

Chapter 2

Schramm–Loewner Evolution (SLE)

Abstract We consider the Loewner chain, which is a time evolution of a conformal transformation defined on the upper-half complex plane. The chain is driven by a given continuous real function of time t and it determines a path γ in the upper half-plane parameterized by t . Schramm–Loewner evolution (SLE) is a stochastic version of the Loewner chain such that the driving function is given by a time change of one-dimensional Brownian motion and thus the path becomes stochastic. We introduce the SLE as a complexification of the Bessel flow studied in Chap. 1. Then the parameter κ of SLE, which is originally introduced to control the time change of Brownian motion driving the SLE, is related to the dimension D of Bessel process. Corresponding to the existence of two critical dimensions $D_c = 2$ and $\bar{D}_c = 3/2$, the appearance of three different phases of the SLE path is clarified. Moreover, based on the detailed analysis of the Bessel flow in $\bar{D}_c < D < D_c$ given in Chap. 1, Cardy’s formula for the critical percolation model is derived. We give a list showing correspondence (up to a conjecture) between lattice paths studied in statistical mechanics and SLE paths describing their scaling limits.

2.1 Complexification of Bessel Flow

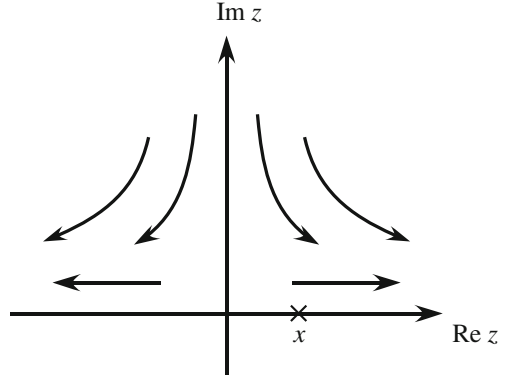
In the sequel we consider an extension of the Bessel flow $\{R^x(t) : t \geq 0\}_{x>0}$ defined on \mathbb{R}_+ to a flow on the upper-half complex plane $\mathbb{H} = \{z = x + \sqrt{-1}y : x \in \mathbb{R}, y > 0\}$ and its boundary $\partial\mathbb{H} = \mathbb{R}$. Let $\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$. We set $Z^z(t) = X^z(t) + \sqrt{-1}Y^z(t) \in \bar{\mathbb{H}} \setminus \{0\}$, $t \geq 0$ and complexificate (1.60) as

$$dZ^z(t) = dB(t) + \frac{D-1}{2} \frac{dt}{Z^z(t)} \quad (2.1)$$

with the initial condition

$$Z^z(0) = z = x + \sqrt{-1}y \in \bar{\mathbb{H}} \setminus \{0\}.$$

Fig. 2.1 A schematic picture of ‘complexified Bessel flow’ on $\mathbb{H} \setminus \{0\}$ for $D > 2$



The crucial point of this complexification of Bessel flow is that the BM remains real, $B(t) \in \mathbb{R}$, $t \geq 0$. Then, there is an asymmetry between the real part and the imaginary part of the flow in \mathbb{H} ,

$$dX^z(t) = dB(t) + \frac{D-1}{2} \frac{X^z(t)}{(X^z(t))^2 + (Y^z(t))^2} dt, \quad (2.2)$$

$$dY^z(t) = -\frac{D-1}{2} \frac{Y^z(t)}{(X^z(t))^2 + (Y^z(t))^2} dt. \quad (2.3)$$

Assume $D > 1$. Then as indicated by the minus sign in the RHS of (2.3), the flow is downward in \mathbb{H} . If the flow goes down and arrives at the real axis, the imaginary part vanishes, $Y^z(t) = 0$, then Eq. (2.2) is reduced to be the same equation as Eq. (1.60) for the $\text{BES}^{(D)}$, which is now considered for $\mathbb{R} \setminus \{0\} = \mathbb{R}_+ \cup \mathbb{R}_-$. If $D > D_c = 2$, by Theorem 1.1 (ii), the flow on $\mathbb{R} \setminus \{0\}$ is asymptotically outward, $X^z(t) \rightarrow \pm\infty$ as $t \rightarrow \infty$. Therefore, the flow on \mathbb{H} will be described as shown by Fig. 2.1. The behavior of flow should be, however, more complicated when $\bar{D}_c = 3/2 < D < D_c$ and $1 < D < \bar{D}_c$.

For $z \in \mathbb{H} \setminus \{0\}$, $t \geq 0$, let

$$g_t(z) = Z^z(t) - B(t). \quad (2.4)$$

Since $B^x(t)$ and $Z^z(t)$ are stochastic processes, they are considered as functions of time $t \geq 0$, where the initial values x and z are put as superscripts ($B(t) \equiv B^0(t)$). On the other hand, as explained below, g_t is considered as a conformal transformation from a domain $\mathbb{H}_t \subset \mathbb{H}$ to \mathbb{H} , and thus it is described as a function of $z \in \mathbb{H}_t$; $g_t(z)$, where time t is a parameter and put as a subscript.

