Chapter 2
Rods and Trusses

Abstract This chapter starts with the analytical description of rod/bar members. Based on the three basic equations of continuum mechanics, i.e. the kinematics relationship, the constitutive law and the equilibrium equation, the partial differential equation, which describes the physical problem, is derived. The weighted residual method is then used to derive the principal finite element equation for rod elements. Assembly of elements and the consideration of boundary conditions is treated in detail. The chapter concludes with the spatial arrangements of rod elements in a plane to form truss structures.

2.1 Introduction

A rod is defined as a prismatic body whose axial dimension is much larger than its transverse dimensions. This structural member is only loaded in the direction of the main body axes, see Fig. 2.1a. As a result of this loading, the deformation occurs only along its main axis.
The following derivations are restricted to some simplifications:

- only applying to straight rods,
- displacements are (infinitesimally) small,
- strains are (infinitesimally) small,
- material is linear-elastic.

The ultimate goal of the finite element approach is to replace the continuum description of the structural member (partial differential equation) by a discretized description based on finite elements (denoted by Roman numerals) where the nodes (denoted by Arabic numbers) now play a major role for the evaluation of the primary quantities, see Fig. 2.1b. It should be noted here that the alternatively nomenclature 'bar' is also found in scientific literature to describe a rod member.

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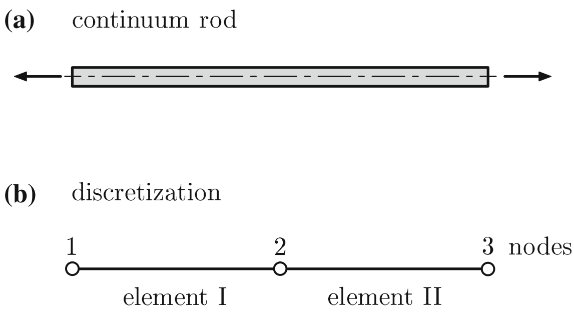


Fig. 2.1 **a** Continuum rod and **b** discretization with two finite elements

2.2 Derivation of the Governing Differential Equation

2.2.1 Kinematics

To derive the strain-displacement relation (kinematics relation), an axially loaded rod is considered as shown in Fig. 2.2. The length of the member is equal to L and the constant axial tensile stiffness is equal to EA. The load is either given as a single force F_x and/or as a distributed load $p_x(x)$.

This distributed load has the unit of force per unit length. In the case of a body force f_x (unit: force per unit volume), the distributed load takes the form $p_x(x) = f_x(x)A(x)$ where A is the cross-sectional area of the rod. A typical example for a body force would be the dead weight, i.e. the mass under the influence of gravity. In the case of a traction force t_x (unit: force per unit area), the distributed load can be written as $p_x(x) = t_x(x)U(x)$ where $U(x)$ is the perimeter of the cross section. Typical examples are frictional resistance, viscous drag and surface shear.

Let us now consider a differential element $\mathrm{d}x$ of such a rod as shown in Fig. 2.3. Under an acting load, this element deforms as indicated in Fig. 2.3b where the initial point at the position x is displaced by u_x and the end point at the position $x + \mathrm{d}x$ is displaced by $u_x + \mathrm{d}u_x$. Thus, the differential element which has a length of $\mathrm{d}x$ in the unloaded state elongates to a length of $\mathrm{d}x + (u_x + \mathrm{d}u_x) - u_x$.
The engineering strain, i.e. the increase in length related to the original length, can be expressed as

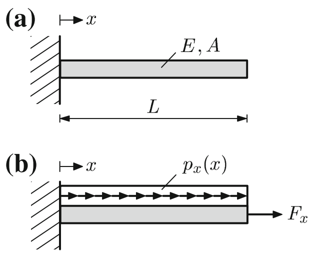


Fig. 2.2 General configuration of an axially loaded rod: **a** geometry and material property; **b** prescribed loads

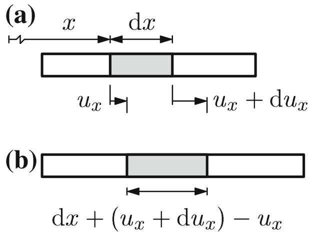


Fig. 2.3 Elongation of a differential element of length dx: **a** undeformed configuration; **b** deformed configuration

$$\varepsilon_x = \frac{(\mathrm{d}x + (u_x + \mathrm{d}u_x) - u_x) - (\mathrm{d}x)}{\mathrm{d}x}, \tag{2.1}$$

or finally as:

$$\varepsilon_x(x) = \frac{\mathrm{d}u_x(x)}{\mathrm{d}x}. \tag{2.2}$$

The last equation is often expressed in a less mathematical way (non-differential) as $\varepsilon_x = \frac{\Delta L}{L}$ where ΔL is the change in length of the entire rod element.

2.2.2 *Constitutive Equation*

The constitutive equation, i.e. the relation between stress σ_x and strain ε_x, is given in its simplest form as HOOKE's law[1]

$$\sigma_x(x) = E\varepsilon_x(x), \tag{2.3}$$

where the YOUNG's modulus[2] E is in the case of linear elasticity a material constant. For the considered rod element, the normal stress and strain is constant over the cross section as shown in Fig. 2.4.

2.2.3 *Equilibrium*

The equilibrium equation between the external forces and internal reactions can be derived for a differential element of length dx as shown in Fig. 2.5. It is assumed for simplicity that the distributed load p_x and the cross-sectional area A are constant in

[1]Robert HOOKE (1635–1703), English natural philosopher, architect and polymath.

[2]Thomas YOUNG (1773–1829), English polymath.

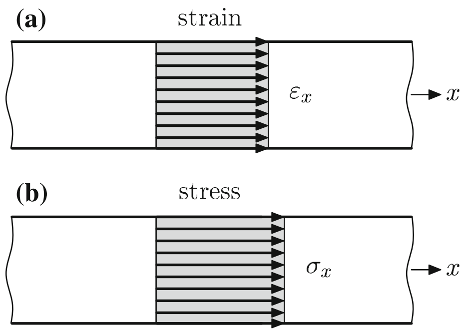


Fig. 2.4 Axially loaded rod: **a** strain and **b** stress distribution

this figure. The internal reactions N_x are drawn in their positive directions, i.e. at the left-hand face in the negative and at the right-hand face in the positive x-direction. The force equilibrium in the x-direction for a static configuration requires that

$$-N_x(x) + p_x\mathrm{d}x + N_x(x + \mathrm{d}x) = 0 \tag{2.4}$$

holds. A first order TAYLOR's[3] series expansion (cf. Appendix A.10) of the normal force $N_x(x + \mathrm{d}x)$ around point x, i.e.

$$N_x(x + \mathrm{d}x) \approx N_x(x) + \left.\frac{\mathrm{d}N_x}{\mathrm{d}x}\right|_x \mathrm{d}x, \tag{2.5}$$

allows to finally express Eq. (2.4) as:

$$\frac{\mathrm{d}N_x(x)}{\mathrm{d}x} = -p_x(x). \tag{2.6}$$

The three fundamental equations to describe the behavior of a rod element are summarized in Table 2.1.

A slightly different derivation of the equilibrium equation is obtained as follows: Equation (2.4) can be expressed based on the normal stresses as:

$$-\sigma_x(x)A + p_x\mathrm{d}x + \sigma_x(x + \mathrm{d}x)A = 0. \tag{2.7}$$

A first order TAYLOR's series expansion of the stress $\sigma_x(x + \mathrm{d}x)$ around point x, i.e.

$$\sigma_x(x + \mathrm{d}x) \approx \sigma_x(x) + \left.\frac{\mathrm{d}\sigma_x}{\mathrm{d}x}\right|_x \mathrm{d}x, \tag{2.8}$$

allows to finally express Eq. (2.7) as:

[3] Brook TAYLOR (1685–1731), English mathematician.

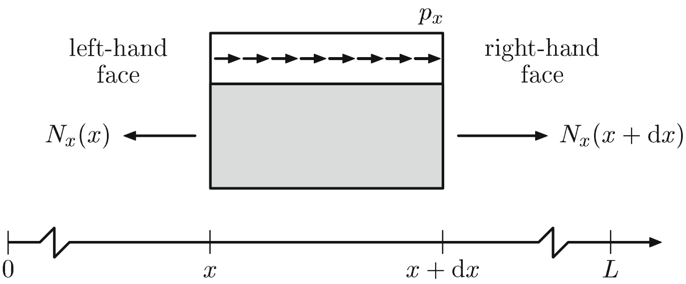


Fig. 2.5 Differential element of a rod with internal reactions and constant external distributed load

Table 2.1 Fundamental governing equations of a rod for deformation along the x-axis

Expression	Equation
Kinematics	$\varepsilon_x(x) = \dfrac{\mathrm{d}u_x(x)}{\mathrm{d}x}$
Equilibrium	$\dfrac{\mathrm{d}N_x(x)}{\mathrm{d}x} = -p_x(x)$
Constitution	$\sigma_x(x) = E\varepsilon_x(x)$

$$\frac{\mathrm{d}\sigma_x(x)}{\mathrm{d}x} + \frac{p_x(x)}{A} = 0. \tag{2.9}$$

The last equation with $\sigma_x = \frac{N_x}{A}$ immediately gives Eq. (2.6).

2.2.4 *Differential Equation*

To derive the governing partial differential equation, the three fundamental equations given in Table 2.1 must be combined. Introducing the kinematics relation (2.2) into HOOKE's law (2.3) gives:

$$\sigma_x(x) = E\frac{\mathrm{d}u_x}{\mathrm{d}x}. \tag{2.10}$$

Considering that a normal stress in the last equation is defined as an acting force N_x over a cross-sectional area A:

$$\frac{N_x}{A} = E\frac{\mathrm{d}u_x}{\mathrm{d}x}. \tag{2.11}$$

The last equation can be differentiated with respect to the x-coordinate to give:

$$\frac{\mathrm{d}N_x}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u_x}{\mathrm{d}x}\right), \tag{2.12}$$

where the derivative of the normal force can be replaced by the equilibrium equation (2.6) to obtain in the general case:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)A(x)\frac{\mathrm{d}u_x(x)}{\mathrm{d}x}\right) = -p_x(x). \tag{2.13}$$

If the axial tensile stiffness *EA* is constant, the last formulation can be simplified to:

$$EA\frac{\mathrm{d}^2u_x(x)}{\mathrm{d}x^2} = -p_x(x). \tag{2.14}$$

Some common formulations of the governing partial differential equation are collected in Table 2.2. It should be noted here that some of the different cases given in Table 2.1 can be combined. The last case in Table 2.1 refers to the case of elastic embedding of a rod where the embedding modulus k has the unit of force per unit area.

If we replace the common formulation of the first order derivative, i.e. $\frac{\mathrm{d}(\ldots)}{\mathrm{d}x}$, by a formal operator symbol, i.e. $\mathcal{L}_1(\ldots)$, the basic equations can be stated in a more formal way as given in Table 2.3.

Table 2.2 Different formulations of the partial differential equation for a rod (x-axis: right facing)

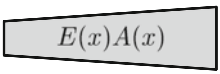
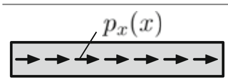
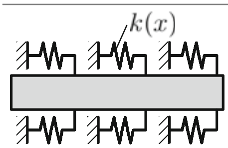
Configuration	Partial differential equation
	$EA\frac{\mathrm{d}^2u_x}{\mathrm{d}x^2} = 0$
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)A(x)\frac{\mathrm{d}u_x}{\mathrm{d}x}\right) = 0$
	$EA\frac{\mathrm{d}^2u_x}{\mathrm{d}x^2} = -p_x(x)$
	$EA\frac{\mathrm{d}^2u_x}{\mathrm{d}x^2} = k(x)u_x$

Table 2.3 Different formulations of the basic equations for a rod (x-axis along the principal rod axis)

Specific formulation	General formulation
Kinematics	
$\varepsilon_x(x) = \dfrac{\mathrm{d}u_x(x)}{\mathrm{d}x}$	$\varepsilon_x(x) = \mathcal{L}_1\,(u_x(x))$
Constitution	
$\sigma_x(x) = E\varepsilon_x(x)$	$\sigma_x(x) = C\varepsilon_x(x)$
Equilibrium	
$\dfrac{\mathrm{d}\sigma_x(x)}{\mathrm{d}x} + \dfrac{p_x(x)}{A} = 0$	$\mathcal{L}_1\,(\sigma_x(x)) + b = 0$
PDE ($A = \text{const.}$)	
$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(E(x)\dfrac{\mathrm{d}u_x}{\mathrm{d}x}\right) + \dfrac{p_x(x)}{A} = 0$	$\mathcal{L}_1\,(C\mathcal{L}_1\,(u_x(x))) + b = 0$

2.3 Finite Element Solution

2.3.1 Derivation of the Principal Finite Element Equation

Let us consider in the following the governing differential equation according to Eq. (2.14). This formulation assumes that the axial tensile stiffness EA is constant and we obtain

$$EA\frac{\mathrm{d}^2u^0(x)}{\mathrm{d}x^2} + p(x) = 0, \tag{2.15}$$

where $u^0(x)$ represents the *exact* solution of the problem. The last equation which contains the exact solution of the problem is fulfilled at each location x of the rod and is called the *strong formulation* of the problem. Replacing the exact solution in Eq. (2.15) by an approximate solution $u(x)$, a residual r is obtained:

$$r(x) = EA\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + p(x) \neq 0. \tag{2.16}$$

As a consequence of the introduction of the approximate solution $u(x)$, it is in general no longer possible to satisfy the differential equation at each location x of the rod. It is alternatively requested in the following that the differential equation is fulfilled over a certain length (and no longer at each location x) and the following integral statement is obtained

$$\int_0^L W(x)\left(EA\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + p(x)\right)\mathrm{d}x \stackrel{!}{=} 0, \tag{2.17}$$

which is called the *inner product*.[4] The function $W(x)$ in Eq. (2.17) is called the weight function which distributes the error or the residual in the considered domain.

Integrating by parts[5] of the first expression in the brackets of Eq. (2.17) gives

$$\int_0^L \underbrace{W}_{f} EA \underbrace{\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}}_{g'} \mathrm{d}x = EA \left[W \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^L - EA \int_0^L \frac{\mathrm{d}W(x)}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \mathrm{d}x. \qquad (2.18)$$

Under consideration of Eq. (2.17), the so-called *weak formulation* of the problem is obtained as:

$$EA \int_0^L \frac{\mathrm{d}W(x)}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \mathrm{d}x = EA \left[W(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^L + \int_0^L W(x) p(x)\, \mathrm{d}x. \qquad (2.19)$$

Looking at the weak formulation, it can be seen that the integration by parts shifted one derivative from the approximate solution to the weight function and a symmetrical formulation with respect to the derivatives is obtained. This symmetry with respect to the derivatives of the approximate solution and the weight function will guarantee in the following that a symmetric stiffness matrix is derived for the rod element. Figure 2.6 illustrates some common approximation methods in the context of the weighted residual method.

In order to continue the derivation of the principal finite element equation, the displacement $u(x)$ and the weight function $W(x)$ must be expressed by some functions. The common way to express the unknown function $u(x)$ in the scope of the finite element method is the so-called nodal approach. This approach states that the unknown function within an element (superscript 'e') is given by

$$u^{\mathrm{e}}(x) = \boldsymbol{N}^{\mathrm{T}}(x)\, \boldsymbol{u}_{\mathrm{p}} = \begin{bmatrix} N_1 & N_2 & \ldots & N_n \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \qquad (2.20)$$

where $\boldsymbol{u}_{\mathrm{p}}$ is the column matrix of n nodal unknowns and $\boldsymbol{N}(x)$ is the column matrix of the *interpolation functions*. Thus, the displacement at any point inside an element

[4]The general formulation of the inner product states the integration over the volume V, see Eq. (7.20). For this integration, the strong form (2.15) must be written as $E\frac{\mathrm{d}^2 u^0(x)}{\mathrm{d}x^2} + \frac{p(x)}{A}$ at which the distributed load is now given as force per unit volume.

[5]A common representation of integration by parts of two functions $f(x)$ and $g(x)$ is: $\int f g' \mathrm{d}x = f g - \int f' g\, \mathrm{d}x$.

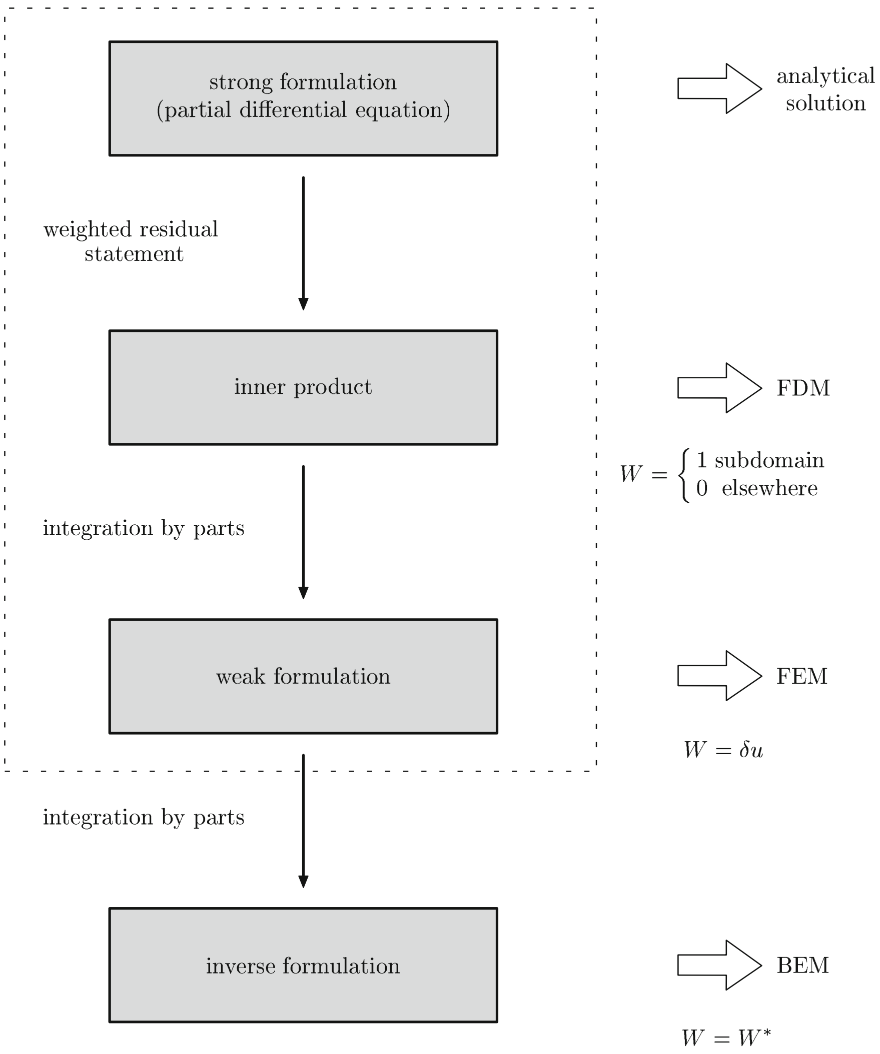


Fig. 2.6 Some classical approximation methods in the context of the weighted residual method

is approximated based on nodal values and interpolation functions which distribute these displacements between the nodes in a certain way. Equation (2.20) illustrates a basic idea of the finite element method where the unknown function is not approximated over the entire domain of the problem (in general Ω) but in a sub-domain (Ω^{e}), the so-called finite element. In a similar way as the unknown function, the weight function is approximated as

$$W(x) = \left(\boldsymbol{N}^{\mathrm{T}}(x)\,\boldsymbol{u}_{\mathrm{p}}\right)^{\mathrm{T}} = \delta\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}\boldsymbol{N}(x) = \left[\delta u_1\ \delta u_2\ \ldots\ \delta u_n\right] \times \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}, \tag{2.21}$$

where δu_i represents the so-called arbitrary or virtual displacements. It will be shown in the following that the virtual displacements occur on both sides of Eq. (2.19) and can be eliminated. Thus, these virtual displacements do not need a deeper consideration at this point of the derivation. Equation (2.19) requires the derivatives of $u(x)$ and $W(x)$ which can be written on the element level as:

$$\frac{\mathrm{d}u^{\mathrm{e}}(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\boldsymbol{N}^{\mathrm{T}}(x)\,\boldsymbol{u}_{\mathrm{p}}\right) = \frac{\mathrm{d}\boldsymbol{N}^{\mathrm{T}}(x)}{\mathrm{d}x}\boldsymbol{u}_{\mathrm{p}}, \tag{2.22}$$

$$\frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\delta\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}\boldsymbol{N}(x)\right) = \delta\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}\frac{\mathrm{d}\boldsymbol{N}(x)}{\mathrm{d}x}. \tag{2.23}$$

It should be noted here that the nodal unknowns and their virtual counterparts are constant values, i.e. not a function of x, and are therefore not affected by the differential operator. It is common in some references (e.g. [58, 12]) to introduce the matrix which contains the derivatives of the interpolation functions as a matrix denoted by $\boldsymbol{B} = \frac{\mathrm{d}\boldsymbol{N}(x)}{\mathrm{d}x}$. Thus, the derivatives can be be written as:

$$\frac{\mathrm{d}u^{\mathrm{e}}(x)}{\mathrm{d}x} = \boldsymbol{B}^{\mathrm{T}}\,\boldsymbol{u}_{\mathrm{p}}, \tag{2.24}$$

$$\frac{\mathrm{d}W(x)}{\mathrm{d}x} = \delta\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}\boldsymbol{B}. \tag{2.25}$$

2.3.1.1 Linear Element Formulation

Let us consider in the following a rod element which is composed of two nodes as schematically shown in Fig. 2.7. Each node has only one degree of freedom, i.e. a displacement in the direction of the principal axis (cf. Fig. 2.7a) and each node can be only loaded by a single force acting in x-direction (cf. Fig. 2.7b).

