Chapter 2
Unbiased Estimation

If the average estimate of several random samples is equal to the population parameter then the estimate is unbiased. For example, if credit card holders in a city were repetitively random sampled and questioned what their account balances were as of a specific date, the average of the results across all samples would equal the population parameter. If, however, only credit card holders in one specific business were sampled, the average of the sample estimates would be biased estimator of all account balances for the city and would not equal the population parameter.

If the mean value of an estimator in a sample equals the true value of the population mean then it is called an unbiased estimator. If the mean value of an estimator is either less than or greater than the true value of the quantity it estimates, then the estimator is called a biased estimator. For example, suppose you decide to choose the smallest or largest observation in a sample to be the estimator of the population mean. Such an estimator would be biased because the average of the values of this estimator would be always less or more than the true population mean.

2.1 Unbiased Estimates and Mean Square Error

Definition 2.1.1 A statistics $T(X)$ is called an unbiased estimator for a function of the parameter $g(\theta)$, provided that for every choice of θ,

$$\mathrm{E}T(X) = g(\theta) \tag{2.1.1}$$

Any estimator that is not unbiased is called biased. The bias is denoted by $b(\theta)$.

$$b(\theta) = \mathrm{E}T(X) - g(\theta) \tag{2.1.2}$$

U.J. Dixit, *Examples in Parametric Inference with R*,
DOI 10.1007/978-981-10-0889-4_2

We will now define mean square error (mse)

$$\begin{aligned}
\mathrm{M}SE[T(X)] &= \mathrm{E}[T(X) - g(\theta)]^2 \\
&= \mathrm{E}[T(X) - \mathrm{E}T(X) + b(\theta)]^2 \\
&= \mathrm{E}[T(X) - \mathrm{E}T(X)]^2 + 2b(\theta)\mathrm{E}[T(X) - \mathrm{E}T(X)] + b^2(\theta) \\
&= \mathrm{V}[T(X)] + b^2(\theta) \\
&= \text{Variance of } [T(X)] + [\text{bias of } T(X)]^2
\end{aligned}$$

Example 2.1.1 Let $(X_1, X_2, \ldots, X_n)$ be Bernoulli rvs with parameter θ, where θ is unknown. $\bar{X}$ is an estimator for θ. Is it unbiased ?

$$\mathrm{E}\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{n\theta}{n} = \theta$$

Thus, $\bar{X}$ is an unbiased estimator for θ.
We denote it as $\hat{\theta} = \bar{X}$.

$$\mathrm{Var}(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{V}(X_i) = \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n}$$

Example 2.1.2 Let $X_i (i = 1, 2, \ldots, n)$ be iid rvs from $N(\mu, \sigma^2)$, where μ and σ^2 are unknown.

Define $nS^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$ and $n\sigma^2 = \sum_{i=1}^{n}(X_i - \mu)^2$
Consider

$$\begin{aligned}
\sum_{i=1}^{n}(X_i - \mu)^2 &= \sum_{i=1}^{n}(X_i - \bar{X} + \bar{X} - \mu)^2 \\
&= \sum_{i=1}^{n}(X_i - \bar{X})^2 + 2\sum_{i=1}^{n}(X_i - \mu)(\bar{X} - \mu) + n(\bar{X} - \mu)^2 \\
&= \sum_{i=1}^{n}(X_i - \bar{X})^2 + n(\bar{X} - \mu)^2
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X} - \mu)^2$$

$$\mathrm{E}\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right]=\mathrm{E}\left[\sum_{i=1}^{n}(X_i-\mu)^2\right]-n\mathrm{E}[(\bar{X}-\mu)^2]$$
$$=n\sigma^2-\frac{n\sigma^2}{n}=n\sigma^2-\sigma^2$$

Hence,

$$\mathrm{E}(S^2)=\sigma^2-\frac{\sigma^2}{n}=\sigma^2\left(\frac{n-1}{n}\right)$$

Thus, S^2 is a biased estimator of σ^2.
Hence

$$b(\sigma^2)=\sigma^2-\frac{\sigma^2}{n}-\sigma^2=-\frac{\sigma^2}{n}$$

Further, $\frac{nS^2}{n-1}$ is an unbiased estimator of σ^2.

