# Boundedness and Stability of Leslie–Gower Model with Sokol–Howell Functional Response

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**Abstract** In this chapter, a three-species model of Leslie–Gower predator–prey food chain model with Sokol–Howell functional response is proposed. The boundedness of the solution of the model is discussed. Local and global stability analyses of the system are carried out. The dynamics of the predator–prey food chain model with Sokol–Howell functional response is investigated theoretically as well as numerically.

**Keywords** Food chain · Chaotic · Leslie–Gower · Functional response · Sokol–Howell

## 1 Introduction

The work of May [1] exploring the chaotic behaviors of population dynamics inspired much research work in the predator–prey system [2–11]. Alaoui [12] proposed and studied the dynamics of a modified Leslie–Gower predator–prey food chain model with Holling type II functional response. Naji et al. [5] studied a modified Leslie–Gower food chain model with Bendigton–DeAnglis functional response and the

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model exhibited chaotic dynamics. Gakkhar with Priyadarshi [13] studied the Leslie–Gower food web system. The numerical works of Alaoui, Naji, and Gakkhar are brilliant and perfect.

In their experiments about the kinetics of phenol oxidation, Sokol and Howell [14] suggested a simplified Holling type IV function of the form $\frac{wx}{d+x^2}$ and found that it is simpler and better than the original function of Holling type IV. The Holling type IV response represents a situation in which the predation of the predator decreases at sufficiently high prey densities [10] and about how this functional response is obtained, see [15–17]. Ruan [18] and Hu [19] both studied the dynamics and bifurcation analysis of continuous-time and discrete-time models of modified Holling type IV, respectively. Investigations on Leslie–Gower type model [12, 13, 20, 21] and Sokol–Howell functional response [14, 18, 19, 22, 23] are relatively less than the other types like Lotka–Volterra and Bendigton–DeAnglis functional responses.

This paper is organized as follows: in Sect. 2, the mathematical model is proposed and each parameter in the model is described. In Sect. 3, the boundedness of the solution of the model is established. Stability analyses of the equilibrium points of the model are derived in Sect. 4. In Sect. 5, numerical study is carried out to obtain the behavior of the model. Finally, the paper ends with a conclusion in Sect. 6.

## 2 The Mathematical Model

Consider the three-species food chain model at time $t$ consisting of the prey population density denoted by $x(t)$, the middle predator population density denoted by $y(t)$, and the top predator whose population density denoted by $z(t)$. The middle predator $y$ preys on its sole food $x$ at the lower level according to simplified Holling type IV functional response, while the top predator $z$ preys on $y$ at the second level according to the modified Leslie–Gower type.

The dynamics of the model described above can be represented by the following set of differential equations:

$$
\begin{aligned}
\frac{dx}{dt} &= x\left(a_0 - b_0 x\right) - \frac{v_0 x y}{d_0 + x^2} : \quad x(0) \geq (0),\\
\frac{dy}{dt} &= \frac{v_1 x y}{d_1 + x^2} - a_1 y - \frac{v_2 y z}{d_2 + y} : \quad y(0) \geq (0),\\
\frac{dz}{dt} &= c_3 z^2 - \frac{v_3 z^2}{d_3 + y} : \quad z(0) \geq 0. \qquad (1)
\end{aligned}
$$

Here the positive constants $a_0$, $b_0$, $v_0$, $d_0$, $v_1$, $a_1$, $d_1$, $v_2$, $d_2$, $c_3$, $v_3$, and $d_3$ denote: $a_0$ is the growth rate of the prey $x$, $b_0$ represents the intraspecific competition among individuals of prey $x$, $v_i$'s are the maximum values attainable by each per capita rate, $d_0$ and $d_1$ measure the extent to which the environment provides protection to the prey $x$ and predator $y$, respectively, $a_1$ represent the death rate of $y$ in the absence of $x$, $d_2$ is the value of $y$ at which the per capita removal rate of $y$ becomes $\frac{v_2}{2}$, $c_3$

represents the growth rate of $z$ by sexual reproduction, the number of males and females being assumed to be equal, $d_3$ represents the residual loss in $z$ population due to serve scarcity of its favorite food $y$; the second term on the right-hand side in the third equation of system (1) depicts the loss in the top predator population.