Then, Eq. (2.1) is rewritten for $g_t(z)$ as

$$\frac{\partial g_t(z)}{\partial t} = \frac{D-1}{2} \frac{1}{g_t(z) + B(t)}, \quad t \geq 0 \quad (2.5)$$

with the initial condition

$$g_0(z) = z \in \overline{\mathbb{H}} \setminus \{0\}. \quad (2.6)$$

For each $z \in \overline{\mathbb{H}} \setminus \{0\}$, set

$$\begin{aligned} T^z &= \inf\{t > 0 : Z^z(t) = 0\} \\ &= \inf\{t > 0 : g_t(z) + B(t) = 0\}, \end{aligned} \quad (2.7)$$

and then the solution of Eq. (2.5) exists up to time T^z . For $t \geq 0$ we put

$$\mathbf{H}_t = \{z \in \mathbb{H} : T^z > t\}. \quad (2.8)$$

This ordinary differential equation (2.5) involving the BM is nothing but the celebrated *Schramm–Loewner evolution* (SLE) [13, 17]. It is known that [13], for each $t \geq 0$, the solution $g_t(z)$ of (2.5) gives a unique *conformal transformation* from \mathbf{H}_t to \mathbb{H} :

$$g_t(z) : \mathbf{H}_t \mapsto \mathbb{H}, \quad \text{conformal},$$

such that

$$g_t(z) = z + \frac{a(t)}{z} + \mathcal{O}\left(\frac{1}{|z|^2}\right), \quad z \rightarrow \infty$$

with

$$a(t) = \frac{D-1}{2}t.$$

The usual parameter for the SLE is given by $\kappa > 0$ [13, 17], which is related to D by

$$\kappa = \frac{4}{D-1} \iff D = 1 + \frac{4}{\kappa}. \quad (2.9)$$

If we set $\widehat{g}_t(z) = \sqrt{\kappa} g_t(z)$ in (2.5), we have the equation in the form [17]

$$\frac{\partial \widehat{g}_t(z)}{\partial t} = \frac{2}{\widehat{g}_t(z) - U_t} \quad (2.10)$$

with

$$U_t = -\sqrt{\kappa} B(t), \quad t \geq 0. \quad (2.11)$$

In the complex analysis, given a real function U_t of $t \geq 0$, a one-parameter family of conformal transformations $(g_t)_{t \geq 0}$ defined by the unique solution of (2.10) under $g_0(z) \equiv z \in \mathbb{H}$ is called the *Loewner chain* driven by $(U_t)_{t \geq 0}$. Note that

$$(-\sqrt{\kappa} B(t))_{t \geq 0} \stackrel{(\text{law})}{=} (B(\kappa t))_{t \geq 0}$$

by the left-right symmetry and the scaling property (1.4) of BM. The parameter $\kappa > 0$ is the diffusion constant and it ‘speeds up’ (as $\kappa \uparrow$) and ‘slows down’ (as $\kappa \downarrow$) the one-dimensional Brownian motion which drives the stochastic Loewner chain. In the present book, however, we will discuss the SLE using the parameter $D > 1$, since we would like to discuss it as a complexification of the D -dimensional Bessel flow.

The inverse map

$$f_t(z) \equiv g_t^{-1}(t), \quad : \mathbb{H} \mapsto \mathbf{H}_t, \quad t \geq 0, \quad (2.12)$$

is also conformal. The equation of $(f_t(z))_{t \geq 0}$ is then obtained as (Exercise 2.1)

$$\frac{\partial f_t(z)}{\partial t} = -\frac{D-1}{2} \frac{\partial f_t(z)}{\partial z} \frac{1}{z + B(t)}, \quad t \geq 0. \quad (2.13)$$

We call this partial differential equation the *backward SLE*. For (2.10), the inverse map $\hat{f}_t(z) \equiv \hat{g}_t^{-1}(z)$ satisfies

$$\frac{\partial \hat{f}_t(z)}{\partial t} = -\frac{\partial \hat{f}_t(z)}{\partial z} \frac{2}{z - U_t}, \quad t \geq 0 \quad (2.14)$$

with (2.11).

By the definition of T^z , (2.7), for each $z \in \mathbb{H}$, $Z^z(t) = g_t(z) + B(t) \rightarrow 0$ as $t \uparrow T^z$. (In this limit the Eq. (2.5) becomes ill-defined.) Set $\zeta = g_t(z) + B(t)$ provided $t < T^z \iff z \in \mathbf{H}_t$. In this case $g_t(z) \in \mathbb{H}$, $B(t) \in \mathbb{R}$, and hence $\zeta \in \mathbb{H}$. Therefore, an approach $Z^z(t) \rightarrow 0$ corresponds to a limit $\zeta \rightarrow 0$, $\zeta \in \mathbb{H}$. Since $\zeta = g_t(z) + B(t) \iff z = g_t^{-1}(\zeta - B(t))$, the behavior of $Z^z(t) \rightarrow 0$ will be represented by the limit

$$\gamma(t) \equiv \lim_{\substack{\zeta \rightarrow 0, \\ \zeta \in \mathbb{H}}} g_t^{-1}(\zeta - B(t)). \quad (2.15)$$

Using properties of BM and the conformal transformation generalized by the Loewner chain (2.5), Rohde and Schramm [16] proved that $\gamma = \gamma[0, \infty) \equiv \{\gamma(t) : t \in [0, \infty)\} \in \overline{\mathbb{H}}$ is a continuous path with probability 1 running from $\gamma(0) = 0$ to $\gamma(\infty) = \infty$. The path γ obtained from the SLE with the parameter $D > 1$ is called the *SLE^(D) path*. (See Exercise 2.2.)

2.2 Schwarz–Christoffel Formula and Loewner Chain

In mathematical physics, the *Schwarz–Christoffel formula* may be more popular than the Loewner chain, when conformal transformations are studied. In this section, we discuss the Loewner chain from the viewpoint of the Schwarz–Christoffel formula using a simple example of conformal transformation. We will use the Schwarz–Christoffel transformation in Sect. 2.4, where Cardy’s formula in Carleson’s form is given for an equilateral triangular domain.