Example 2.1.3 Further, if $(n-1)S^2=\sum_{i=1}^{n}(X_i-\bar{X})^2$, then $\frac{(n-1)S^2}{\sigma^2}$ has χ^2 with $(n-1)$ df. Here, we examine whether S is an unbiased estimator of σ.

Let $\frac{(n-1)S^2}{\sigma^2}=w$
Then

$$\mathrm{E}(\sqrt{w})=\int_0^\infty \frac{w^{\frac{1}{2}}e^{-\frac{w}{2}}w^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}}dw$$

$$=\frac{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}}=\frac{\Gamma\left(\frac{n}{2}\right)2^{\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$\mathrm{E}\left[\frac{(n-1)^{\frac{1}{2}}S}{\sigma}\right]=\frac{2^{\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

Hence

$$\mathrm{E}(S)=\frac{2^{\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{\sigma}{(n-1)^{\frac{1}{2}}}=\left(\frac{2}{n-1}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\sigma$$

Therefore,

$$\mathrm{E}\left(\frac{S}{\sigma}\right)=\left(\frac{2}{n-1}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

Therefore,

$$\text{Bias}(S) = \sigma\left[\left(\frac{2}{n-1}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - 1\right]$$

Example 2.1.4 For the family (1.5.4), $\hat{p}$ is U-estimable and $\hat{\theta}$ is not U-estimable. For (p, θ), it can be easily seen that $\hat{p} = \frac{r}{n}$ and $E\hat{p} = p$. Next, we will show $\hat{\theta}$ is not U-estimable.

Suppose there exist a function $h(r, z)$ such that

$$Eh(r, z) = \theta \quad \forall \quad (p, \theta) \in \Theta.$$

Since

$$EE[h(r, z)|r] = \theta$$

We get

$$\sum_{r=1}^{n}\binom{n}{r}p^r q^{n-r}\int\limits_0^\infty h(r, z)\frac{e^{-\frac{z}{\theta}}z^{r-1}dz}{\theta^r\Gamma(r)} + q^n h(0, 0) = \theta$$

Substituting $\frac{p}{q} = \Psi$, and dividing q^n on both sides

$$\sum_{r=1}^{n}\Psi^r\binom{n}{r}\int\limits_0^\infty h(r, z)\frac{e^{-\frac{z}{\theta}}z^{r-1}dz}{\theta^r\Gamma(r)} + h(0, 0) = \theta(1+\Psi)^n, \quad \text{Since} \quad q = (1+\Psi)^{-1}$$

Comparing the coefficients of Ψ^r in both sides, we get, $h(0, 0) = \theta$, which is a contradiction.
Hence, there does not exist any unbiased estimator of θ. Thus θ is not U-estimable.

Example 2.1.5 Let X is $N(0, \sigma^2)$ and assume that we have one observation. What is the unbiased estimator of σ^2?

$$E(X) = 0$$

$$V(X) = EX^2 - (EX)^2 = \sigma^2$$

Therefore,

$$E(X^2) = \sigma^2$$

Hence X^2 is an unbiased estimator of σ^2.

Example 2.1.6 Sometimes an unbiased estimator may be absurd.

Let the rv X be $P(\lambda)$ and we want to estimate $\Psi(\lambda)$, where

$$\Psi(\lambda) = \exp[-k\lambda]; \quad k > 0$$

Let $T(X) = [-(k-1)]^x; k > 1$

$$\begin{aligned} E[T(X)] &= \sum_{x=0}^{\infty} [-(k-1)]^x \frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{[-(k-1)\lambda]^x}{x!} \\ &= e^{-\lambda} e^{[-(k-1)\lambda]} \\ &= e^{-k\lambda} \end{aligned}$$

$$T(x) = \begin{cases} [-(k-1)]^x > 0; & x \text{ is even and } k > 1 \\ [-(k-1)]^x < 0; & x \text{ is odd and } k > 1 \end{cases}$$

which is absurd since $\Psi(\lambda)$ is always positive.

Example 2.1.7 Unbiased estimator is not unique.