**Remark**: The origin of the model of system (1) is standard in first two equations, but the third equation is absolutely not standard. About how the third equation obtained, see [12, 13, 20, 21].

Now according to the third equation of system (1):

$$\frac{dz}{dt} = z\left(c_3 z - \frac{v_3 z}{d_3 + y}\right),$$

if the middle predator $y$ is absence ($y = 0$), the top predator goes extinct if

$$c_3 d_3 < v_3, \tag{2}$$

and increase without bound if $c_3 d_3 > v_3$. In this paper, we will suppose that condition (2) holds.

Then, the system (1) when $d_0 = d_1$ can be written as follows:

$$\begin{aligned}
\frac{dx}{dt} &= x\left(a_0 - b_0 x - \frac{v_0 y}{d_1 + x^2}\right) = G_1(x, y, z),\\
\frac{dy}{dt} &= y\left(\frac{v_1 x}{d_1 + x^2} - a_1 - \frac{v_2 z}{d_2 + y}\right) = G_2(x, y, z),\\
\frac{dz}{dt} &= z\left(c_3 z - \frac{v_3 z}{d_3 + y}\right) = G_3(x, y, z),
\end{aligned} \tag{3}$$

with $x(0) \geq 0$, $y(0) \geq 0$, and $z(0) \geq 0$. Obviously, the interaction functions $G_i$ ($i = 1, 2, 3$) of system (3) are continuous and have continuous partial derivatives on the positive octant $R_+^3 = \left\{(x, y, z) \in R_+^3 : x \geq 0, y \geq 0, z \geq 0\right\}$. Therefore, these functions are Lipschitzian on $R_+^3$, and hence the solution of the system (3) exists and is unique.

## 3 Boundedness of the Model

In this section, the boundedness of the solution of the system (3) in $R_+^3$ is established in the next theorem.

**Theorem 1** *All the solutions of the three-species food chain system (3) are uniformly bounded, provided*

$$\frac{a_0 v_1}{b_0 v_0} + \frac{a_0^2 v_1}{4 a_1 b_0 v_0} + d_3 < \frac{v_3}{c_3}, \tag{4}$$

*and let $\Omega$ be the set defined by*

$$\Omega = \left\{(x, y, z) \in R_+^3 : 0 \le x \le \frac{a_0}{b_0},\ 0 \le x + \frac{v_0}{v_1} y \le \frac{a_0}{b_0} + \frac{a_0^2}{4a_1 b_0}, \right.$$
$$\left. 0 \le x + \frac{v_0}{v_1} y + \alpha z \le \frac{a_0}{b_0} + \frac{a_0^2}{4a_1 b_0} + \frac{N}{a_1} \right\}$$

*where*

$$\alpha = \frac{1}{a_1^2 \left(\frac{a_0 v_1}{b_0 v_0} + \frac{a_0^2 v_1}{4a_1 b_0 v_0} + d_3\right)} \tag{5}$$

$$N = \frac{1}{4\left(v_3 - \left(\frac{a_0 v_1}{b_0 v_0} + \frac{a_0^2 v_1}{4a_1 b_0 v_0} + d_3\right) c_3\right)} \tag{6}$$

*Proof* Let $(x(t), y(t), z(t))$ be any solution of the system with non-negative initial condition. Now there are three cases about the boundedness of the solutions.

- Case 1: To prove that x(t) is bounded $\forall t \ge 0$.

Since we have

$$\frac{dx}{dt} \le x\,(a_0 - b_0 x)\,, \tag{7}$$

then according to comparison theorem [24], we obtain that

$$\lim_{t\to\infty} Sup\ x(t) \le \frac{a_0}{b_0} \tag{8}$$

implies that $x(t) \le \frac{a_0}{b_0}$, $\forall t \ge 0$. Now as $t \to \infty$.

$$\Rightarrow 0 \le x(t) \le \frac{a_0}{b_0}.$$

- Case 2: To prove that x(t) and y(t) are bounded $\forall t \ge 0$.