Since there are only two nodes with two unknowns, the equation for the unknown displacement in the element and its virtual counterpart (cf. Eqs. (2.20) and (2.21)) are simplified to the following expressions:

$$u^{\mathrm{e}}(x) = \boldsymbol{N}^{\mathrm{T}}(x)\,\boldsymbol{u}_{\mathrm{p}} = \left[N_1\ N_2\right] \times \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{2.26}$$

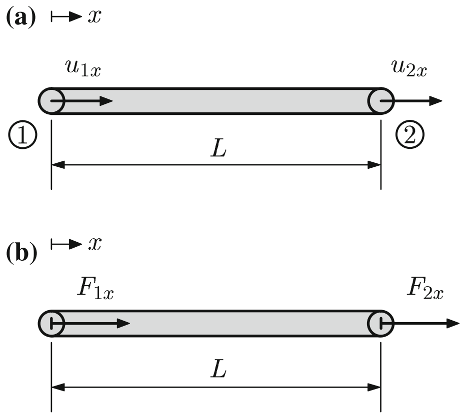


Fig. 2.7 Definition of the one-dimensional linear rod element: **a** deformations; **b** external loads. The nodes are symbolized by the two circles at the ends (◯)

and

$$W(x) = \delta \boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}} \boldsymbol{N}(x) = \begin{bmatrix} \delta u_1 & \delta u_2 \end{bmatrix} \times \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}. \tag{2.27}$$

Let us first consider in the following only the left-hand side of Eq. (2.19) in order to derive the expression for the elemental stiffness matrix $\boldsymbol{K}^{\mathrm{e}}$ of the linear rod element. Introduction of expressions (2.26) and (2.27) in the weak form gives

$$EA \int_0^L \left(\delta \boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}} \frac{\mathrm{d}\boldsymbol{N}(x)}{\mathrm{d}x} \right) \left(\frac{\mathrm{d}\boldsymbol{N}^{\mathrm{T}}(x)}{\mathrm{d}x} \boldsymbol{u}_{\mathrm{p}} \right) \mathrm{d}x, \tag{2.28}$$

or under consideration that the column matrix of the nodal unknowns can be considered as constant as:

$$\delta \boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}} \underbrace{EA \int_0^L \left(\frac{\mathrm{d}\boldsymbol{N}(x)}{\mathrm{d}x} \right) \left(\frac{\mathrm{d}\boldsymbol{N}^{\mathrm{T}}(x)}{\mathrm{d}x} \right) \mathrm{d}x}_{\boldsymbol{K}^{\mathrm{e}}} \, \boldsymbol{u}_{\mathrm{p}}. \tag{2.29}$$

It will be seen in the following that the expression $\delta \boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}$ can be 'canceled' with an identical expression on the right-hand side of Eq. (2.19) and $\boldsymbol{u}_{\mathrm{p}}$ represents the column matrix of the unknown nodal displacements. Under consideration of the $\boldsymbol{B}$-matrix, the stiffness matrix can be expressed in a more general way for constant tensile stiffness EA as:

$$\boldsymbol{K}^{\mathrm{e}} = EA \int_0^L \boldsymbol{B}\boldsymbol{B}^{\mathrm{T}} \mathrm{d}x. \tag{2.30}$$

In order to further evaluate Eq. (2.29), we can introduce the components of the derivatives to give:

$$EA\int_0^L \begin{bmatrix} \dfrac{\mathrm{d}N_1(x)}{\mathrm{d}x} \\ \dfrac{\mathrm{d}N_2(x)}{\mathrm{d}x} \end{bmatrix} \begin{bmatrix} \dfrac{\mathrm{d}N_1(x)}{\mathrm{d}x} & \dfrac{\mathrm{d}N_2(x)}{\mathrm{d}x} \end{bmatrix} \mathrm{d}x, \tag{2.31}$$

or after the matrix multiplication as:

$$EA\int_0^L \begin{bmatrix} \dfrac{\mathrm{d}N_1(x)}{\mathrm{d}x}\dfrac{\mathrm{d}N_1(x)}{\mathrm{d}x} & \dfrac{\mathrm{d}N_1(x)}{\mathrm{d}x}\dfrac{\mathrm{d}N_2(x)}{\mathrm{d}x} \\ \dfrac{\mathrm{d}N_2(x)}{\mathrm{d}x}\dfrac{\mathrm{d}N_1(x)}{\mathrm{d}x} & \dfrac{\mathrm{d}N_2(x)}{\mathrm{d}x}\dfrac{\mathrm{d}N_2(x)}{\mathrm{d}x} \end{bmatrix} \mathrm{d}x. \tag{2.32}$$

Any further evaluation of this equation requires now that the functional expressions $N_1(x)$ and $N_2(x)$ are known. The simplest assumption that can be done is that the nodal values are linearly distributed within the element, from its value at the node to zero at the opposite node. For such a linear superposition, the interpolation functions can be assumed as shown in Fig. 2.8a.

The derivatives of the interpolation functions can easily be calculated as

$$\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} = -\frac{1}{L}, \quad \frac{\mathrm{d}N_2(x)}{\mathrm{d}x} = \frac{1}{L}, \tag{2.33}$$

$$\frac{\mathrm{d}N_1(\xi)}{\mathrm{d}\xi} = -\frac{1}{2}, \quad \frac{\mathrm{d}N_2(\xi)}{\mathrm{d}\xi} = \frac{1}{2}. \tag{2.34}$$

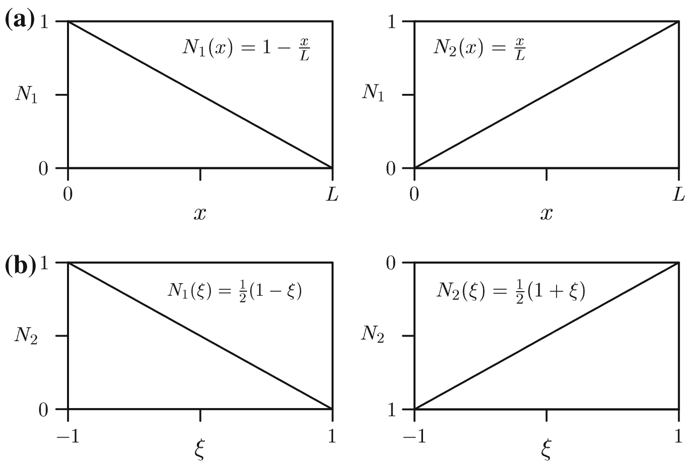


Fig. 2.8 Interpolation functions for the linear rod element: **a** physical coordinate (x); **b** natural coordinate (ξ)

Thus, the $\boldsymbol{B}$-matrix given in Eq. (2.24) takes the form:

$$\boldsymbol{B} = \frac{1}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{2.35}$$

The derivatives introduced into Eq. (2.36) give

$$EA\int_0^L \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} \mathrm{d}x = \frac{EA}{L^2}\int_0^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathrm{d}x. \tag{2.36}$$

The integral in the last equation can be analytically integrated to obtain

$$\frac{EA}{L^2}\begin{bmatrix} x & -x \\ -x & x \end{bmatrix}\Bigg|_0^L = \frac{EA}{L^2}\begin{bmatrix} L & -L \\ -L & L \end{bmatrix} \tag{2.37}$$

and the stiffness matrix for a linear rod element is given by:

$$\boldsymbol{K}^{\mathrm{e}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{2.38}$$

It must be noted here that an analytical integration as performed to obtain Eq. (2.37) cannot be performed in commercial finite element codes since they are written in traditional programming languages such as FORTRAN. Instead of the analytical integration, a numerical integration is performed (cf. Appendix A.9) where the integral is approximated by the evaluation and weighting of functional values at so-called integration or GAUSS[6] points. To this end, the Cartesian coordinate x is transformed to the natural coordinate ξ ranging from -1 to 1. Depending on the origin of the Cartesian coordinate system, the transformation can be performed based on the relations given in Table 2.4.

The integral in Eq. (2.36) can be written in terms of the natural coordinate ξ and approximated in terms of a GAUSS–LEGENDRE[7] quadrature as:

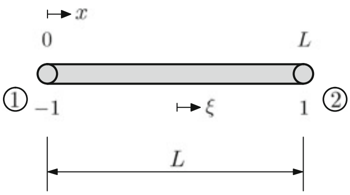
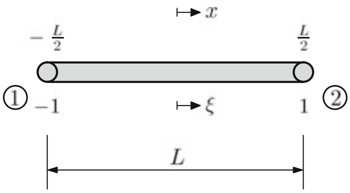
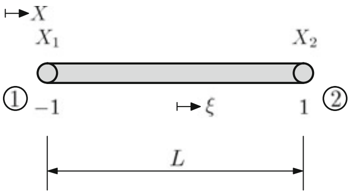
$$\boldsymbol{K}^{\mathrm{e}} = \frac{EA}{L^2}\int_{-1}^{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{L}{2}\,\mathrm{d}\xi \approx \frac{EA}{L^2}\sum_{i=1}^{n} \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}(\xi_i)\,w(\xi_i), \tag{2.39}$$

where the matrix is to be evaluated at the n integration points and multiplied by certain weights w, cf. Appendix A.9. Since the matrix is in this simple case only

[6] Johann Carl Friedrich GAUSS (1777–1855), German mathematician and physical scientist.

[7] Adrien–Marie LEGENDRE (1752–1833), French mathematician.

Table 2.4 Transformation between Cartesian (x) and natural coordinates (ξ)

Configuration	Transformation
	$\xi = \frac{2x}{L} - 1,$ $\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{2}{L}.$
	$\xi = \frac{2x}{L},$ $\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{2}{L}.$
	$\xi = \frac{2}{X_2 - X_1}(X - X_1) - 1,$ $\frac{\mathrm{d}\xi}{\mathrm{d}X} = \frac{2}{L}.$

composed of constant values, it is sufficient to consider a one-point integration rule ($\xi = 0$, $w = 2$) to achieve the analytical result[8] as:

$$\boldsymbol{K}^{\mathrm{e}} = \frac{EA}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\Bigg|_{\xi=0} \times \underbrace{2}_{w} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{2.40}$$

The transformation between Cartesian (x) and natural coordinates (ξ) as indicated in Table 2.4 can be further generalized. Let us assume for this purpose that the Cartesian coordinate can be interpolated in the following manner:

$$x(\xi) = \overline{N}_1(\xi)x_1 + \overline{N}_2(\xi)x_2, \tag{2.41}$$

where x_1 and x_2 are the coordinates of the start and end node in the elemental Cartesian coordinate system. The interpolation functions $\overline{N}_i(\xi)$ are—in the case of the coordinate approximation—called shape functions because they describe the geometry or shape of the element. Considering the shape functions in natural coordinates as given in Fig. 2.8 for the displacement interpolation (a so-called isoparametricformulation),

[8] It must be noted here that in the general case only an *approximation* of the integral can be obtained and that the exact, i.e. analytical solution, is reserved for simple cases.

the following expression for the derivative of the Cartesian coordinate with respect to the natural coordinate is obtained:

$$\frac{\mathrm{d}x(\xi)}{\mathrm{d}\xi} = \frac{\mathrm{d}\overline{N}_1(\xi)}{\mathrm{d}\xi}x_1 + \frac{\mathrm{d}\overline{N}_2(\xi)}{\mathrm{d}\xi}x_2 = -\frac{1}{2}x_1 + \frac{1}{2}x_2. \tag{2.42}$$

The last equation allows to reproduce the geometrical derivatives given in Table 2.4 or for any other location of the elemental Cartesian coordinate system. Equation (2.42) is also known as the general form of the Jacobian determinant and allows to perform the numerical integration of the stiffness matrix in natural coordinates as outlined in Eq. (A.40). The choice of the shape functions in Eq. (2.41) allows to distinguish different element formulations. If the degree of the shape functions is equal to the degree of the interpolation functions, i.e. $\deg(\overline{N}) = \deg(N)$, a so-called isoparametric element formulation is obtained. If the degree of the shape functions is smaller than the degree of the interpolation functions, i.e. $\deg(\overline{N}) < \deg(N)$, a so-called subparametric element formulation is obtained. A larger degree of the shape functions compared to the interpolation functions, i.e. $\deg(\overline{N}) > \deg(N)$, gives a so-called superparametric element formulation.

Let us summarize here in a systematic manner the major steps which are required to calculate the elemental stiffness matrix of a linear rod element.

❶ Introduce an elemental coordinate system (x).

❷ Express the coordinates (x_i) of the corner nodes i ($i = 1, 2$) in this elemental coordinate system.

❸ Calculate the partial derivative of the Cartesian (x) coordinate with respect to the natural (ξ) coordinate, see Eq. (2.42):

$$\frac{\mathrm{d}x(\xi)}{\mathrm{d}\xi} = J = -\frac{1}{2}x_1 + \frac{1}{2}x_2.$$

❹ Calculate the partial derivative of the natural (ξ) coordinate with respect to the Cartesian (x) coordinate, see Eq. (A.49):

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{1}{J}.$$

❺ Calculate the $\boldsymbol{B}$-matrix and its transposed, see Eqs. (2.30) and (2.31):

$$\boldsymbol{B}^{\mathrm{T}} = \left[\frac{\mathrm{d}N_1(x)}{\mathrm{d}x}\ \frac{\mathrm{d}N_2(x)}{\mathrm{d}x}\right]$$

where the partial derivatives are $\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} = \frac{\mathrm{d}N_1(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x}, \ldots$ and the derivatives of the interpolation functions are given in Eq. (2.34), i.e., $\frac{\partial N_1(\xi)}{\partial \xi} = -\frac{1}{2}, \ldots$

❻ Calculate the triple matrix product $\boldsymbol{B}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}$, where the elasticity matrix $\boldsymbol{C}$ is given in this special case as the scalar Young's modulus E.
❼ Perform the numerical integration based on a 1-point integration rule:

$$\int_V (\boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{\mathrm{T}})\mathrm{d}V = \boldsymbol{B}E\boldsymbol{B}^{\mathrm{T}}J \times 2 \times A\Big|_{(0)}.$$

❽ $\boldsymbol{K}$ obtained.

Let us now consider the right-hand side of Eq. (2.19) in order to derive the expression for the elemental load column matrix $\boldsymbol{f}^{\mathrm{e}}$ of the linear rod element. The first part of the right-hand side, i.e.

$$EA\left[W(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right]_0^L \tag{2.43}$$

results with the definition of the weight function according to Eq. (2.27) in

$$EA\left[\delta\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}\boldsymbol{N}(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right]_0^L, \tag{2.44}$$

or in components

$$\delta\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}EA\left[\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right]_0^L. \tag{2.45}$$

The virtual displacements $\delta\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}$ in the last equation can be 'canceled' with a corresponding expression in Eq. (2.29). Furthermore, the last equation constitutes a system of two equations which must be evaluated at the integration boundaries, i.e. at $x = 0$ and $x = L$. The first equation reads:

$$\left(N_1 EA\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=L} - \left(N_1 EA\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=0}. \tag{2.46}$$

This gives under consideration of the boundary values of the interpolation functions, i.e. $N_1(L) = 0$ and $N_1(0) = 1$, the following statement:

$$-EA\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_{x=0} \overset{(2.11)}{=} -N_x(x=0). \tag{2.47}$$

A corresponding expression can be derived for the second equation as:

$$EA\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=L} \overset{(2.11)}{=} N_x(x=L). \tag{2.48}$$

It must be noted here that the forces N_x are the internal reactions according to Fig. 2.5. The external loads with their positive directions according to Fig. 2.7b can be obtained from the internal loads by inverting the sign at the left-hand boundary and by maintaining the positive direction of the internal reaction at the right-hand boundary. This can easily be shown by balancing the internal and external forces at each boundary node. Thus, the contribution to the load matrix due to single *external* forces F_i at the nodes is expressed by:

$$\boldsymbol{f}_F^{\mathrm{e}} = \begin{bmatrix} F_{1x} \\ F_{2x} \end{bmatrix}. \tag{2.49}$$

The second part of Eq. (2.19), i.e. after 'canceling' of the virtual displacements $\delta\boldsymbol{u}^{\mathrm{T}}$

$$\int_0^L \boldsymbol{N}(x)\, p(x)\, \mathrm{d}x \tag{2.50}$$

presents the general rule to determine equivalent nodal loads in the case of arbitrarily distributed loads $p(x)$. As an example, the evaluation of Eq. (2.50) for a constant load p results in the following load matrix:

$$\boldsymbol{f}_p^{\mathrm{e}} = p\int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \mathrm{d}x = \frac{pL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.51}$$

Further expressions for equivalent nodal loads can be taken from Table 2.5. Let us remind ourselves at this step that in the scope of the finite element method any type of load can be only introduced at nodes into the discretized structure.

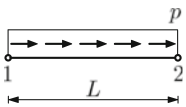
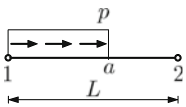
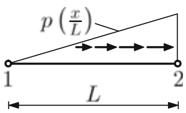
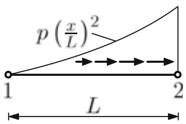
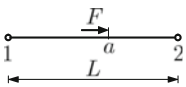
Based on the derived results, the principal finite element equation for a single linear rod element with constant axial tensile stiffness EA can be expressed in a general form as

$$\boldsymbol{K}^{\mathrm{e}}\boldsymbol{u}^{\mathrm{e}} = \boldsymbol{f}^{\mathrm{e}}, \tag{2.52}$$

or in components as:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \end{bmatrix} + \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} p_x(x)\, \mathrm{d}x. \tag{2.53}$$

Table 2.5 Equivalent nodal loads for a linear rod element (x-axis: right facing)

Loading	Axial force
	$F_{1x} = \dfrac{pL}{2}$ $F_{2x} = \dfrac{pL}{2}$
	$F_{1x} = -\dfrac{pa^2}{2L} + pa$ $F_{2x} = \dfrac{pa^2}{2L}$
	$F_{1x} = \dfrac{pL}{6}$ $F_{2x} = \dfrac{pL}{3}$
	$F_{1x} = \dfrac{pL}{12}$ $F_{2x} = \dfrac{pL}{4}$
	$F_{1x} = \dfrac{F(L-a)}{L}$ $F_{2x} = \dfrac{Fa}{L}$

At the end of this section, a few comments on the accuracy of a linear rod element should be given, cf. Table 2.6. As can be seen, the linear rod element gives under certain conditions the exact, i.e. the analytical solution. This is illustrated by several examples in the section 'Solved Rod Problems' and 'Supplementary Problems'.

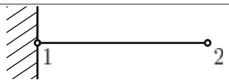
2.3.1.2 Quadratic Element Formulation

Let us consider now a rod element which is composed of three nodes as schematically shown in Fig. 2.9. Each node has again only one degree of freedom, i.e. a displacement in x-direction and each node can be only loaded by a single force acting along the x-axis. It is assumed in the following that the second node is exactly located in the middle, i.e. at $x = \frac{L}{2}$, of the element.

Since there are now three nodes with three unknowns, the equation for the unknown displacement in the element and its virtual counterpart (cf. Eqs. (2.20) and (2.21)) are now given by the expressions:

Table 2.6 Comments on the accuracy of the finite element solution for a single cantilevered linear rod element

Configuration



Axial Tensile Stiffness and Loading	Accuracy of $u(x)$
$EA =$ const.; loaded by single force F at node 2	FE gives analytical solution at nodes and between nodes
$EA =$ const.; displacement BC u at node 2	FE gives exact nodal values and analytical solution between nodes
$EA =$ const.; distributed load p	FE gives analytical solution at nodes but only approximate solution between nodes
$EA \neq$ const.; loaded by single force F at node 2	FE gives approximate solution at nodes and approximate solution between nodes
$EA \neq$ const.; displacement BC u at node 2	FE gives exact nodal values but only approximate solution between nodes

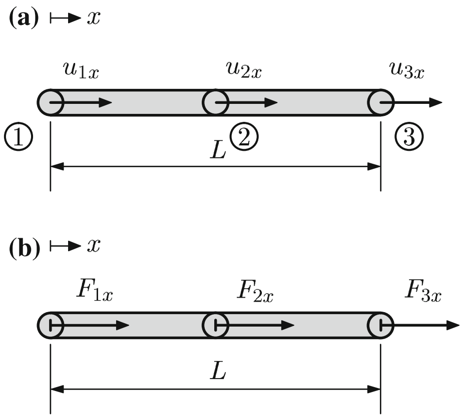


Fig. 2.9 Definition of the one-dimensional quadratic rod element: **a** deformations; **b** external loads. The nodes are symbolized by *circles* at the ends and in the *middle* (◯)

$$u^{\mathrm{e}}(x) = \boldsymbol{N}^{\mathrm{T}}(x)\,\boldsymbol{u}_{\mathrm{p}} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \tag{2.54}$$

and

$$W(x) = \delta\boldsymbol{u}_{\mathrm{p}}^{\mathrm{T}}\boldsymbol{N}(x) = \begin{bmatrix} \delta u_1 & \delta u_2 & \delta u_3 \end{bmatrix} \times \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}. \tag{2.55}$$

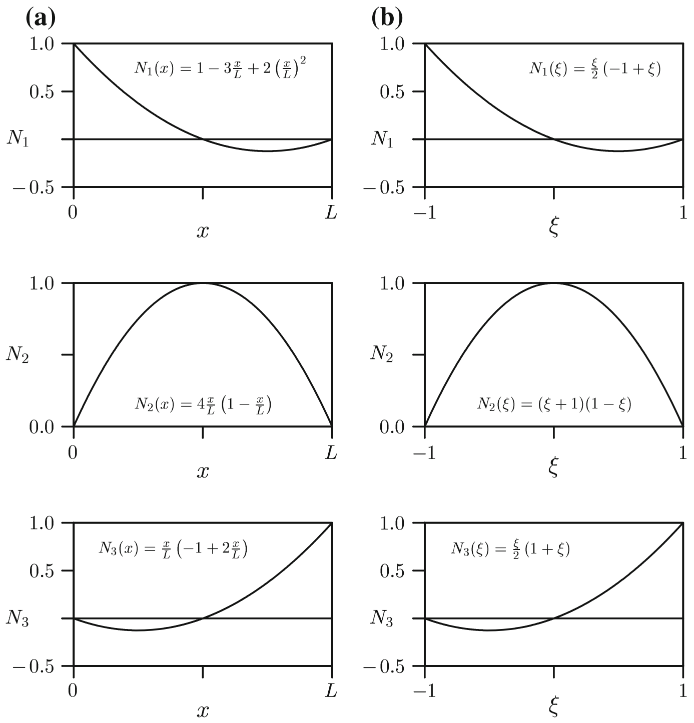


Fig. 2.10 Interpolation functions for the quadratic rod element with equidistant nodes: **a** physical coordinate (x); **b** natural coordinate (ξ)

Similar as in Eq. (2.36), the elemental stiffness matrix can be expressed before evaluating the integral as:

$$\boldsymbol{K}^{\mathrm{e}} = EA \int_0^L \begin{bmatrix} \frac{\mathrm{d}N_1(x)}{\mathrm{d}x}\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} & \frac{\mathrm{d}N_1(x)}{\mathrm{d}x}\frac{\mathrm{d}N_2(x)}{\mathrm{d}x} & \frac{\mathrm{d}N_1(x)}{\mathrm{d}x}\frac{\mathrm{d}N_3(x)}{\mathrm{d}x} \\ \frac{\mathrm{d}N_2(x)}{\mathrm{d}x}\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} & \frac{\mathrm{d}N_2(x)}{\mathrm{d}x}\frac{\mathrm{d}N_2(x)}{\mathrm{d}x} & \frac{\mathrm{d}N_2(x)}{\mathrm{d}x}\frac{\mathrm{d}N_3(x)}{\mathrm{d}x} \\ \frac{\mathrm{d}N_3(x)}{\mathrm{d}x}\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} & \frac{\mathrm{d}N_3(x)}{\mathrm{d}x}\frac{\mathrm{d}N_2(x)}{\mathrm{d}x} & \frac{\mathrm{d}N_3(x)}{\mathrm{d}x}\frac{\mathrm{d}N_3(x)}{\mathrm{d}x} \end{bmatrix} \mathrm{d}x. \tag{2.56}$$

The interpolation functions N_i in this case[9] are given by quadratic equations as shown in Fig. 2.10 in physical and natural coordinates.

