with the flow as shown in Fig. 2.4.

Let N be the total thermodynamical property in our system. Because of the flow field, there will be a change with time in the property N as DN/Dt. Let η be the specific and local value of property N, which is distributed throughout the control volume. The total value of this property can be represented as an integral as follows: $N = \int \eta \rho \, d\forall$. Here, $d\forall$ shows the infinitesimal volume element in the control volume. Now, we can relate the time rate of change of property η in the control volume in terms of its flux through the control surface as the control volume coincides with the

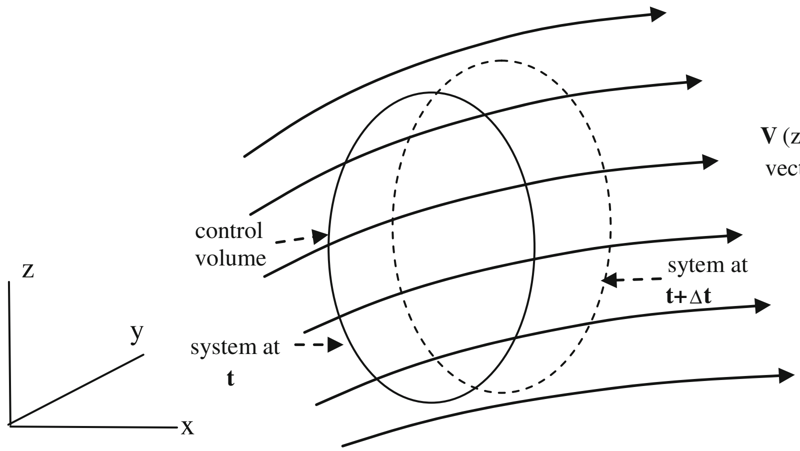

Fig. 2.4 The velocity vector field **V**(x,y,z,t), the system and the control volume

system as Δt approaches zero. Under this condition, the flux of η from the control surface will be $\oint \eta\rho(\vec{V}.d\vec{A})$, (Fox and McDonald 1992). If we consider the limiting case as the system coinciding with the control volume, the total derivative of the property N in the system can be related to the control volume as follows

$$\frac{DN}{Dt} = \frac{\partial}{\partial t}\iiint \eta\rho\, d\forall + \oiint \eta\rho(\vec{V}.d\vec{A}) \tag{2.27}$$

where $\mathbf{V} = \vec{V}$. Now, we can apply the conservation laws of mechanics to Eq. 2.27 and obtain the strong forms of the governing equations.

2.2.2 Global Continuity and the Continuity of the Species

Continuity equation: If M is the total mass in the system then N = M and for the system DN/Dt = DM/Dt = 0. In addition, since η = M/N = 1 Eq. 2.27 reads

$$0 = \frac{\partial}{\partial t}\iiint \rho\, d\forall \; + \; \oiint \rho(\vec{V}.d\vec{A}). \tag{2.28}$$

Using the divergence theorem, the second term at the right hand side of Eq. 2.28 reads as (Hildebrand 1976),

$$\iiint \nabla.(\rho\vec{V})\, d\forall \; = \; \oiint \rho(\vec{V}.d\vec{A}) \tag{2.29}$$

The new form of Eq. 2.28 becomes

$$\frac{\partial}{\partial t}\iiint \rho\, d\forall \; + \; \iiint \; \vec{\nabla}.(\rho\vec{V})\, d\forall \; = \iiint (\frac{\partial\rho}{\partial t} \; + \; \vec{\nabla}.(\rho\vec{V}))d\forall = 0 \tag{2.30}$$

In Eq. 2.30, the control volume does not change with time therefore, the time derivative can be taken into inside of the first term without causing any alteration. Since the volume element d ∀ is arbitrary and different from 0, to satisfy Eq. 2.30 the integrand must be zero to give the differential form, strong form, of the continuity equation.

$$\frac{\partial\rho}{\partial t} + \; \vec{\nabla}.(\rho\vec{V}) \; = 0 \tag{2.31}$$

At high temperatures when the real gas effects take place, the air starts to disassociate and chemical reactions create new species. Because of this, we may need to write continuity of the species for each specie separately. If we consider specie i whose density is ρ_i and its production rate is $\dot{w}_i$ in a control volume, then we have to have a source term at the left hand side of Eq. 2.27.

$$\iiint \dot{w}_i \, d\forall = \frac{\partial}{\partial t} \iiint \rho_i \, d\forall \; + \; \oiint \rho_i (\vec{V}_i . d\vec{A}) \tag{2.32}$$

Here, the velocity $\mathbf{V}_i$ is the mass velocity of specie i. The differential form of Eq. 2.32 reads as

$$\frac{\partial \rho_i}{\partial t} + \vec{\nabla}.(\rho_i \vec{V}_i) \; = \dot{w}_i \tag{2.33}$$

Defining the mass fraction or the concentration of a specie with $c_i = \rho_i / \rho$, the total density then becomes $\rho = \Sigma \, c_i \, \rho_i$. The mass velocity $\mathbf{V}_i$ of a specie in a mixture is related with the global velocity as follows: $\mathbf{V} = \Sigma \, c_i \, \mathbf{V}_i$. A mass velocity of a specie is found with adding its diffusion velocity $\mathbf{U}_i$ to the global velocity $\mathbf{V}$ i.e., $\mathbf{V}_i = \mathbf{V} + \mathbf{U}_i$. According to the Ficks law of diffusion, the diffusion speed of a specie is proportional with its concentration. If we denote the proportionality constant with D_{mi} the diffusion velocity of i reads

$$\vec{U}_i = -c_i D_{mi} \vec{\nabla} c_i \tag{2.34}$$

If we combine Eq. 2.34 with 2.31 and use it in 2.31, we obtain the continuity of the species in terms of their concentrations as follows, (Anderson 1989),

$$\rho \, \frac{D \, c_i}{D \, t} = \vec{\nabla}.(\rho \, D_{mi} \vec{\nabla} c_i) + \dot{w}_i \tag{2.35}$$

2.2.3 Momentum Equation

The Newton's second law of motion, based on the conservation of momentum, is applicable only on the systems. According to this law, the forces acting on the system cause a change in their momentum. For a system which is not under the influence of any non-inertial force, let $\mathbf{F}_S$ be the surface force acting at time t. This surface force changes the $\mathbf{N} = \mathbf{MV}$ momentum of the system. Here, if we let the momentum be independent of mass, then we find for the relevant property $\eta = \mathbf{N}/\mathbf{M} = \mathbf{V}$. We can now write the balance between the surface forces and the corresponding moment changes at the system which coincides with the control volume at time t.

$$\vec{F}_S \; = \frac{\partial}{\partial t} \iiint \rho \, \vec{V} \, d\forall \; + \; \oiint \rho \, \vec{V} (\vec{V} . d\vec{A}) \tag{2.36}$$

The forces at the surface of the system can be considered as the integral effect of the stress tensor $\boldsymbol{\tau}$ over the entire surface of the control volume: $\vec{F}_S = \oint \vec{\tau} . d\vec{A}$. If we use this on the left hand side of Eq. 2.36 and change the surface integrals to volume integrals with the aid of divergence theorem we obtain

$$\iiint \vec{\nabla}.(\vec{\vec{\tau}})\, d\forall \;= \frac{\partial}{\partial t}\iiint \rho\, \vec{V}\, d\forall \;+\; \iiint \vec{\nabla}.(\rho\, \vec{V}\vec{V})\, d\forall \tag{2.37}$$

Here, the double arrow and the velocity vector multiplied by itself indicate the tensor quantities. Equation 2.37 can also be expressed in differential form to give the local expression of the momentum equation as

$$\frac{\partial \rho\, \vec{V}}{\partial t} + \;\vec{\nabla}.(\rho\, \vec{V}\vec{V} - \vec{\vec{\tau}}) = 0 \tag{2.38}$$

In Eq. 2.38, the stress tensor includes in itself the pressure, velocity gradient and for the turbulent flows the Reynolds stresses and reads like

$$\vec{\vec{\tau}} = (-p + \lambda\;\vec{\nabla}.\vec{V})\vec{\vec{I}} + \mu\, sim\vec{V} - <\rho\, \vec{v}'\vec{v}'> \tag{2.39}$$

Here, **I** is the unit tensor and sim**V** is the symmetric part of the gradient of the velocity vector. According to Stoke's hypothesis, the coefficient $\lambda = -2/3\ \mu$, wherein the average viscosity of the species is denoted by μ. Equation 2.38 is valid only for the inertial reference frame. If we include the inertial forces, we consider a control volume in a local non-inertial coordinate system xyz accelerating in a fixed reference frame XYZ. Let the non-inertial coordinate system xyz move with a linear acceleration $\mathbf{R}''$ and rotate with angular speed $\mathbf{\Omega}$ and the angular acceleration $\mathbf{\Omega}'$ in the fixed coordinate system XYZ as shown in Fig. 2.5.