Consider

$$M_1(t) = x(t) + \frac{v_0 y(t)}{v_1}, \quad M_1(0) \ge 0$$

Then

$$\frac{dM_1(t)}{dt} \le x(a_0 - b_0 x) - \frac{a_1 v_0}{v_1} y$$

Since in $\Omega$, $0 \leq x \leq \frac{a_0}{b_0}$ and simplification using $Max_{[0,\frac{a_0}{b_0}]}x(a_0 - b_0x) = \frac{a_0^2}{4b_0}$ gives

$$\frac{dM_1(t)}{dt} + a_1M_1(t) \leq \frac{a_0}{b_0} + \frac{a_0^2}{4a_1b_0} \tag{9}$$

Therefore for all $t \geq 0$

$$M_1(t) \leq \left(\frac{a_0}{b_0} + \frac{a_0^2}{4a_1b_0}\right) - \left[\left(\frac{a_0}{b_0} + \frac{a_0^2}{4a_1b_0}\right) - M_1(0)\right]e^{-a_1M_1t} \tag{10}$$

Hence as $t \to \infty$, since $(x(0), y(0), z(0)) \in \Omega$

$$x(t) + \frac{v_0}{v_1}y(t) \leq \frac{a_0}{b_0} + \frac{a_0^2}{4a_1b_0} \ \ \forall t \geq 0. \tag{11}$$

Similarly for Case 3

$$0 \leq x + \frac{v_0}{v_1}y + \alpha z \leq \frac{a_0}{b_0} + \frac{a_0^2}{4a_1b_0} + \frac{N}{a_1}.$$

Therefore, every solution initiated in non-negative octant are attracted in a bounded set $\Omega$ defined above, which implies to the uniformly bounded of $y(t)$ and $z(t)$. Thus the proof is complete.

## 4 Stability of the Model

There are at most three non-negative equilibrium points of system (3) in addition to the positive equilibrium point $E_3$ in $R+^3$ existence and stability conditions of them are given as follows:

- The trivial equilibrium point $E_0 = (0, 0, 0)$ always exits.
- The equilibrium point $E_1 = (\frac{a_0}{b_0}, 0, 0)$ always exists on the boundary of the first octant.
- The middle predator can exist and survive depending on its prey. Therefore, the equilibrium point $E_2 = (\bar{x}, \bar{y}, 0)$ exists uniquely in the positive quadrant of $x$–$y$ plane where $\bar{x}$ and $\bar{y}$ are given by:

$$\bar{x} = \frac{v_1}{2a_1}, \ \ \bar{y} = \frac{1}{v_0}\left(a_0 - b_0\frac{v_1}{2a_1}\right)(d_1 + \bar{x}^2), \tag{12}$$

provided that the following conditions hold

$$\frac{v_1}{2a_1} < \frac{a_0}{b_0}, \ \ v_1^2 - 4{a_1}^2d_1 = 0 \tag{13}$$

- In the absence of prey $x$, then both $y$ and $z$ cannot survive, so there is no equilibrium point in the $y$–$z$ plane. In addition to that, if the middle predator $y$ is absent, then there is no equilibrium point in the $x$–$z$ plane.
- The positive equilibrium point $E_3 = (x^*, y^*, z^*)$ exists in the interior of the first octant if and only if there is a positive solution to the following equations:

$$
\begin{aligned}
f_1 &= a_0 - b_0 x - \frac{v_0 y}{d_1 + x^2} = 0, \\
f_2 &= \frac{v_1 x}{d_1 + x^2} - a_1 - \frac{v_2 z}{d_2 + y} = 0, \\
f_3 &= c_3 z - \frac{v_3 z}{d_3 + y} = 0.
\end{aligned} \tag{14}
$$

Straightforward computation shows that

$$
y^* = \frac{v_3}{c_3} - d_3, \tag{15}
$$

while $x^*$ is the positive root of the following equation

$$
x^3 - \frac{a_0}{b_0} x^2 + d_1 x + \left[ \frac{1}{b_0} \left( v_0 y^* - a_0 d_1 \right) \right] = 0,
$$

this equation can be rewritten as

$$
f(x) = Ax^3 + Bx^2 + Cx + D = 0, \tag{16}
$$

where $A = 1$, $B = -\frac{a_0}{b_0}$, $C = d_1$ and $D = \left[ \frac{1}{b_0} \left( v_0 y^* - a_0 d_1 \right) \right]$.