Let Γ be a polygon having vertices w_1, w_2, \dots, w_n and interior angles $\alpha_1\pi, \alpha_2\pi, \dots, \alpha_n\pi$ in the counterclockwise direction and D be the interior of Γ as shown in Fig. 2.2. The following theorem is known as the Schwarz–Christoffel formula [8].

Theorem 2.1 *Let \hat{f} be any conformal map from \mathbb{H} to D with $\hat{f}(x_i) = w_i, 1 \leq i \leq n-1$, and $\hat{f}(\infty) = w_n$, where $x_i \in \mathbb{R}, 1 \leq i \leq n-1$. Then*

$$\frac{d\hat{f}(z)}{dz} = C \prod_{i=1}^{n-1} (z - x_i)^{\alpha_i - 1}, \quad (2.16)$$

where C is a complex constant.

As an application of this formula, we consider a conformal map \hat{f} from \mathbb{H} to the upper-half complex plane with a straight slit starting from the origin: $\mathbb{H} \setminus \{a \text{ slit}\}$. Let $0 < \alpha < 1$. As shown by Fig. 2.3, the angle between the slit and the positive

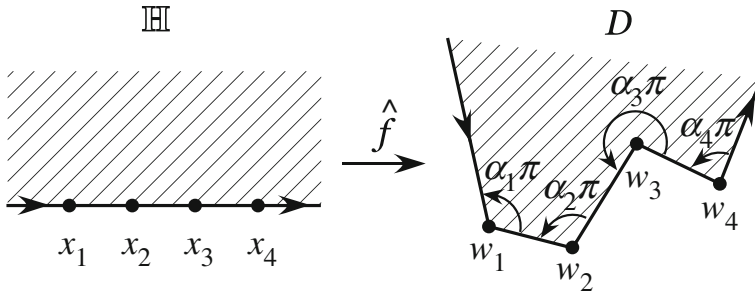


Fig. 2.2 The conformal map \hat{f} from \mathbb{H} to D with $\hat{f}(x_i) = w_i, 1 \leq i \leq 4$ and $\hat{f}(\infty) = w_5 \equiv \infty$

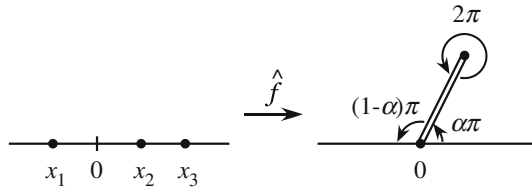


Fig. 2.3 The conformal map \hat{f} from \mathbb{H} to $\mathbb{H} \setminus \{a \text{ slit}\}$, where the angle between the slit and the positive direction of the real axis is $\alpha\pi, \alpha \in (0, 1)$

direction of the real axis is supposed to be $\alpha\pi$. Since the region $\mathbb{H} \setminus \{\text{a slit}\}$ can be regarded as a polygon with the interior angles $(1 - \alpha)\pi$ on the left side of the origin, 2π around the tip of the slit, and $\alpha\pi$ on the right side of the origin, for any length of a slit, the formula (2.16) gives

$$\frac{d\widehat{f}(z)}{dz} = C(z - x_1)^{-\alpha}(z - x_2)(z - x_3)^{\alpha-1}, \quad (2.17)$$

where $x_1 < 0$, $x_1 < x_2 < x_3$, and $x_3 > 0$. We assume that $\widehat{f}(x_1) = \widehat{f}(x_3) = 0$ and $\widehat{f}(x_2)$ gives the tip of the slit. If we impose the condition on the asymptotics as

$$\frac{\widehat{f}(z)}{z} \rightarrow 1 \quad \text{as } z \rightarrow \infty,$$

the solution of (2.17) is uniquely determined as

$$\widehat{f}(z) = (z - x_1)^{1-\alpha}(z - x_3)^\alpha, \quad (2.18)$$

where the following relation should be satisfied,

$$x_3 - x_2 = \alpha(x_3 - x_1). \quad (2.19)$$

Using (2.18), the Schwarz–Christoffel differential equation (2.17) is rewritten as

$$\frac{d\widehat{f}(z)}{dz} \frac{2}{z - x_2} = \frac{2\widehat{f}(z)}{(z - x_1)(z - x_3)}. \quad (2.20)$$

We then introduce a parameter $t \geq 0$ and assume $x_i = x_i(t)$, $i = 1, 2, 3$, and put $\widehat{f}_t(z) = (z - x_1(t))^{1-\alpha}(z - x_3(t))^\alpha$. The differential of \widehat{f}_t with respect to t is given as

$$\frac{\partial \widehat{f}_t(z)}{\partial t} = -\frac{2A_t(z)}{(z - x_1(t))(z - x_3(t))} \widehat{f}_t(z) \quad (2.21)$$

with

$$A_t(z) = \frac{1}{2} \left\{ (1 - \alpha)(z - x_3(t)) \frac{dx_1(t)}{dt} + \alpha(z - x_1(t)) \frac{dx_3(t)}{dt} \right\}.$$

Let $x_1(t) = -2ct^\beta$, $x_3(t) = (2/c)t^\beta$ with constants $c, \beta > 0$. Then we find that, if and only if

$$c = \sqrt{\frac{\alpha}{1 - \alpha}}, \quad \beta = \frac{1}{2}, \quad (2.22)$$

$A_t(z)$ becomes independent both of z and t ; $A_t(z) \equiv 1$. In this case (2.20) and (2.21) give the equation

$$\frac{\partial \widehat{f}_t(z)}{\partial t} = -\frac{\partial \widehat{f}_t(z)}{\partial z} \frac{2}{z - x_2(t)}, \quad t \geq 0 \quad (2.23)$$

with

$$x_2(t) = \begin{cases} \sqrt{\kappa t}, & \text{if } \alpha \leq 1/2, \\ -\sqrt{\kappa t}, & \text{if } \alpha > 1/2, \end{cases} \quad (2.24)$$

where

$$\kappa = \kappa(\alpha) = \frac{4(1 - 2\alpha)^2}{\alpha(1 - \alpha)}. \quad (2.25)$$