Let the control volume in Fig. 2.5 be attached to the non-inertial frame of reference xyz. The infinitesimal mass element $\rho\, d\forall$ considered in the control volume in the fixed reference frame XYZ has the acceleration $\mathbf{a}_{XYZ}$. At this stage, the relation between the acceleration $\mathbf{a}_{xyz}$ in the non-inertial frame and the acceleration $\mathbf{a}_{XYZ}$ in the inertial frame in terms of linear acceleration: $\mathbf{R}''$, Coriolis **force**: $2\mathbf{\Omega}$ $\mathbf{x}\mathbf{V}_{xyz,}$ centripetal force: $\mathbf{\Omega}$ **x(**$\mathbf{\Omega}$ **xr)** and $\mathbf{\Omega}'$ **xr** reads as given in (Shames 1969)

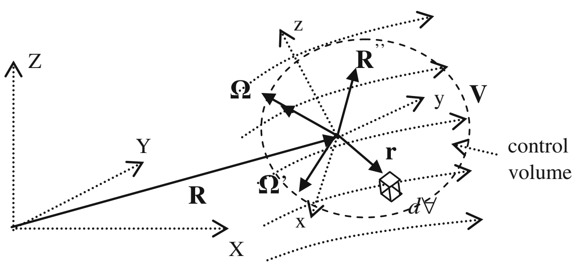

Fig. 2.5 The non-inertial coordinate system xyz in the inertial system XYZ

$$\mathbf{a}_{\mathrm{XYZ}} = \mathbf{a}_{\mathrm{xyz}} + \mathbf{R}'' + 2\boldsymbol{\Omega}\mathbf{x}\mathbf{V}_{\mathrm{xyz}} + \boldsymbol{\Omega}\mathbf{x}(\boldsymbol{\Omega}\mathbf{x}\mathbf{r}) + \boldsymbol{\Omega}'\mathbf{x}\,\mathbf{r} \tag{2.40}$$

Here, $\mathbf{V}_{\mathrm{xyz}}$ is the velocity vector in xyz and $\mathbf{r}$ is the position of the infinitesimal mass $\rho\, d\forall$ in xyz coordinate system. If we write the Newton's second law of motion in the fixed reference frame for the infinitesimal mass at time t using Eq. 2.40 we obtain

$$\mathrm{d}\mathbf{F} = \rho\, d\forall\, \mathbf{a}_{\mathrm{XYZ}} = \rho\, d\forall \left[\mathbf{a}_{\mathrm{xyz}} + \mathbf{R}'' + 2\boldsymbol{\Omega}\mathbf{x}\mathbf{V}_{\mathrm{xyz}} + \boldsymbol{\Omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r}) + \boldsymbol{\Omega}'\mathbf{x}\,\mathbf{r}\right] \tag{2.41}$$

Equation 2.41 can be written for the acceleration in the non-inertial reference frame in terms of the inertial forces

$$\rho d\forall\, \mathbf{a}_{\mathrm{XYZ}} = \mathrm{d}\mathbf{F} - \rho d\forall \left[\mathbf{R}'' + 2\Omega\, \mathbf{x}\mathbf{V}_{\mathrm{xyz}} + \Omega\, \mathbf{x}(\Omega\mathbf{x}\mathbf{r}) + \Omega'\, \mathbf{x}\, \mathbf{r}\right] \tag{2.41a}$$

We know that $\mathbf{F} = \int \mathrm{d}\mathbf{F}$. As the new form of the momentum equation expressed in the non-inertial reference frame xyz we obtain

$$\vec{F}_S - \iiint \left[\vec{R}'' + 2\,\vec{\Omega}\,x\vec{V} + \vec{\Omega}\,x(\vec{\Omega}\,x\,\vec{r}) + \vec{\Omega}'x\,r\right]\rho\, d\forall = \frac{\partial}{\partial t}\iiint \rho\vec{V}\, d\forall + \oiint \rho\vec{V}\,(\vec{V}.dA) \tag{2.42}$$

If we consider the surface forces expressed in terms of stress tensor the differential form of the momentum equation becomes

$$\frac{\partial\rho\,\vec{V}}{\partial t} + \vec{\nabla}.(\rho\,\vec{V}\vec{V} - \vec{\tau}) = -\rho\left[(\vec{R}'' + 2\,\vec{\Omega}\,x\vec{V} + \vec{\Omega}\,x(\vec{\Omega}\,x\,\vec{r}) + \vec{\Omega}'x\,r\right] \tag{2.43}$$

Equation 2.43, can be used, in general, for studying the pitching and heaving-plunging airfoils and finite wings in roll and viscous analysis for drag prediction of fuselages.

2.2.4 Energy Equation

The conservation of energy can be formulated with applying the first law of thermodynamics on systems. The system here is in the flow field and receives the heat rate of $\dot{Q}$. If the work done by the system to the surroundings is $\dot{W}$ then the change of total energy in the system becomes

$$\frac{DE}{Dt} = \dot{Q} - \dot{W} \tag{2.44}$$

At a given time t, let the system under consideration coincide with the control volume we choose. If we let E_i denote the internal energy and $E_k = ½MV^2$ the kinetic energy of the total mass in the system, then as the mass independent transferable quantities the specific internal energy becomes $e = E_i/M$ and the specific kinetic energy reads as $E_k/M = ½V^2$. Which means the total specific energy in the control volume is $\eta = e + ½\ V^2$. Now, we can relate the energy changes of the system and the control volume using Eq. 2.44 in Eq. 2.27 to obtain the integral form of the energy equation

$$\dot{Q} - \dot{W} = \frac{\partial}{\partial t}\iiint (e + V^2/2)\rho\, d\forall + \oiint (e + V^2/2)\rho\, \vec{V}.d\vec{A} \tag{2.45}$$

During the flow if we do not provide heat from outside, the system will heat the surroundings by the flux of internal heat from the control surface as follows $\dot{Q} = -\oint \vec{q}.\,d\vec{A}$. On the other hand, the work of the stress tensor throughout the whole control surface will become $\dot{W} = -\oint (\vec{V}.\vec{\vec{\tau}}).d\vec{A}$. Now, if we substitute the integral forms of the heat flux to the surroundings and the work done by the system on the surrounding, Eq. 2.45 becomes

$$-\oiint \vec{q}\,.\,d\vec{A} + \oiint (\vec{V}.\vec{\vec{\tau}}).d\vec{A} = \frac{\partial}{\partial t}\iiint (e + V^2/2)\rho\, d\forall + \oiint (e + V^2/2)\rho\, \vec{V}.d\vec{A} \tag{2.46}$$

In Eq. 2.46 we have three surface integral terms. If all three area integrals are changed to volume integrals using the divergence theorem, and all the all volume integrals are collected together over the same control volume, we can write the differential form of the energy equation as follows

$$\frac{\partial(\rho\,\varepsilon)}{\partial t} + \ \vec{\nabla}.(\rho\,\varepsilon\vec{V} - \vec{V}.\vec{\vec{\tau}} + \vec{q}) = 0 \tag{2.47}$$

Here, $\varepsilon = e + ½V^2$ denotes the specific total energy and Eq. 2.39 defines the stress tensor. The heat flux from a unit surface area reads as

$$\vec{q} = -k\ \vec{\nabla}T + \sum_{i} \rho_i \vec{U}_i h_i + \vec{q}_R + < \varepsilon'\, \vec{v}' > \tag{2.48}$$

Wherein, k denotes the heat conduction coefficient, the second term indicates the heat of diffusion, the third term represents radiative heat flux and the last term shows the turbulence heating. In summary, the global continuity is given by Eq. 2.31, continuity of species by 2.35, global momentum by 2.38 and the energy Equation by 2.47. Let us express these equations in Cartesian coordinates in conservative forms.