Now since $0 \leq x^* \leq \frac{a_0}{b_0}$, then $f(0) = D < 0$, if

$$
y^* < \frac{a_0}{v_0} d_1, \tag{17}
$$

$f(\frac{a_0}{b_0}) = \frac{v_0}{b_0} y^* > 0$. Thus, $f(0) f(\frac{a_0}{b_0}) < 0$, and then there is a positive root of Eq. (16) lies in $(0, \frac{a_0}{b_0})$ when $y^* < \frac{a_0}{v_0} d_1$, is satisfied.

The second equation of (14) gives

$$
z^* = \frac{(d_2 + y^*)}{v_2} \left( \frac{v_1 x^*}{d_1 + x^{*2}} - a_1 \right) \tag{18}
$$

Therefore, the positive equilibrium point $E_3 = (x^*, y^*, z^*)$ exists if in addition to conditions (2) and (17), the following condition holds:

$$
a_1 < \frac{v_1 x^*}{(d_1 + x^{*2})}. \tag{19}
$$

Now in order to investigate the dynamical behavior of the three species food chain system (3) near the above equilibrium points, the variational matrix $V$ of system (3) at $(x, y, z)$ is computed as:

$$V(x, y, z) = \begin{bmatrix} x\frac{\partial f_1}{\partial x} + f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y} + f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z} + f_3 \end{bmatrix}$$

where $\frac{\partial f_1}{\partial x} = -b_0 + \frac{2v_0xy}{(d_1+x^2)}$, $\frac{\partial f_1}{\partial y} = -\frac{v_0}{(d_1+x^2)}$, $\frac{\partial f_1}{\partial z} = 0$, $\frac{\partial f_2}{\partial x} = \frac{v_1(d_1-x^2)}{(d_1+x^2)}$, $\frac{\partial f_2}{\partial y} = \frac{v_2z}{(d_2+y)^2}$, $\frac{\partial f_2}{\partial z} = -\frac{v_2}{(d_2+y)}$, $\frac{\partial f_3}{\partial x} = 0$, $\frac{\partial f_3}{\partial y} = \frac{v_3z}{(d_3+y)^2}$, $\frac{\partial f_3}{\partial z} = c_3 - \frac{v_3}{d_3+y}$.

Further, the stability analysis of the system (3) is carried out and according to the variational matrix $V_i; i = 0, 1, 2, 3$ of $E_i; i = 0, 1, 2, 3$ respectively, the following results are obtained:

$$V_0 = \begin{bmatrix} a_0 & 0 & 0 \\ 0 & -a_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -a_0 & -\frac{v_0}{d_1+1} & 0 \\ 0 & \frac{v_1}{d_1+1} - a_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} \bar{x}\left(-b_0 + \frac{2\bar{x}(a_0-b_0\bar{x})}{(d_1+\bar{x}^2)}\right) & -\frac{v_0\bar{x}}{(d_1+\bar{x}^2)} & 0 \\ \frac{v_1\bar{y}(d_1-\bar{x}^2)}{(d_1+\bar{x}^2)^2} & \frac{v_1\bar{x}}{(d_1+\bar{x}^2)} - a_1 & \frac{-v_2\bar{y}}{d_1+y} \\ 0 & 0 & 0 \end{bmatrix}.$$

- Here the trivial equilibrium point $E_0$ is a nonhyperbolic saddle-node, having an unstable manifold along $x$-direction.
- From variational matrix $V_1$, it is observed that the nonhyperbolic equilibrium point $E_1$ is a saddle point having stable manifold along $x$-direction if the following condition holds

$$\frac{v_1}{d_1+1} > a_1, \tag{20}$$

while nonhyperbolic equilibrium point $E_1$ having stable manifold along $x$ and $y$-direction if

$$\frac{v_1}{d_1+1} < a_1. \tag{21}$$

- The equilibrium point $E_2 = (\bar{x}, \bar{y}, 0)$ is a nonhyperbolic point having stable manifold along $x$ and $y$-direction if the following conditions hold.

$$\frac{2\bar{x}(a_0 - b_0\bar{x})}{(d_1 + \bar{x}^2)^2} < b_0, \quad \frac{v_1\bar{x}}{d_1 + \bar{x}^2} < a_1 \tag{22}$$

while $E_2$ is unstable saddle if the opposite of any part of condition (22) hold.