From the functional expressions given in Fig. 2.10, the derivatives are obtained as $\frac{\mathrm{d}N_1}{\mathrm{d}x} = -\frac{3}{L} + \frac{4x}{L^2}$, $\frac{\mathrm{d}N_2}{\mathrm{d}x} = \frac{4}{L} - \frac{8x}{L^2}$, and $\frac{\mathrm{d}N_3}{\mathrm{d}x} = -\frac{1}{L} + \frac{4x}{L^2}$ and Eq. (2.56) can be evaluated

[9] A formal derivation of the functional expressions is presented in Sect. 2.3.2.

by analytical or numerical integration to give the elemental stiffness matrix of the quadratic rod element as:

$$\boldsymbol{K}^{\mathrm{e}} = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}. \tag{2.57}$$

Furthermore, it should be noted that the $\boldsymbol{B}$-matrix, cf. Eq. (2.24), takes the following form for the quadratic rod element:

$$\boldsymbol{B} = \frac{1}{L}\begin{bmatrix} -3 + \frac{4x}{L} \\ 4 - \frac{8x}{L} \\ -1 + \frac{4x}{L} \end{bmatrix} = \frac{1}{L}\begin{bmatrix} -1 + 2\xi \\ -4\xi \\ 1 + 2\xi \end{bmatrix}. \tag{2.58}$$

The right-hand side of Eq. (2.19) can be treated in a similar way as in Sect. 2.3.1.1 to obtain the elemental load vector in the form of:

$$\boldsymbol{f}^{\mathrm{e}} = \boldsymbol{f}^{\mathrm{e}}_{F} + \boldsymbol{f}^{\mathrm{e}}_{p} = \begin{bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{bmatrix} + \int\limits_{0}^{L} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} p(x)\,\mathrm{d}x. \tag{2.59}$$

Based on the derived results, the principal finite element equation for a single quadratic rod element with constant axial tensile stiffness EA can be expressed in components as:

$$\frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{bmatrix} u_{1x} \\ u_{2x} \\ u_{3x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{bmatrix} + \int\limits_{0}^{L} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} p(x)\,\mathrm{d}x. \tag{2.60}$$

This formulation of the principal finite element equation can be alternatively expressed by eliminating the expression for the second node. The system given in Eq. (2.60) can be written in the form of single equations as:

$$\frac{EA}{3L}(7u_{1x} - 8u_{2x} + 1u_{3x}) = F_{1x} + I_1, \tag{2.61}$$

$$\frac{EA}{3L}(-8u_{1x} + 16u_{2x} - 8u_{3x}) = F_{2x} + I_2, \tag{2.62}$$

$$\frac{EA}{3L}(1u_{1x} - 8u_{2x} + 7u_{3x}) = F_{3x} + I_3, \tag{2.63}$$

where I_i is the abbreviation for the integral with the distributed load, e.g. $I_1 = \int N_1(x)p(x)\mathrm{d}x$. The second equation can be rearranged for u_{2x}, i.e.

$$u_{2x} = \frac{1}{2}u_{1x} + \frac{1}{2}u_{3x} + \frac{1}{16}\frac{3L}{EA}(F_{2x} + I_2)\,, \tag{2.64}$$

which can be introduced into Eqs. (2.61) and (2.63) to obtain:

$$\frac{EA}{3L}(3u_{1x} - 3u_{3x}) = F_{1x} + I_1 + \frac{1}{2}(F_{2x} + I_2)\,, \tag{2.65}$$

$$\frac{EA}{3L}(-3u_{1x} + 3u_{3x}) = F_{3x} + I_3 + \frac{1}{2}(F_{2x} + I_2)\,. \tag{2.66}$$

The last two equations can be written again in matrix form as:

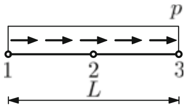
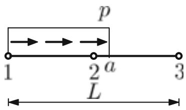
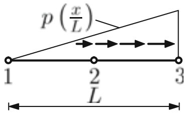
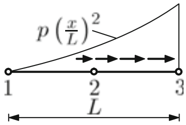
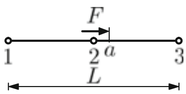
$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_{1x} \\ u_{3x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{3x} \end{bmatrix} + \begin{bmatrix} I_1 \\ I_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}F_{2x} \\ \frac{1}{2}F_{2x} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}I_2 \\ \frac{1}{2}I_2 \end{bmatrix}. \tag{2.67}$$

This formulation looks similar to the expression for the linear rod element in Eq. (2.53). However, the right-hand side in addition here contains the contribution of the load from the middle load and it should be not forgotten that the distribution of the displacement $u^{\mathrm{e}}(x)$ inside the element is of quadratic shape. The values of equivalent nodal loads, i.e. the evaluation of the integral in Eq. (2.60), is given for some standard cases in Table 2.7. The reader should here pay attention to the fact that these equivalent nodal loads are different to those in the case of the linear rod element, cf. Table 2.5.

At the end of this section again a few words on the accuracy of the quadratic rod element will be given. As can be seen in Table 2.8, the accuracy is for the investigated cases at least in the range of the linear element if we compare the general statements without investigating specific numerical values. For a constant distributed load, the quadratic element reproduces not only at the nodes but also between the nodes the analytical solution. However, it must be highlighted here that these results are element specific and that the finite element method calculates in the general case—even at nodes—only approximate solutions. Nevertheless, the comments presented in Table 2.8 can be helpful in special cases where a mesh refinement would not increase the accuracy but the computation time and the size of the results file. If the problem is such that the exact solution is obtained at the nodes, a mesh refinement is in all likelihood not required in this case.

Let us summarize at the end of this section the major steps that were undertaken to transform the partial differential equation into the principal finite element equation, see Table 2.9.

Table 2.7 Equivalent nodal loads for a quadratic rod element (x-axis: right facing)

Loading	Axial force	
	$F_{1x} = \frac{pL}{6}$, $F_{3x} = \frac{pL}{6}$	$F_{2x} = \frac{2pL}{3}$,
	$F_{1x} = \frac{2pa^3}{3L^2} - \frac{3pa^3}{2L} + pa$, $F_{3x} = \frac{2pa^3}{3L^2} - \frac{pa^2}{2L}$	$F_{2x} = -\frac{4pa^3}{3L^2} + \frac{2pa^2}{L}$,
	$F_{1x} = 0$, $F_{3x} = \frac{pL}{6}$	$F_{2x} = \frac{pL}{3}$,
	$F_{1x} = -\frac{pL}{60}$, $F_{3x} = \frac{3pL}{20}$	$F_{2x} = \frac{pL}{5}$,
	$F_{1x} = FN_1(a)$, $F_{3x} = FN_3(a)$	$F_{2x} = FN_2(a)$,

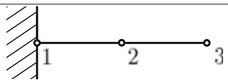
2.3.2 Derivation of Interpolation Functions

A more general concept based on basis functions will be introduced in the following in order to derive the complete set of interpolation functions.[10] To this end, let us just assume that the shape of the displacement distribution $u^{\mathrm{e}}(\xi)$ within an element is without reference to the nodal values. It is obvious that this choice must be conform to the physical problem under consideration. We may assume that the distribution is given by a polynomial of the form

$$u^{\mathrm{e}}(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_2\xi^2 + \cdots + a_n\xi^n, \tag{2.68}$$

[10]This approach is presented in Ref. [33] in a general way.

Table 2.8 Comments on the accuracy of the finite element solution for a single cantilevered quadratic rod element

Configuration 

Axial tensile stiffness and loading	Accuracy of $u(x)$
$EA =$ const.; loaded by single force F at node 3	FE gives analytical solution at nodes and between nodes
$EA =$ const.; displacement BC u at node 3	FE gives exact nodal values and analytical solution between nodes
$EA =$ const.; distributed load $p =$ const.	FE gives analytical solution at nodes and between nodes
$EA =$ const.; distributed load $p(x) =$ linear	FE gives analytical solution at nodes but only approximate solution between nodes
$EA \neq$ const.; loaded by single force F at node 3	FE gives approximate solution at nodes and approximate solution between nodes

Table 2.9 Summary: derivation of principal finite element equation for rod elements

Strong formulation
$EA\dfrac{\mathrm{d}^2u^0(x)}{\mathrm{d}x^2} + p(x) = 0$
Inner product
$\int\limits_0^L W(x)\left(EA\dfrac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + p(x)\right)\mathrm{d}x \stackrel{!}{=} 0$
Weak formulation
$EA\int\limits_0^L \dfrac{\mathrm{d}W(x)}{\mathrm{d}x}\dfrac{\mathrm{d}u(x)}{\mathrm{d}x}\mathrm{d}x = EA\left[W(x)\dfrac{\mathrm{d}u(x)}{\mathrm{d}x}\right]_0^L + \int\limits_0^L W(x)p(x)\,\mathrm{d}x$
Principal finite element equation
$\dfrac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \end{bmatrix} + \int\limits_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} p_x(x)\,\mathrm{d}x \text{ (lin.)}$
$\dfrac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{bmatrix} u_{1x} \\ u_{2x} \\ u_{3x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{bmatrix} + \int\limits_0^L \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} p(x)\,\mathrm{d}x \text{ (quad.)}$

which can be expressed in matrix form as:

$$u^{\mathrm{e}}(\xi) = \boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{a} = \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 & \dots & \xi^n \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}. \tag{2.69}$$

The elements of $\boldsymbol{\chi}$ will be called *basis functions* and the elements of $\boldsymbol{a}$ will be called *basis coefficients*. If we assume that the number of basis functions equals the number of nodal variables associated with u, then the relationship between the basis coefficients $\boldsymbol{a}$ and the nodal values $\boldsymbol{u}_\mathrm{p}$ can be expressed as

$$\boldsymbol{a} = \boldsymbol{A}\boldsymbol{u}_\mathrm{p}, \tag{2.70}$$

where $\boldsymbol{A}$ is a square matrix of constants. Equalizing the nodal approach given in Eq. (2.20) with the new expression in (2.69) and considering (2.70) results in:

$$\boldsymbol{N}^\mathrm{T}\boldsymbol{u}_\mathrm{p} = \boldsymbol{\chi}^\mathrm{T}\boldsymbol{a} \quad \text{or} \quad \boldsymbol{N}^\mathrm{T} = \boldsymbol{\chi}^\mathrm{T}\boldsymbol{A}. \tag{2.71}$$

Thus, the row matrix of the interpolation functions $\boldsymbol{N}^\mathrm{T}$ can be factored into a row vector of basis functions $\boldsymbol{\chi}^\mathrm{T}$ and a square matrix $\boldsymbol{A}$ of constant coefficients.

To illustrate the procedure, let us have a look at a linear rod element as shown in Fig. 2.11 where the natural coordinate is used.

If the physical problem supports the assumption of a linear distribution of the displacement, the following linear description of the displacement field can be introduced:

$$u^\mathrm{e}(\xi) = a_0 + a_1\xi, \tag{2.72}$$

where the column matrix of the basis functions is given by $\boldsymbol{\chi} = \begin{bmatrix}1 & \xi\end{bmatrix}^\mathrm{T}$ and the column matrix of the basis coefficients by $\boldsymbol{a} = \begin{bmatrix}a_0 & a_1\end{bmatrix}^\mathrm{T}$. Evaluation of this function at both nodes gives:

$$\text{Node 1:} \;\; u_1 = u^\mathrm{e}(\xi = -1) = a_0 - a_1, \tag{2.73}$$

$$\text{Node 2:} \;\; u_2 = u^\mathrm{e}(\xi = +1) = a_0 + a_1. \tag{2.74}$$

The last two equations can be expressed in matrix form according to Eq. (2.70) as:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\boldsymbol{A}^{-1}} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}. \tag{2.75}$$

Solving this system of equations for the unknown basis functions a_i gives

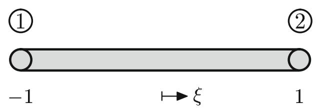


Fig. 2.11 Linear rod element described based on the natural coordinate (ξ)

$$\underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{\boldsymbol{a}} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\boldsymbol{u}_\mathrm{p}}, \tag{2.76}$$

and the matrix of the interpolation functions results according to Eq. (2.71) as:

$$\boldsymbol{N}^\mathrm{T} = \boldsymbol{\chi}^\mathrm{T}\boldsymbol{A} = \begin{bmatrix} 1 & \xi \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}. \tag{2.77}$$

Alternatively, one may use the Cartesian coordinate (x) to derive the interpolation functions based on the same approach. Assuming that the x-coordinate is in the range $0 \leq x \leq L$ and that the same ordinate values as given by Eq. (2.72) are maintained at the nodes, the following linear description of the displacement field can be introduced:

$$u^\mathrm{e}(x) = (a_0 - a_1) + \frac{2a_1}{L} \times x, \tag{2.78}$$

where the column matrix of the basis functions is given by $\boldsymbol{\chi} = \begin{bmatrix} 1 & x \end{bmatrix}^\mathrm{T}$ and the column matrix of the basis coefficients by $\boldsymbol{a} = \begin{bmatrix} (a_0 - a_1) & \frac{2a_1}{L} \end{bmatrix}^\mathrm{T}$. Evaluation of this function at both nodes gives:

$$\text{Node 1:} \;\; u_1 = u^\mathrm{e}(x=0) = a_0 - a_1, \tag{2.79}$$

$$\text{Node 2:} \;\; u_2 = u^\mathrm{e}(x=L) = a_0 + a_1 = (a_0 - a_1) + \tfrac{2a_1}{L}\, L. \tag{2.80}$$

The last two equations can be expressed in matrix form according to Eq. (2.70) as:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}}_{\boldsymbol{A}^{-1}} \begin{bmatrix} a_0 - a_1 \\ \frac{2a_1}{L} \end{bmatrix}. \tag{2.81}$$

Solving this system of equations for the unknown basis functions a_i gives

$$\underbrace{\begin{bmatrix} a_0 - a_1 \\ \frac{2a_1}{L} \end{bmatrix}}_{\boldsymbol{a}} = \underbrace{\frac{1}{L}\begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\boldsymbol{u}_\mathrm{p}}, \tag{2.82}$$

and the matrix of the interpolation functions results according to Eq. (2.71) as:

$$\boldsymbol{N}^\mathrm{T} = \boldsymbol{\chi}^\mathrm{T}\boldsymbol{A} = \begin{bmatrix} 1 & x \end{bmatrix} \frac{1}{L}\begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{L}(L-x) & \frac{1}{L}(x) \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}. \tag{2.83}$$

If the Cartesian coordinate (x) is used based on a different set of ordinate values, the following linear description of the displacement field can be introduced:

$$u^{\mathrm{e}}(x) = a_0 + a_1 \times x, \tag{2.84}$$

where the column matrix of the basis functions is given by $\boldsymbol{\chi} = \begin{bmatrix} 1 & x \end{bmatrix}^{\mathrm{T}}$ and the column matrix of the basis coefficients by $\boldsymbol{a} = \begin{bmatrix} a_0 & a_1 \end{bmatrix}^{\mathrm{T}}$. Evaluation of this function at both nodes gives:

$$\text{Node 1:}\ \ u_1 = u^{\mathrm{e}}(x = 0) = a_0, \tag{2.85}$$

$$\text{Node 2:}\ \ u_2 = u^{\mathrm{e}}(x = L) = a_0 + a_1 L, \tag{2.86}$$

which can be expressed as in Eq. (2.81) and the same interpolation functions as presented in Eq. (2.83) are obtained.

2.3.3 Assembly of Elements and Consideration of Boundary Conditions

Real structures of complex geometry (cf. Figs. 1.2b and 1.3b) require the application of many finite elements in order to discretize the geometry. Thus, it is necessary to assemble the single elemental equations $\boldsymbol{K}^{\mathrm{e}}\boldsymbol{u}_{\mathrm{p}}^{\mathrm{e}} = \boldsymbol{f}^{\mathrm{e}}$ to a global system of equations which can be symbolically written as $\boldsymbol{K}\boldsymbol{u}_{\mathrm{p}} = \boldsymbol{f}$, where $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{u}_{\mathrm{p}}$ the global column matrix of unknowns, and $\boldsymbol{f}$ the global column matrix of loads.

Let us illustrate the process to assemble the global system of equations for a three-element axial structure as shown in Fig. 2.12. As can be seen in Fig. 2.12a, each element has its own coordinate system x_i with $i = \mathrm{I, II, III}$ and its own nodal displacements u_{1x}^{i} and u_{2x}^{i}. In order to assemble the single elements to a connected structure as shown in Fig. 2.12b, it is useful to introduce a global coordinate X and global nodal displacements denoted by u_{iX}. Comparing the elemental and global nodal displacements shown in Fig. 2.12, the following mapping between the local and global displacements can be derived:

$$u_{1X} = u_{1x}^{\mathrm{I}}, \tag{2.87}$$

$$u_{2X} = u_{2x}^{\mathrm{I}} = u_{1x}^{\mathrm{II}}, \tag{2.88}$$

$$u_{3X} = u_{2x}^{\mathrm{II}} = u_{1x}^{\mathrm{III}}, \tag{2.89}$$

$$u_{4X} = u_{2x}^{\mathrm{III}}. \tag{2.90}$$

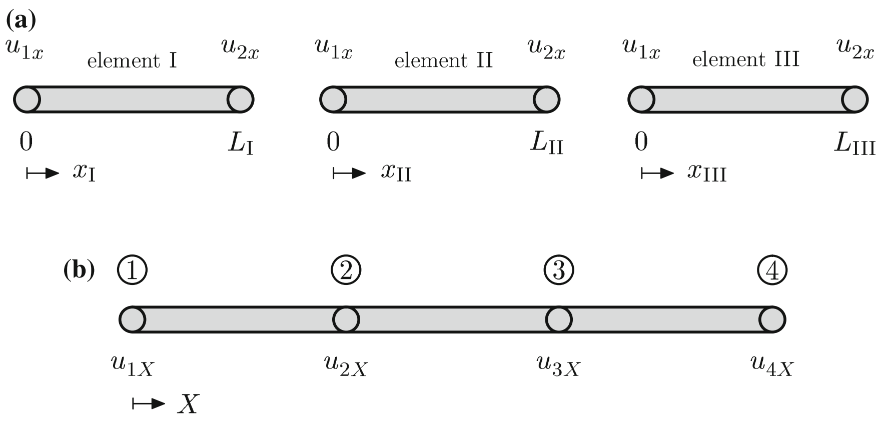


Fig. 2.12 Relationship between **a** elemental and **b** global nodes and displacements in a horizontal rod structure

One possible way to assemble the elemental stiffness matrices to the global system will be illustrated in the following. In a first step, each single element is considered separately and its elemental stiffness matrix is written as, for example, given in Eq. (2.38). In addition, the corresponding *global* nodal displacements are written over the matrix and on the right-hand side which gives the following expressions:

$$\boldsymbol{K}_{\mathrm{I}}^{\mathrm{e}} = \frac{EA}{L} \begin{array}{c} \begin{array}{cc} u_{1X} & u_{2X} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ u_{1X} \\ u_{2X} \end{array}, \tag{2.91}$$

$$\boldsymbol{K}_{\mathrm{II}}^{\mathrm{e}} = \frac{EA}{L} \begin{array}{c} \begin{array}{cc} u_{2X} & u_{3X} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ u_{2X} \\ u_{3X} \end{array}, \tag{2.92}$$

$$\boldsymbol{K}_{\mathrm{III}}^{\mathrm{e}} = \frac{EA}{L} \begin{array}{c} \begin{array}{cc} u_{3X} & u_{4X} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ u_{3X} \\ u_{4X} \end{array}. \tag{2.93}$$

By indicating the global unknowns in the described manner at each elemental stiffness matrix, it is easy to assign to each element in a matrix a unique index. For example, the upper right element of the stiffness matrix $\boldsymbol{K}_{\mathrm{I}}^{\mathrm{e}}$ has the index[11] (u_{1X}, u_{2X}) and the

[11] We follow here the convention where the first expression specifies the row and the second one the column: (row, column).

value $-\frac{EA}{L}$. The next step consists in indicating the structure of the global stiffness matrix with its correct dimension. To this end, the total number of global unknowns[12] must be determined. In general, the global number of unknowns is given by the number of nodes multiplied by the degrees of freedom per node. Thus, the number of global unknowns for a structure of rod elements is simply the total number of nodes in the assembled structure. It should be noted here that the determination of the unknowns at this step of the procedure is without any consideration of boundary conditions. For the problem shown in Fig. 2.12b, the number of nodes is four which equals the number of unknowns. Thus, the dimension of the global stiffness matrix is given by (number global unknowns × number global unknowns) or for our example as (4 × 4) and the structure can be written as:

$$\boldsymbol{K} = \begin{array}{c} \begin{array}{cccc} \;u_{1X}\; & \;u_{2X}\; & \;u_{3X}\; & \;u_{4X}\; \end{array} \\ \left[\begin{array}{c|c|c|c} \quad & \quad & \quad & \quad \\ \hline \quad & & & \\ \hline \quad & & & \\ \hline \quad & & & \end{array}\right] \end{array} \begin{array}{l} u_{1X} \\ u_{2X} \\ u_{3X} \\ u_{4X} \end{array} . \tag{2.94}$$

It is now required to indicate the global unknowns over the empty global stiffness matrix and on its right-hand side. Any order can be chosen but it is common for the problem under consideration to start with u_{1X} and simply move to the next node. The scheme for this consecutive use of the global unknowns from the lowest to the highest number is drawn on the matrix in Eq. (2.94). Each cell of the global stiffness matrix has now its unique index expressed by the global unknowns. Or in other words, each cell of each elemental stiffness matrix has a cell in the global stiffness matrix with the same index and each element of the elemental stiffness matrix must be placed in the global matrix based on this unique index scheme. As an example, the upper right element of the stiffness matrix $\boldsymbol{K}_{\mathrm{I}}^{\mathrm{e}}$ with the index (u_{1X}, u_{2X}) must be placed in the global stiffness matrix in the first row and the second column. If each entry of the elemental stiffness matrices is inserted into the global matrix based on the described index scheme, the assembly of the global stiffness matrix is completed. The process for the consecutive use of global unknowns is illustrated in Fig. 2.13a. As can be seen in this figure, there is an interaction at nodes where elements are connected and the corresponding entries of the elemental stiffness matrices are summed up. This interaction is illustrated in a different way in Fig. 2.13b where it can be seen that at each inner node two interpolation functions are acting, i.e. one from the left element and one from the right element.