Let the rvs X_1 and X_2 are $N(\theta, 1)$. X_1, X_2, and $\alpha X_1 + (1-\alpha)X_2$ are unbiased estimators of θ, $0 \le \alpha \le 1$.

Example 2.1.8 Let $X_1, X_2, \ldots, X_n$ be iid rvs from Cauchy distribution with parameter θ. Find an unbiased estimator of θ.

Let

$$f(x|\theta) = \frac{1}{\pi[1 + (x-\theta)^2]}; \quad -\infty < x < \infty, -\infty < \theta < \infty$$

$$\begin{aligned} F(x|\theta) &= \int_{-\infty}^{x} \frac{dy}{\pi[1 + (y-\theta)^2]} \\ &= \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x-\theta) \end{aligned}$$

Let $g(x_{(r)})$ be the pdf of $X_{(r)}$, where $X_{(r)}$ is the rth order statistics.

$$g(x_{(r)}) = \frac{n!}{(n-r)!(r-1)!} f(x_{(r)})[F(x_{(r)})]^{r-1}[1-F(x_{(r)})]^{n-r}$$

$$= \frac{n!}{(n-r)!(r-1)!}\left[\frac{1}{\pi}\frac{1}{[1+(x_{(r)}-\theta)^2]}\right]\left[\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x_{(r)}-\theta)\right]^{r-1}\left[\frac{1}{2}-\frac{1}{\pi}\tan^{-1}(x_{(r)}-\theta)\right]^{n-r}$$

$$\mathrm{E}(X_{(r)}-\theta) = \frac{n!}{(n-r)!(r-1)!}\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{x_{(r)}-\theta}{[1+(x_{(r)}-\theta)^2]}\left[\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(x_{(r)}-\theta)\right]^{r-1}$$
$$\times\left[\frac{1}{2}-\frac{1}{\pi}\tan^{-1}(x_{(r)}-\theta)^{n-r}\right]dx_{(r)}$$

Let $(x_{(r)}-\theta) = y$

$$\mathrm{E}(X_{(r)}-\theta) = C_{rn}\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{y}{1+y^2}\left[\frac{1}{2}+\frac{1}{\pi}\tan^{-1}y\right]^{r-1}\left[\frac{1}{2}-\frac{1}{\pi}\tan^{-1}y\right]^{n-r}dy,$$

where $C_{rn} = \frac{n!}{(n-r)!(r-1)!}$
Let

$$u = \frac{1}{2}+\frac{1}{\pi}\tan^{-1}y \Rightarrow u-\frac{1}{2} = \frac{1}{\pi}\tan^{-1}y$$

$$\Rightarrow \left(u-\frac{1}{2}\right)\pi = \tan^{-1}y \Rightarrow y = \tan\left(u-\frac{1}{2}\right)\pi \Rightarrow y = -\cot\pi u$$

$$dy = \pi\left[\frac{(\cos\pi u)(\cos\pi u)}{\sin^2\pi u}+\frac{\sin\pi u}{\sin\pi u}\right]du$$

$$= \pi[\cot^2\pi u+1] = \pi[y^2+1]du$$

$$\mathrm{E}(X_{(r)}-\theta) = -\frac{n!}{(n-r)!(r-1)!}\int\limits_0^1 u^{r-1}(1-u)^{n-r}\cot\pi u du$$

$$= -C_{rn}\int\limits_0^1 u^{r-1}(1-u)^{n-r}\cot\pi u du$$

Replace r by $n-r+1$

$$\mathrm{E}(X_{(n-r+1)}-\theta) = -\frac{n!}{(n-r)!(r-1)!}\int_0^1 \cot(\pi u)u^{n-r}(1-u)^{r-1}du$$

Let $1-u=w$

$$= -\frac{n!}{(n-r)!(r-1)!}\int_0^1 (-1)\cot[\pi(1-w)](1-w)^{n-r}w^{r-1}dw$$

$$= \frac{n!}{(n-r)!(r-1)!}\int_0^1 \cot(\pi w)(1-w)^{n-r}w^{r-1}dw$$

Now

$$\int_0^1 u^{r-1}(1-u)^{n-r}\cot\pi u du = \int_0^1 \cot(\pi w)(1-w)^{n-r}w^{r-1}dw$$

$$\mathrm{E}[(x_{(r)}-\theta)+(x_{(n-r+1)}-\theta)] = 0$$

$$\mathrm{E}[X_{(r)}+X_{(n-r+1)}] = 2\theta$$

$$\hat{\theta} = \frac{x_{(r)}+x_{(n-r+1)}}{2}$$

Therefore, $\frac{x_{(r)}+x_{(n-r+1)}}{2}$ is an unbiased estimator of θ.
Note: Moments of Cauchy distribution does not exist but still we get an unbiased estimator of θ.