Equation (2.23) can be regarded as the backward Loewner evolution (2.14) driven by (2.24). The obtained conformal transformation

$$\widehat{f}_t(z) = \left(z + 2\sqrt{\frac{\alpha}{1 - \alpha}}\sqrt{t} \right)^{1 - \alpha} \left(z - 2\sqrt{\frac{1 - \alpha}{\alpha}}\sqrt{t} \right)^{\alpha} \quad (2.26)$$

is a solution of the Schwarz–Christoffel equation (2.17) and the backward Loewner evolution (2.23). The corresponding Loewner path is a straight slit starting from the origin growing upward in \mathbb{H} with a tip

$$\gamma(t) = \widehat{f}_t(x_2(t)) = 2 \left(\frac{1 - \alpha}{\alpha} \right)^{1/2 - \alpha} e^{\sqrt{-1}\alpha\pi}\sqrt{t}, \quad t \geq 0. \quad (2.27)$$

Note that the quadratic variation of (2.11) is

$$\langle U, U \rangle_t = \kappa \langle B, B \rangle_t = \kappa t, \quad t \geq 0.$$

It is identified with $x_2(t)^2$, if κ is given by (2.25). SLE will be considered as a randomization of the time-dependent conformal map (2.26).

2.3 Three Phases of SLE

The dependence on D of the Bessel flow given by Theorems 1.1 and 1.2 is mapped to the feature of the $\text{SLE}^{(D)}$ paths so that they exhibit three phases.

[Phase 1] When $D \geq D_c = 2$ (i.e., $0 < \kappa \leq \kappa_c \equiv 4$), the $\text{SLE}^{(D)}$ path is a *simple curve*, i.e., $\gamma(s) \neq \gamma(t)$ for any $0 \leq s \neq t < \infty$, and $\gamma(0, \infty) \in \mathbb{H}$ (i.e., $\gamma(0, \infty) \cap \mathbb{R} = \emptyset$). In this phase,

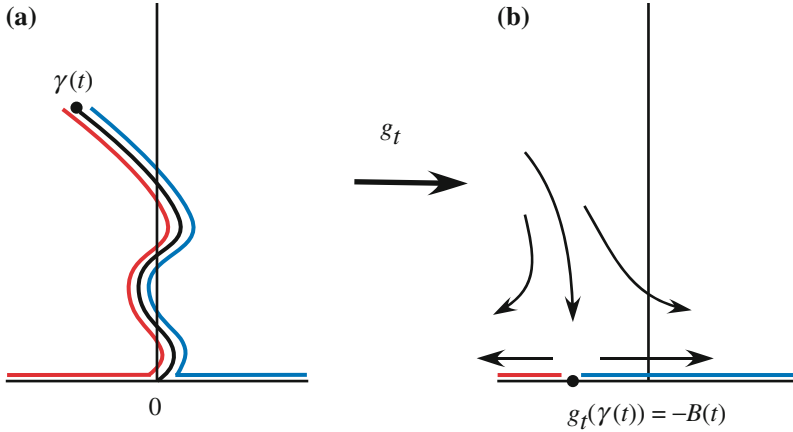


Fig. 2.4 **a** When $D \geq 2$, the $\text{SLE}^{(D)}$ path is simple. **b** By g_t , the $\text{SLE}^{(D)}$ path is erased from \mathbb{H} . The tip of the $\text{SLE}^{(D)}$ path, $\gamma(t)$, is mapped to $g_t(\gamma(t)) = -B(t) \in \mathbb{R}$. The flow associated with this conformal transformation is represented by arrows

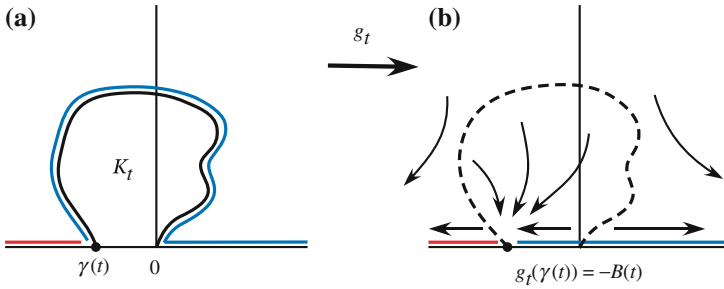


Fig. 2.5 **a** When $3/2 < D < 2$, the $\text{SLE}^{(D)}$ path can osculate the real axis. The SLE hull is denoted by K_t . **b** The SLE hull is swallowed. This means that all the points in K_t are simultaneously mapped to a single point $-B(t) \in \mathbb{R}$, which is the image of the tip of $\text{SLE}^{(D)}$ path, $\gamma(t)$

$$\mathbb{H}_t = \mathbb{H} \setminus \gamma(0, t], \quad t \geq 0.$$

For each $t \geq 0$, g_t gives a map, which conformally erases a simple curve $\gamma(0, t]$ from \mathbb{H} , and the image of the tip $\gamma(t)$ of the SLE path is a BM, $-B(t) \in \mathbb{R} = \partial\mathbb{H}$, as given by (2.15). As shown by Fig. 2.4, it implies that the ‘SLE flow’ in \mathbb{H} is downward in the vertical (imaginary-axis) direction and outward from the position $-B(t)$ in the horizontal (real-axis) direction. Since $Z^z(t) = g_t(z) + B(t)$ by (2.4), if we shift this figure by $B(t)$, we will have a similar picture to Fig. 2.1 for the complicated version of Bessel flow for $D > 2$.