2.2.5 Equation of Motion in General Coordinates

Continuum equations of motion written in vector form are suitable for implementing the numerical solution of aerodynamical problems. In these equations the unknown vector **U** the flux vectors **F**, **G** and **H,** and the right hand side vector **R** are written as follows

$$\vec{U} = \begin{pmatrix} \rho \\ \rho\, u \\ \rho\, v \\ \rho\, w \\ \rho\, \varepsilon \\ \rho\, c_i \end{pmatrix}, \vec{F} = \begin{pmatrix} \rho\, u \\ \rho\, uu + \tau_{xx} \\ \rho\, uv + \tau_{yx} \\ \rho\, uw + \tau_{zx} \\ \rho\, u\varepsilon + q_x + u\tau_{xx} + v\tau_{xy} + w\tau_{xz} \\ \rho\, u\, c_i + D_{mi} \frac{\partial c_i}{\partial x} \end{pmatrix}, \vec{G} = \begin{pmatrix} \rho\, v \\ \rho\, uv + \tau_{xy} \\ \rho\, vv + \tau_{yy} \\ \rho\, vw + \tau_{zy} \\ \rho\, v\varepsilon + q_y + u\tau_{yx} + v\tau_{yy} + w\tau_{yz} \\ \rho\, v\, c_i + D_{mi} \frac{\partial c_i}{\partial y} \end{pmatrix}$$

$$\vec{H} = \begin{pmatrix} \rho\, w \\ \rho\, uw + \tau_{xz} \\ \rho\, vw + \tau_{yz} \\ \rho\, ww + \tau_{zz} \\ \rho\, w\varepsilon + q_z + u\tau_{zx} + v\tau_{zy} + w\tau_{zz} \\ \rho\, v\, c_i + D_{mi} \frac{\partial c_i}{\partial z} \end{pmatrix}, \vec{R} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{w}_i \end{pmatrix}$$

Here, τ_{xx}, τ_{xy}, …, τ_{zz} are the components of the stress tensor and q_x, q_y and q_z are the components of the heat flux vector. Now, we can write the equation of motion in compact form as follows

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial y} + \frac{\partial \vec{H}}{\partial z} = \vec{R} \tag{2.49}$$

In many aerospace applications the Cartesian coordinates are not adequate to represent the surface equations of the body on which the boundary conditions are imposed. For this reason we have to write the equation of motion in body fitted coordinates which are generally referred as the generalized coordinates. Let the transformation from Cartesian coordinates xyz to the generalized coordinates $\xi\eta\varsigma$ be given as

$$x = x(\xi, \eta, \varsigma), \quad y = y(\xi, \eta, \varsigma), \quad z = z(\xi, \eta, \varsigma)$$

With this information in hand, Eq. 2.49 is written in generalized coordinates in terms of the product of flux vectors with the metrics of transformation as follows, (Anderson et al. 1984).

$$\frac{\partial \vec{U}}{\partial t} + \left(\frac{\partial \xi}{\partial x}\,\frac{\partial \xi}{\partial y}\,\frac{\partial \xi}{\partial z}\right)\begin{pmatrix} \frac{\partial \vec{F}}{\partial \xi} \\ \frac{\partial \vec{G}}{\partial \xi} \\ \frac{\partial \vec{H}}{\partial \xi} \end{pmatrix} + \left(\frac{\partial \eta}{\partial x}\,\frac{\partial \eta}{\partial y}\,\frac{\partial \eta}{\partial z}\right)\begin{pmatrix} \frac{\partial \vec{F}}{\partial \eta} \\ \frac{\partial \vec{G}}{\partial \eta} \\ \frac{\partial \vec{H}}{\partial \eta} \end{pmatrix} + \left(\frac{\partial \varsigma}{\partial x}\,\frac{\partial \varsigma}{\partial y}\,\frac{\partial \varsigma}{\partial z}\right)\begin{pmatrix} \frac{\partial \vec{F}}{\partial \varsigma} \\ \frac{\partial \vec{G}}{\partial \varsigma} \\ \frac{\partial \vec{H}}{\partial \varsigma} \end{pmatrix} = \vec{R} \tag{2.50}$$

Shown in Fig. 2.6a, b are two different external flow regions: (a) wing upper surface and the boundaries of its computational domain, and (b) half a fuselage and the computational domain transformed from xyz, Cartesian coordinates to $\xi\eta\zeta$, generalized coordinate system. Both flow domains, after the transformation in $\xi\eta\zeta$ coordinate system, are mapped into the cube denoted by 12345678 for which the discretization of the computational domain becomes straight forward.

In Fig. 2.6, the $\xi\eta$ surfaces of physical domain transforms into the square denoted with 1234, wherein, ζ coordinate of the physical domain is inclined with the body surface, i.e. it is not necessarily normal to the surface. After knowing one to one correspondence of the discrete points of both domains, we can numerically calculate the derivative terms for ξ_x, ξ_y, …, ζ_z to be used for solving Eq. 2.50 in the discretized cube 12345678. There are quite a few numbers of literature published about the mesh generation and coordinate transformation techniques, however, two separate works by Anderson and Hoffman can be recommended for beginners and the intermediate level users, (Anderson et al. 1984) and (Hoffman 1992).

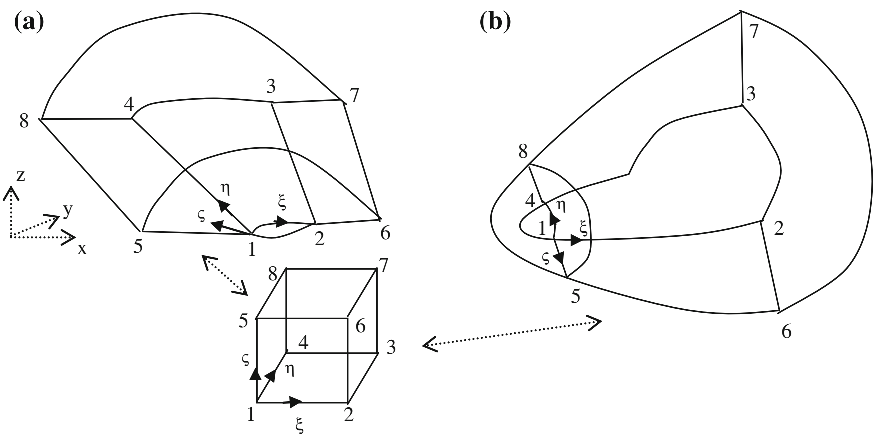

Fig. 2.6 The coordinate transformation (**a**) the wing, (**b**) the fuselage: ξ-η: surface coordinates, ς: the coordinate which is inclined with the surface

2.2.6 Navier-Stokes Equations

In its most general form, including the chemical reactions at high temperatures, Eq. 2.49 was introduced as the set of equations for external flows. Global continuity equation and the conservation of momentum equations deal with the average values of flow parameters, therefore they are of mechanical nature, whereas the energy equation deals with the effect of heating as well the enthalpy increase caused by the diffusion of species. If we do not consider the chemical reactions, then there will not be diffusion terms present and the related specie conservation terms disappear. Therefore, Eq. 2.49 reduces to the well known Navier-Stokes Equations, (Schlichting 1968). Since the Navier-Stokes equations can model all laminar and turbulent flows, they have a wide range of their implementation in aerodynamical applications. For the case of turbulent flows, we have to include the effective viscosity μ_T into the constitutive relations to model the Reynolds stresses. Now, we can re-write the constitutive relation 2.39 and the heat flux term 2.48 with the turbulent Prandtl number P_{rT} as follows

$$\begin{aligned} \vec{\vec{\tau}} &= (-p + \lambda \vec{\nabla} . \vec{V}) \vec{\vec{I}} + (\mu + \mu_T)\, sim\vec{V}, \\ \vec{q} &= -(k + c_p \mu_T / P_{tT}) \;\; \vec{\nabla} T + \sum_i \rho_i \vec{U}_i h_i + \vec{q}_R \end{aligned} \tag{2.51a, b}$$

Let us separate the molecular viscosity and the heat transfer terms to rearrange Eq. 2.49 for chemically non-reacting flows to give the new right hand side vectors

$$\vec{S}_1 = \begin{pmatrix} 0 \\ 2\mu(\frac{\partial u}{\partial x}) - \frac{2}{3}\mu \;\; \vec{\nabla} . \vec{V} \\ \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\ k\frac{\partial T}{\partial x} + u\tau'_{xx} + v\tau'_{xy} + w\tau'_{xz} \end{pmatrix}, \vec{S}_2 = \begin{pmatrix} 0 \\ \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ 2\mu(\frac{\partial v}{\partial y}) - \frac{2}{3}\mu \;\; \vec{\nabla} . \vec{V} \\ \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \\ k\frac{\partial T}{\partial x} + u\tau'_{xy} + v\tau'_{yy} + w\tau'_{yz} \end{pmatrix},$$

$$\vec{S}_1 = \begin{pmatrix} 0 \\ \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\ \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \\ 2\mu(\frac{\partial w}{\partial z}) - \frac{2}{3}\mu \;\; \vec{\nabla} . \vec{V} \\ k\frac{\partial T}{\partial x} + u\tau'_{xz} + v\tau'_{yz} + w\tau'_{zz} \end{pmatrix}$$

and to obtain the in final form of the equations

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}_1}{\partial x} + \frac{\partial \vec{G}_1}{\partial y} + \frac{\partial \vec{H}_1}{\partial z} = \frac{\partial \vec{S}_1}{\partial x} + \frac{\partial \vec{S}_2}{\partial y} + \frac{\partial \vec{S}_3}{\partial z} \tag{2.52}$$

Here, $\vec{\vec{\tau}}' = \vec{\vec{\tau}} - p\vec{\vec{I}}$ is the pressure free stress tensor, $\mathbf{F}_1$, $\mathbf{G}_1$ and $\mathbf{H}_1$ are the flux terms which are free of viscous effects. That is if we let the right hand side of Eq. 2.52 be zero, we obtain the Euler equations which are already given by Eqs. 2.1–2.3.