A further important property of the global stiffness matrix can be seen in Fig. 2.13a. If an appropriate node numbering is chosen,[13] the global stiffness matrix reveals a

[12] The total number of unknowns is alternatively named the total number of degrees of freedom (DOF).

[13] Commercial finite element codes offer an option which is called ‘bandwidth optimization’ to achieve this structure. This is important if a direct solver is used in order to minimize the solution time and the amount of storage.

strong band structure where all entries are grouped around the main diagonale and major parts grouped in the form of triangles contain only zeros. If there is such a clear boundary between the non-zero and the zero components, the border line is called the *skyline* of the matrix. As the elemental stiffness matrices, the global stiffness matrix is symmetric and commercial finite element codes store only half of the entries in order to reduce the requirements for data storage.

In order to complete the assembly of the global finite element equation, the global load vector $\boldsymbol{f}$ must be composed. Here, it is more advantageous to look from the beginning at the assembled structure and fill the external single loads F_i, which are acting at nodes, in the proper order in the column matrix $\boldsymbol{f}$. A bit care must be taken if distributed loads were converted to equivalent nodal loads. For this case, components f_i from both elements must be summed up at inner nodes:

$$\boldsymbol{f} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} + \begin{bmatrix} f_{1,\mathrm{I}} \\ f_{2,\mathrm{I}} + f_{2,\mathrm{II}} \\ f_{3,\mathrm{II}} + f_{3,\mathrm{III}} \\ f_{4,\mathrm{III}} \end{bmatrix}. \tag{2.95}$$

The global system of equations for the problem shown in Fig. 2.12b is finally obtained as:

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{bmatrix}, \tag{2.96}$$

where the right-hand side is not specified since nothing on the loading is indicated in Fig. 2.12. This system of equations without consideration of any boundary conditions is called the non-reduced system. For this system, the global stiffness matrix $\boldsymbol{K}$ is still singular and cannot be inverted in order to solve the global system of equations. Boundary conditions must be introduced in order to make this matrix regular and thus invertible.

For the rod elements under consideration, two types of boundary conditions must be distinguished. The DIRICHLET boundary condition[14] specifies the displacement u at a node while the NEUMANN boundary condition[15] assigns a force F (i.e., $EA\frac{\mathrm{d}u}{\mathrm{d}x}$) at a node. The different ways to handle these different types of boundary conditions will be explained in the following based on the problem shown in Fig. 2.14 where a cantilevered rod structure has different boundary conditions at its right-hand end node.

---

[14] Alternatively known as 1st kind, essential, geometric or kinematic boundary condition.

[15] Alternatively known as 2nd kind, natural or static boundary condition.

(a)

$$\underbrace{\begin{array}{cccc} u_{1X} & u_{2X} & u_{3X} & u_{4X} \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{array}}_{\text{global } \boldsymbol{K}} \cdot \begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix}$$

$\boldsymbol{K}^{\mathrm{e}}_{\mathrm{I}}$, $\boldsymbol{K}^{\mathrm{e}}_{\mathrm{II}}$, $\boldsymbol{K}^{\mathrm{e}}_{\mathrm{III}}$

(b)

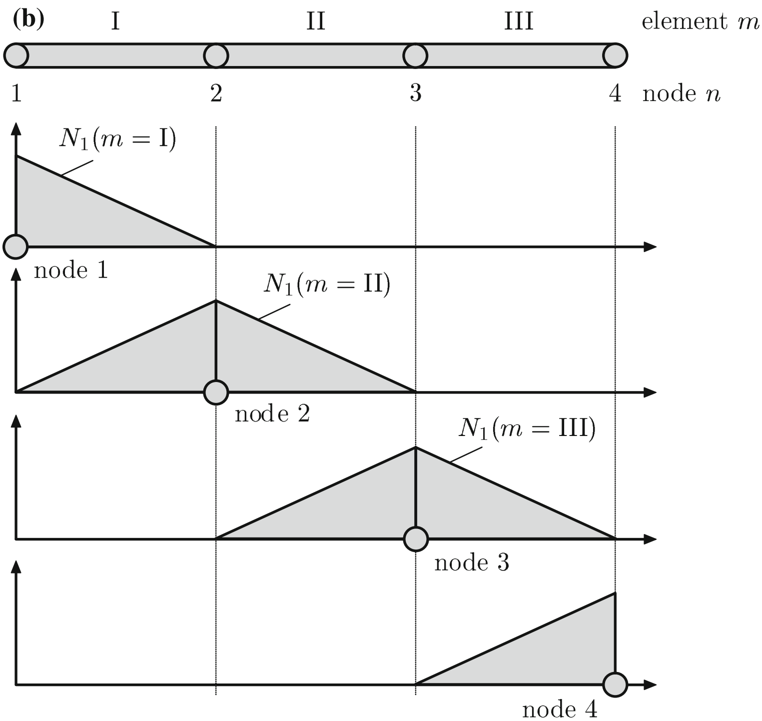


Fig. 2.13 Assembly process to the global stiffness matrix: **a** composition of the elemental stiffness matrices to the global system; **b** interaction of interpolation functions at common nodes

The consideration of the homogeneous DIRICHLET boundary condition, i.e. $u_{1X} = u(X = 0) = 0$, is the simplest case. To incorporate this boundary condition in the system (2.96), the first row and column can be canceled to obtain a reduced system as:

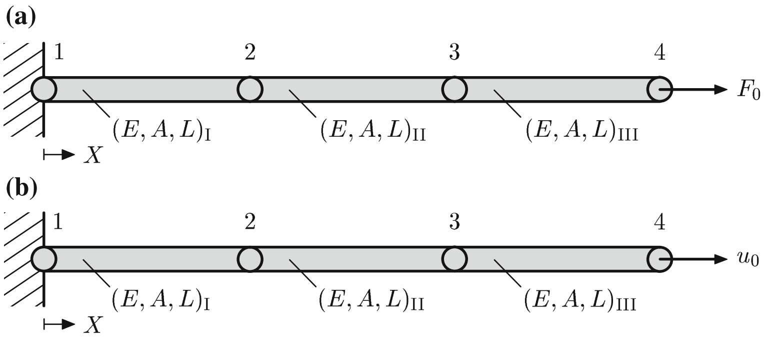


Fig. 2.14 Consideration of boundary conditions for a cantilevered rod structure: **a** force boundary condition; **b** displacement boundary condition at the right-hand boundary node

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}. \tag{2.97}$$

In general we can state that a homogenous DIRICHLET boundary condition at node n ($u_{nX} = 0$) can be considered in the non-reduced system of equations by eliminating the nth row and nth column of the system. Let us consider next the case shown in Fig. 2.14a where the right-hand end node is subjected to a force F_0. This external force can simply be specified on the right-hand side and since no other external forces are acting, the reduced system of equations is finally obtained as:

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_0 \end{bmatrix}. \tag{2.98}$$

This system of equations can be solved, e.g. by inverting the reduced stiffness matrix and solving for the unknown nodal displacements in the form $\boldsymbol{u}_{\mathrm{p}} = \boldsymbol{K}^{-1}\boldsymbol{f}$:

$$\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \frac{LF_0}{EA}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \tag{2.99}$$

To incorporate a non-homogeneous DIRICHLET boundary condition ($u \neq 0$) as shown in Fig. 2.14b, three different strategies can be mentioned. The first one modifies the system shown in Eq. (2.98) in such a way that the boundary condition, i.e., $u_{4X} = u_0$, is directly introduced:

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \times \frac{L}{EA} \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_0 \end{bmatrix}, \tag{2.100}$$

where the last equation gives immediately the boundary condition as $u_{4X} = u_0$. The solution of the system of equations given in Eq. (2.113) can be obtained by inverting the coefficient matrix and multiplying it with the vector on the right-hand side as:

$$\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\,u_0 \\ \frac{2}{3}\,u_0 \\ u_0 \end{bmatrix}. \tag{2.101}$$

In general we can state that a non-homogeneous DIRICHLET boundary condition at node n can be introduced in the system of equations by modifying the nth line in such a way that at the position of the nth column a '1' is obtained while all other entries of the nth line are set to zero. On the right-hand side, the given value is introduced at the nth position of the column matrix.

The second way of considering a non-homogenous DIRICHLET boundary condition consists in the following step: The column of the stiffness matrix, which corresponds to the node where the boundary condition is given, is multiplied by the given displacement. In other words, if the boundary condition is specified at node n, the nth column of the stiffness matrix is multiplied by the given value u_0:

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \times u_0 \\ -1 & 2 & -1 \times u_0 \\ 0 & -1 & 1 \times u_0 \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \end{bmatrix}. \tag{2.102}$$

Now we bring the nth column of the stiffness matrix to the right-hand side of the system

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \end{bmatrix} - \frac{EA}{L}\begin{bmatrix} 0 \times u_0 \\ -1 \times u_0 \\ 1 \times u_0 \end{bmatrix}, \tag{2.103}$$

and delete the nth row of the system:

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{EA}{L}\begin{bmatrix} 0 \times u_0 \\ -1 \times u_0 \end{bmatrix}. \tag{2.104}$$

As a result of this second approach, the dimension of the system of equations could be reduced compared to the first approach. However, this smaller matrix was not obtained for free since more steps have to be performed compared to the first possibility. The solution of Eq. (2.104) can be stated as:

$$\begin{bmatrix} u_{2X} \\ u_{3X} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\,u_0 \\ \frac{2}{3}\,u_0 \end{bmatrix}. \tag{2.105}$$

A third possible approach should be mentioned here since often the question arises by students why not simply replace in the column matrix of unknowns, i.e. on the left-hand side, the variable of the nodal value with the given value. This can be done but requires that the corresponding reaction force[16] is introduced on the right-hand side:

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} u_{2X} \\ u_{3X} \\ u_0 \end{bmatrix}}_{\boldsymbol{u}_{\mathrm{p}}} = \begin{bmatrix} 0 \\ 0 \\ -R_4 \end{bmatrix}. \tag{2.106}$$

However, the column matrix of the nodal displacements $\boldsymbol{u}_{\mathrm{p}}$ contains now unknown quantities (u_{2X}, u_{3X}) and the given nodal boundary condition (u_0). On the other hand, the right-hand side contains the unknown reaction force R_4. Thus, the structure of the linear system of equations is unfavorable for the solution. To rearrange the system to the classical structure where all unknowns are collected on the left and given quantities on the right-hand side, it is advised to write out the three single equations as:

$$\frac{EA}{L}(2u_{2X} - u_{3X}) = 0, \tag{2.107}$$

$$\frac{EA}{L}(-u_{2X} + 2u_{3X} - u_0) = 0, \tag{2.108}$$

$$\frac{EA}{L}(-u_{3X} + u_0) = -R_4. \tag{2.109}$$

After collecting unknown quantities on the left-hand side and known quantities on the right-hand side, one gets

$$\frac{EA}{L}(2u_{2X} - u_{3X}) = 0, \tag{2.110}$$

$$\frac{EA}{L}(-u_{2X} + 2u_{3X}) = \frac{EA}{L}u_0, \tag{2.111}$$

$$\frac{EA}{L}\left(-u_{3X} + \frac{L}{EA}R_4\right) = -\frac{EA}{L}u_0, \tag{2.112}$$

or in matrix notation:

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & \frac{L}{EA} \end{bmatrix}\underbrace{\begin{bmatrix} u_{2X} \\ u_{3X} \\ R_4 \end{bmatrix}}_{\text{unknown}} = \underbrace{\frac{EA}{L}\begin{bmatrix} 0 \\ u_0 \\ -u_0 \end{bmatrix}}_{\text{given}}. \tag{2.113}$$

[16] Let us assume in the following that the reaction force R_4 is oriented in the negative X-direction.

Table 2.10 Different types of boundary conditions

DIRICHLET	NEUMANN
$u = 0$ (homogeneous)	F
$u \neq 0$ (non-homogeneous)	

The solution of the last system of equations is obtained as:

$$\begin{bmatrix} u_{2X} \\ u_{3X} \\ R_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\,u_0 \\ \frac{2}{3}\,u_0 \\ -\frac{1}{3}\frac{EA}{L}u_0 \end{bmatrix}. \tag{2.114}$$

It should be noted here that this third approach is not the common way within the finite element method and is only shown for the sake of completeness. At this stage, let us summarize the considered boundary conditions, see Table 2.10.

A special type of 'boundary condition' can be realized by attaching a spring to a rod element as shown in Fig. 2.15. Let us have first a look at the configuration where the spring is attached to node 1 as shown in Fig. 2.15a. Assuming that node 2 is moved to the positive x-direction, the spring will cause a force on the rod element which can be expressed as $F_s = -ku_1$, where k is the spring constant and u_1 the displacement of node 1, i.e. where the spring is attached to the rod element.[17] It should be mentioned here that the required force to elongate the spring by u_1 in the positive x-direction is equal to ku_1 but the force acting on the rod is oriented in the negative x-direction.

Thus, the principal finite element equation for the rod element can be written as:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_s \\ F_2 \end{bmatrix} = \begin{bmatrix} -ku_1 \\ F_2 \end{bmatrix}. \tag{2.115}$$

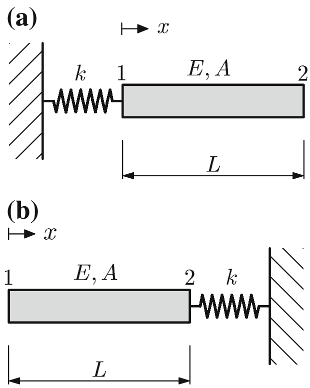


Fig. 2.15 Consideration of a spring in a rod structure: **a** spring attached to node 1 or; **b** to node 2

[17] It is assumed here that the spring is in its unstrained state in the sketched configuration, i.e. without the application of any force or displacement boundary conditions at the nodes of the rod.

Looking at Eq. (2.115), it can be concluded that the expression $-ku_1$ on the right-hand side should be shifted to the left-hand side where the expressions with the nodal unknowns are collected. Thus, one can obtain the following expression:

$$\frac{EA}{L}\left[\begin{array}{c|c} 1+\frac{L}{EA}k & -1 \\ \hline -1 & 1 \end{array}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}. \tag{2.116}$$

It can be seen from the last equation that a spring can simply be considered by adding the spring constant in the cell of the stiffness matrix with the index of the degree of freedom where the spring is attached, i.e. in our example the cell (u_1, u_1). If the spring would be attached at the second node, cf. Fig. 2.15b, the spring constant should be added in the cell (u_2, u_2) and the principal finite element equation for this case would finally read:

$$\frac{EA}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1+\frac{L}{EA}k \end{array}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}. \tag{2.117}$$

If we like to consider that the springs shown in Fig. 2.15 are pre-strained,[18] i.e. elongated or compressed by a displacement of magnitude u_s, the force which acts on the rod element is given[19] by $F_\mathrm{s} = -k(u_1 - u_\mathrm{s})$ or $F_\mathrm{s} = -k(u_2 - u_\mathrm{s})$ and the principal finite element equations given in (2.116) and (2.117) are modified to:

$$\frac{EA}{L}\left[\begin{array}{c|c} 1+\frac{L}{EA}k & -1 \\ \hline -1 & 1 \end{array}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} ku_\mathrm{s} \\ F_2 \end{bmatrix}, \tag{2.118}$$

$$\frac{EA}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1+\frac{L}{EA}k \end{array}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ ku_\mathrm{s} \end{bmatrix}. \tag{2.119}$$

2.3.4 Post-Computation: Determination of Strain, Stress and Further Quantities

The previous section explained how to compose the global system of equations from which the primary unknowns, i.e. the nodal displacements, can be obtained. After the solution for the nodal unknowns, further quantities can be calculated in a

[18] Such a pre-strained spring has its analogon in one-dimensional heat conduction in the form of a convective boundary condition: NEWTON's cooling law, i.e. $\dot{q} = h(T_\infty - T)$ where $\dot{q}$ is the heat flux in $\frac{\mathrm{W}}{\mathrm{m}^2}$, h is the heat transfer coefficient in $\frac{\mathrm{W}}{\mathrm{m}^2\,\mathrm{K}}$, T_∞ is the temperature of the environment and T is the temperature of the object's surface, is in a similar manner treated as this type of spring. See also Table 2.12.

[19] Setting $u_\mathrm{s} = 0$ results in an unstrained spring.

post-computational step. Based on the kinematics relationship for the continuum rod according to Eq. (2.2) together with the nodal approach (2.20) and the definition of the $\boldsymbol{B}$-matrix (2.24), the following expression for the strain distribution inside a rod can be obtained:

$$\varepsilon_x^{\mathrm{e}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} u^{\mathrm{e}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \boldsymbol{N}^{\mathrm{T}}(x)\, \boldsymbol{u}_{\mathrm{p}} = \boldsymbol{B}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{p}}. \tag{2.120}$$

Considering the specific formulations of the $\boldsymbol{B}$-matrices for a linear and a quadratic rod element according to Eqs. (2.35) and (2.58), the strain distribution can be expressed as

$$\varepsilon_x^{\mathrm{e}}(x) = \frac{1}{L}(-u_1 + u_2) \quad \text{(lin.)}, \tag{2.121}$$

$$\varepsilon_x^{\mathrm{e}}(x) = \frac{1}{L}\left(\left(-3 + \frac{4x}{L}\right)u_1 + \left(4 - \frac{8x}{L}\right)u_2 + \left(-1\frac{4x}{L}\right)u_3\right) \quad \text{(quad.)}, \tag{2.122}$$

or expressed in the natural coordinate ξ:

$$\varepsilon_x^{\mathrm{e}}(\xi) = \frac{1}{L}(-u_1 + u_2) \quad \text{(lin.)}, \tag{2.123}$$

$$\varepsilon_x^{\mathrm{e}}(\xi) = \frac{1}{L}((-1 + 2\xi)\, u_1 + (-4\xi)\, u_2 + (1 + 2\xi)\, u_3) \quad \text{(quad.)}. \tag{2.124}$$

Based on the obtained strain distribution, HOOKE's law (2.3) permits the calculation of the stress distribution inside a rod element as

$$\sigma_x^{\mathrm{e}}(x) = E\frac{\mathrm{d}}{\mathrm{d}x} u^{\mathrm{e}}(x) = E\frac{\mathrm{d}}{\mathrm{d}x} \boldsymbol{N}^{\mathrm{T}}(x)\, \boldsymbol{u}_{\mathrm{p}} = E\boldsymbol{B}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{p}}, \tag{2.125}$$

or based on the nodal displacements for a linear and quadratic rod element as a function of the natural coordinate ξ:

$$\varepsilon_x^{\mathrm{e}}(\xi) = \frac{E}{L}(-u_1 + u_2) \quad \text{(lin.)}, \tag{2.126}$$

$$\varepsilon_x^{\mathrm{e}}(\xi) = \frac{E}{L}((-1 + 2\xi)\, u_1 + (-4\xi)\, u_2 + (1 + 2\xi)\, u_3) \quad \text{(quad.)}. \tag{2.127}$$

A final task is often to calculate the reaction forces at the supports or nodes of prescribed displacements. To explain the procedure, let us return to the example shown in Fig. 2.14. The free body diagram of the problem can be sketched as shown in Fig. 2.16.

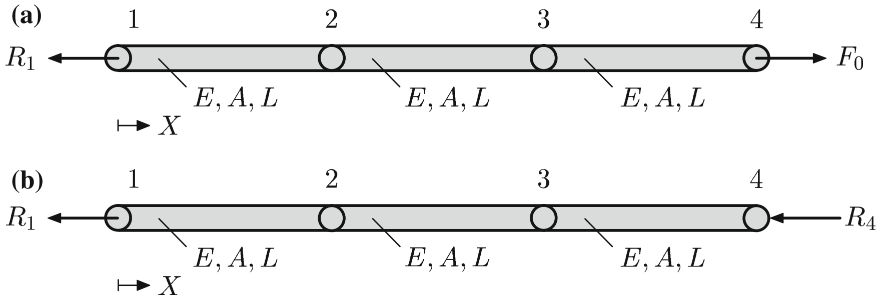


Fig. 2.16 Free body diagram of the cantilevered rod structure shown in Fig. 2.14

Based on the indicated reaction forces, the global (non-reduced) system of equations can be stated for the configuration in Fig. 2.16a as

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} -R_1 \\ 0 \\ 0 \\ F_0 \end{bmatrix}, \tag{2.128}$$

or for Fig. 2.16b as:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_{1X} \\ u_{2X} \\ u_{3X} \\ u_{4X} \end{bmatrix} = \begin{bmatrix} -R_1 \\ 0 \\ 0 \\ -R_4 \end{bmatrix}. \tag{2.129}$$

Knowing all nodal displacements, the support reaction R_1 can be obtained for both cases by evaluating the first equation of the linear system as:

$$R_1 = -\frac{EA}{L}(u_{1X} - u_{2X}). \tag{2.130}$$

For the second case as shown in Fig. 2.16b, the reaction force R_4 is obtained by evaluating the fourth equation of the the linear system (2.129) as:

$$R_4 = -\frac{EA}{L}(-u_{3X} + u_{4X}). \tag{2.131}$$

In general we can state that reactions forces are obtained from the non-reduced system of equations based on the prior to this calculated nodal displacements. Special attention must be given to the consideration of the reactions on the right-hand side of the system of equations since the pure calculation of the nodal displacements did not require an exact mentioning of these quantities. At the end of this section, let us highlight the different nature of the evaluated quantities as indicated in Table 2.11.

Table 2.11 Evaluation of different quantities

Quantity	Nodal Value	Elemental Value
Displacement	X	
Strain		X
Stress		X
Reaction force	X	

It is important to realize that the elemental values are evaluated at integration points of the element.