However, for the positive equilibrium point $E_3 = (x^*, y^*, z^*)$, the variational matrix is:

$$V_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{23}$$

The characteristic equation of the variational matrix (23) can be written as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$$

where $A_1 = -(a_{11} + a_{22})$, $A_2 = a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32}$, and $A_3 = a_{11}a_{23}a_{32}$. According to Routh–Hurwitz criterion, $E_3 = (x^*, y^*, z^*)$ is locally asymptotically stable provided $A_1 > 0$, $A_3 > 0$, and $\Delta = A_1A_2 - A_3 > 0$.

Now straightforward computations show that, $A_1 > 0$ and $A_3 > 0$ if and only if the following conditions are satisfied:

$$a_0 < 2b_0x + \frac{v_0y^*(d_1 - x^{*2}) - v_1x^*R}{R^2} + \left(a_1 + \frac{v_2z^*}{Q_1^2}\right), \tag{24}$$

with

$$\frac{v_1}{v_0} < \frac{y^*(d_1 - x^{*2})}{x^*R}, \quad x^{*2} < d_1. \tag{25}$$

In addition to that, since

$$A_1A_2 - A_3 = (a_{11} + a_{22})(a_{12}a_{21} - a_{11}a_{22}) + a_{22}a_{23}a_{32}.$$

Hence, the necessary condition for $A_1A_2 - A_3 > 0$ is

$$(a_{12}a_{21} - a_{11}a_{22}) < 0,$$

which is equivalent to the following condition

$$\left[a_0 - \left(2b_0x^* + \frac{v_0y^*(d_1 - x^{*2})}{R^2}\right)\right] \left[\frac{v_1x^*}{R} - \left(a_1 + \frac{v_2d_2z^*}{Q_1^2}\right)\right] + \frac{v_0v_1x^*y^*(d_1 - x^{*2})}{R^3} > 0. \tag{26}$$

Therefore, depending on the above analysis, the locally asymptotically stable in $Int R^3_+$ of the positive equilibrium point $E_3 = (x^*, y^*, z^*)$ is discussed in the following theorem

**Theorem 2** *Suppose that positive equilibrium point* $E_3 = (x^*, y^*, z^*)$ *exists in* $Int R^3_+$*, then* $E_3$ *is locally asymptotically stable provided conditions (24–26) hold.*

*Proof* Follows directly from Routh–Hurwitz criterion [25].

## 5 Numerical Simulation

In this section, the global dynamics of system (3) are studied numerically. The food chain system is solved numerically using predictor–corrector method with six order Runge–Kutta method [26]. System (3) run for 60,000 time steps and the first 30,000