The non-dimensional form of the Navier-Stokes equations are usually more convenient to apply to problems of aerodynamics. For this purpose, we use characteristic parameters of the flow. The free stream values for the density, speed, pressure, viscosity, conductivity and the temperature which are $\rho_\infty,\ V_\infty, p_\infty, \mu_\infty,\ k_\infty$ *and* T_∞ respectively. The corresponding non dimensional quantities become

$$\begin{array}{ccccc} \hat{\rho} = \rho/\rho_\infty & \hat{p} = p/p_\infty & \hat{e} = e/V_\infty^2 & \hat{\mu} = \mu/\mu_\infty & \hat{k} = k/k_\infty \\ \hat{T} = T/T_\infty & \hat{t} = t\,V_\infty/c & \hat{x} = x/c & \hat{y} = y/c & \hat{z} = z/c \end{array}$$

The non dimensional form of the Navier Stokes equations reads as

$$\frac{\partial \widehat{U}}{\partial t} + \frac{\partial \widehat{F}_1}{\partial x} + \frac{\partial \widehat{G}_1}{\partial y} + \frac{\partial \widehat{H}_1}{\partial z} = \widehat{S}_1 + \widehat{S}_2 + \widehat{S}_3 \tag{2.53}$$

The non dimensional quantities in Eq. 2.53

$$\widehat{U} = \begin{pmatrix} \hat{\rho} \\ \hat{\rho}\,\hat{u} \\ \hat{\rho}\,\hat{v} \\ \hat{\rho}\,\hat{w} \\ \hat{\rho}\,\hat{\varepsilon} \end{pmatrix}, \quad \widehat{F}_1 = \begin{pmatrix} \hat{\rho}\,\hat{u} \\ \hat{\rho}\,\hat{u}\hat{u} + \hat{p} \\ \hat{\rho}\,\hat{u}\hat{v} \\ \hat{\rho}\,\hat{u}\overleftarrow{w} \\ (\hat{\rho}\,\hat{\varepsilon} + \hat{p})\overleftarrow{u} \end{pmatrix}, \quad \widehat{G}_1 = \begin{pmatrix} \hat{\rho}\,\hat{v} \\ \hat{\rho}\,\hat{u}\hat{v} \\ \hat{\rho}\,\hat{v}\hat{v} + \hat{p} \\ \hat{\rho}\,\hat{v}\overleftarrow{w} \\ (\hat{\rho}\,\hat{\varepsilon} + \hat{p})\hat{v} \end{pmatrix},$$
$$\widehat{H}_1 = \begin{pmatrix} \hat{\rho}\,\hat{w} \\ \hat{\rho}\,\hat{u}\hat{w} \\ \hat{\rho}\,\hat{v}\hat{w} \\ \hat{\rho}\,\hat{w}\overleftarrow{w} + \hat{p} \\ (\hat{\rho}\,\hat{\varepsilon} + \hat{p})\hat{w} \end{pmatrix}$$

Here, the total non dimensional specific energy is $\hat{\varepsilon} = \hat{e} + (\hat{u}^2 + \hat{v}^2 + \hat{w}^2)/2$. The viscous terms on the other hand becomes

$$\widehat{S}_1 = \begin{pmatrix} 0 \\ \hat{\tau}_{xx} \\ \hat{\tau}_{xy} \\ \hat{\tau}_{xz} \\ \hat{u}\hat{\tau}_{xx} + \hat{v}\hat{\tau}_{xy} + \hat{w}\hat{\tau}_{xz} - \hat{q}_x \end{pmatrix}, \quad \widehat{S}_2 = \begin{pmatrix} 0 \\ \hat{\tau}_{xz} \\ \hat{\tau}_{yy} \\ \hat{\tau}_{yz} \\ \hat{u}\hat{\tau}_{xy} + \hat{v}\hat{\tau}_{yy} + \hat{w}\hat{\tau}_{yz} - \hat{q}_y \end{pmatrix},$$
$$\widehat{S}_3 = \begin{pmatrix} 0 \\ \hat{\tau}_{xz} \\ \hat{\tau}_{yz} \\ \hat{\tau}_{zz} \\ \hat{u}\hat{\tau}_{xz} + \hat{v}\hat{\tau}_{yz} + \hat{w}\hat{\tau}_{zz} - \hat{q}_z \end{pmatrix}$$

The open form of these viscous terms in terms of velocity components reads

$$\hat{\tau}_{xx} = \frac{\hat{\mu}}{R_e}\left[2\left(\frac{\partial \hat{u}}{\partial \hat{x}}\right) - \frac{2}{3}\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial z}\right)\right], \quad \hat{\tau}_{xy} = \frac{\hat{\mu}}{R_e}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\hat{\tau}_{yy} = \frac{\hat{\mu}}{R_e}\left[2\left(\frac{\partial \hat{v}}{\partial \hat{y}}\right) - \frac{2}{3}\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial z}\right)\right], \quad \hat{\tau}_{xz} = \frac{\hat{\mu}}{R_e}\left(\frac{\partial \hat{u}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{x}}\right)$$
$$\hat{\tau}_{zz} = \frac{\hat{\mu}}{R_e}\left[2\left(\frac{\partial \hat{w}}{\partial \hat{z}}\right) - \frac{2}{3}\left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial z}\right)\right], \quad \hat{\tau}_{yz} = \frac{\hat{\mu}}{R_e}\left(\frac{\partial \hat{v}}{\partial \hat{z}} + \frac{\partial \hat{w}}{\partial \hat{y}}\right)$$

Heat conduction terms become

$$\hat{q}_x = -\frac{\hat{\mu}}{(\gamma-1)M_\infty^2 R_e P_r}\frac{\partial \widehat{T}}{\partial \hat{x}}, \quad \hat{q}_y = -\frac{\hat{\mu}}{(\gamma-1)M_\infty^2 R_e P_r}\frac{\partial \hat{T}}{\partial \hat{y}},$$
$$\hat{q}_z = -\frac{\hat{\mu}}{(\gamma-1)M_\infty^2 R_e P_r}\frac{\partial \widehat{T}}{\partial \hat{z}}$$

The non dimensional similarity parameters appearing in the equations are well known Reynolds, Mach and Prandtl numbers which are defined with their physical meanings attached as follows

Reynolds number: $R_e = \rho_\infty V_\infty c/\mu_\infty$, (inertia forces / viscous forces)
Mach number: $M_\infty = V_\infty/a_\infty$, (kinetic energy of the flow / internal energy)
Prandtl number: $P_r = c_{p\infty}\mu_\infty/k_\infty$, (energy dissipation / heat conduction).

From the perfect gas assumption: $\hat{p} = (\gamma - 1)\hat{\rho}\,\hat{e}$ and $\hat{T} = \gamma M_\infty^2 \hat{p}/\hat{\rho}$ relations among the non dimensional parameters are obtained.

In most of the aerodynamics applications there is high free stream speed involved. For the classical applications usually unseparated flows are considered. Regardless of flow being attached or separated, for the flows with high free stream speeds we can apply some approximations to Eq. 2.53 to obtain simpler solutions. Let us now, see this approximations and conditions for their applicability.

2.2.7 *Thin Shear Layer Navier-Stokes Equations*

In the open form of Navier-Stokes Eqs. (2.53), we observe the existence of second derivatives for the velocity and the temperature. This implies that the Navier-Stokes equations are second order partial differential equations. When the freestream speed is high, the Reynolds number is high. This makes the gradients of the flow parameters to be high normal to the surface as compared to the gradients parallel to the surface. Therefore, we can neglect the effect of the viscous terms which are parallel to the flow surface and simplify Eqs. 2.53. Let us now, perform some order

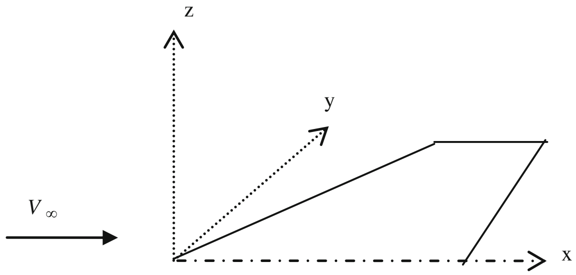

Fig. 2.7 Thin wing in a high freestream speed

of magnitude analysis for the simplification process on a simple wing surface immersed in a high free stream speed given in Fig. 2.7.