Example 2.1.9 Let X be rv with $B(1,p)$. We examine whether p^2 is U-estimable.

Let $T(x)$ be an unbiased estimator of p^2

$$\sum_{x=0}^{1} T(x)p^x(1-p)^{1-x} = p^2$$

$$T(0)(1-p)+T(1)p = p^2$$

$$p[T(1) - T(0)] + T(0) = p^2$$

Coefficient of p^2 does not exist.
Hence, an unbiased estimator of p^2 does not exist.

Empirical Distribution Function

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous population with df F and pdf f. Then the order statistics $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ is a sufficient statistics. Define $\hat{F}(x) = \frac{\text{Number of } X_i'\text{s } \le x}{n}$, same thing can be written in terms of order statistics as,

$$\hat{F}(x) = \begin{cases} 0 \ ; \ X_{(1)} > x \\ \frac{k}{n} \ ; \ X_{(k)} \le x < X_{(k+1)} \\ 1 \ ; \ x \ge X_{(n)} \end{cases}$$

$$= \frac{1}{n}\sum_{j=1}^{n} \mathbf{I}(x - X_{(j)})$$

where

$$I(y) = \begin{cases} 1; \ y \ge 0 \\ 0; \ \text{otherwise} \end{cases}$$

Example 2.1.10 Show that empirical distribution function is an unbiased estimator of $F(x)$

$$\hat{F}(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{I}(x - X_{(j)})$$

$$\begin{aligned} \mathrm{E}\hat{F}(x) &= \frac{1}{n}\sum_{j=1}^{n} P[X_{(j)} \le x] \\ &= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=j}^{n}\binom{n}{k}[F(x)]^k[1 - F(x)]^{n-k} \text{ (see (Eq. 20 in “Prerequisite”))} \\ &= \frac{1}{n}\sum_{j=1}^{k}\sum_{k=1}^{n}\binom{n}{k}[F(x)]^k[1 - F(x)]^{n-k} \end{aligned}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}[F(x)]^k[1-F(x)]^{n-k}\sum_{j=1}^{k}(1)$$

$$= \frac{1}{n}\sum_{k=1}^{n}k\binom{n}{k}[F(x)]^k[[1-F(x)]^{n-k}$$

$$= \frac{1}{n}[nF(x)] = F(x)$$

Note: One can see that $\mathbf{I}(x-X_{(j)})$ is a Bernoulli random variable. Then $\mathrm{E}\mathbf{I}(x-X_{(j)}) = F(x)$, so that $\mathrm{E}\hat{F}(x) = F(x)$. We observe that $\hat{F}(x)$ has a Binomial distribution with mean $F(x)$ and variance $\frac{F(x)[1-F(x)]}{n}$. Using central limit theorem, for iid rvs, we can show that as $n \to \infty$

$$\sqrt{n}\left[\frac{\hat{F}(x)-F(x)}{\sqrt{F(x)[1-F(x)]}}\right] \quad \to \quad N(0,1).$$

2.2 Unbiasedness and Sufficiency

Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ. Then $T = \sum X_i$ is sufficient for λ. Also $\mathrm{E}(X_1) = \lambda$ then X_1 is unbiased for λ but it is not based on T. Moreover, we can say that it is not a function of T.
(i) Let $T_1 = \mathrm{E}(X_1|T)$. We will prove that T_1 is better than X_1 as an estimate of λ. The distribution of X_1 given T as

$$f(X_1|T=t) = \begin{cases} \binom{t}{x_1}\left(\frac{1}{n}\right)^{x_1}\left(1-\frac{1}{n}\right)^{t-x_1}; & x_1 = 0,1,2,\ldots,t \\ 0; & \text{otherwise} \end{cases} \tag{2.2.1}$$