2.3.5 Analogies to Other Field Problems

Further analogies to other field problems can be found, for example, in [44]. A comparison between solid mechanics and heat conduction is presented in Table 2.12

Table 2.12 Comparison of analogous properties in one-dimensional heat conduction and solid mechanics. p_0: load per unit length in $\frac{\mathrm{N}}{\mathrm{m}}$; γ_0: load per unit volume in $\frac{\mathrm{N}}{\mathrm{m}^3}$; $\dot{\eta}_0$: rate of energy generation per unit volume in $\frac{\mathrm{W}}{\mathrm{m}^3}$; $\o_0$: rate of energy generation per unit length in $\frac{\mathrm{W}}{\mathrm{m}}$; $\dot{q}_x$: heat flux in $\frac{\mathrm{W}}{\mathrm{m}^2}$; $\dot{Q}_x$: heat transfer rate in W; k: thermal conductivity in $\frac{\mathrm{W}}{\mathrm{m\,K}}$

Solid Mechanics	Heat Conduction
Partial differential equation	
$EA\dfrac{\mathrm{d}^2u_x}{\mathrm{d}x^2} = -p_0$ $\left(E\dfrac{\mathrm{d}^2u_x}{\mathrm{d}x^2} = -\gamma_0\right)$	$k\dfrac{\mathrm{d}^2T}{\mathrm{d}x^2} = -\dot{\eta}_0$ $\left(kA\dfrac{\mathrm{d}^2T}{\mathrm{d}x^2} = -\dot{\o}_0\right)$
Primary variable	
Displacement u_x	Temperature T
Derivative of primary variable	
Strain $\varepsilon_x = \frac{\mathrm{d}u_x}{\mathrm{d}x}$ Stress $\sigma_x = E\frac{\mathrm{d}u_x}{\mathrm{d}x}$ Force $F_x = EA\frac{\mathrm{d}u_x}{\mathrm{d}x}$	Temperature gradient $\frac{\mathrm{d}T}{\mathrm{d}x}$ Heat flux $\dot{q}_x = -k\frac{\mathrm{d}T}{\mathrm{d}x}$ Heat transfer rate $\dot{Q}_x = -kA\frac{\mathrm{d}T}{\mathrm{d}x}$
Principal finite element equation	
$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \end{bmatrix}$ $\left(\frac{E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix} = \begin{bmatrix} \sigma_{1x} \\ \sigma_{2x} \end{bmatrix}\right)$	$\frac{k}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_{1x} \\ \dot{q}_{2x} \end{bmatrix}$ $\left(\frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \dot{Q}_{1x} \\ \dot{Q}_{2x} \end{bmatrix}\right)$

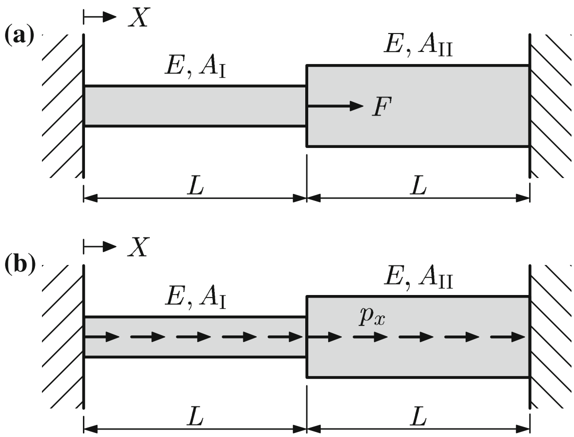


Fig. 2.17 Rod structure fixed at both ends: **a** axial point load; **b** load per length

2.3.6 *Solved Rod Problems*

2.1 Example: Rod structure fixed at both ends

Given is a rod structure as shown in Fig. 2.17. The structure is composed of two rods of different cross-sectional areas A_{I} and A_{II}. Length L and YOUNG's modulus E are the same for both rods. The structure is fixed at both ends and loaded by

(a) a point load F in the middle and
(b) a uniform distributed load p_x, i.e. a force per unit length.

Model the rod structure with two linear finite elements and determine for both cases

- the displacement $u_2 = u(X = L)$ in the middle of the structure,
- the stresses and strains in both elements,
- the average stress and strain in the middle of the structure at $X = L$,
- the reaction forces at the supports and check the global force equilibrium.

Simplify all the results obtained for the special case of $A_{\mathrm{I}} = A_{\mathrm{II}} = A$.

2.1 Solution

The finite element discretization and all acting forces are shown in Fig. 2.18.
Case (a) point load:

- Displacement in the middle of the structure

The elemental stiffness matrix for each element is given by

$$\frac{EA_i}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{with} \quad i = \mathrm{I}, \mathrm{II} \tag{2.132}$$

and can be assembled to obtain the global finite element equation:

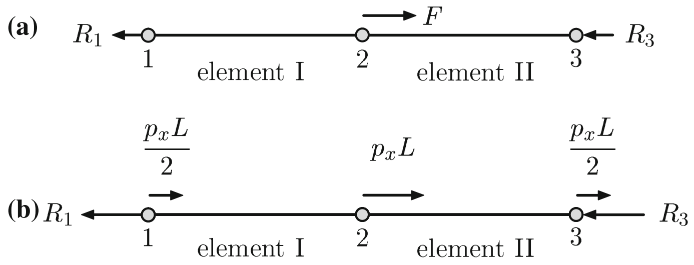


Fig. 2.18 Discretized rod structure: **a** point load and reaction forces; **b** equivalent nodal loads and reaction forces

$$\frac{E}{L}\begin{bmatrix} A_{\text{I}} & -A_{\text{I}} & 0 \\ -A_{\text{I}} & A_{\text{I}}+A_{\text{II}} & -A_{\text{II}} \\ 0 & -A_{\text{II}} & A_{\text{II}} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -R_1 \\ F \\ -R_3 \end{bmatrix}. \tag{2.133}$$

Consideration of the boundary conditions, i.e. $u_1 = u_3 = 0$, in the last system of equations allows to solve for the unknown displacement in the middle of the structure:

$$u_2 = \frac{FL}{E(A_{\text{I}} + A_{\text{II}})}. \tag{2.134}$$

• Stresses and strains in both elements
Based on the general definition of the strain in a rod element, i.e. $\varepsilon = \frac{1}{L}(u_{\text{right}} - u_{\text{left}})$, the constant strains in both elements can be derived under consideration of the boundary conditions as:

$$\varepsilon_{\text{I}} = \frac{1}{L}(u_2 - 0) = \frac{F}{E(A_{\text{I}} + A_{\text{II}})}, \tag{2.135}$$

$$\varepsilon_{\text{II}} = \frac{1}{L}(0 - u_2) = -\frac{F}{E(A_{\text{I}} + A_{\text{II}})}. \tag{2.136}$$

Application of HOOKE's law, i.e. $\sigma = E\varepsilon$, gives the constant stresses in each element:

$$\sigma_{\text{I}} = E\varepsilon_{\text{I}} = \frac{F}{(A_{\text{I}} + A_{\text{II}})}, \tag{2.137}$$

$$\sigma_{\text{II}} = E\varepsilon_{\text{II}} = -\frac{F}{(A_{\text{I}} + A_{\text{II}})}. \tag{2.138}$$

• Average stress and strain in the middle of the structure
As in the case of many finite element codes, the average stress and strain at the middle node can be calculated by the following averaging rule as:

$$\varepsilon_2 = \frac{\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{II}}}{2} = 0, \tag{2.139}$$

$$\sigma_2 = \frac{\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}}{2} = 0. \tag{2.140}$$

As can be seen from this result, stress and strain values displayed at nodes should be taken with care.

• Reaction forces at the supports and check of the global force equilibrium

Evaluation of the first and third equation of the system (2.133) for known nodal displacements gives:

$$R_1 = \frac{EA_{\mathrm{I}}}{L} \times u_2 = \frac{A_{\mathrm{I}}}{A_{\mathrm{I}} + A_{\mathrm{II}}} \times F, \tag{2.141}$$

$$R_3 = \frac{EA_{\mathrm{II}}}{L} \times u_2 = \frac{A_{\mathrm{II}}}{A_{\mathrm{I}} + A_{\mathrm{II}}} \times F, \tag{2.142}$$

and the global force equilibrium

$$F - \frac{A_{\mathrm{I}}}{A_{\mathrm{I}} + A_{\mathrm{II}}} \times F - \frac{A_{\mathrm{II}}}{A_{\mathrm{I}} + A_{\mathrm{II}}} \times F = 0 \tag{2.143}$$

is fulfilled.

Case (b) distributed load:

• Displacement in the middle of the structure

The global finite element equation results under consideration of the equivalent nodal loads, cf. Fig. 2.18, as

$$\frac{E}{L}\begin{bmatrix} A_{\mathrm{I}} & -A_{\mathrm{I}} & 0 \\ -A_{\mathrm{I}} & A_{\mathrm{I}} + A_{\mathrm{II}} & -A_{\mathrm{II}} \\ 0 & -A_{\mathrm{II}} & A_{\mathrm{II}} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -R_1 + \frac{p_x L}{2} \\ p_x L \\ -R_3 + \frac{p_x L}{2} \end{bmatrix}, \tag{2.144}$$

from which the displacement at node 2 follows under consideration of the boundary conditions:

$$u_2 = \frac{(p_x L)L}{E(A_{\mathrm{I}} + A_{\mathrm{II}})}. \tag{2.145}$$

• Stresses and strains in both elements and at the middle node

Based on the procedure given in (a), the constant strains and stresses are given by:

Table 2.13 Results of the problem shown in Fig. 2.17 for the special case $A_{\text{I}} = A_{\text{II}} = A$

Quantity	Point load F	Distributed load p_x
u_2	$\frac{1}{2}\frac{FL}{EA}$	$\frac{1}{2}\frac{(p_x L)L}{EA}$
ε_{I}	$\frac{F}{2EA}$	$\frac{p_x L}{2EA}$
ε_{II}	$-\frac{F}{2EA}$	$-\frac{p_x L}{2EA}$
σ_{I}	$\frac{F}{2A}$	$\frac{p_x L}{2A}$
σ_{II}	$-\frac{F}{2A}$	$-\frac{p_x L}{2A}$
σ_2	0	0
ε_2	0	0
R_1	$\frac{F}{2}$	$p_x L$
R_3	$\frac{F}{2}$	$p_x L$

$$\varepsilon_{\text{I}} = \frac{p_x L}{E(A_{\text{I}} + A_{\text{II}})} \ , \ \sigma_I = \frac{p_x L}{A_{\text{I}} + A_{\text{II}}}, \tag{2.146}$$

$$\varepsilon_{\text{II}} = -\frac{p_x L}{E(A_{\text{I}} + A_{\text{II}})} \ , \ \sigma_{II} = -\frac{p_x L}{A_{\text{I}} + A_{\text{II}}}, \tag{2.147}$$

$$\varepsilon_2 = 0 \ , \ \sigma_2 = 0. \tag{2.148}$$

• Reaction forces at the supports and check of the global force equilibrium
Evaluation of the first and third equation of the system (2.144) for known nodal displacements gives:

$$R_1 = \frac{p_x L}{2} + \frac{EA_{\text{I}}}{L} \times u_2 = \left(\frac{1}{2} + \frac{A_{\text{I}}}{A_{\text{I}} + A_{\text{II}}} \right) \times p_x L, \tag{2.149}$$

$$R_3 = \frac{p_x L}{2} + \frac{EA_{\text{II}}}{L} \times u_2 = \left(\frac{1}{2} + \frac{A_{\text{II}}}{A_{\text{I}} + A_{\text{II}}} \right) \times p_x L, \tag{2.150}$$

and the global force equilibrium is fulfilled. It can be concluded from this exercise that the equivalent loads applied at the supports do not influence the strains and stresses inside the rods but contribute to the reaction forces at the supports. Results for the special case $A_{\text{I}} = A_{\text{II}} = A$ are summarized in Table 2.13.

2.2 Example: Rod structure with gap

Given is a rod structure as shown in Fig. 2.19. The structure is composed of a rod with cross-sectional area A, length L, and YOUNG's modulus E. The structure is

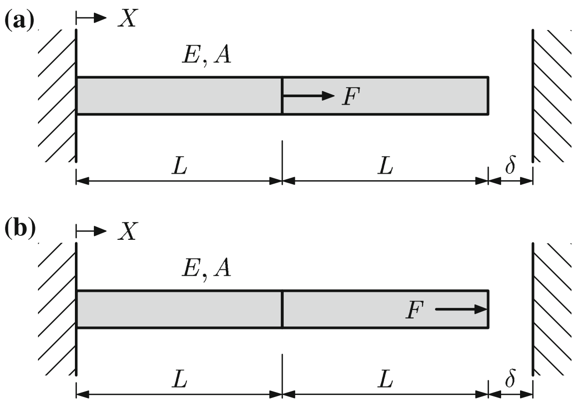


Fig. 2.19 Rod structure with a gap at the *right* end: **a** axial point load in the *middle*; **b** axial point load at the *right* end

fixed at the left-hand end and a gap of distance δ is between the right-hand end and a rigid wall. The structure is loaded by

(a) a point load F in the middle and
(b) a point load F at the right end.

Model the rod structure with two linear finite elements and determine for both cases

- the displacement $u_2 = u(X = L)$ in the middle of the structure for the case of no contact and contact,
- the reaction forces at the supports and check the global force equilibrium,
- the stress distribution in the rod structure for increasing force F.

2.2 Solution

The finite element discretization and all acting forces are shown in Fig. 2.20.

Case (a) point load in the middle:

- Displacement in the middle of the structure

In the case that there is no contact, element II is not acting, i.e. $u_2 = u_3$, or contributing to the global stiffness matrix and the problem can be described by

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -R_1 \\ F \end{bmatrix}, \tag{2.151}$$

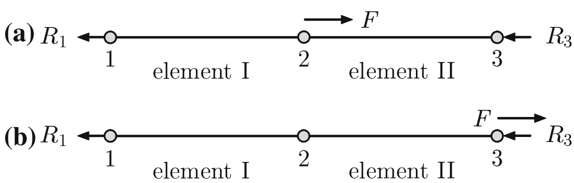


Fig. 2.20 Discretized rod structure: **a** axial point load in the middle; **b** axial point load at the right end. The reaction force R_3 is only acting in the case of contact

from which the displacement at node 2 can be obtained under consideration of the boundary condition ($u_1 = 0$) as:

$$u_2 = \frac{FL}{EA}. \tag{2.152}$$

If the force F is further increased to a value of $F = \frac{EA\delta}{L}$, contact occurs, i.e. $u_2 = u_3 = \delta$, and the situation for the global system is different. Now, both elements contribute to the global system:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -R_1 \\ F \\ -R_3 \end{bmatrix}. \tag{2.153}$$

Consideration of the boundary conditions at the left- and right-hand end, i.e. $u_1 = 0$ and $u_3 = \delta$, gives

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F \\ \frac{EA\delta}{L} \end{bmatrix}, \tag{2.154}$$

and the displacement in the middle of the structure is obtained as:

$$u_2 = \frac{LF}{2EA} + \frac{\delta}{2}. \tag{2.155}$$

• Reaction forces
Based on the known values of the nodal displacements, Eq. (2.153) can be evaluated for the reaction forces:

$$R_1 = \left(\frac{F}{2} + \frac{EA\delta}{2L}\right), \tag{2.156}$$

$$R_3 = \left(\frac{F}{2} - \frac{EA\delta}{2L}\right). \tag{2.157}$$

It should be noted that both reaction forces are directed to the negative X-direction.
• Stress distribution in the rod structure for increasing force F
Since the nodal displacements are known, the strains can be obtained based on the general definition $\varepsilon = \frac{1}{L}(u_{\text{right}} - u_{\text{left}})$ and HOOKE's law gives the stresses. The results for the stress σ in both elements as a function of the applied external force F are shown in Fig. 2.21. As can be seen from this figure, the global stiffness changes as soon as the gap is closed.

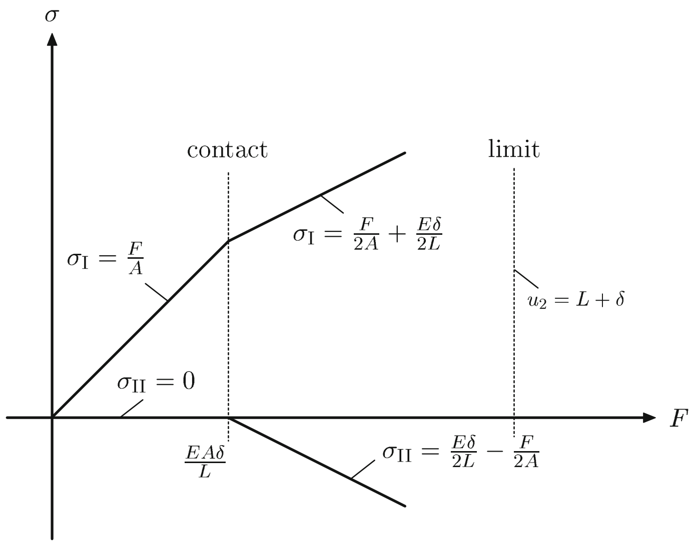


Fig. 2.21 Stress distribution in the rod structure as a function of the external point load F

Case (b) point load at the right end:

- Displacement in the middle of the structure

In the case that there is no contact, the displacement of the middle node is simply half of the displacement obtained at the node of the right-hand end. Under the condition of contact, the global system of equations reads as:

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -R_1 \\ 0 \\ F - R_3 \end{bmatrix}. \tag{2.158}$$

Consideration of the boundary conditions at the left- and right-hand end, i.e. $u_1 = 0$ and $u_3 = \delta$, gives

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{EA\delta}{L} \end{bmatrix}, \tag{2.159}$$

and the displacement in the middle of the structure is obtained as:

$$u_2 = \frac{\delta}{2}. \tag{2.160}$$

- Reaction forces

Based on the known values of the nodal displacements, Eq. (2.158) can be evaluated for the reaction forces:

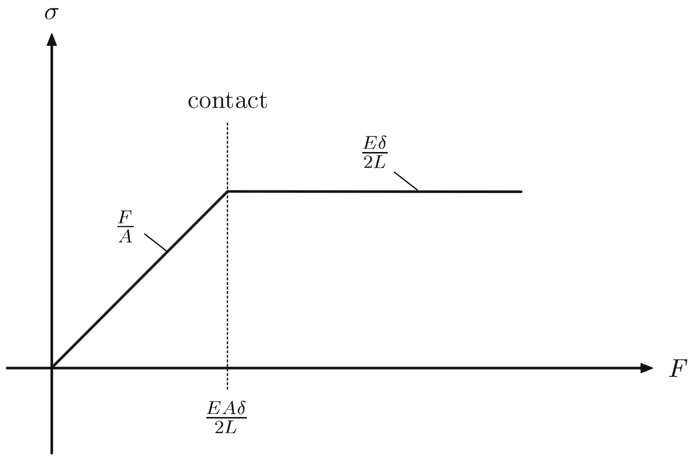


Fig. 2.22 Stress distribution in the rod structure as a function of the external point load F

$$R_1 = \frac{EA\delta}{2L}, \tag{2.161}$$

$$R_3 = F - \frac{EA\delta}{2L}. \tag{2.162}$$

• Stress distribution in the rod structure for increasing force F.
As mentioned in (a), strains and stresses can be calculated based on the known nodal displacements. The graphical representation of the stress in the rod is given in Fig. 2.22. It can be concluded that any additional force after closing the gap is absorbed by the support and does not affect the stress state in the rod.

2.3 Example: Rod with changing cross-sectional area

Given is a rod structure as shown in Fig. 2.23. The structure reveals a linear changing cross-sectional area $A(x)$ while the Young's modulus E is assumed to be constant. The structure is fixed at the left-hand end and loaded by a single force F_0 at the right-hand end. The ratio between the area A_5 and A_1 is given by the factor a.

Model the rod structure with four linear finite elements of constant cross-sectional area and determine for the stepped rod the nodal displacements. Each element should have the same length $\frac{L}{4}$ and the cross section should be the average of the cross-sectional area at the left- and right-hand end of each step.

2.3 Solution

Given the ratio between the cross section at the right- and left-hand end as $\frac{A_5}{A_1} = a$ and the functional dependency of the cross-sectional area as

$$A(X) = A_1 + \frac{X}{L}(A_5 - A_1)\,, \tag{2.163}$$

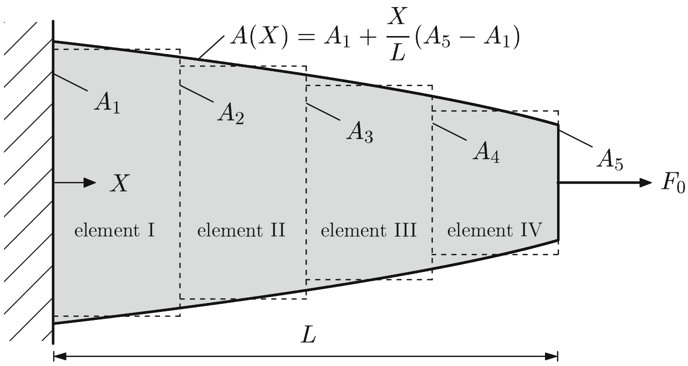


Fig. 2.23 Rod with changing cross-sectional area $A = A(x)$

the areas A_i $(i = 2, \ldots, 5)$ can be expressed as:

$$A_2 = \frac{3+a}{4} A_1 \ , \ A_3 = \frac{1+a}{2} A_1 \ , \ A_4 = \frac{1+3a}{4} A_1 \ , \ A_5 = a A_1. \tag{2.164}$$

Based on these area relations, the averaged area for each element $A_{ij} = \frac{A_i + A_j}{2}$ is obtained as:

$$A_{12} = \frac{7+a}{8} A_1 \ , \ A_{23} = \frac{5+3a}{8} A_1 \ , \ A_{34} = \frac{3+5a}{8} A_1 \ , \ A_{45} = \frac{1+7a}{8} A_1, \tag{2.165}$$

and the global stiffness matrix can be assembled to:

$$\frac{EA_1}{2L} \begin{bmatrix} 7+a & -(7+a) & 0 & 0 & 0 \\ -(7+a) & 12+4a & -(5+3a) & 0 & 0 \\ 0 & -(5+3a) & 8+8a & -(3+5a) & 0 \\ 0 & 0 & -(3+5a) & 4+12a & -(1+7a) \\ 0 & 0 & 0 & -(1+7a) & 1+7a \end{bmatrix}. \tag{2.166}$$

Considering the boundary condition at the left-hand end, i.e. $u_1 = 0$, the system of equations is obtained as

$$\frac{EA_1}{2L} \begin{bmatrix} 12+4a & -(5+3a) & 0 & 0 \\ -(5+3a) & 8+8a & -(3+5a) & 0 \\ 0 & -(3+5a) & 4+12a & -(1+7a) \\ 0 & 0 & -(1+7a) & 1+7a \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F_0 \end{bmatrix}, \tag{2.167}$$

from which the vector of nodal displacements can be calculated:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \frac{LF_0}{EA_1} \begin{bmatrix} 0 \\ \dfrac{2}{7+a} \\ \dfrac{8(3+a)}{3a^2+26a+35} \\ \dfrac{2(71+98a+23a^2)}{105+253a+139a^2+15a^3} \\ \dfrac{32(11+53a+53a^2+11a^3)}{(1+7a)(105+253a+139a^2+15a^3)} \end{bmatrix}. \tag{2.168}$$

2.4 Example: Rod with linearly increasing distributed load

The following Fig. 2.24a shows a cantilevered rod structure of length L which is loaded with a triangular shaped distributed load (maximum value of q_0 at $X = L$).