Since we consider the air flowing over the wing as a real gas, the boundary conditions on the surface will be (i) no slip condition and (ii) the wall temperature specification. According to Fig. 2.7, the wing surface is almost parallel to xy plane where the molecular diffusion parallel to the xy plane is negligible compared to the diffusion taking place normal to the surface. This is because of high free stream speed transporting the properties in the parallel direction much faster than the molecular diffusion. On the other hand, because of no slip condition, the gradients which are normal to the surface are much higher than the gradients parallel to the surface. The order of magnitude analysis performed on the terms of Eq. 2.53 gives

$$\frac{1}{R_e}\frac{\partial\hat{\mu}}{\partial\hat{z}}\left(\frac{\partial}{\partial\hat{z}}\right) \gg \frac{1}{R_e}\frac{\partial\hat{\mu}}{\partial\hat{x}}\left(\frac{\partial}{\partial\hat{x}}\right),\ldots,\frac{1}{R_e}\frac{\partial\hat{\mu}}{\partial\hat{y}}\left(\frac{\partial}{\partial\hat{y}}\right).$$

The approximate form of the equations result in modeling an external real gas flow which takes place in a thin shear layer around the wing surface. Therefore, the first approximate form of Eq. 2.53 is called 'Thin Shear Layer Navier-Stokes Equations' which are to be introduced next

$$\frac{\partial}{\partial\hat{t}}\begin{pmatrix}\hat{\rho}\\ \hat{\rho}\,\hat{u}\\ \hat{\rho}\,\hat{v}\\ \hat{\rho}\,\hat{w}\\ \hat{\rho}\,\hat{\varepsilon}\end{pmatrix}+\frac{\partial}{\partial\hat{x}}\begin{pmatrix}\hat{\rho}\,\hat{u}\\ \hat{\rho}\,\hat{u}\hat{u}+\hat{p}\\ \hat{\rho}\,\hat{u}\hat{v}\\ \hat{\rho}\,\hat{u}\overleftarrow{w}\\ (\hat{\rho}\,\hat{\varepsilon}+\hat{p})\overleftarrow{u}\end{pmatrix}+\frac{\partial}{\partial\hat{y}}\begin{pmatrix}\hat{\rho}\,\hat{v}\\ \hat{\rho}\,\hat{u}\hat{v}\\ \hat{\rho}\,\hat{v}\hat{v}+\hat{p}\\ \hat{\rho}\,\hat{v}\overleftarrow{w}\\ (\hat{\rho}\,\hat{\varepsilon}+\hat{p})\hat{v}\end{pmatrix}+\frac{\partial}{\partial\hat{z}}\begin{pmatrix}\hat{\rho}\,\hat{w}\\ \hat{\rho}\,\hat{u}\hat{w}\\ \hat{\rho}\,\hat{v}\hat{w}\\ \hat{\rho}\,\hat{w}\overleftarrow{w}+\hat{p}\\ (\hat{\rho}\,\hat{\varepsilon}+\hat{p})\hat{w}\end{pmatrix}$$

$$=\frac{1}{R_e}\frac{\partial}{\partial\hat{z}}\begin{pmatrix}0\\ \hat{\mu}\frac{\partial\hat{u}}{\partial\hat{z}}\\ \hat{\mu}\frac{\partial\hat{v}}{\partial\hat{z}}\\ \frac{4}{3}\hat{\mu}\frac{\partial\hat{w}}{\partial z}\\ \hat{\mu}(\ddot{u}\frac{\partial\hat{u}}{\partial\hat{z}}+\hat{v}\frac{\partial\hat{v}}{\partial\hat{y}}+\frac{4}{3}\frac{\partial\hat{w}}{\partial\hat{z}})+\frac{\hat{\mu}}{(\gamma-1)M_\infty^2P_r}\frac{\partial\hat{T}}{\partial\hat{z}}\end{pmatrix} \tag{2.54}$$

Equations 2.54 are written in Cartesian coordinates without considering the wing thickness effect. If we consider the thickness effect and high angles of attack, Eqs. 2.54 can be written in ξηζ coordinates where only the viscous terms in ζ coordinate, which is normal to the wing surface are retained. With these assumptions and with the additional assumption that the general coordinate system changes with time, the transformation of coordinates from Cartesian to generalized reads

$$\xi = \xi(x,y,z,t), \quad \eta = \eta(x,y,z,t), \quad \zeta = \zeta(x,y,z,t), \quad \tau = t \tag{2.55}$$

Using 2.55, we can write the open form of the non-dimensional Thin Shear Layer Navier-Stokes equations in generalized coordinates where ς is the direction normal to the wing surface

$$\frac{\partial}{\partial\tau}\frac{1}{J}\begin{pmatrix}\hat\rho\\ \hat\rho\,\hat u\\ \hat\rho\,\hat v\\ \hat\rho\,\hat w\\ \hat\rho\,\hat\varepsilon\end{pmatrix} + \frac{\partial}{\partial\xi}\frac{1}{J}\begin{pmatrix}\hat\rho\,U\\ \hat\rho\,\hat uU+\xi_x\hat p\\ \hat\rho\,\hat vU+\xi_y\hat p\\ \hat\rho\,\hat wU+\xi_z\hat p\\ (\hat\rho\,\hat\varepsilon+\hat p)U-\xi_t\hat p\end{pmatrix} + \frac{\partial}{\partial\eta}\frac{1}{J}\begin{pmatrix}\hat\rho\,V\\ \hat\rho\,\hat uV+\eta_x\hat p\\ \hat\rho\,\hat vV+\eta_y\hat p\\ \hat\rho\,\hat wV+\eta_z\hat p\\ (\hat\rho\,\hat\varepsilon+\hat p)V-\xi_t\hat p\end{pmatrix} + \frac{\partial}{\partial\varsigma}\frac{1}{J}\begin{pmatrix}\hat\rho\,W\\ \hat\rho\,\hat uW+\varsigma_x\hat p\\ \hat\rho\,\hat vW+\varsigma_y\hat p\\ \hat\rho\,\hat wW+\varsigma_z\hat p\\ (\hat\rho\,\hat\varepsilon+\hat p)W-\varsigma_t\hat p\end{pmatrix} = \frac{1}{R_e}\frac{\partial S}{\partial\varsigma} \tag{2.56}$$

Here, $J = \frac{\partial(\xi,\eta,\varsigma,\tau)}{\partial(x,y,z,t)}$ is the Jakobian determinant of the transformation, U, V and W are the contravariant velocity components which are normal to the curvilinear surfaces given with constant ξ, η and ς coordinates respectively. They read

$$U = \xi_t + \xi_x\hat u + \xi_y\hat v + \xi_z\hat w,\ V = \eta_t + \eta_x\hat u + \eta_y\hat v + \eta_z\hat w,\ W = \varsigma_t + \varsigma_x\hat u + \varsigma_y\hat v + \varsigma_z\hat w \tag{2.57}$$

The viscous terms at the right hand side of Eq. 2.56 become

$$\hat S = \begin{pmatrix} 0\\ \hat\mu(\varsigma_x^2+\varsigma_y^2+\varsigma_z^2)\hat u_\varsigma + \frac{\hat\mu}{3}(\varsigma_x\hat u_\varsigma+\varsigma_y\hat v_\varsigma+\varsigma_z\hat w_\varsigma)\eta_x\\ \hat\mu(\varsigma_x^2+\varsigma_y^2+\varsigma_z^2)\hat v_\varsigma + \frac{\hat\mu}{3}(\varsigma_x\hat u_\varsigma+\varsigma_y\hat v_\varsigma+\varsigma_z\hat w_\varsigma)\eta_y\\ \hat\mu(\varsigma_x^2+\varsigma_y^2+\varsigma_z^2)\hat w_\varsigma + \frac{\hat\mu}{3}(\varsigma_x\hat u_\varsigma+\varsigma_y\hat v_\varsigma+\varsigma_z\hat w_\varsigma)\eta_z\\ \hat\mu(\varsigma_x^2+\varsigma_y^2+\varsigma_z^2)\left[\frac{1}{2}(\hat u^2+\hat v^2+\hat w^2)_\varsigma + \frac{1}{(\gamma-1)M_\infty^2P_r}\hat T_\varsigma\right] + \frac{\hat\mu}{3}(\varsigma_x\hat u+\varsigma_y\hat v+\varsigma_z\hat w)(\varsigma_x\hat u_\varsigma+\varsigma_y\hat v_\varsigma+\varsigma_z\hat w_\varsigma)\end{pmatrix}$$

The convective terms in Eq. 2.56 contain the Jacobian determinant in the denominator. This form of the equations are called 'strong conservative forms' and their derivations are provided in Appendix 1.