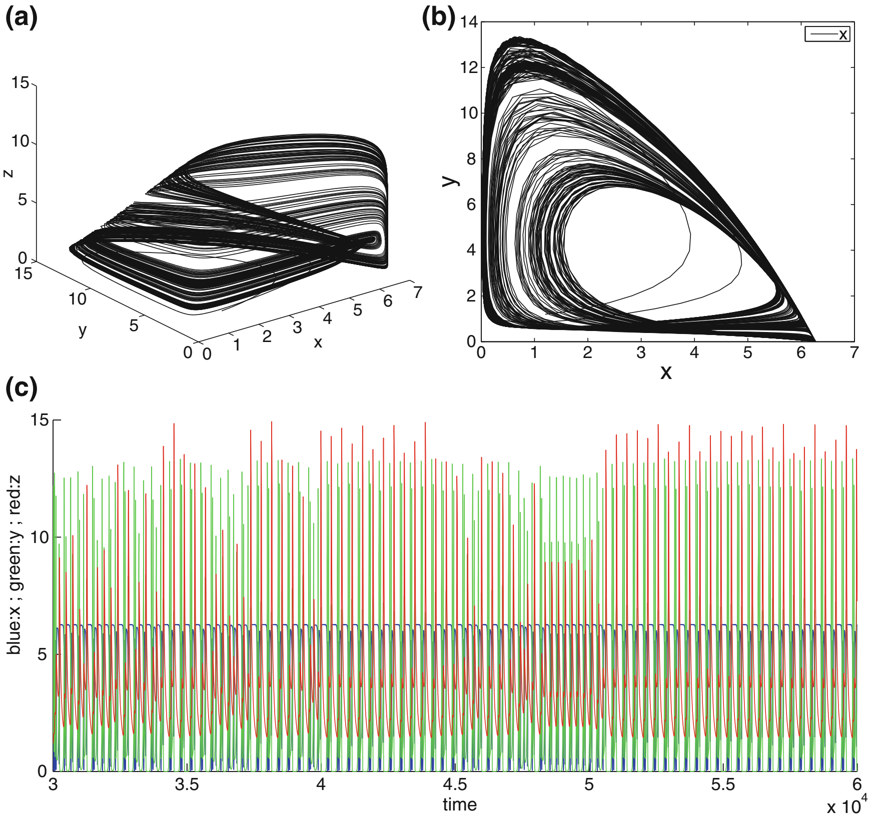


**Fig. 1** **a** 3D of system (3) chaotic attractor for data (27) with $a_0 = 0.47$ and $c_3 = 0.047$, **b** 2D $x$–$y$ plane of Fig. 1a. **c** Time series of Fig. 1a

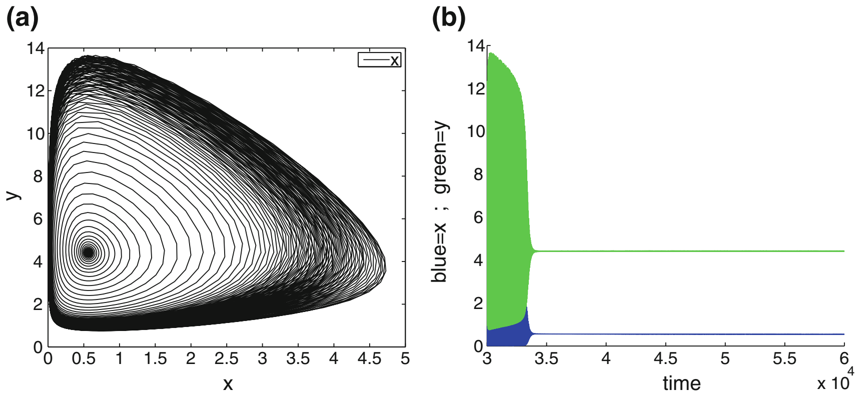


**Fig. 2** **a** 2D $x$–$y$ plane of system (3) asymptotically stable for data (27), $a_0 = 0.47$ and $c_3 = 0.040$, **b** Time series of Fig. 2a

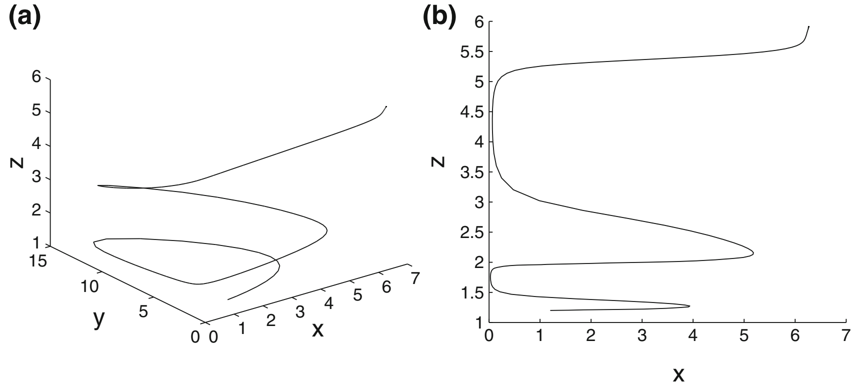


**Fig. 3** **a** 3D of system (3) asymptotically stable for data set (27), $a_0 = 0.47$ and $c_3 = 0.050$, **b** 2D $x$–$z$ plane of Fig. 3a

time steps are deleted to eliminate the transient effect. For the following set of fixed parameter values

$$
\begin{aligned}
&a_1 = 0.105,\ b_0 = 0.075,\ d_1 = 10.0,\ d_2 = 10.0,\\
&d_3 = 20.0,\ v_0 = 1.0,\ v_1 = 2.0,\ v_2 = 0.405,\ v_3 = 1.0.
\end{aligned}
\tag{27}
$$

The attractor of system (3) in 3D and 2D with their time series are drawn in Fig. 1 for the initial condition (1.2, 1.2, 1.2). The main objective is to explore the possibility of chaotic behavior of system (3) by depending on the controlling parameter and keeping other parameters of (27) fixed.