Use two linear rod elements of equal length $\frac{L}{2}$ (see Fig. 2.24b) and:

- Calculate for each element separately the vector of the equivalent nodal loads based on the general statement $\int N q(x) \mathrm{d}x$.
- Assemble the global system of equations without consideration of the boundary conditions at the fixed support.
- Obtain the reduced system of equations (the solution of the system of equations is not required).

2.4 Solution

The separated elements and the corresponding distributed loads are shown in Fig. 2.25.

Special consideration requires the elemental length $\frac{L}{2}$ since the interpolation functions and integrals are defined from $0 \ldots L$. Thus, let us calculate the equivalent nodal loads first for the length L and at the end we substitute $L := \frac{L}{2}$.

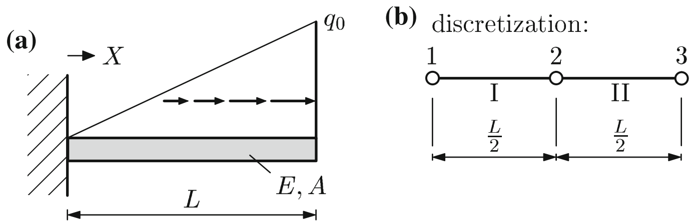


Fig. 2.24 Cantilevered rod with *triangular* shaped distributed load: **a** geometry and boundary conditions and **b** discretization

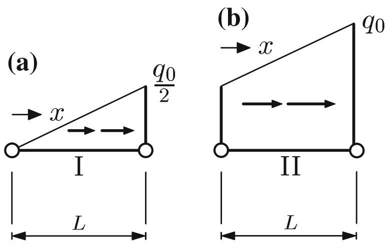


Fig. 2.25 Single elements and corresponding distributed loads

• Let us look in the following first separately at each element. The load vector for element I can be written as:

$$\boldsymbol{f}_{\mathrm{I}} = \int_0^L \boldsymbol{N}(x)q(x)\,\mathrm{d}x = \int_0^L \begin{bmatrix} N_{1u}(x) \\ N_{2u}(x) \end{bmatrix} \left(\frac{q_0}{2} \frac{x}{L} \right) \mathrm{d}x = \frac{q_0}{2L} \int_0^L \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} x\,\mathrm{d}x$$
$$= \frac{q_0}{2L} \begin{bmatrix} \frac{x^2}{2} - \frac{x^3}{3L} \\ \frac{x^3}{3L} \end{bmatrix}_0^L = \frac{q_0}{2} \begin{bmatrix} \frac{L}{6} \\ \frac{L}{3} \end{bmatrix}. \tag{2.169}$$

$$L := \frac{L}{2} \;\; \Rightarrow \;\; \boldsymbol{f}_{\mathrm{I}} = \frac{q_0}{4} \begin{bmatrix} \frac{L}{6} \\ \frac{L}{3} \end{bmatrix}. \tag{2.170}$$

In a similar way we obtain for element II:

$$\boldsymbol{f}_{\mathrm{II}} = \int_0^L \boldsymbol{N}(x)q(x)\,\mathrm{d}x = \int_0^L \begin{bmatrix} N_{1u}(x) \\ N_{2u}(x) \end{bmatrix} \left(q_0 \left[\frac{1}{2} + \frac{x}{2L} \right] \right) \mathrm{d}x$$
$$= \frac{q_0}{2L} \int_0^L \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \left(1 + \frac{x}{L} \right) \mathrm{d}x = \frac{q_0}{2} \begin{bmatrix} x - \frac{x^3}{3L^2} \\ \frac{x^2}{2L} + \frac{x^3}{3L^2} \end{bmatrix}_0^L = \frac{q_0}{2} \begin{bmatrix} \frac{2L}{3} \\ \frac{5L}{6} \end{bmatrix}. \tag{2.171}$$

$$L := \frac{L}{2} \;\; \Rightarrow \;\; \boldsymbol{f}_{\mathrm{II}} = \frac{q_0}{4} \begin{bmatrix} \frac{2L}{3} \\ \frac{5L}{6} \end{bmatrix}. \tag{2.172}$$

Check: Simple superposition based on tabled values (see Table 2.5) for an element of length L, see Fig. 2.26.

$$\begin{bmatrix} \frac{qL}{4} \\ \frac{qL}{4} \end{bmatrix} + \begin{bmatrix} \frac{qL}{12} \\ \frac{qL}{6} \end{bmatrix} = \begin{bmatrix} \frac{qL}{3} \\ \frac{5qL}{12} \end{bmatrix} \checkmark \tag{2.173}$$

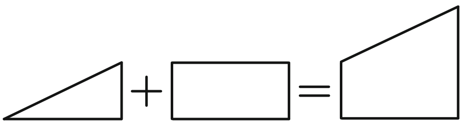


Fig. 2.26 Superposition of simple load cases

- The principal finite element equation for element I reads:

$$\frac{EA}{\frac{L}{2}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_{1X}\\ u_{2X}\end{bmatrix} = \frac{q_0}{4}\begin{bmatrix}\frac{L}{6}\\ \frac{L}{3}\end{bmatrix}. \tag{2.174}$$

and for element II:

$$\frac{EA}{\frac{L}{2}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_{2X}\\ u_{3X}\end{bmatrix} = \frac{q_0}{4}\begin{bmatrix}\frac{2L}{3}\\ \frac{5L}{6}\end{bmatrix}. \tag{2.175}$$

Global system of equations:

$$\frac{EA}{\frac{L}{2}}\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_{1X}\\ u_{2X}\\ u_{3X}\end{bmatrix} = \frac{q_0}{4}\begin{bmatrix}\frac{L}{6}\\ \frac{L}{3}+\frac{2L}{3}\\ \frac{5L}{6}\end{bmatrix}. \tag{2.176}$$

- Reduced system of equations: $u_{1X} = 0$

$$\frac{EA}{\frac{L}{2}}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_{2X}\\ u_{3X}\end{bmatrix} = \frac{q_0}{4}\begin{bmatrix}L\\ \frac{5L}{6}\end{bmatrix}. \tag{2.177}$$

2.4 Assembly of Elements to Plane Truss Structures

2.4.1 Rotational Transformation in a Plane

Let us consider in the following a rod element which can deform in the global X–Y plane. The local x-coordinate is rotated by an angle α against the global coordinate system (X, Y), cf. Fig. 2.27.

Each node has now in the global coordinate system two degrees of freedom, i.e. a displacement in the X- and a displacement in the Y-direction. These two global displacements at each node can be used to calculate the displacement in the direction of the rod axis, i.e. in the direction of the local x-axis. Based on the right-angled triangles shown in Fig. 2.27, the displacements in the local coordinate system are given based on the global displacements as:

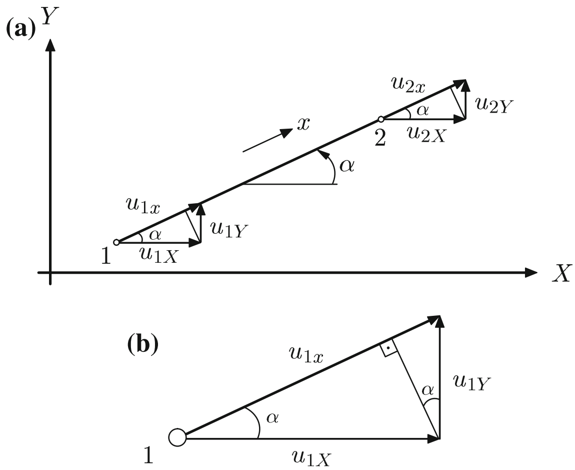


Fig. 2.27 Rotational transformation of a rod element in the X–Y plane: **a** total view and **b** detail for node 1

$$u_{1x} = \cos\alpha\, u_{1X} + \sin\alpha\, u_{1Y}, \tag{2.178}$$

$$u_{2x} = \cos\alpha\, u_{2X} + \sin\alpha\, u_{2Y}. \tag{2.179}$$

It is possible to derive in a similar way the global displacements based on the local displacements as:

$$u_{1X} = \cos\alpha\, u_{1x}, \qquad u_{2X} = \cos\alpha\, u_{2x}, \tag{2.180}$$

$$u_{1Y} = \sin\alpha\, u_{1x}, \qquad u_{2Y} = \sin\alpha\, u_{2x}. \tag{2.181}$$

The last relationships between the global and local displacements can be written in matrix form as

$$\begin{bmatrix} u_{1X} \\ u_{1Y} \\ u_{2X} \\ u_{2Y} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \\ 0 & \cos\alpha \\ 0 & \sin\alpha \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix}, \tag{2.182}$$

or in abbreviated matrix notation as:

$$\boldsymbol{u}_{XY} = \boldsymbol{T}^{\mathrm{T}} \boldsymbol{u}_{xy}, \tag{2.183}$$

where $\boldsymbol{u}_{XY}$ is the displacement column matrix in the global coordinate system and $\boldsymbol{u}_{xy}$ the local displacement column matrix. The last equation can be solved for the displacements in the local coordinate system and inverting[20] the transformation matrix gives

[20] Since the transformation matrix is orthogonal, it follows that $\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{T}^{-1}$.

$$\boldsymbol{u}_{xy} = \boldsymbol{T}\,\boldsymbol{u}_{XY}, \tag{2.184}$$

or in components:

$$\begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}}_{\boldsymbol{T}} \begin{bmatrix} u_{1X} \\ u_{1Y} \\ u_{2X} \\ u_{2Y} \end{bmatrix}. \tag{2.185}$$

It is possible to transform in a similar way the vector of the external loads as:

$$\boldsymbol{f}_{XY} = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{f}_{xy}, \tag{2.186}$$

$$\boldsymbol{f}_{xy} = \boldsymbol{T}\,\boldsymbol{f}_{XY}. \tag{2.187}$$

Considering the transformation of the local displacements and loads in the principal finite element equation according to Eq. (2.52), the transformation of the stiffness matrix into the global coordinate system is given as:

$$\underbrace{(\boldsymbol{T}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{e}}_{xy}\boldsymbol{T})}_{\boldsymbol{K}^{\mathrm{e}}_{XY}}\,\boldsymbol{T}^{\mathrm{T}}\boldsymbol{u}_{xy} = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{f}_{xy}, \tag{2.188}$$

or in components

$$\begin{bmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \\ 0 & \cos\alpha \\ 0 & \sin\alpha \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}. \tag{2.189}$$

The evaluation of this triple product results finally in the stiffness matrix in the global X–Y coordinate system as:

$$\frac{AE}{L}\begin{bmatrix} \cos^2\alpha & \cos\alpha\,\sin\alpha & -\cos^2\alpha & -\cos\alpha\,\sin\alpha \\ \cos\alpha\,\sin\alpha & \sin^2\alpha & -\cos\alpha\,\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\,\sin\alpha & \cos^2\alpha & \cos\alpha\,\sin\alpha \\ -\cos\alpha\,\sin\alpha & -\sin^2\alpha & \cos\alpha\,\sin\alpha & \sin^2\alpha \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{1Y} \\ u_{2X} \\ u_{2Y} \end{bmatrix} = \begin{bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \end{bmatrix}. \tag{2.190}$$

To simplify the solution of simple truss structures, Table 2.14 collects expressions for the global stiffness matrix for some common angles α.

Table 2.14 Elemental stiffness matrices for truss elements given for different rotation angles α, cf. Eq. (2.190)

$0°$	$180°$
$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$-30°$	$30°$
$\frac{EA}{L}\begin{bmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} & -\frac{3}{4} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} & \frac{1}{4}\sqrt{3} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4}\sqrt{3} & \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & -\frac{1}{4} & -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{bmatrix}$	$\frac{EA}{L}\begin{bmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} & -\frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} & -\frac{1}{4}\sqrt{3} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{4}\sqrt{3} & \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & -\frac{1}{4} & \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{bmatrix}$
$-45°$	$45°$
$\frac{EA}{L}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$\frac{EA}{L}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
$-90°$	$90°$
$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

Let us consider in the following a slightly different configuration in which a rod element can now deform in the global X–Z plane, Fig. 2.28. The local x-coordinate is rotated by an angle α against the global coordinate system (X, Z).
For this case, the global displacements can be expressed based on the local displacements as:

$$\underbrace{u_{1X}}_{>0} = \underbrace{\cos\alpha}_{>0}\ \underbrace{u_{1x}}_{>0}, \qquad \underbrace{u_{2X}}_{>0} = \underbrace{\cos\alpha}_{>0}\ \underbrace{u_{2x}}_{>0}, \tag{2.191}$$

$$\underbrace{u_{1Z}}_{>0} = -\underbrace{\sin\alpha}_{<0}\ \underbrace{u_{1x}}_{>0}, \qquad \underbrace{u_{2Z}}_{>0} = -\underbrace{\sin\alpha}_{<0}\ \underbrace{u_{2x}}_{>0}. \tag{2.192}$$

The last relationships between the global and local displacements can be written in matrix form as

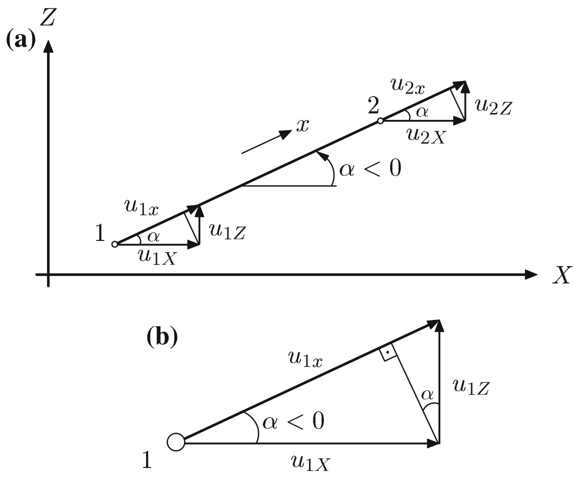


Fig. 2.28 Rotational transformation of a rod element in the X–Z plane: **a** total view and **b** detail for node 1

$$\begin{bmatrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ -\sin\alpha & 0 \\ 0 & \cos\alpha \\ 0 & -\sin\alpha \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix}, \tag{2.193}$$

or in abbreviated matrix notation as:

$$\boldsymbol{u}_{XZ} = \boldsymbol{T}^{\mathrm{T}} \boldsymbol{u}_{xz}. \tag{2.194}$$

The last equation can be solved for the displacements in the local coordinate system and inverting[21] the transformation matrix gives

$$\boldsymbol{u}_{xz} = \boldsymbol{T} \boldsymbol{u}_{XZ}, \tag{2.195}$$

or in components:

$$\begin{bmatrix} u_{1x} \\ u_{2x} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \end{bmatrix}}_{\boldsymbol{T}} \begin{bmatrix} u_{1X} \\ u_{1Z} \\ u_{2X} \\ u_{2Z} \end{bmatrix}. \tag{2.196}$$

It is possible to transform in a similar way the vector of the external loads as:

$$\boldsymbol{f}_{XZ} = \boldsymbol{T}^{\mathrm{T}} \boldsymbol{f}_{xz}, \tag{2.197}$$

$$\boldsymbol{f}_{xz} = \boldsymbol{T} \boldsymbol{f}_{XZ}. \tag{2.198}$$

[21] Since the transformation matrix is orthogonal, it follows that $\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{T}^{-1}$.

Considering the transformation of the local displacements and loads in the principal finite element equation according to Eq. (2.52), the transformation of the stiffness matrix into the global coordinate system is given as:

$$\underbrace{(\boldsymbol{T}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{e}}_{xz}\boldsymbol{T})}_{\boldsymbol{K}^{\mathrm{e}}_{XZ}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{u}_{xz} = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{f}_{xz}, \tag{2.199}$$

or in components

$$\begin{bmatrix} \cos\alpha & 0 \\ -\sin\alpha & 0 \\ 0 & \cos\alpha \\ 0 & -\sin\alpha \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \end{bmatrix}. \tag{2.200}$$

The evaluation of this triple product results finally in the stiffness matrix in the global X–Z coordinate system as:

$$\frac{AE}{L} \begin{bmatrix} \cos^2\alpha & \cos\alpha\,\sin\alpha & -\cos^2\alpha & -\cos\alpha\,\sin\alpha \\ \cos\alpha\,\sin\alpha & \sin^2\alpha & -\cos\alpha\,\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\,\sin\alpha & \cos^2\alpha & \cos\alpha\,\sin\alpha \\ -\cos\alpha\,\sin\alpha & -\sin^2\alpha & \cos\alpha\,\sin\alpha & \sin^2\alpha \end{bmatrix} \begin{bmatrix} u_{1X} \\ u_{1Y} \\ u_{2X} \\ u_{2Y} \end{bmatrix} = \begin{bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \end{bmatrix}. \tag{2.201}$$

2.4.2 Solved Truss Problems

2.5 Example: Truss structure arranged as an equilateral triangle

Given is the two-dimensional truss structure as shown in Fig. 2.29 where the trusses are arranged in the form of an equilateral triangle (all internal angles $\beta = 60°$). The three trusses have the same length L, the same YOUNG's modulus E, and the same cross-sectional area A. The structure is loaded by

(a) a horizontal force F at node 2,
(b) a prescribed displacement u at node 2.

Determine for both cases

- the global system of equations,
- the reduced system of equations,
- all nodal displacements,
- all reaction forces,
- the force in each rod.

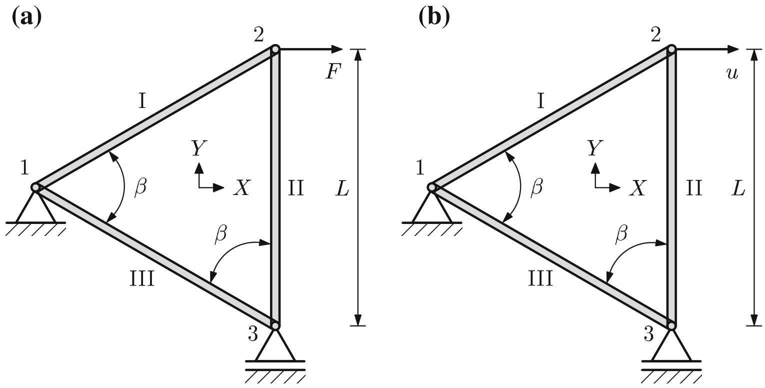


Fig. 2.29 Truss structure in the form of an equilateral triangle: **a** force boundary condition; **b** displacement boundary condition

2.5 Solution

The free-body diagram and the local coordinate axes of each element are shown in Fig. 2.30. From this figure, the rotational angles from the global to the local coordinate system can be determined and the sine and cosine values calculated as given in Table 2.15.

(a) Force boundary condition
Based on Eq. (2.190) and the values given in Table 2.15, the elemental stiffness matrices can be calculated as:

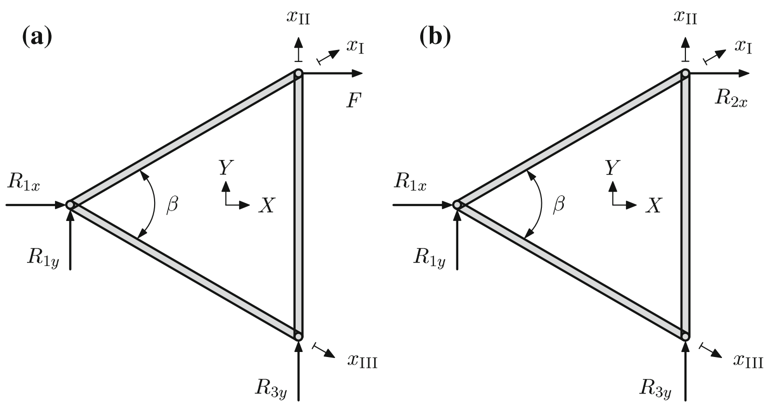


Fig. 2.30 Free body diagram of the truss structure: **a** force boundary condition; **b** displacement boundary condition

Table 2.15 Angles of rotation α_i and sine and cosine values for the problem shown in Fig. 2.30

Element	Angle of Rotation	sine	cosine
I	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
II	90°	1	0
III	330°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

$$\boldsymbol{K}_{\mathrm{I}}^{\mathrm{e}} = \underbrace{\frac{EA}{L}}_{k_{\mathrm{I}}} \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}, \tag{2.202}$$

$$\boldsymbol{K}_{\mathrm{II}}^{\mathrm{e}} = \underbrace{\frac{EA}{L}}_{k_{\mathrm{II}}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \tag{2.203}$$

$$\boldsymbol{K}_{\mathrm{III}}^{\mathrm{e}} = \underbrace{\frac{EA}{L}}_{k_{\mathrm{III}}} \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}, \tag{2.204}$$

which can be assembled to the global stiffness matrix as:

$$\frac{EA}{L} \begin{bmatrix} \frac{3}{4}+\frac{3}{4} & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & \frac{1}{4}+\frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & 0 & 0 \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4}+1 & 0 & -1 \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & 0 & 0 & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & -1 & -\frac{\sqrt{3}}{4} & 1+\frac{1}{4} \end{bmatrix} \begin{array}{l} u_{1X} \\ u_{1Y} \\ u_{2X} \\ u_{2Y} \\ u_{3X} \\ u_{3Y} \end{array}. \tag{2.205}$$

Introducing the boundary conditions, i.e. $u_{1X} = u_{1Y} = u_{3Y} = 0$, gives the reduced system of equations as:

$$\frac{EA}{L}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & 0 \\ \frac{\sqrt{3}}{4} & \frac{5}{4} & 0 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{2Y} \\ u_{3X} \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}. \tag{2.206}$$

The solution of this system can be obtained, for example, by inverting the reduced stiffness matrix to give the reduced result vector as:

$$\begin{bmatrix} u_{2X} \\ u_{2Y} \\ u_{3X} \end{bmatrix} = \frac{L}{EA}\begin{bmatrix} \frac{5}{3} & -\frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{3}}{3} & 1 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}\begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix} = \frac{LF}{EA}\begin{bmatrix} \frac{5}{3} \\ -\frac{\sqrt{3}}{3} \\ 0 \end{bmatrix}. \tag{2.207}$$

The reaction forces can be obtained by multiplying the stiffness matrix according to Eq. (2.205) with the total displacement vector, i.e.

$$\boldsymbol{u}^{\mathrm{T}} = \left[0\ 0\ u_{2X}\ u_{2Y}\ u_{3X}\ 0\right], \tag{2.208}$$

to give:

$$R_{1X} = -F\ ,\ \ R_{1Y} = -\frac{\sqrt{3}\,F}{3}\ ,\ \ R_{3X} = 0\ ,\ \ R_{3Y} = \frac{\sqrt{3}\,F}{3}. \tag{2.209}$$

The rod forces can be obtained from the global coordinates as:

$$F_{\mathrm{I}} = k_{\mathrm{I}}\left(-\cos\alpha_{\mathrm{I}} u_{1X} - \sin\alpha_{\mathrm{I}} u_{1Y} + \cos\alpha_{\mathrm{I}} u_{2X} + \sin\alpha_{\mathrm{I}} u_{2Y}\right) = \frac{2\sqrt{3}\,F}{3}, \tag{2.210}$$

$$F_{\mathrm{II}} = k_{\mathrm{II}}\left(-\cos\alpha_{\mathrm{II}} u_{3X} - \sin\alpha_{\mathrm{II}} u_{3Y} + \cos\alpha_{\mathrm{II}} u_{2X} + \sin\alpha_{\mathrm{II}} u_{2Y}\right) = \frac{\sqrt{3}\,F}{3}, \tag{2.211}$$

$$F_{\mathrm{III}} = k_{\mathrm{III}}\left(-\cos\alpha_{\mathrm{III}} u_{1X} - \sin\alpha_{\mathrm{III}} u_{1Y} + \cos\alpha_{\mathrm{III}} u_{3X} + \sin\alpha_{\mathrm{III}} u_{3Y}\right) = 0. \tag{2.212}$$

(b) Displacement boundary condition
Considering the displacement boundary condition u at node 2, the reduced system of equations reads:

$$\frac{EA}{L}\begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ \frac{\sqrt{3}}{4} & \frac{5}{4} & 0 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{2Y} \\ u_{3X} \end{bmatrix} = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}. \tag{2.213}$$

Inverting the reduced stiffness matrix can be used to calculate the unknown displacements as:

$$\begin{bmatrix} u_{2X} \\ u_{2Y} \\ u_{3X} \end{bmatrix} = \frac{L}{EA}\begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ -\frac{\sqrt{3}EA}{5L} & \frac{4}{5} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}\begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} = u\begin{bmatrix} 1 \\ -\frac{\sqrt{3}}{5} \\ 0 \end{bmatrix}. \tag{2.214}$$

Reaction and rod forces can be obtained as described in part (a) as:

$$R_{1X} = -\frac{3}{5} \times \frac{EAu}{L} \ , \ R_{1Y} = -\frac{\sqrt{3}}{5} \times \frac{EAu}{L} \ , \ R_{2X} = \frac{3}{5} \times \frac{EAu}{L}, \tag{2.215}$$

$$R_{3X} = 0 \ , \ R_{3Y} = \frac{\sqrt{3}}{5} \times \frac{EAu}{L}. \tag{2.216}$$

$$F_{\mathrm{I}} = \frac{2\sqrt{3}}{5} \times \frac{EAu}{L} \ , \ F_{\mathrm{II}} = \frac{\sqrt{3}}{5} \times \frac{EAu}{L} \ , \ F_{\mathrm{III}} = 0. \tag{2.217}$$

2.6 Example: Plane truss structure with two rod elements

The following Fig. 2.31 shows a two-dimensional truss structure. The two rod elements have the same cross-sectional area A and YOUNG's modulus E. The length of each element can be calculated based on the given dimensions in the figure. The structure is loaded by prescribed displacements u_X and u_Y at node 2.