$\mathrm{E}[X_1|T=t] = \frac{t}{n}$ and distribution of T is $P(n\lambda)$

$$\mathrm{V}\left(\frac{T}{n}\right) = \frac{1}{n^2}\mathrm{V}(T) = \frac{n\lambda}{n^2} = \frac{\lambda}{n}$$

$$\mathrm{V}(X_1) > \mathrm{V}\left(\frac{T}{n}\right) \tag{2.2.2}$$

(ii) Let $T_2 = \left(X_n, \sum_{i=1}^{n-1} X_i\right)$ is also sufficient for λ.

$T_0 = \sum_{i=1}^{n-1} X_i$. We have to find the distribution of X_1 given T_2

$$
\begin{aligned}
P[X_1|T_2] &= \frac{P[X_1 = x_1, T_2 = t_2]}{P[T_2 = t_2]} \\
&= \frac{P[X_1 = x_1, X_n = x_n, \sum_{i=2}^{n-1} X_i = t_0 - x_1]}{P[X_n = x_n, \sum_{i=1}^{n-1} X_i = t_0]} \\
&= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \frac{e^{-\lambda}\lambda^{x_n}}{x_n!} \frac{e^{-(n-2)\lambda}[(n-2)\lambda]^{t_0-x_1}}{(t_0-x_1)!} \frac{x_n!}{e^{-\lambda}\lambda^{x_n}} \frac{t_0!}{e^{-(n-1)\lambda}[(n-1)\lambda]^{t_0}} \\
&= \frac{t_0!}{x_1!(t_0-x_1)!} \frac{(n-2)^{t_0-x_1}}{(n-1)^{t_0}} \\
&= \binom{t_0}{x_1} \left(\frac{n-2}{n-1}\right)^{t_0} \left(\frac{1}{n-2}\right)^{x_1} \\
&= \binom{t_0}{x_1} \left(\frac{1}{n-1}\right)^{x_1} \left(\frac{n-2}{n-1}\right)^{t_0-x_1} \quad ; \; x_1 = 0, 1, 2, \ldots, t_0
\end{aligned} \tag{2.2.3}
$$

Now X_1 given T_2 has $B(t_0, \frac{1}{n-1})$

$$
\mathrm{E}[X_1|T_2] = \frac{t_0}{n-1} = \frac{\sum_{i=1}^{n-1} X_i}{n-1}
$$

$$
\mathrm{V}\left[\frac{T_0}{n-1}\right] = \frac{(n-1)\lambda}{(n-1)^2} = \frac{\lambda}{n-1} \tag{2.2.4}
$$

We conclude that $\frac{\sum_{i=1}^{n-1} X_i}{n-1}$ is unbiased for λ and has smaller variance than X_1. Comparing the variance of X_1, $\bar{X}$, and $\frac{\sum_{i=1}^{n-1} X_i}{n-1}$, we have

$$
\mathrm{V}(X_1) > \mathrm{V}\left(\frac{\sum_{i=1}^{n-1} X_i}{n-1}\right) > \mathrm{V}(\bar{X})
$$

This implies $\lambda > \frac{\lambda}{n-1} > \frac{\lambda}{n}$.

Hence, we prefer $\bar{X}$ to $\frac{\sum_{i=1}^{n-1} X_i}{n-1}$ and X_1.

Note:

1. One should remember that $\mathrm{E}(X_1|T = t)$ and $\mathrm{E}(X_1|T_2 = t_2)$ are the unbiased estimators for λ.
2. Even though sufficient statistic reduce the data most we have to search for the minimal sufficient statistic.