[Phase 2] When $\bar{D}_c = 3/2 < D < D_c = 2$ (i.e., $\kappa_c = 4 < \kappa < \bar{\kappa}_c \equiv 8$), the $\text{SLE}^{(D)}$ path can osculate the real axis, $\mathbb{P}(\gamma(0, t] \cap \mathbb{R} \neq \emptyset) > 0, \forall t > 0$. Figure 2.5a illustrates the moment $t > 0$ such that the tip of $\text{SLE}^{(D)}$ path just

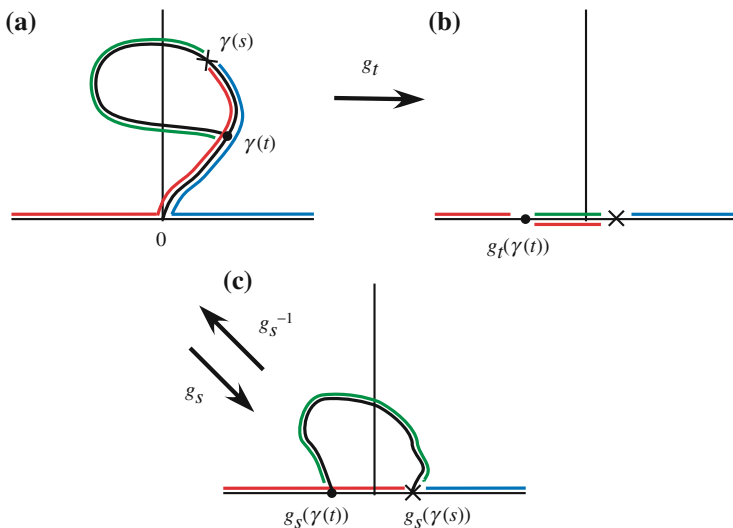


Fig. 2.6 The event that the $\text{SLE}^{(D)}$ path osculates \mathbb{R} is equivalent to the event that the $\text{SLE}^{(D)}$ path makes a loop

osculates the real axis. The closed region encircled by the path $\gamma(0, t)$ and the line $[\gamma(t), 0] \in \mathbb{R}$ is called an *SLE hull* at time t and denoted by K_t . In this phase

$$H_t = \mathbb{H} \setminus K_t, \quad t \geq 0.$$

That is, $g_t(z)$ is a map which erases conformally the SLE hull from \mathbb{H} . We can think that by this transformation all the points in K_t are simultaneously mapped to a single point $-B(t) \in \mathbb{R}$, which is the image of the tip $\gamma(t)$. (We say that the hull K_t is *swallowed*. See Fig. 2.5b.) By the definition (2.8), the moment when K_t is swallowed is the time T^z at which the equality $Z^z(t) = g_t(z) + B(t) = 0$ holds $\forall z \in K_t$. (Then the RHS of (2.5) diverges and all the points $z \in K_t$ are lost from the domain of the map g_t .) Theorem 1.2 (ii) states that, when $\overline{D}_c < D < D_c$, two $\text{BES}^{(D)}$'s starting from different points $0 < x < y < \infty$ can simultaneously return to the origin. In the complexified version, all $Z^z(t)$ starting from $z \in K_t + B(t)$ can arrive at the origin simultaneously (i.e., they are all swallowed).

Osculation of the SLE path with \mathbb{R} means that the SLE path has loops. Figure 2.6a shows the event that the SLE path makes a loop at time $t > 0$. The SLE hull K_t consists of the closed region encircled by the loop and the segment of the SLE path between the origin and the osculating point, and it is completely erased by the conformal transformation g_t from \mathbb{H} as shown by Fig. 2.6b. Let $0 < s < t$ and consider the map g_s , which is the solution of (2.5) at time s . Assume that $\gamma(s)$ is located in the middle of the loop of $\gamma[0, t]$ as shown by Fig. 2.6a. The segment $\gamma[0, s]$ of the SLE path is mapped by g_s to a part of \mathbb{R} . Since $\gamma(t)$ osculates a point in $\gamma[0, s]$, its image $g_s(\gamma(t))$ should osculate the real axis \mathbb{R} as shown by Fig. 2.6c.

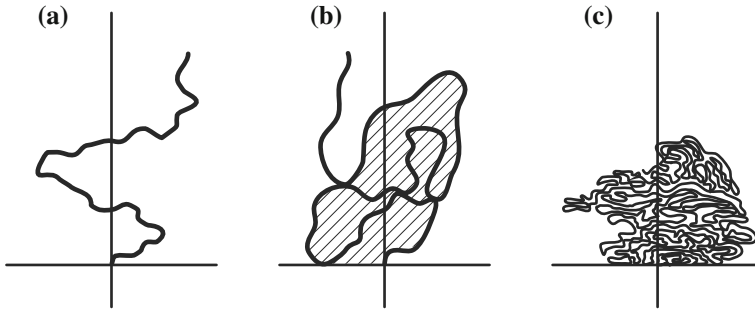


Fig. 2.7 Schematic pictures of $\text{SLE}^{(D)}$ paths in **a** Phase 1 ($D \geq D_c = 2 \Leftrightarrow 0 < \kappa \leq \kappa_c = 4$), **b** Phase 2 ($\overline{D}_c = 3/2 < D < D_c = 2 \Leftrightarrow \kappa_c = 4 < \kappa < \overline{\kappa}_c = 8$), and **c** Phase 3 ($1 < D \leq \overline{D}_c = 3/2 \Leftrightarrow \kappa \geq \overline{\kappa}_c = 8$)

This is the same situation as the one shown in Fig. 2.5a. Since g_s^{-1} is uniquely determined from g_s , the above argument can be reversed. Then the equivalence between the osculation of the SLE path with \mathbb{R} and the self-intersection of the SLE path is concluded.