2.2.8 Parabolized Navier-Stokes Equations

In numerous aerospace applications we encounter the steady flow cases for which the time dependent terms of the equations are discarded. The thin shear layer equations written for steady flows without time dependent terms are called 'Parabolized Navier-Stokes Equations', (Anderson 1989). According to this definition, from Eqs. 2.54 we write the parabolized Navier-Stokes equations in Cartesian coordinates as follows

$$\frac{\partial}{\partial \hat{x}}\begin{pmatrix} \hat{\rho}\,\hat{u} \\ \hat{\rho}\,\hat{u}\hat{u}+\hat{p} \\ \hat{\rho}\,\hat{u}\hat{v} \\ \hat{\rho}\,\hat{u}\overleftarrow{w} \\ (\hat{\rho}\,\hat{\varepsilon}+\hat{p})\overleftarrow{u} \end{pmatrix} + \frac{\partial}{\partial \hat{y}}\begin{pmatrix} \hat{\rho}\,\hat{v} \\ \hat{\rho}\,\hat{u}\hat{v} \\ \hat{\rho}\,\hat{v}\hat{v}+\hat{p} \\ \hat{\rho}\,\hat{v}\overleftarrow{w} \\ (\hat{\rho}\,\hat{\varepsilon}+\hat{p})\hat{v} \end{pmatrix} + \frac{\partial}{\partial \hat{z}}\begin{pmatrix} \hat{\rho}\,\hat{w} \\ \hat{\rho}\,\hat{u}\hat{w} \\ \hat{\rho}\,\hat{v}\hat{w} \\ \hat{\rho}\,\hat{w}\overleftarrow{w}+\hat{p} \\ (\hat{\rho}\,\hat{\varepsilon}+\hat{p})\hat{w} \end{pmatrix}$$

$$= \frac{1}{R_e}\frac{\partial}{\partial \hat{z}}\begin{pmatrix} 0 \\ \hat{\mu}\frac{\partial \hat{u}}{\partial \hat{z}} \\ \hat{\mu}\frac{\partial \hat{v}}{\partial \hat{z}} \\ \frac{4}{3}\hat{\mu}\frac{\partial \hat{w}}{\partial z} \\ \hat{\mu}(\ddot{u}\frac{\partial \hat{u}}{\partial \hat{z}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} + \frac{4}{3}\frac{\partial \hat{w}}{\partial \hat{z}}) + \frac{\hat{\mu}}{(\gamma-1)M_\infty^2 P_r}\frac{\partial \hat{T}}{\partial \hat{z}} \end{pmatrix}$$

In curvilinear coordinates, we neglect the $\partial(\,)/\partial t$ terms as well as the time dependency of ξ, η and ζ coordinates. Thus, we obtain the parabolized Navier-Stokes equations in curvilinear coordinates. In addition if we can, somehow, impose the pressure from the outside of shear layer then we obtain the well known boundary layer equations.

2.2.9 Boundary Layer Equations

In the attached or slightly detached external flow cases, we can obtain the surface pressure distribution using the methods described in Sect. 2.1 and further simplify set of Eqs. 2.49 and 2.54. In these simplifications we again resort to the order of magnitude analysis. Assuming again that the viscous effects are only in the vicinity of the surface of the body, we can consider the gradients and the diffusion normal to the surface we obtain

$$\text{Continuity}: \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0 \tag{2.58}$$

$$\text{Continuity of the species}: \rho\frac{\partial c_i}{\partial t} + \rho u\frac{\partial c_i}{\partial x} + \rho w\frac{\partial c_i}{\partial z} = \frac{\partial}{\partial z}(\rho D_{12}\frac{\partial c_i}{\partial z}) + \dot{w}_i \tag{2.59}$$

$$\text{x}-\text{momentum}: \rho\frac{\partial \text{u}}{\partial \text{t}} + \rho \text{u}\frac{\partial \text{u}}{\partial \text{x}} + \rho \text{w}\frac{\partial \text{u}}{\partial \text{z}} = -\frac{\partial \text{p}}{\partial \text{x}} + \frac{\partial}{\partial \text{z}}(\mu\frac{\partial \text{u}}{\partial \text{z}}) \tag{2.60}$$

$$\text{z-momentum} : \frac{\partial \mathrm{p}}{\partial \mathrm{z}} = 0 \tag{2.61}$$

$$\begin{aligned}\text{Energy} : \ &\rho \frac{\partial h}{\partial t} + \rho u \frac{\partial h}{\partial x} + \rho w \frac{\partial h}{\partial z} \\ &= \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \mu \left(\frac{\partial u}{\partial z}\right)^2 + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial z}\left(\rho D_{12} \sum_i h_i \frac{\partial c_i}{\partial z}\right)\end{aligned} \tag{2.62}$$

Here, x is the direction parallel to the surface, z is the normal direction and h_i in Eq. 2.62 is the enthalpy of species i.

The real gas effect in an external flow can be measured with the change caused in the stagnation enthalpy. If we neglect the effect of vertical velocity component, the stagnation enthalpy of the boundary layer flow reads: $h_o = h + u^2/2$. The normal gradient of the stagnation enthalpy at a point then reads

$$\frac{\partial h_0}{\partial z} = \frac{\partial h}{\partial z} + u \frac{\partial u}{\partial z}$$

Hence the new form of the energy equation becomes

$$\rho \frac{\partial h_o}{\partial t} + \rho u \frac{\partial h_o}{\partial x} + \rho w \frac{\partial h_o}{\partial z} = \frac{\partial p}{\partial t} + \mu \left(\frac{\partial u}{\partial z}\right)^2 + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial z}\left(\rho D_{12} \sum_i h_i \frac{\partial c_i}{\partial z}\right) \tag{2.63}$$

During the non dimensionalization process of the boundary layer equations, we introduce the Lewis number to represent the magnitude of diffusion in terms of heat conduction as a non dimensional number: $L_e = \rho D_{12} c_p / k$. The non dimensional form of Eq. 2.63 reads as

$$\rho \frac{\partial h_o}{\partial t} + \rho u \frac{\partial h_o}{\partial x} + \rho w \frac{\partial h_o}{\partial z} = \frac{\partial p}{\partial t} + \frac{\partial}{\partial z}\left[\left(1 - \frac{1}{P_r}\right) \mu u \frac{\partial u}{\partial z} + \frac{\mu}{P_r} \frac{\partial h_0}{\partial z} + \left(1 - \frac{1}{L_e}\right) \rho D_{12} \sum_i h_i \frac{\partial c_i}{\partial z}\right] \tag{2.64}$$

In Eq. 2.64 the local value 1 for the Lewis number makes the contribution of diffusion vanish and as the Lewis number gets higher the diffusion gets stronger. The c_p value in the Lewis number is obtained from the average c_{pi} values of the species involved in the boundary layer under the frozen flow assumption.

2.2.10 Incompressible Flow Navier-Stokes Equations

In a wide region of aerodynamical applications low subsonic speeds are encountered. Since the free stream Mach number for these types of are very low, the flow is assumed incompressible. The continuity equation for the incompressible flow becomes

$$\vec{\nabla}.\vec{V} = 0 \tag{2.65}$$

Equation 2.65 implies that the flow is divergenless which in turn simplifies the constitutive relations, Eq. 2.51a, b. In addition, because of low speeds the temperature changes in the flow field will also be low which makes the viscosity remain constant. Since the viscosity is constant, the momentum equation is simplified also to take the following form

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla}p + \mu\nabla^2\vec{V} \tag{2.66}$$

In case of turbulent flows, we use the effective viscosity: $\mu_e = \mu + \mu_T$ in Eq. 2.66 which undergoes an averaging process after Reynolds decomposition which makes the final form of the equations to be called 'Reynolds Averaged Navier-Stokes Equations'.

Another convenient form of incompressible Navier-Stokes equations is written in terms of a new variable called vorticity. The vorticity vector is derived from the velocity vector as

$$\vec{\omega} \;=\; \vec{\nabla}x\vec{V} \tag{2.67}$$

The vorticity transport equation obtained from two dimensional version of Eq. 2.66 reads as

$$\frac{\partial \omega}{\partial t} \;+\; (\vec{V}.\vec{\nabla})\,\omega = \;\nabla^2\omega \tag{2.68}$$

Here, ω as the third component of the vorticity appears as a scalar quantity in Eq. 2.68, which does not have any pressure term involved. The integral form of Eqs. 2.65 and 2.67 reads as, (Wu and Gulcat 1981),

$$\vec{V}(\vec{r},t) = -\frac{1}{2\pi}\int_R \frac{\vec{\omega}_o x(\vec{r}_o - \vec{r})}{|\vec{r}_o - \vec{r}|^2} dR_0 + \frac{1}{2\pi}\int_B \frac{(\vec{V}_0.\vec{n}_0)(\vec{r}_o - \vec{r}) - (\vec{V}_0 x\vec{n}_0)x(\vec{r}_o - \vec{r})}{|\vec{r}_o - \vec{r}|^2} dB_0 \tag{2.69}$$

Here, R shows the region for vortical flow, B the boundaries, **r** and $\mathbf{r_o}$ the position vectors and $\mathbf{n_o}$ the unit vector pointing outwards to the boundaries. The boundary B contains the airfoil surface and the far field boundary. While solving Eq. 2.68, we only consider the vertical region confined around the airfoil. Same is done for the evaluation of the velocity field via Eq. 2.69. The integro-differential formulation presented here, therefore, enables us to work with small computational domains. Another use of Eq. 2.69 comes into picture while determining the surface vortex sheet strength through the no-slip boundary condition.