Let $T_1(X_1, X_2, \ldots, X_n)$ and $T_2(X_1, X_2, \ldots, X_n)$ be two unbiased estimates of a parameter θ. Further, suppose that $T_1(X_1, X_2, \ldots, X_n)$ be sufficient for θ. Let $T_1 = f(t)$ for some function f. If sufficiency of T for θ is t_0 have any meaning, we should expect T_1 to perform better than T_2 in the sense that $\mathrm{V}(T_1) \le \mathrm{V}(T_2)$. More generally, given an unbiased estimate h for θ, is it possible to improve upon h using a sufficient statistics for θ? We have seen in the above example that the estimator is improved. Therefore, the answer is "Yes."
If T is sufficient for θ then by definition, the conditional distribution of $(X_1, X_2, \ldots, X_n)$ given T does not depend on θ.
Consider $\mathrm{E}\{h(X_1, X_2, \ldots, X_n)|T(X_1, X_2, \ldots, X_n)\}$. Since T is sufficient then this expected value does not depend on θ.
Set $T_1 = \mathrm{E}\{h(X_1, X_2, \ldots, X_n)|T(X_1, X_2, \ldots, X_n)\}$ is itself an estimate of θ.
Using Theorem 5 in "Prerequisite", we can get $\mathrm{E}T_1$

$$\mathrm{E}(T_1) = \mathrm{E}\left[\mathrm{E}\{h(X_1, X_2, \ldots, X_n)|T(X_1, X_2, \ldots, X_n)\}\right]$$

$$= \mathrm{E}\{h(X_1, X_2, \ldots, X_n)\} = \theta$$

Since h is unbiased for θ, hence $\mathrm{E}(T_1)$ is also unbiased for θ.
Thus, we have found out another unbiased estimate of θ that is a function of the sufficient statistic. What about the variance of T_1?
Using Theorem 6 in "Prerequisite"

$$\begin{aligned}\mathrm{V}[h(X_1, X_2, \ldots, X_n)] = {} & \mathrm{E}\{\mathrm{V}(h(X_1, X_2, \ldots, X_n)|T(X_1, X_2, \ldots, X_n))\} \\ & + \mathrm{V}\{\mathrm{E}h(X_1, X_2, \ldots, X_n)|T(X_1, X_2, \ldots, X_n)\}\end{aligned}$$

$$= \mathrm{E}\{\mathrm{V}(h(X_1, X_2, \ldots, X_n)|T(X_1, X_2, \ldots, X_n))\} + \mathrm{V}(T_1) \tag{2.2.5}$$

Since $\mathrm{V}(h|T) > 0$ so that $\mathrm{E}[\mathrm{V}(h|T)] > 0$
From (2.2.5), $\mathrm{V}(T_1) < \mathrm{V}[h(X)]$
If $T(X)$ is minimal sufficient for θ then T_1 is the best unbiased estimate of θ. Sometimes we face the problem of computations of expectation of h given T.
The procedure for finding unbiased estimates with smaller variance can now be summarized.

1. Find the minimal sufficient statistic.
2. Find a function of this sufficient statistic that is unbiased for the parameter.

Remark If you have a minimal sufficient statistic then your unbiased estimate will have the least variance. If not, the unbiased estimate you construct will not be the best possible but you have the assurance that it is based on a sufficient statistic.

Theorem 2.2.1 *Let $h(X)$ be an unbiased estimator of $g(\theta)$. Let $T(X)$ be a sufficient statistics for θ. Define $\Psi(T) = \mathrm{E}(h|T)$. Then $\mathrm{E}[\Psi(T)] = g(\theta)$ and $\mathrm{V}[\Psi(T)] \le \mathrm{V}(h) \;\; \forall \;\; \theta$. Then $\Psi(T)$ is uniformly minimum variance unbiased estimator (UMVUE) of $g(\theta)$.*
This theorem is known as Rao–Blackwell Theorem.

Proof Using Theorem 5 in "Prerequisite",

$$\mathrm{E}[h(X)] = \mathrm{E}[\mathrm{E}h(X)|T = t] = \mathrm{E}[\Psi(T)] = g(\theta) \tag{2.2.6}$$