In this intermediate phase $\overline{D}_c < D < D_c$,

$\text{SLE}^{(D)}$ path γ is *self-intersecting*, and

$$\bigcup_{t>0} \overline{K_t} = \overline{\mathbb{H}} \quad \text{but} \quad \gamma[0, \infty) \cap \mathbb{H} \neq \mathbb{H} \quad \text{with probability 1.}$$

[Phase 3] When $1 < D \leq \overline{D}_c = 3/2$ (i.e., $\kappa \geq \overline{\kappa}_c = 8$), Theorem 1.2 (i) states for the Bessel flow that the ordering $T^x < T^y$ is conserved for any $0 < x < y$. It implies that in this phase the SLE path should be a *space-filling curve*:

$$\gamma[0, \infty) = \overline{\mathbb{H}}.$$

(Otherwise, a swallowing of regions occurs, contradicting Theorem 1.2 (i).)

Figure 2.7 summarizes the three phases of SLE paths.

The SLE paths are fractal curves and their *Hausdorff dimensions* $d_H^{(D)}$ are determined by Beffara [2] as

$$d_H^{(D)} = \begin{cases} 2 & \text{if } 1 < D < \overline{D}_c = \frac{3}{2}, \\ \frac{2D-1}{2(D-1)} & \text{if } D \geq \overline{D}_c = \frac{3}{2}. \end{cases} \quad (2.28)$$

We note that a reciprocity relation is found between D and $d_{\mathbb{H}}^{(D)}$ in **[Phase 1]** and **[Phase 2]**,

$$(D - 1)(d_{\mathbb{H}}^{(D)} - 1) = \frac{1}{2}, \quad D \geq \overline{D}_c = \frac{3}{2}. \quad (2.29)$$

The stochastic Loewner chain $(g_t)_{t \geq 0}$ as well as the SLE path $\gamma = (\gamma(t))_{t \geq 0}$ are functionals of BM. Therefore for each $D > 1$ we have a statistical ensemble of random curves $\{\gamma(\omega)\}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of BM. It is a statistical ensemble of SLE paths $\{\gamma(\omega)\}$ in the upper half plane \mathbb{H} , in which they start from the origin: $\gamma(0, \omega) = 0$, and approach infinity: $\lim_{t \rightarrow \infty} \gamma(t, \omega) = \infty$. We write the probability law of $\{\gamma(\omega)\}$ in such a geometrical setting as $\mathbb{P}_{(\mathbb{H}; 0, \infty)}$. In general, the probability law of SLE paths $\{\gamma(\omega)\}$ in an simply connected domain $D \subset \mathbb{C}$, $D \neq \mathbb{C}$ with $\gamma(0, \omega) = a \in \partial D$ and $\lim_{t \rightarrow \infty} \gamma(t, \omega) = b \in \partial D$ will be denoted by $\mathbb{P}_{(D; a, b)}$. The important consequence from the facts that BM is a strong Markov process with independent increments and that g_t gives a conformal transformation is the following [13].

(SLE1) The SLE path γ has the following kind of stationary Markov property,

$$\mathbb{P}_{(\mathbb{H}; 0, \infty)}[\cdot | \gamma(0, t)] = \mathbb{P}_{(\mathbb{H} \setminus \gamma(0, t]; \gamma(t), \infty)}[\cdot], \quad \forall t \geq 0. \quad (2.30)$$

This is called the *domain Markov property*.

(SLE2) Let f be a conformal transformation which maps \mathbb{H} to a domain $D = f(\mathbb{H})$. Then

$$\mathbb{P}_{(\mathbb{H}; 0, \infty)}[\cdot] = \mathbb{P}_{(D; f(0), f(\infty))}[\cdot]. \quad (2.31)$$

That is, the probability law of γ has *conformal invariance*. Here it should be remarked that the dependence on the geometry of an event should be properly mapped by f . For example, if the event of γ measured by $\mathbb{P}_{(\mathbb{H}; 0, \infty)}$ depends on a domain $A \subset \mathbb{H}$, the corresponding event of γ measured by $\mathbb{P}_{(D; f(0), f(\infty))}$ should be considered so that it depends on the domain $f(A)$.

2.4 Cardy's Formula

Let

$$T_{[1, \infty)} = \inf\{t > 0 : \gamma(t) \in [1, \infty)\}.$$

If $D \geq 2$, the SLE $^{(D)}$ path is in **[Phase 1]** and γ does not touch the real axis \mathbb{R} with probability 1. Therefore

$$T_{[1, \infty)} = \infty, \quad D \geq 2 \quad \text{with probability 1.}$$

If $1 < D \leq 3/2$, the $\text{SLE}^{(D)}$ path is in **[Phase 3]**, in which γ is a space-filling curve in $\overline{\mathbb{H}}$. Then

$$\gamma(T_{[1,\infty)}) = 1 \quad \text{with probability 1.}$$

When $3/2 < D < 2$, which corresponds to **[Phase 2]**, $\gamma(T_{[1,\infty)})$ has a nontrivial distribution on $[1, \infty)$ as follows.