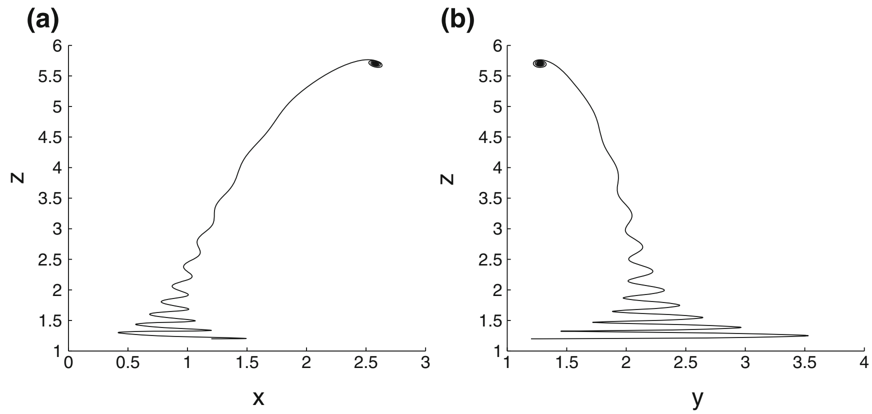


**Fig. 4** **a** 2D $x$–$z$ plane of system (3) asymptotically stable point for data in (27) with $a_0 = 0.27$ and $c_3 = 0.047$, **b** 2D $y$–$z$ plane of system (3) with same data of Fig. 4a

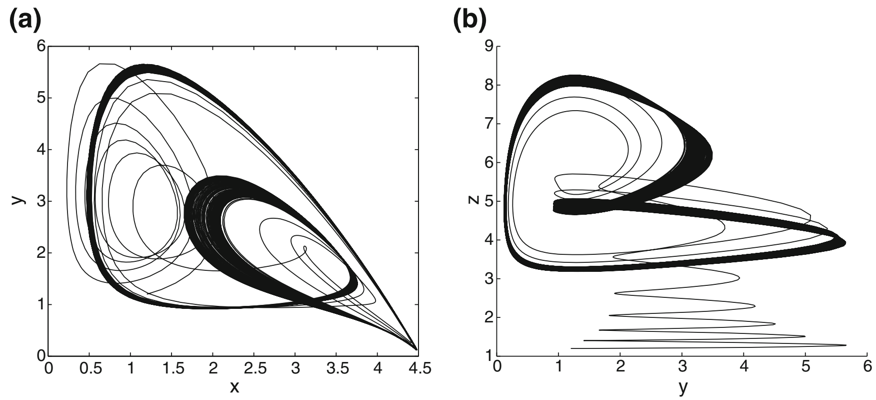


**Fig. 5** **a** 2D $x$–$y$ plane of system (3) period doubling attractor with $a_0 = 0.34$ and $c_3 = 0.047$, **b** 2D $y$–$z$ plane of Fig. 5a

- The first case by fixing $a_0 = 0.47$ and varying the value of $c_3$ in the range 0.041–0.049, it is observed that the system approaches to chaotic dynamics of system (3) as shown in Fig. 1 for $c_3 = 0.047$, while decreasing the value of $c_3$ less than 0.041 stabilizing the system as shown in Fig. 2 for the typical value $c_3 = 0.040$. Further increasing the of $c_3$ more than 0.049 change the dynamics of system (3) from chaotic to stable point (6.265, 1.003e−192, 5.910) as shown in Fig. 3 for the typical value $c_3 = 0.050$.
- The second case is by fixing $c_3 = 0.047$ and varying the value of $a_0$ in the range 0.27–0.047 with data in (27). For the value $a_0 = 0.27$ the system approach to stable at the point (2.577, 1.276, 5.700) as in Fig. 4. Increasing the value of $a_0 = 0.34$,