Determine:

- The global system of equations without consideration of the boundary conditions at node 1 and 3.
- The reduced system of equations.

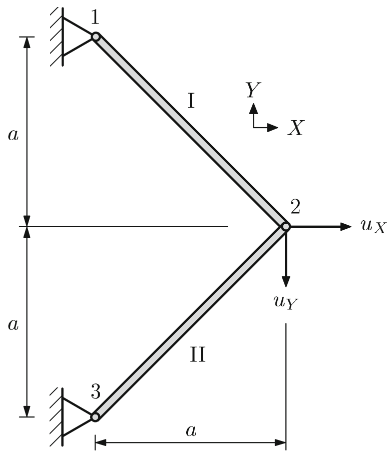


Fig. 2.31 Two-element truss structure with displacement boundary condition

- All nodal displacements.
- The elemental forces in each rod.

2.6 Solution

The free body diagram is shown in Fig. 2.32. Both elements have the same length of $L = \sqrt{2}a$

• Let us look in the following first separately at each element. The stiffness matrix for element I ($\alpha = -45°$) can be written as:

$$\boldsymbol{K}_{\mathrm{I}}^{\mathrm{e}} = \frac{EA}{\sqrt{2}a} \begin{array}{c} \begin{matrix} u_{1X} & u_{1Y} & u_{2X} & u_{2Y} \end{matrix} \\ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} \begin{array}{l} u_{1X} \\ u_{1Y} \\ u_{2X} \\ u_{2Y} \end{array}. \tag{2.218}$$

In the same way, the stiffness matrix for element II ($\alpha = +45°$) reads as:

$$\boldsymbol{K}_{\mathrm{II}}^{\mathrm{e}} = \frac{EA}{\sqrt{2}a} \begin{array}{c} \begin{matrix} u_{3X} & u_{3Y} & u_{2X} & u_{2Y} \end{matrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} \begin{array}{l} u_{3X} \\ u_{3Y} \\ u_{2X} \\ u_{2Y} \end{array}. \tag{2.219}$$

The global system of equations without consideration of the boundary conditions is obtained as:

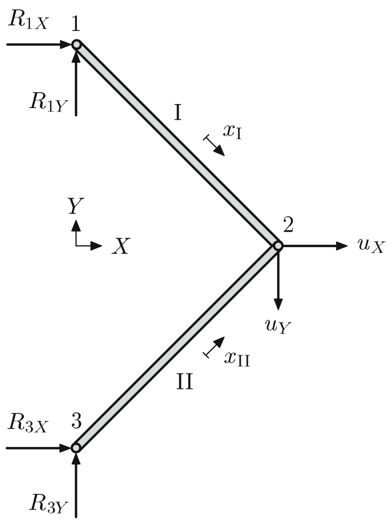


Fig. 2.32 Free body diagram of the truss structure problem

$$\frac{EA}{\sqrt{2}a}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} u_{1X} \\ u_{1Y} \\ u_{2X} \\ u_{2Y} \\ u_{3X} \\ u_{3Y} \end{bmatrix} = \begin{bmatrix} R_{1X} \\ R_{1Y} \\ 0 \\ 0 \\ R_{3X} \\ R_{3Y} \end{bmatrix}. \tag{2.220}$$

• Introduction of the boundary conditions, i.e. $u_{1X} = u_{1Y} = u_{3X} = u_{3Y} = 0$, gives the following reduced system of equations:

$$\frac{EA}{\sqrt{2}a}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{2Y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.221}$$

• All nodal displacements

– $u_{1X} = u_{1Y} = u_{3X} = u_{3Y} = 0$,
– $u_{2X} = u_X, \; u_{2Y} = u_Y$.

• Elemental forces in each rod

General: $\sigma = \dfrac{E}{L}(-u_1 + u_2) \quad \Rightarrow \quad F = \dfrac{EA}{L}(-u_1 + u_2)$.

Thus: $F = \dfrac{EA}{L}(-\cos(\alpha)u_{1X} - \sin(\alpha)u_{1Y} + \cos(\alpha)u_{2X} + \sin(\alpha)u_{2Y})$.

Our case:

$$F_{\mathrm{I}} = \frac{EA}{\sqrt{2}a}\left(+\frac{1}{2}\sqrt{2}u_X - \frac{1}{2}\sqrt{2}u_Y\right) = \frac{EA}{2a}(u_X - u_Y), \tag{2.222}$$

$$F_{\mathrm{II}} = \frac{EA}{\sqrt{2}a}\left(+\frac{1}{2}\sqrt{2}u_X + \frac{1}{2}\sqrt{2}u_Y\right) = \frac{EA}{2a}(u_X + u_Y). \tag{2.223}$$

2.7 Example: Plane truss structure with three rod elements

The following Fig. 2.33 shows a two-dimensional truss structure. The three rod elements have the same YOUNG's modulus E and length L. However, the cross-sectional areas A_i (i = I, II, III) are different from rod to rod. The structure is loaded by a point load F_0 at node 2.

Determine:

• the free body diagram,
• the global stiffness matrix,

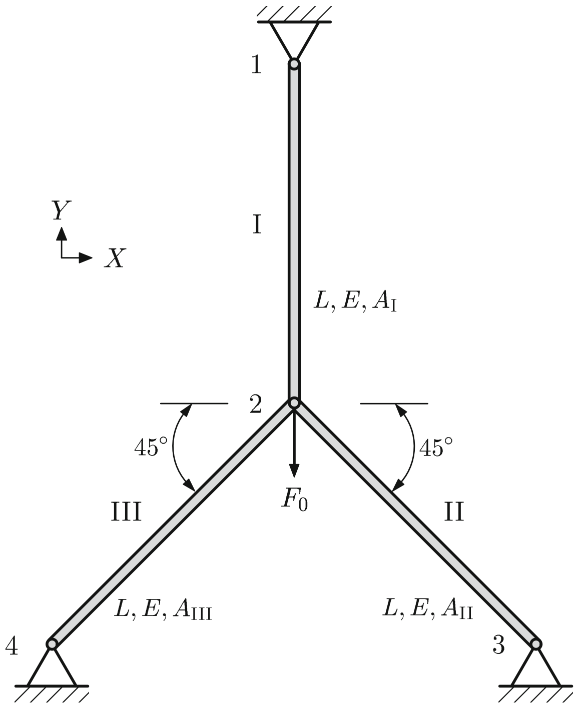


Fig. 2.33 Three-element truss structure with force boundary condition

- the reduced system of equations under consideration of the boundary conditions,
- the nodal displacements at node 2.
- Simplify the nodal displacements at node 2 for the special case $A_{\text{I}} = A_{\text{II}} = A_{\text{III}} = A$.

2.7 Solution

• The free body diagram is shown in Fig. 2.34.
• Let us look in the following first separately at each element. The stiffness matrix for element I ($\alpha = -90°$) can be written as:

$$\boldsymbol{K}_{\text{I}}^{\text{e}} = \frac{EA_{\text{I}}}{L} \begin{array}{c} \begin{array}{cccc} u_{1X} & u_{1Y} & u_{2X} & u_{2Y} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_{1X} \\ u_{1Y} \\ u_{2X} \\ u_{2Y} \end{array}. \tag{2.224}$$

In the same way, the stiffness matrix for element II ($\alpha = -45°$) reads as:

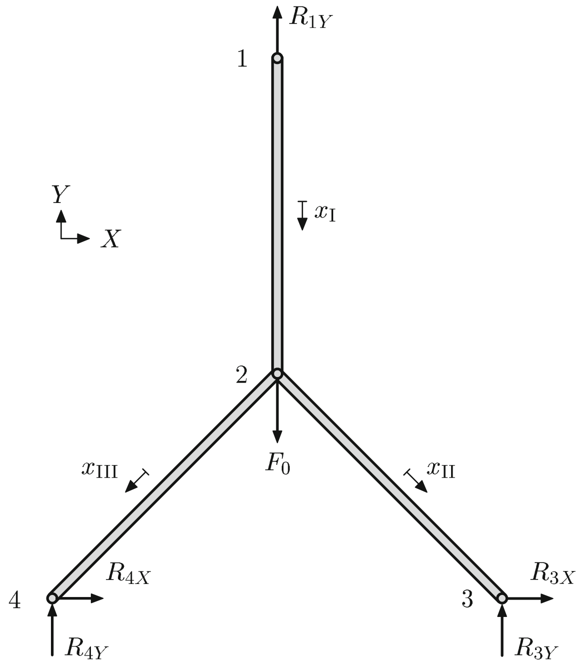


Fig. 2.34 Free body diagram of the truss structure problem

$$\boldsymbol{K}^{\mathrm{e}}_{\mathrm{II}} = \frac{EA_{\mathrm{II}}}{L} \begin{array}{c} \begin{matrix} u_{2X} & u_{2Y} & u_{3X} & u_{3Y} \end{matrix} \\ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} \begin{array}{c} u_{2X} \\ u_{2Y} \\ u_{3X} \\ u_{3Y} \end{array} . \tag{2.225}$$

In the same way, the stiffness matrix for element III ($\alpha = 225°$) reads as:

$$\boldsymbol{K}^{\mathrm{e}}_{\mathrm{III}} = \frac{EA_{\mathrm{III}}}{L} \begin{array}{c} \begin{matrix} u_{2X} & u_{2Y} & u_{4X} & u_{4Y} \end{matrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} \begin{array}{c} u_{2X} \\ u_{2Y} \\ u_{4X} \\ u_{4Y} \end{array} . \tag{2.226}$$

The global stiffness matrix can be assembled as:

$$
\boldsymbol{K} = \frac{E}{L}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{\mathrm{I}} & 0 & -A_{\mathrm{I}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & -\frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & -\frac{A_{\mathrm{II}}}{2} - \frac{A_{\mathrm{III}}}{2} & \frac{A_{\mathrm{II}}}{2} - \frac{A_{\mathrm{III}}}{2} & 0 & 0 \\
0 & -A_{\mathrm{I}} & -\frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & A_{\mathrm{I}} + \frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & \frac{A_{\mathrm{II}}}{2} - \frac{A_{\mathrm{III}}}{2} & -\frac{A_{\mathrm{II}}}{2} - \frac{A_{\mathrm{III}}}{2} & & \\
0 & 0 & -\frac{A_{\mathrm{II}}}{2} - \frac{A_{\mathrm{III}}}{2} & \frac{A_{\mathrm{II}}}{2} - \frac{A_{\mathrm{III}}}{2} & \frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & -\frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & 0 & 0 \\
0 & 0 & \frac{A_{\mathrm{II}}}{2} - \frac{A_{\mathrm{III}}}{2} & -\frac{A_{\mathrm{II}}}{2} - \frac{A_{\mathrm{III}}}{2} & -\frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & \frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{2.227}
$$

• Introduction of the boundary conditions, i.e. $u_{1X} = u_{1Y} = u_{3X} = u_{3Y} = u_{4X} = u_{4Y} = 0$, gives the following reduced system of equations:

$$
\frac{E}{L}\begin{bmatrix}
\frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & -\frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} \\
-\frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2} & A_{\mathrm{I}} + \frac{A_{\mathrm{II}}}{2} + \frac{A_{\mathrm{III}}}{2}
\end{bmatrix}\begin{bmatrix} u_{2X} \\ u_{2Y} \end{bmatrix} = \begin{bmatrix} 0 \\ -F_0 \end{bmatrix}. \tag{2.228}
$$

The solution of this system can be obtained, for example, by inverting the reduced stiffness matrix to give the reduced result vector as:

$$
\begin{bmatrix} u_{2X} \\ u_{2Y} \end{bmatrix} = \begin{bmatrix}
-\dfrac{L\,(A_{\mathrm{II}} - A_{\mathrm{III}})\,F_0}{E\,(A_{\mathrm{II}}\,A_{\mathrm{I}} + A_{\mathrm{III}}\,A_{\mathrm{I}} + 2\,A_{\mathrm{II}}\,A_{\mathrm{III}})} \\
-\dfrac{L\,(A_{\mathrm{II}} + A_{\mathrm{III}})\,F_0}{E\,(A_{\mathrm{II}}\,A_{\mathrm{I}} + A_{\mathrm{III}}\,A_{\mathrm{I}} + 2\,A_{\mathrm{II}}\,A_{\mathrm{III}})}
\end{bmatrix}. \tag{2.229}
$$

• Special case $A_{\mathrm{I}} = A_{\mathrm{II}} = A_{\mathrm{III}} = A$:

$$
\begin{bmatrix} u_{2X} \\ u_{2Y} \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{LF_0}{2EA} \end{bmatrix}. \tag{2.230}
$$

Let us summarize at the end of this section the recommended steps for a linear finite element solution ('hand calculation'):

① Sketch the free-body diagram of the problem, including a global coordinate system.
② Subdivide the geometry into finite elements. Indicate the node and element numbers, and local coordinate systems.
③ Write separately all elemental stiffness matrices expressed in the global coordinate system. Indicate the nodal unknowns on the right-hand sides and over the matrices.
④ Determine the dimension of the global stiffness matrix and sketch the structure of this matrix with global unknowns on the right-hand side and over the matrix.
⑤ Insert step-by-step the values of the elemental stiffness matrices into the global stiffness matrix.
⑥ Add the column matrix of unknowns and external loads to complete the global system of equations.
⑦ Introduce the boundary conditions to obtain the reduced system of equations.

⑧ Solve the reduced system of equations to obtain the unknown nodal deformations.
⑨ Post-computation: determination of reaction forces, stresses and strains.
⑩ Check the global equilibrium between the external loads and the support reactions.

2.5 Supplementary Problems

2.8 Knowledge questions on rods and trusses

- How many material parameters are required for the one-dimensional HOOKE's law?
- State the one-dimensional HOOKE's law for a pure normal stress and strain state.
- HOOKE's law can be written as $\sigma(x) = E\varepsilon(x)$ for a special case. State two assumptions for this formulation.
- Explain the assumptions for (a) an 'isotropic' and (b) a 'homogeneous' material.
- State the major characteristic of an *elastic* material.
- State a common value for the YOUNG's modulus of steel, aluminum, and titanium.
- State the three (3) basic equations of continuum mechanics which are required to derive the partial differential equation of a static problem.
- Explain the meaning of the kinematics, constitutive and equilibrium equations.
- Explain in words the meaning of the strong formulation, the inner product, and the weak formulation in the scope of the weighted residual method.
- Given is a differential equation of the form $\mathrm{d}^2 f(x)/\mathrm{d}x^2 - a = 0$. State (a) the strong formulation and (b) the inner product of the problem.
- Given is a differential equation of the form $\mathrm{d}^2 y(x)/\mathrm{d}x^2 - c(x) = 0$. State (a) the strong formulation and (b) the inner product of the problem.
- State the difference between the (a) analytical and (b) the finite element solution of a problem described based on a partial differential equation.
- State in words the definition of a rod.
- Characterize in words the stress and strain distribution in an elastic rod.
- The following Fig. 2.35 shows a rod of length L and constant cross-sectional area A. The structure is loaded by a point load F_0 and a constant distributed load p_x. State for this problem three boundary conditions and the appropriate differential equation under the assumption that the YOUNG's modulus E is a function of the spatial coordinate x.
- Which general types of 'load conditions' did we distinguish for rod problems?
- State the required (a) geometrical parameters and (b) material parameters to define a rod element.
- Sketch the interpolation functions $N_1(x)$ and $N_2(x)$ of a linear rod element.
- State four (4) characteristics of a finite element stiffness matrix.
- The stiffness matrix for a rod element can be stated as (Fig. 2.36)

$$\boldsymbol{K}^{\mathrm{e}} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

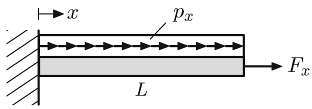


Fig. 2.35 Axially loaded continuum rod

Which assumptions does this equation involve in regards to the (a) material and (b) the geometry?

- State the DOF per node for a truss element in a plane (2D) problem.
- State the DOF per node for a truss element in a 3D problem.
- The following Fig. 2.36a shows a *plane* truss structure which is composed of 15 rod (E, A) elements.
  State the size of the stiffness matrix of the *non-reduced* system of equations, i.e. without consideration of the boundary conditions. What is the size of the stiffness matrix of the *reduced* system of equations, i.e. under consideration of the boundary conditions? Consider now Fig. 2.36b where the rod element 1–2 (length L) has been replaced by a spring of stiffness $k = \frac{EA}{L}$. How does the overall stiffness of the truss structure change?

2.9 Simplified model of a tower under dead weight (analytical approach)

Given is a simplified model of a tower which is deforming under the influence of its dead weight, cf. Fig. 2.37. The tower is of original length L, cross-sectional area A, Youngs modulus E, and mass density ϱ. The standard gravity is given by g.
Calculate

- The reduction in length $\Delta L = L - u_x(L)$ due to acting dead weight.
- The maximum length $L_{\max}$ if a given stress limit $\sigma_{\max}$ at the foundation ($x = 0$) cannot be exceeded.

2.10 Analytical solution for a rod problem

Given is a rod of length L and constant axial tensile stiffness EA as shown in Fig. 2.38. At the left-hand side there is a fixed support and the right-hand side is either elongated by a displacement u_0 (case a) or loaded by a single force F_0 (case b). Determine the analytical solution for the elongation $u(x)$, the strain $\varepsilon(x)$, and stress $\sigma(x)$ along the rod axis.

2.11 Weighted residual method based on general formulation of partial differential equation

Derive the weak formulation for a rod based on the general formulation of the partial differential equation:

$$\mathcal{L}_1 \left(C \mathcal{L}_1 \left(u_x(x) \right) \right) + b = 0, \tag{2.231}$$

where $\mathcal{L}_1 = \frac{\mathrm{d}}{\mathrm{d}x}$, $C = E$ and $b = \frac{p_x(x)}{A}$. Simplify the GREEN- GAUSS theorem as given in Eq. (A.27) to derive the solution.

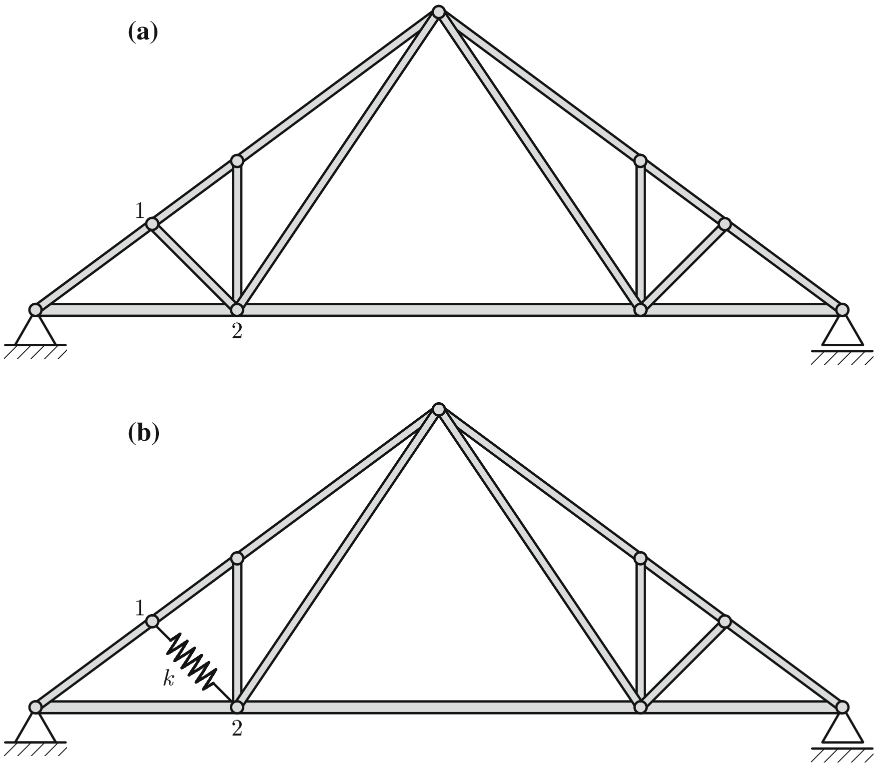
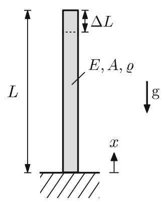


Fig. 2.36 Plane truss structure. Nodes are symbolized by circles (◯)

Fig. 2.37 Simplified model of a tower loaded under its dead weight



2.12 Weighted residual method with arbitrary distributed load for a rod

Derive the principal finite element equation for a rod element based on the weighted residual method. Starting point should be the partial differential equation with an arbitrary distributed load $p_x(x)$. In addition, it can be assumed that the axial tensile stiffness EA is constant.