Proposition 2.1 *Suppose γ is an $\text{SLE}^{(D)}$ path with $3/2 < D < 2$. Then, for $x > 0$*

$$\begin{aligned} \mathbb{P}_{(\mathbb{H};0,\infty)}[\gamma(T_{[1,\infty)}) < 1+x] \\ = \frac{\Gamma(D-1)}{\Gamma(2D-3)\Gamma(2-D)} \int_0^{x/(1+x)} \frac{du}{(1-u)^{D-1}u^{2(2-D)}} \end{aligned} \quad (2.32)$$

$$= \frac{\Gamma(D-1)}{\Gamma(2(D-1))\Gamma(2-D)} \left(\frac{x}{1+x}\right)^{2D-3} F\left(2D-3, D-1, 2(D-1); \frac{x}{1+x}\right). \quad (2.33)$$

Proof By (2.7) and (2.15), we see the equivalence between the events

$$\{\omega : \gamma(T_{[1,\infty)}) < 1+x\} \iff \{\omega : T^1 < T^{1+x}\}.$$

Then the probability is just obtained from (1.99) and (1.100) in Proposition 1.1 by setting $x \rightarrow 1$ and $y \rightarrow 1+x$. Then (2.32) and (2.33) are obtained. \square

When $D = 5/3$ (i.e., $\kappa = 6$), (2.32) gives

$$\mathbb{P}_{(\mathbb{H};0,\infty)}[\gamma^{(5/3)}(T_{[1,\infty)}) < 1+x] = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^{x/(1+x)} \frac{du}{(1-u)^{2/3}u^{2/3}}. \quad (2.34)$$

This formula has the following meaning, and it is called *Cardy's formula* [3, 4].¹

Let Δ be a domain in \mathbb{C} whose boundary is the equilateral triangle with vertices $w_1 = 0$, $w_2 = 1$ and $w_3 = e^{\pi\sqrt{-1}/3}$. The conformal map f_Δ from \mathbb{H} to Δ satisfying the conditions

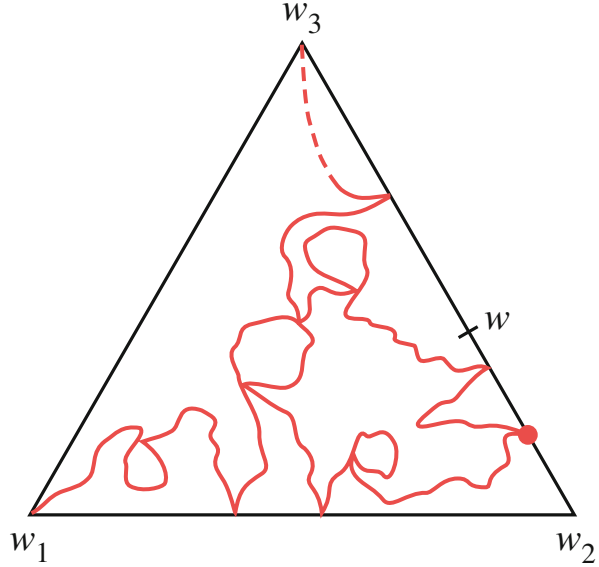
$$f_\Delta(0) = w_1, \quad f_\Delta(1) = w_2, \quad f_\Delta(\infty) = w_3 \quad (2.35)$$

is given by (Exercise 2.3)

$$f_\Delta(z) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^z \frac{du}{u^{2/3}(1-u)^{2/3}}, \quad z \in \mathbb{H}. \quad (2.36)$$

¹The original work by Cardy was given on a rectangular domain, but the statement is conformally invariant [3, 4]. The formula becomes particularly easy for an equilateral triangular domain as shown here, and is called *Cardy's formula in Carleson's form* [19]. See Sects. 6.7 and 6.8 in [13].

Fig. 2.8 Red curve describes an $\text{SLE}^{(5/3)}$ path, $\gamma^{(5/3)} = (\gamma^{(5/3)}(t))_{t \geq 0}$, in Δ starting from w_1 and approaching w_3 as $t \rightarrow \infty$. The point at which $\gamma^{(5/3)}$ first touches the line segment $[w_2, w_3]$ is marked by a red dot, which is given by $\gamma^{(5/3)}(T_{[w_2, w_3]})$. In this case $\gamma^{(5/3)}(T_{[w_2, w_3]}) \in [w_2, w]$ for a given $w \in [w_2, w_3]$



We take the branches in the integrand in (2.36) so that (Exercise 2.4)

$$f_{\Delta}(-x) = e^{\pi\sqrt{-1}/3} \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^{x/(1+x)} \frac{du}{u^{2/3}(1-u)^{2/3}}, \quad 0 < x < \infty, \quad (2.37)$$

$$f_{\Delta}(x) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^x \frac{du}{u^{2/3}(1-u)^{2/3}}, \quad 0 \leq x \leq 1, \quad (2.38)$$

$$f_{\Delta}(1+x) = 1 + e^{2\pi\sqrt{-1}/3} \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^{x/(1+x)} \frac{du}{u^{2/3}(1-u)^{2/3}}, \quad 0 < x < \infty. \quad (2.39)$$

Now we consider the $\text{SLE}^{(5/3)}$ path, $\gamma^{(5/3)}(t)$, $t \in [0, \infty)$ in Δ starting from w_1 and approaching w_3 as $t \rightarrow \infty$. Denote the line segment connecting w_2 and w_3 by $[w_2, w_3]$. Let

$$T_{[w_2, w_3]} = \inf\{t > 0 : \gamma^{(5/3)}(t) \in [w_2, w_3]\}.$$

See Fig. 2.8. For $w \in [w_2, w_3]$, we write the line segment between w_2 and w on $[w_2, w_3]$ as $[w_2, w]$ and the distance between w_2 and w as $|w - w_2|$. By the conformal invariance (**SLE2**) of $\text{SLE}^{(D)}$,

$$\mathbb{P}_{(\Delta; w_1, w_3)} \left[\gamma^{(5/3)}(T_{[w_2, w_3]}) \in [w_2, f_{\Delta}(1+x)] \right] = \mathbb{P}_{(\mathbb{H}; 0, \infty)} \left[\gamma^{(5/3)}(T_{[1, \infty)}) < 1+x \right].$$

Since (2.35) and (2.39) hold, the RHS given by (2.34) is equal to

$$\frac{f_{\Delta}(1+x) - w_2}{e^{2\pi\sqrt{-1}/3}} = |f_{\Delta}(1+x) - w_2|.$$

Then we can conclude the following.