Hence $\Psi(T)$ is unbiased estimator of $g(\theta)$
Using Theorem 6 in "Prerequisite",

$$\mathrm{V}[h(X)] = \mathrm{V}[\mathrm{E}(h(X)|T(X))] + \mathrm{E}[\mathrm{V}(h(X)|T(X))]$$

$$= \mathrm{V}[\Psi(T)] + \mathrm{E}[\mathrm{V}(h(X)|T(X))]$$

Since $\mathrm{V}[h(X)|T(X)] \ge 0$ and $\mathrm{E}[\mathrm{V}(h(X)|T(X))] > 0$
Therefore,

$$\mathrm{V}[\Psi(T)] \le \mathrm{V}[h(X)] \tag{2.2.7}$$

We have to show that $\Psi(T)$ is an estimator,
i.e., $\Psi(T)$ is a function of sample only and independent of θ.
From the definition of sufficiency, we can conclude that the distribution of $h(X)$ given $T(X)$ is independent of θ. Hence $\Psi(T)$ is an estimator.
Therefore, $\Psi(T)$ is UMVUE of $g(\theta)$.

Note: We should remember that conditioning on anything will not result in improving the estimator.

Example 2.2.1 Let X_1, X_2 be iid $N(\theta, 1)$.

Let

$$h(X) = \bar{X} = \frac{X_1 + X_2}{2},$$

$$\mathrm{E}h(X) = \theta \quad \text{and} \quad \mathrm{V}[h(X)] = \frac{1}{2},$$

Now conditioning on X_1, which is not sufficient. Let $\Psi(X_1) = \mathrm{E}(\bar{X})|X_1)$.
Using Theorem 5 in "Prerequisite", $\mathrm{E}[\Psi(X_1)] = \mathrm{E}\bar{X} = \theta$. Using Theorem 6 in "Prerequisite", $\mathrm{V}[\Psi(X_1)] \le \mathrm{V}(\bar{X})$. Hence $\Psi(X_1)$ is better than $\bar{X}$. But question is whether $\Psi(X_1)$ is an estimator?

$$\begin{aligned}\Psi(X_1) &= \mathrm{E}(\bar{X}|X_1)\\ &= \mathrm{E}\left(\frac{X_1+X_2}{2}|X_1\right) = \frac{1}{2}\mathrm{E}(X_1|X_1) + \frac{1}{2}\mathrm{E}(X_2|X_1)\\ &= \frac{1}{2}X_1 + \frac{1}{2}\mathrm{E}(X_2)\ (X_1 \text{ and } X_2 \text{ are independent})\\ &= \frac{1}{2}X_1 + \frac{1}{2}\theta\end{aligned}$$

Hence $\Psi(X_1)$ is not an estimator. This imply that we cannot say that $\Psi(X_1)$ is better than $\bar{X}$.

Theorem 2.2.2 (Lehmann–Scheffe Theorem) *If T is a complete sufficient statistic and there exists an unbiased estimate h of $g(\theta)$, there exists a unique UMVUE of θ, which is given by* $\mathrm{E}h|T$.

Proof Let h_1 and h_2 be two unbiased estimators of $g(\theta)$ Rao–Blackwell theorem, $\mathrm{E}(h_1|T)$ and $\mathrm{E}(h_2|T)$ are both UMVUE of $g(\theta)$.
Hence $\mathrm{E}[\mathrm{E}(h_1|T) - \mathrm{E}(h_2|T)] = 0$
But T is complete therefore

$$[\mathrm{E}(h_1|T) - \mathrm{E}(h_2|T)] = 0$$

This implies $\mathrm{E}(h_1|T) = \mathrm{E}(h_2|T)$.
Hence, UMVUE is unique.
Even if we cannot obtain sufficient and complete statistic for a parameter, still we can get UMVUE for a parameter. Therefore, we can see the following theorem:

Theorem 2.2.3 *Let T_0 be the UMVUE of $g(\theta)$ and v_0 be the unbiased estimator of 0. Then T_0 is UMVUE if and only if* $\mathrm{E}v_0T_0 = 0 \quad \forall \quad \theta \in \Theta$. *Assume that the second moment exists for all unbiased estimators of $g(\theta)$.*