2.2.11 Aerodynamic Forces and Moments

The aim in performing the real gas flow analysis over bodies is to determine the aerodynamic forces, moments and the heat loads acting. For this purpose the computed pressure and stress fields are integrated over whole surface of the body. The surface stresses are obtained from the velocity gradients calculated at the surface. Let us now write down the x,y and z components of the infinitesimal surface force d**F** acting on the infinitesimal area dA of the surface

$$\begin{aligned} dF_x &= (n_x\tau_{xx} + n_y\tau_{xy} + n_z\tau_{xz})\, dA \\ dF_y &= (n_x\tau_{yx} + n_y\tau_{yy} + n_z\tau_{yz})\, dA \\ dF_z &= (n_x\tau_{zx} + n_y\tau_{zy} + n_z\tau_{zz})\, dA \end{aligned} \tag{2.70}$$

Here, $n_{x,}$ n_y and n_z are the direction cosines of the vector normal to the infinitesimal surface dA Let us now express the area dA in curvilinear coordinates. We can express the integral relations which give the total force components in xyz in terms of the differential area given in curvilinear coordinates ξη as shown in Fig. 2.8.

As seen in Fig. 2.8 the differential area dA can be computed in terms of the product of two infinitesimal vectors given as the changes of the position vector **r** = x**i** + y**j** + z**k** in directions of ξ and η coordinates as dA = |(d**r**/dξ) x (d**r**/dη)| dξ dη. The vector product of these two vectors also give the direction of the unit normal **n** of dA. In explicit form we find

$$\begin{aligned} dA &= \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_\xi & y_\xi & z_\xi \\ x_\eta & y_\eta & z_\eta \end{vmatrix} \right\| d\xi\, d\eta \\ &= \sqrt{(y_\xi z_\eta - z_\xi y_\eta)^2 + (x_\xi z_\eta - z_\xi x_\eta)^2 + (y_\xi y_\eta - x_\xi y_\eta)^2}\, d\xi\, d\eta \end{aligned} \tag{2.71}$$

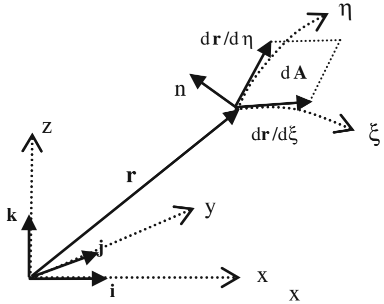

Fig. 2.8 Expressing dA in curvilinear coordinates ξη

Here, the term under the square root is named reduced Jacobian I. The unit normal vector in open form becomes

$$\vec{n} = \left[(y_\xi z_\eta - z_\xi y_\eta)\vec{i} - (x_\xi z_\eta - z_\xi x_\eta)\vec{j} + (y_\xi y_\eta - x_\xi y_\eta)\vec{k}\right] \tag{2.72}$$

We can write the components of the stress tensor in terms of the velocity gradients expressed in curvilinear coordinates as follows for example for τ_{xy}

$$\tau_{xy} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \mu(\frac{\partial u}{\partial \xi}\xi_y + \frac{\partial u}{\partial \eta}\eta_y + \frac{\partial u}{\partial \varsigma}\varsigma_y + \frac{\partial v}{\partial \xi}\xi_x + \frac{\partial v}{\partial \eta}\eta_x + \frac{\partial v}{\partial \varsigma}\varsigma_x) \tag{2.73}$$

If we consider Eqs. 2.71–2.73 to form the differential force elements and integrate them numerically over the differential area, we obtain the total force components as follows

$$\begin{aligned} F_x &= \int_A dF_x = \int_A (n_x\tau_{xx} + n_y\tau_{xy} + n_z\tau_{xz})\, I\, d\xi\, d\eta \\ F_y &= \int_A dF_y = \int_A (n_x\tau_{yx} + n_y\tau_{yy} + n_z\tau_{yz}) I\, d\xi\, d\eta \\ F_z &= \int_A dF_z = \int_A (n_x\tau_{zx} + n_y\tau_{zy} + n_z\tau_{zz}) I\, d\xi\, d\eta \end{aligned} \tag{2.74}$$

Computations of the moments with respect to a point can be performed similarly with considering the moment arm of the point to the differential area dA.

In case of two dimensional incompressible external flows if we know the vorticity field $\boldsymbol{\omega}$, first the surface vortex sheet strength $\gamma = \int_0^\delta \omega\, dy$ is determined. Afterwards, we can compute the aerodynamic force acting on an airfoil as follows, (Wu)

$$\vec{F} = -\rho\frac{d}{d\,t}\int_{B_s} \vec{r}\,x\,(\gamma - \vec{V}\,x\,\vec{n}_s)\,dB_s - \rho\frac{d}{d\,t}\int_W \vec{r}\,x\,\vec{\omega}\,dR \tag{2.75}$$

Here, $\mathbf{n_s}$ is the unit normal to the airfoil surface and $\mathbf{Vxn_s}$ is the velocity tangent to the surface. For a pitching and plunging airfoil, the value of the tangential velocity is computed at every discrete point on the surface and used in Eq. 2.75.

2.2.12 Turbulence Modeling

At high free stream speeds external flows are likely to go through a transition from laminar to turbulence on the airfoil surface close to the leading edge. Depending on the value of the Reynolds number most of the flow on the airfoil becomes turbulent. The Reynolds decomposition technique applied to the Navier-Stokes equations results in new unknowns of the flow field called Reynolds stresses. These new unknowns introduce more unknowns than the existing equations which is called the closure problem of turbulence. In order to close the problem, the Reynolds stresses are empirically modeled in terms of the velocity gradients. All these models aim at finding the suitable value of turbulence viscosity μ_T applicable for different flow cases. The empirical turbulence models are in general based on the wind tunnel tests and some numerical verification. The simplest models of turbulence are the algebraic models. More complex models are based on differential equations. Although so many models have been introduced, there has not been a satisfactory model developed to reflect the main characteristics of a turbulent flow. Now, we present the well known Baldwin-Lomax model which is used for the numerical solution of attached or separated, incompressible or compressible flows of aerodynamics. This model is a simple algebraic model which assumes the turbulent region to be composed of two different layers. Accordingly the turbulence viscosity reads

$$\mu_T = \begin{cases} (\mu_T)_i, \, for\, z \geq z_c \\ (\mu_T)_o, \, for\, z < z_c \end{cases}. \tag{2.76}$$

Here, z is the normal distance to the surface, z_c is the shortest distance where inner and outer viscosity values are equal. The inner viscosity value in terms of the mixing length l and the vorticity ω reads as

$$(\mu_T)_i = \rho\, l^2 |\omega| R_e \quad \text{and} \quad l = \kappa\, z[1 - \exp(-z^+/A^+)] \tag{2.77a, b}$$

Here, κ = 0.41 is the von Karman constant, $A^+ = 26$ damping coefficient and $z^+ = z\sqrt{|\,\omega\,|\, R_e}$. The outer viscosity, on the other hand

$$(\mu_T)_o = K\, C_{cp} F_w F_{kl}(z), \quad F_w = z_{max} F_{max} \tag{2.78a, b}$$

Here, $K = 0.0168$ is the Clauser constant and $C_{cp} = 1.6$. F_{max} maximum of F(z) where z_{max} is the z value at which F_{max} is found. For this purpose,

$$F(z) = z\,|\,\omega\,|\,[1 - \exp(-z^+/A^+)] \quad \text{and} \quad F_{kl}(z) = \left[1 + 5.5\,(\frac{z\, C_{kl}}{z_{max}})^6\right]^{-1} \tag{2.79a, b}$$

Here, $C_{kl} = 0.3$ (Baldwin and Lomax 1978).

The research on turbulence models are of interest to many branches of fluid mechanics. The Baldwin-Lomax model is implemented for the aerodynamic applications of attached or separated flows considered here. More complex models based on the differential equation solutions are utilized even in commercial softwares of CFD together with the necessary documentations. Detailed information, scientific basis and their application areas for different turbulent models are provided by Wilcox (1998).