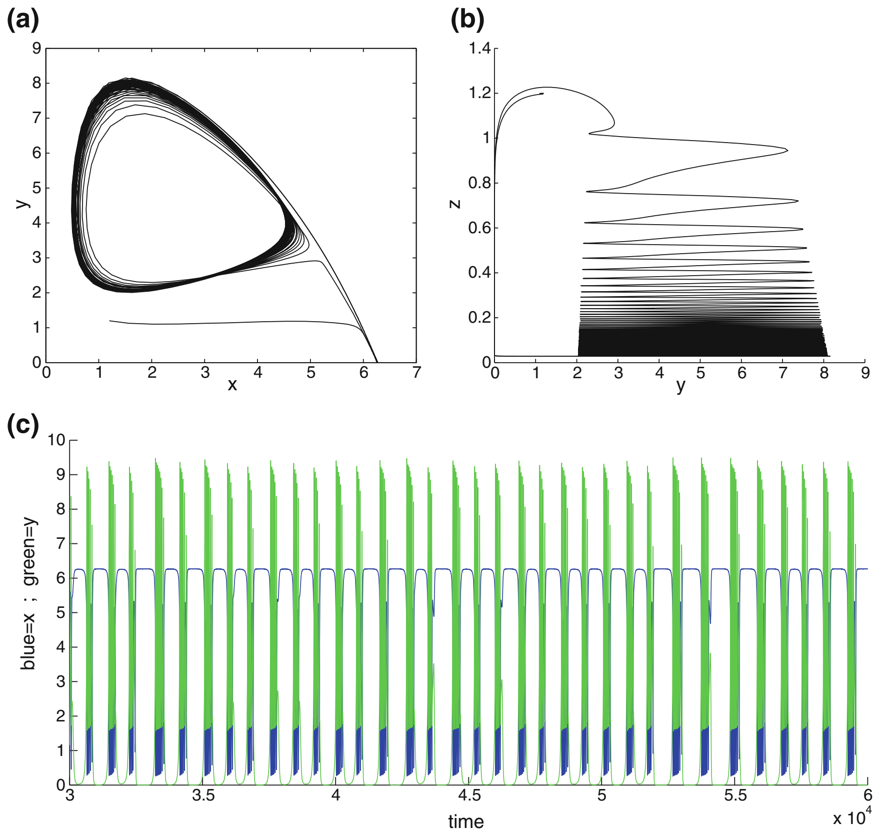


**Fig. 6** **a** 2D $x$–$y$ plane of system (3) strange attractor approach to periodic for data in (27) with $a_0 = 0.47$, $c_3 = 0.047$ and $a_1 = 0.25$, **b** 2D $y$–$z$ plane tea-cup attractor for data set Fig. 6a, **c** time series of Fig. 6a

change the dynamics of system (3) to period doubling as shown in Fig. 5, while increasing $a_0$ more than 0.034 change the dynamics of the system to chaotic for the typical value of $a_0 = 0.47$ as shown in Fig. 1.

- The third case about fixing $a_0 = 0.47$ and $c_3 = 0.047$ with same data in (27), it is observed by increasing the death rate of the middle predator $a_1 = 0.25$ then system (3) behavior is chaotic approach to periodic as shown in Fig. 6. Further increasing of $a_1$ more than 0.25 change system (3) dynamics from chaotic to asymptotically stable.

## 6 Conclusion

In order to explain the dynamical behavior of the proposed food chain systems (3), local as well as global stability analysis are carried out. Boundedness of the system is discussed.

In addition to that to confirm the analytical results, the system is solved numerically for different sets of biologically feasible parameter values, and then the attracting sets with their time series are drown in order to explain the dynamical behavior of the model as in Figs. 1–6 with data the same as in (27). According to our study, the following results are obtained:

(1) The intrinsic growth rate of the top predator $c_3$ is a sensitive parameter which lead to sensitivity of system (3) dynamics. Decreasing $c_3$ less than 0.041 and increasing $c_3$ more than 0.049 lead to change the dynamics of the food chain model from chaotic to asymptotically stable as shown in Figs. 2 and 3, which prove that a small change in $c_3$ will lead to major change in the dynamics of system (3).
(2) System (3) has a chaotic dynamics as in Fig. 1, but if we decreasing the values of intrinsic growth of the prey species $a_0$, then system (3) approaches to periodic attractor as in Fig. 5 and decreasing $a_0$ more change the behavior of the system to an asymptotically stable as in Fig. 4, so decreasing $a_0$ has a stabilizing effect on the dynamics of system (3) and $a_0$ is control parameter of system (3).
(3) Increasing the death rate $a_1$ in the middle predator $y$ has a stabilizing effect in the dynamics of system (3) as shown in Fig. 6.

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