Proposition 2.2 *Let $\gamma^{(5/3)}$ be the $SLE^{(5/3)}$ path in Δ from w_1 to w_3 . Then the distribution of $\gamma^{(5/3)}(T_{[w_2, w_3]})$ is uniform on $[w_2, w_3]$. That is,*

$$\mathbb{P}_{(\Delta; w_1, w_3)}\left[\gamma^{(5/3)}(T_{[w_2, w_3]}) \in [w_2, w]\right] = |w - w_2| \quad \text{for any point } w \in [w_2, w_3]. \quad (2.40)$$

2.5 SLE and Statistical Mechanics Models

The highlight of the theory of SLE would be that, if the value of D is properly chosen, the probability law of γ realizes that of the scaling limit of important lattice paths studied in a *statistical mechanics model* exhibiting *critical phenomena* or describing interesting *fractal geometry* defined on an infinite discrete lattice.

The following is a list of the correspondence (up to a conjecture) between the $SLE^{(D)}$ paths with specified values of D , and the names of lattice paths (with the names of models studied in statistical mechanics and fractal physics), whose scaling limits are described by the $SLE^{(D)}$ paths.

- $SLE^{(3/2)} \iff$ random Peano curve (uniform spanning tree) [14]
- $SLE^{(5/3)} \iff$ percolation exploration process (critical percolation model) [19]
- $SLE^{(7/4)} \iff$ FK–Ising interface (critical Ising model) [6, 20]
- $SLE^{(2)} \iff$ random contour curve (Gaussian free surface model) [18]
- $SLE^{(7/3)} \iff$ Ising interface (critical Ising model) [6, 7]
- $SLE^{(5/2)} \iff$ self-avoiding walk [conjecture]
- $SLE^{(3)} \iff$ loop-erased random walk [14]

It is obvious from (2.9) that $D = 3/2, 5/3, 7/4, 2, 7/3, 5/2$, and 3 correspond to $\kappa = 8, 6, 16/3, 4, 3, 8/3$, and 2, respectively.

The $SLE^{(D)}$ path has the following special property if and only if $D = 5/3$: for any $A \subset \mathbb{H}$ such that $0 \notin \partial A, \infty \notin \partial A$,

$$\mathbb{P}_{(\mathbb{H}; 0, \infty)}[\cdot, \gamma^{(5/3)}(0, t) \cap A = \emptyset] = \mathbb{P}_{(\mathbb{H} \setminus A; 0, \infty)}[\cdot], \quad t \geq 0.$$

This is called the *locality property*. The lattice path called the *percolation exploration process* $\{\gamma^{\text{per}}\}$ defined on the Bernoulli site percolation model [11, 12] studied in statistical mechanics has this property. Cardy conjectured that the scaling limit of $\{\gamma^{\text{per}}\}$ obtained from the *critical percolation model* satisfies (2.40) [3]. It was proved

by Smirnov [19] for the critical site percolation model on a triangular lattice by showing the conformal invariance of its scaling limit.

The $\text{SLE}^{(D)}$ path has another special property called the *restriction property*, if and only if $D = 5/2$ ($\kappa = 8/3$) [13]: if $D \subset \mathbb{H}$, $0 \in \partial D$, $\infty \in \partial D$, then

$$\mathbb{P}_{(\mathbb{H};0,\infty)}[\cdot, \gamma^{(5/2)}(0, \infty) \subset D] = \mathbb{P}_{(D;0,\infty)}[\cdot].$$

We can see that the *self-avoiding walk* (SAW) [15], which is defined on a lattice and has been studied as a model for polymers, has this property. The conformal invariance of the scaling limit of SAW is, however, not yet proved. If this holds true, then it would imply the equivalence in probability law between the scaling limit of SAW and the $\text{SLE}^{(5/2)}$ path. See [9, 10] for more details and other conjectures.

The relationship between the SLE and the *conformal field theory* (CFT) is discussed in [1, 5]. The *central charge* c and the *scaling dimension* h of the CFT are given as functions of D as

$$c = \frac{(3D-5)(5-2D)}{D-1}, \quad h = \frac{3D-5}{4}, \quad D > 1. \quad (2.41)$$

Exercises

2.1 Derive (2.13) from (2.5).

2.2 Consider the deterministic case where $B(t) \equiv 0$ in (2.5) and

$$\frac{\partial g_t(z)}{\partial t} = \frac{D-1}{2} \frac{1}{g_t(z)}, \quad t \geq 0. \quad (2.42)$$

(i) Solve the equation (2.42) under the initial condition (2.6).

(ii) Determine $\gamma(t)$, $t \geq 0$ by setting $-B(t) \equiv 0$ in (2.15).

2.3 As an application of the Schwarz–Christoffel formula given by Theorem 2.1, prove that (2.36) is the conformal map from \mathbb{H} to Δ satisfying (2.35).

2.4 Derive the expressions (2.37) and (2.39) from (2.36).

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