2.2.13 Initial and Boundary Conditions

The study of aerodynamical problems with real gas effects requires solution of a system of partial differential equations which are first order in time and second order in space coordinates. In order to solve Eq. 2.49 to determine the flow field, all dependent variables must be prescribed at time t = 0, and for all times t at the boundaries of the computational domain. All the prescribed values must be in accordance with the physics of the problem. As the initial conditions for the unknown values of **U** we prescribe the undisturbed flow conditions, i.e., u = 1, v = w = 0 which represents the impulsive start of the flow. Under these conditions the initial values for the unknown vector in generalized coordinates become

$$\vec{U}(t = 0,\ \xi, \eta, \varsigma) = \begin{pmatrix} \rho^0 \\ \rho^0 \\ 0 \\ 0 \\ \varepsilon^0 \\ c_i^0 \end{pmatrix} \tag{2.80}$$

Here, ρ^o is the initial value for the density, ε^o is the initial value for the energy and c_i^o is the initial value of the ith specie.

As for the boundary conditions: (i) the unknowns at the surface, and (ii) farfield boundary conditions must be provided.

Accordingly:

(i) As the no slip condition at the surface: $\mathbf{U}(t,\xi,\eta,\varsigma = 0) = \mathbf{0}$ is prescribed. (In Fig. 2.6, $\varsigma = 0$ prescribes the surface). In reactive flows the catalicity of the surface determines the value of the concentration gradients,

(ii) At the farfield: for $\varsigma = \varsigma_{max}$ $\mathbf{U}(t,\xi,\eta,\varsigma = \varsigma_{max}) = \vec{U}_\infty$ is prescribed, and the flux condition at $\xi = \xi_{max}$ is $\frac{\partial \vec{U}}{\partial \xi} = \left(\vec{0}\right)$,

(iii) If there is a symmetry condition as shown in Fig. 2.6b, we prescribe the flux normal to the symmetry as $\frac{\partial \vec{U}}{\partial \eta} = \left(\vec{0}\right)$.

Summary: *In order to analyze the problems of aerodynamics we have constructed mathematical models of the physical phenomena. For this purpose, conservation equations are used for the development of the potential theory. The equation which is satisfied by the velocity potential is developed using the state, continuity, momentum and the energy equations with assuming the flow of a perfect fluid without any viscous and body forces and without any chemical reactions. The potential equation will be solved with proper boundary conditions. The boundary* *conditions for an external flow are applied at the infinity and at the body surface. We assumed that at the infinity all disturbances die out, only the free stream conditions prevail and on the body surface the boundary conditions are given according to the time dependent motion of the body surface. The material derivative of the surface equation is utilized to get the downwash which is the vertical velocity component solely responsible for the generation of the lift from the body. Both the potential equation and the boundary conditions are nonlinear in nature. It is very difficult to solve this nonlinear equation except for a few cases where the series solutions are obtained. Therefore, the equations are linearized through the concept of small disturbances. A linear equation satisfied by the perturbation potential and a linear form of the downwash expression are obtained. It is now possible to solve this linear equation with a linear boundary condition for variety of external flow cases using the superposition techniques. Another useful potential function is the acceleration potential derived from the definition of the acceleration vector. It is related to the velocity potential and gives a convenient expression for the surface pressure coefficient. Finally, the concept of body fixed and the flow fixed coordinate systems are introduced. The coordinate transformation from body fixed to flow fixed coordinate systems render the potential equation into the form of classical wave equation which has variety of known solutions.*

Real gas effects are also considered in deriving the new form of equations for the viscous, thermally conducting, chemically reacting and diffusing flows which are of interest to aerodynamics. In derivation, the Reynolds transport theorem which relates the system to the control volume for the conservation laws is utilized. Then, the differential form of these conservation laws are derived from the original integral representations. The conservation of species is added to the system of equations for the case of reacting flows which occur at very high speeds. The equations are cast in conservative form. In order to impose the surface boundary conditions properly the equations are expressed in generalized curvilinear coordinates. The Navier-Stokes equations and their simplified versions like thin shear layer N-S and boundary layer equations are derived through order of magnitude analysis. Incompressible Navier- Stokes equations are presented for the flows having low free stream velocities. The formulas for the evaluation of aerodynamic forces and moments involving curved surfaces are provided. For high-speed aerodynamic flows the turbulence is inevitably present. Therefore, an algebraic turbulence modeling which is suitable for the separated and unseparated flows is given. Finally, the initial and the boundary conditions for the aerodynamic flow are presented.

2.3 Questions and Problems

2.1 In a barotropic flow show that $\frac{1}{\rho}\nabla\, p = \nabla \int \frac{dp}{\rho}$.

2.2 Equation 2.15 is written in terms of the velocity potential. Express the same equation with partial derivatives of velocity potential.

2.3 An oblate ellipsoid is undergoing vertical simple harmonic motion with amplitude $\bar{a}$. Express the equation of upper and lower surfaces of the airfoil.

2.4 The ellipsoid given in Problem 2.3 is also undergoing a pulsative major axis change with the same period but with phase difference ϕ. Express the equation of surfaces.

2.5 Comment on the steady or unsteady lift generation by referring to the downwash expression given by 2.19.

2.6 The equation of a paraboloid of length l and whose axis is in line with x axis is given as $c(x/l) = (y^2 + z^{2)}/a^2$, $0 \leq x \leq l$ and $0 \leq y, z \leq a$. Obtain the downwash expression at the surface. If a slender paraboloid undergoes SHM about its nose in a vertical y-z plane, find the unsteady downwash expression at the surface.

2.7 A lighter than air prolate ellipsoid moves in air with constant speed U. If this air vehicle oscillates simple harmonically about its center with a small amplitude A in a vertical plane then find the time dependent surface downwash expression at the (i) shoulders, and (ii) at the front end rear ends.

2.8 We do not need to define perturbation potential for the acceleration potential. Why?

2.9 From the non linear relation between the velocity and the acceleration potential, obtain the linear relation given by Eq. 2.25.

2.10 Obtain the surface pressure and downwash expressions in terms of acceleration potential.

2.11 Derive the Reynolds Transport theorem, 2.27, which interlaces the system and control volume approaches.

2.12 Obtain Eq. 2.35 which gives the continuity of the species.

2.13 Express the conservation of momentum in open form in Cartesian coordinates.

2.14 Obtain the expression given by 2.50 by means of the transformation from Cartesian to generalized coordinates.

2.15 For a tapered wing with half-span of 4 units let x be the chordwise and y be the spanwise directions. The equation for the leading edge is given as: $x = 0.15y$, $0 < y < 4$ and the trailing edge: $x = -0.025y + 4$, $0 < y < 4$. Using the two dimensional numerical transformation with $0 < \xi < 1 \times 0 < \eta < 1$ for 11×11 equally spaced discrete points transform the wing surface from x-y coordinates to ξ–η generalized coordinates. Find the metrics of transformation and Jacobian determinant at each discrete location.

2.16 In generalized coordinates, obtain the Navier-Stokes equations for the thin shear layer case in terms of the contravariant velocity components.

2.17 Express the components of stress tensor in generalized coordinates in terms of velocity gradients.

References

Anderson DA, Tannehill JC, Pletcher RH (1984) Computational fluid mechanics and heat transfer. Hemisphere, New York
Anderson JD (1989) Hypersonic and high temperature gas dynamics. McGraw-Hill, New York
Bisplinghoff RL, Ashley H, Halfman RL (1996) Aeroelasticity. Dover Publications Inc., New York
Baldwin BS, Lomax H (1978) Thin layer approximations and algebraic model for separated flows, AIAA Paper 78-0257
Batchelor GK (1979) An introduction to fluid dynamics. Cambridge University Press, London
Fox RF, McDonald AT (1992) Introduction to fluid mechanics, 4th edn. John Wiley, New York
Hildebrand FB (1976) Advanced calculus for applications. Prentice-Hall Inc., Engelwood Cliffs, New Jersey
Hoffman JD (1992) Numerical methods for engineers and scientists. McGraw-Hill, New York
Liepmann HW, Roshko A (1963) Elements of gas dynamics. Wiley, New York
Schlichting H (1968) Boundary-layer theory, 6th edn. McGraw-Hill, New York
Shames IH (1969) Engineering mechanics: statics and dynamics. Prentice-Hall, New Delhi
Wilcox David C (1998) Turbulence modelling for CFD. DCW Industries, California
Wu JC, Gulcat U (1981) Separate treatment of attached and detached flow regions in general viscous flows. AIAA J 19(1):20–27

http://www.springer.com/978-981-10-0016-4

Fundamentals of Modern Unsteady Aerodynamics
Gülçat, Ü.
2016, XIV, 395 p. 199 illus., 194 illus. in color.,
Hardcover
ISBN: 978-981-10-0016-4