2.13 Numerical integration and coordinate transformation

The derivation of the principal finite element equation involves numerical integration and coordinate transformation. The Cartesian coordinate range $x_1 \leq x \leq x_2$ is transformed to the natural coordinate range $-1 \leq \xi \leq 1$. The general transformation

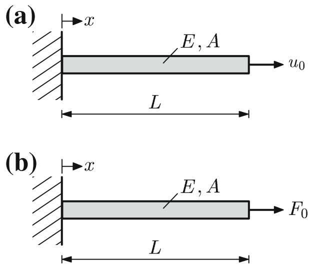


Fig. 2.38 Rod under different loading conditions: **a** displacement and **b** force

between these two coordinates is illustrated in Table 2.4 and given by:

$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1. \tag{2.232}$$

Derive this relationship between the Cartesian and the natural coordinate.

2.14 Finite element solution for a rod problem

Given is a rod of length L and constant axial tensile stiffness EA as shown in Fig. 2.39. At the left-hand side there is a fixed support and the right-hand side is either loaded by a single force F_0 (case a) or elongated by a displacement u_0 (case b). Determine the finite element solution based on a single rod element for the elongation $u(x)$, the strain $\varepsilon(x)$, and stress $\sigma(x)$ along the rod axis.

2.15 Finite element approximation with a single linear rod element

Given is a rod with different load cases as shown in Fig. 2.40. The axial tensile stiffness EA is constant and the length is equal to L. Derive the finite element solution based on a single linear element and compare the elongation $u_x(x)$ and $u_x(L)$ with the analytical solution.

2.16 Finite element approximation with a single quadratic rod element

Solve problem 2.15 with a single quadratic rod element.

2.17 Equivalent nodal loads for a quadratic distribution (linear rod element)

Given are the following two formulations for a distributed quadratic load. Calculate the equivalent nodal loads for a linear rod element, cf. Fig. 2.41.

(a) $p_x(x) = p_0^* x^2$,

(b) $p_x(x) = p_0 \left(\frac{x}{L}\right)^2$.

The dimension of the constant p_0^* is equal to force per unit length to power 3 while the dimension of p_0 is force per unit length.

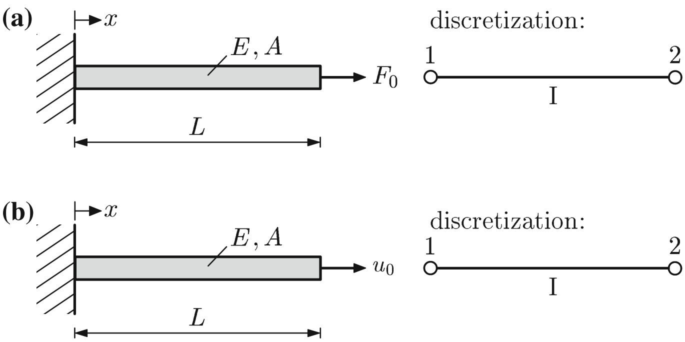
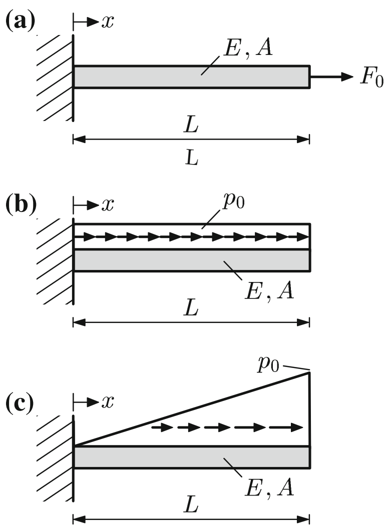


Fig. 2.39 Rod under different loading conditions: **a** force and **b** displacement boundary condition

Fig. 2.40 Finite element approximation with a single element for different load cases: **a** single force, **b** constant distributed load, and **c** linear distributed load



2.18 Derivation of interpolation functions for a quadratic rod element

Derive the three interpolation functions for a quadratic rod element under the following assumption for the displacement field:

$$u^{\mathrm{e}}(\xi) = a_0 + a_1\xi + a_2\xi^2. \tag{2.233}$$

Assume for the derivation that the third node is exactly in the middle of the element ($\xi = 0$). Plot the three interpolation functions in dependence of the natural coordinate ξ.

2.19 Derivation of the Jacobian determinant for a quadratic rod element

Consider a quadratic rod element with the second node located exactly in the middle of the element. Use the following nodal approach for the Cartesian coordinate to calculate the Jacobian determinant for the case that the elemental coordinate system

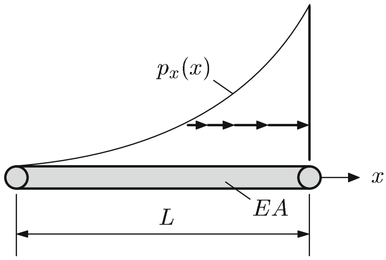


Fig. 2.41 Distributed load with quadratic function

is located in node 1:

$$x(\xi) = \overline{N}_1(\xi)x_1 + \overline{N}_2(\xi)x_2 + \overline{N}_3(\xi)x_3. \tag{2.234}$$

2.20 Comparison of the stress distribution for a linear and quadratic rod element with linear increasing load

Given is a rod with linear increasing load as shown in Fig. 2.42. The axial tensile stiffness EA is constant and the length is equal to L. Calculate and compare the stress distribution based on:

(a) the analytical solution,
(b) a single *linear* rod element and
(c) a single *quadratic* rod element.

2.21 Derivation of interpolation functions and stiffness matrix for a quadratic rod element with unevenly distributed nodes

Derive the three interpolation functions for a quadratic rod element with unevenly distributed nodes (cf. Fig. 2.43) under the following assumption for the displacement field:

$$u^{\mathrm{e}}(\xi) = a_0 + a_1\xi + a_2\xi^2. \tag{2.235}$$

Derive the stiffness matrix as a function of position b for an axial stiffness EA and length L of the element. Wich problems can occur if the second node is close to the boundary, e.g. $\xi = -0.9$?

2.22 Derivation of interpolation functions for a cubic rod element

Derive the four interpolation functions for a cubic rod element under the following assumption for the displacement field:

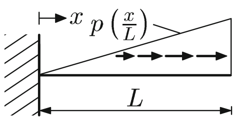


Fig. 2.42 Rod element with linear increasing load

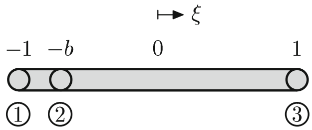


Fig. 2.43 Quadratic rod element with unevenly distributed nodes

$$u^{\mathrm{e}}(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3. \tag{2.236}$$

Assume for the derivation that the nodes are equally spaced.

2.23 Structure composed of three linear rod elements

Calculate for the two structures shown in Fig. 2.44 the unknown displacement vector and the reaction forces for $L_{\mathrm{I}} = L_{\mathrm{II}} = L_{\mathrm{III}} = L$ and

$$(EA)_{\mathrm{I}} = 3EA, \tag{2.237}$$

$$(EA)_{\mathrm{II}} = 2EA, \tag{2.238}$$

$$(EA)_{\mathrm{III}} = 1EA. \tag{2.239}$$

2.24 Finite element approximation of a rod with four elements: comparison of displacement, strain and stress distribution with analytical solution

Given is a rod of length L and tensile stiffness EA which is loaded by a constant distributed load p_0 as shown in Fig. 2.45. Use four linear rod elements of length $\frac{L}{4}$ to discretize the rod structure and calculate the nodal displacements, strains and stresses. Compare the results with the analytical solution and sketch the normalized finite element and analytical solutions $u(X)/\frac{p_0L^2}{EA}$, $\varepsilon(X)/\frac{p_0L}{EA}$, and $\sigma(X)/\frac{p_0L}{A}$ over the normalized coordinate X/L.

2.25 Elongation of a bi-material rod: finite element solution and comparison with analytical solution

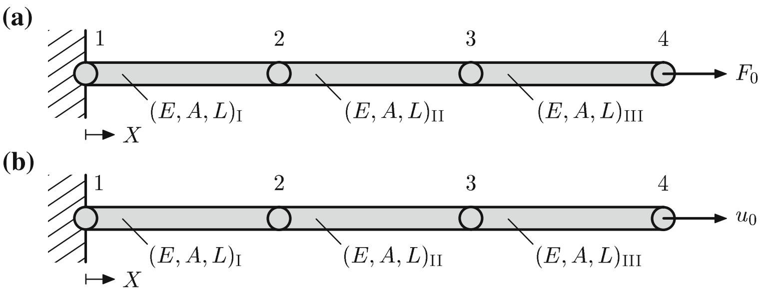


Fig. 2.44 Structure composed of three rod elements: **a** force boundary condition; **b** displacement boundary condition

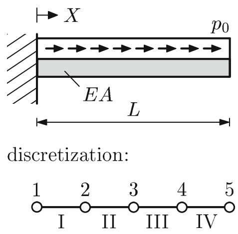


Fig. 2.45 Rod structure discretized by four elements

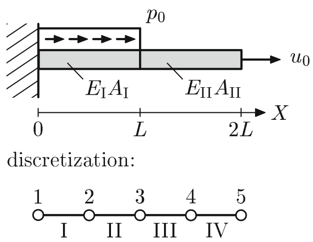


Fig. 2.46 Bi-material rod discretized by four elements

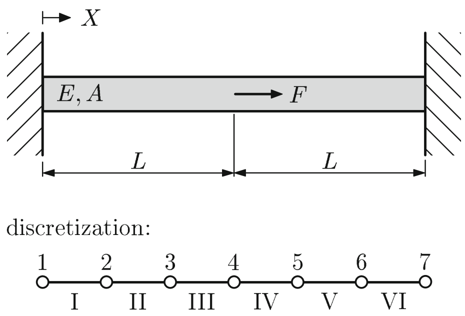


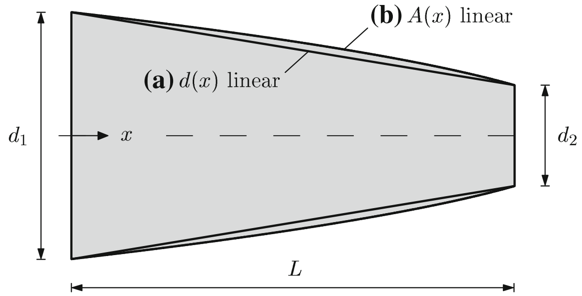
Fig. 2.47 Rod structure fixed at both ends

Given is a rod as shown in Fig. 2.46 which is made of two different sections with axial stiffness $k_\mathrm{I} = E_\mathrm{I}A_\mathrm{I}$ and $k_\mathrm{II} = E_\mathrm{II}A_\mathrm{II}$. Each section is of length L and in the left-hand section, i.e. $0 \leq x \leq L$, is a constant distributed load p_0 acting while the right-hand end is elongated by u_0. Use four linear rod elements of length $\frac{L}{4}$ to discretize the rod and calculate the nodal displacements, strains, and stresses. Compare the results with the analytical solution and sketch the distributions $u(x)$, $\varepsilon(x)$ and $\sigma(x)$ for for the case $k_\mathrm{I} = 2k_\mathrm{II} = 1$, $L_\mathrm{I} = L_\mathrm{II} = 1$, $p_0 = 1$, $u_0 = 1$, and $E_\mathrm{I} = 2E_\mathrm{II} = 1$.

2.26 Stress distribution for a fixed-fixed rod structure

Given is a rod structure as shown in Fig. 2.47. The structure is of length $2L$, cross-sectional area A, and Youngs modulus E. The structure is fixed at both ends and loaded by a point load F in the middle, i.e. $X = L$. Calculate the stress distribution based on six finite elements of length $\frac{L}{3}$. Show the difference between the elemental stress values and the averaged nodal values.

Fig. 2.48 Rod element with variable cross section: **a** linear changing diameter; **b** linear changing cross-sectional area



2.27 Linear rod element with variable cross section: derivation of stiffness matrix

Determine the elemental stiffness matrix for a linear rod element with changing cross-sectional area as shown in Fig. 2.48. Consider the following two relationships for a linear changing diameter and a linear changing area:

$$\text{a)}\ d(x) = d_1 + \frac{x}{L}(d_2 - d_1), \tag{2.240}$$

$$\text{b)}\ A(x) = A_1 + \frac{x}{L}(A_2 - A_1). \tag{2.241}$$

Use analytical integration to obtain the stiffness matrix and compare the results with a two-point GAUSS integration rule. A circular cross section can be assumed in case (a).

2.28 Quadratic rod element with variable cross section: derivation of stiffness matrix

Solve problem 2.27 with a single quadratic rod element.

2.29 Linear rod element with variable cross-section: comparison of displacements between FE and analytical solution for a single element

Determine for the rod element shown in Fig. 2.49 the end displacement based on a single finite element. The elemental stiffness matrix from problem 2.27 can be used. Distinguish two different cases of boundary conditions, i.e. a single force F_0 or a prescribed displacement u_0 at the right-hand end. Compare the results obtained with the analytical solution.

2.30 Quadratic rod element with variable cross section: comparison of end displacement between FE and analytical solution for single element

Recalculate problem 2.29 (a) for a single quadratic rod element.

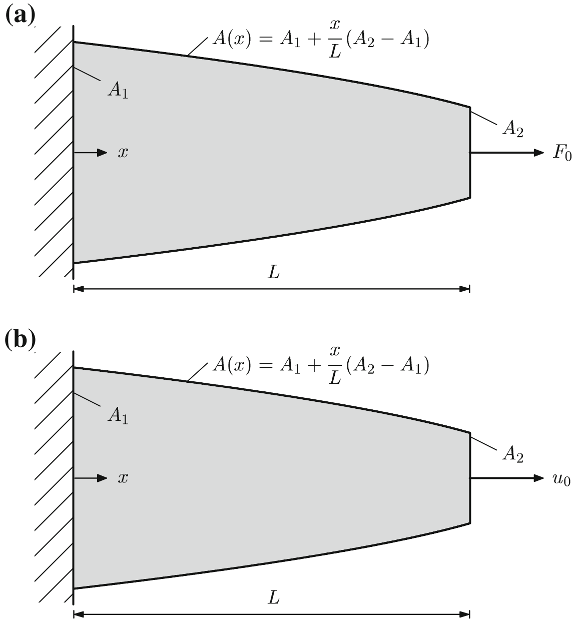


Fig. 2.49 Rod element with variable cross-section and different boundary conditions at the right end: **a** external force F_0; **b** displacement u_0

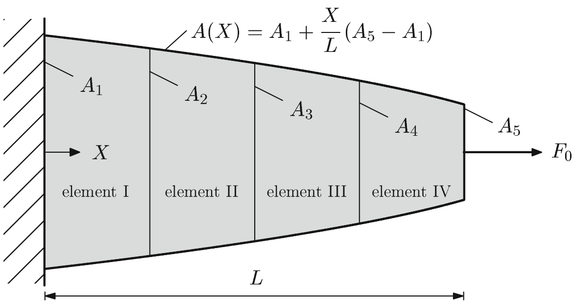


Fig. 2.50 Rod element with variable cross-section discretized by four finite elements

2.31 Subdivided structure with variable cross section: comparison of displacements and stresses between FE and analytical solution for four elements

Calculate for the rod shown in Fig. 2.50 the distribution of the elongation $u(X)$ and the stress $\sigma(X)$. To this end, subdivide the structure in four elements of length $\frac{L}{4}$ and use the expression for the elemental stiffness matrix which was derived in problem 2.27. The ratio between the end and initial cross section area is equal to $\frac{A_5}{A_1} = 0.2$. Compare the results obtained with the analytical solution.

2.32 Submodel of a structure with variable cross section

To increase the accuracy of a stress approximation, the submodeling technique can be applied. Let us come back to Problem 2.31 and assume that the area of interest is near the left support, i.e. $X \to 0$. In the first step of a submodeling analysis,

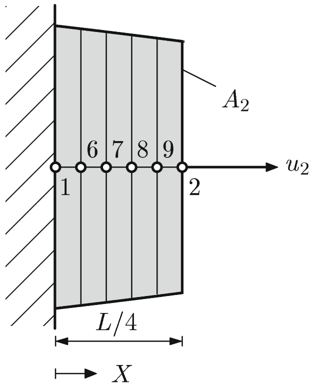


Fig. 2.51 Submodel of the structure with variable cross section, cf. Fig. 2.50

the structure is simulated with a coarse mesh as indicated in Fig. 2.50. In the next step, the area of element I is separately considered and discretized with a finer mesh. The nodal displacement u_2 from Problem 2.31 is applied as boundary condition at the right-hand end of the submodel. Calculate the stress distribution based on the submodel and compare with the analytical solution and the result from the coarse mesh (Fig. 2.51).

2.33 Rod with elastic embedding: stiffness matrix

A rod with elastic embedding is schematically shown in Fig. 2.52. Derive the elemental stiffness matrix for a rod element with (a) linear and (b) quadratic interpolation functions under the assumption that the elastic modulus k is constant. The describing partial differential equation can be taken from Table 2.2.

2.34 Rod with elastic embedding: single force case

A cantilevered rod with elastic embedding is loaded by a single force F as shown in Fig. 2.53. Assume that the elastic modulus k and the axial tensile stiffness EA are constant. Use a single (a) linear and (b) quadratic rod element to determine

- the reduced system of equations,
- the elongation of the rod at $x = L$,
- simplify your result for the special case $k = 0$,
- simplify your result for the special case $EA = 0$.
- Compare the finite element solution with the analytical solution for the case $k = 3$, $EA = 1$ and $L = 1$.

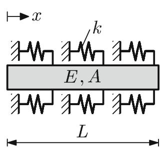


Fig. 2.52 Rod with elastic embedding

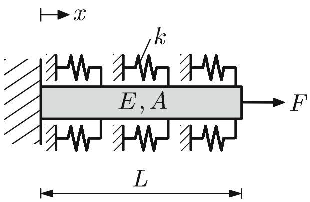


Fig. 2.53 Rod with elastic embedding loaded by a single force

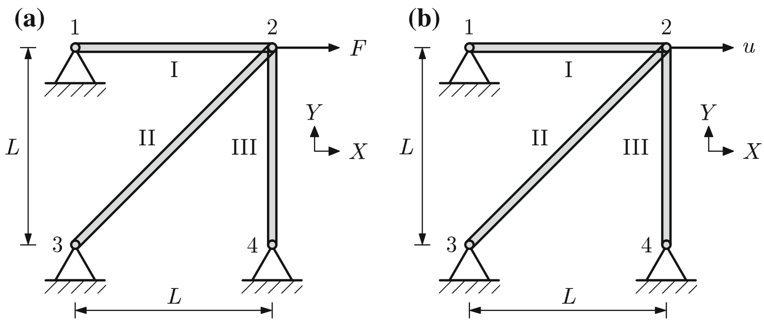


Fig. 2.54 Three-element truss structure with different external loading: **a** force boundary condition; **b** displacement boundary condition

2.35 Plane truss structure arranged in a square

Given is the two-dimensional truss structure as shown in Fig. 2.54. The three truss elements have the same cross-sectional area A and YOUNG's modulus E. The length of each element can be taken from the dimensions given in the figure. The structure is loaded by

(a) a horizontal force F at node 2,
(b) a prescribed horizontal displacement u at node 2.

Determine for both cases

- the global system of equations,
- the reduced system of equations,
- all nodal displacements,
- all reaction forces,
- the force in each rod.

2.36 Plane truss structure arranged in a triangle

Given is the two-dimensional truss structure as shown in Fig. 2.55. The three truss elements have the same cross-sectional area A and YOUNG's modulus E. The length of each element can be taken from the dimensions given in the figure. The structure is loaded by

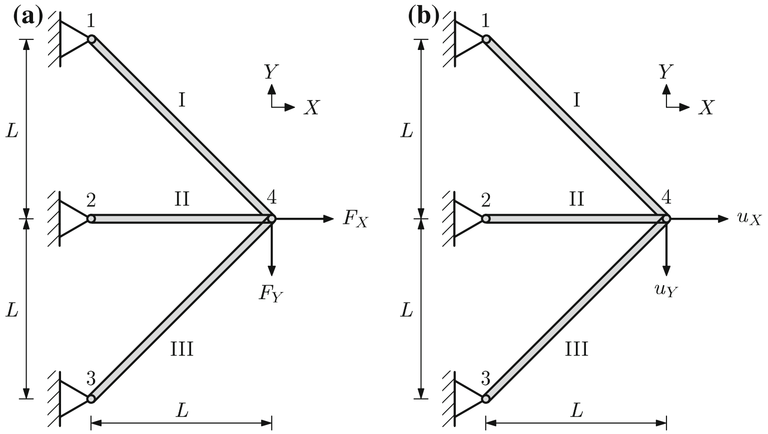


Fig. 2.55 Three-element truss structure with different external loading: **a** force boundary conditions; **b** displacement boundary conditions

(a) single forces F_X and F_Y at node 4,
(b) prescribed displacements u_X and u_Y at node 4.

Determine for both cases

- the global system of equations,
- the reduced system of equations,
- all nodal displacements,
- all reaction forces,
- the force in each rod.

2.37 Plane truss structure with two rod elements

Given is the two-dimensional truss structure as shown in Fig. 2.56. The two truss elements have the same cross-sectional area A and YOUNG's modulus E. The length

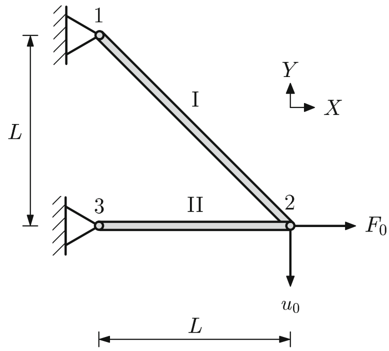


Fig. 2.56 Two-element truss structure with 'mixed' boundary conditions

of each element can be taken from the dimensions given in the figure. The structure is loaded by

- a single forces F_0 at node 2 in X-direction and
- a prescribed displacements u_0 at node 2 in Y-direction.

Consider two linear truss (bar) finite elements and determine

- the free body diagram,
- the global system of equations,
- the reduced system of equations under consideration of the boundary conditions,
- the nodal displacements at node 2,
- all reaction forces.
- Check if the global force equilibrium is fulfilled.

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