Proof (i) Suppose T_0 is UMVUE and $\mathrm{E}v_0T_0 \neq 0$ for some θ_0 and v_0 where $\mathrm{E}v_0 = 0$. Then $T_0 + \alpha v_0$ is unbiased for all real α. If $\mathrm{E}v_0^2 = 0$ then v_0 is degenerate rv. Hence $\mathrm{E}v_0T_0 = 0$. This implies $P[v_0 = 0] = 1$.
Let $\mathrm{E}v_0^2 > 0$

$$\begin{aligned}\mathrm{E}[T_0 + \alpha v_0 - g(\theta)]^2 &= \mathrm{E}(T_0 + \alpha v_0)^2 - 2g(\theta)\mathrm{E}(T_0 + \alpha v_0) + g^2(\theta)\\ &= \mathrm{E}(T_0 + \alpha v_0)^2 - g^2(\theta)\\ &= \mathrm{E}(T_0)^2 + 2\alpha\mathrm{E}(T_0v_0) + \alpha^2\mathrm{E}v_0^2 - g^2(\theta) \qquad (2.2.8)\end{aligned}$$

Choose α such that (2.2.8) is equal to zero, then differentiating (2.2.8) with respect to α, we get

$$= 2\mathrm{E}(T_0v_0) + 2\alpha\mathrm{E}v_0^2 = 0$$

Hence

$$\alpha_0 = -\frac{\mathrm{E}(T_0 v_0)}{\mathrm{E}v_0^2} \tag{2.2.9}$$

$$\begin{aligned} \mathrm{E}(T_0 + \alpha v_0)^2 &= \mathrm{E}(T_o)^2 + 2\alpha \mathrm{E}(T_0 v_0) + \alpha^2 \mathrm{E}v_0^2 \\ &= \mathrm{E}(T_0)^2 - \frac{(\mathrm{E}(T_0 v_0))^2}{\mathrm{E}v_0^2} \\ &< \mathrm{E}(T_0)^2 \end{aligned} \tag{2.2.10}$$

Because $\frac{(\mathrm{E}T_0 v_0)^2}{\mathrm{E}v_0^2} > 0$ (our assumption $\mathrm{E}(T_0 v_0) \neq 0$)
Then we can conclude that

$$\mathrm{V}(T_0 + \alpha v_0) < \mathrm{E}(T_0)^2$$

which is a contradiction, because T_0 is UMVUE.
Hence $\mathrm{E}vT_0 = 0$
(ii) Suppose that

$$\mathrm{E}vT_0 = 0 \quad \forall \quad \theta \in \Theta \tag{2.2.11}$$

Let T be an another unbiased estimator of θ, then $\mathrm{E}(T - T_0) = 0$.
Now T_0 is unbiased estimator and $(T - T_0)$ is unbiased estimator of 0, then by (2.2.11),

$$\mathrm{E}T_0(T - T_0) = 0$$

$$\mathrm{E}T_0 T - \mathrm{E}{T_0}^2 = 0$$

This implies $\mathrm{E}{T_0}^2 = \mathrm{E}T_0 T$
Using Cauchy–Schwarz's inequality

$$\mathrm{E}T_0 T \leq (\mathrm{E}{T_0}^2)^{\frac{1}{2}} (\mathrm{E}T^2)^{\frac{1}{2}}$$

Therefore,

$$\mathrm{E}T_0^2 \leq (\mathrm{E}{T_0}^2)^{\frac{1}{2}} (\mathrm{E}T^2)^{\frac{1}{2}}$$

$$(\mathrm{E}{T_0}^2)^{\frac{1}{2}} \leq (\mathrm{E}T^2)^{\frac{1}{2}} \tag{2.2.12}$$

Now if $ET_0{}^2 = 0$ then $P[T_0 = 0] = 1$
Then (2.2.12) is true.
Next, if $ET_0{}^2 > 0$ then also (2.2.12) is true
Hence $V(T_0) \leq V(T) \Rightarrow T_0$ is UMVUE.

Remark We would like to mention the comment made by Casella and Berger (2002). "An unbiased estimator of 0 is nothing more than random noise; that is there is no information in an estimator of 0. It makes sense that most sensible way to estimate 0 is with 0, not with random noise. Therefore, if an estimator could be improved by adding random noise to it, the estimator probably is defective."
Casella and Berger (2002) gave an interesting characterization of best unbiased estimators.

Example 2.2.2 Let X be an rv with $\cup(\theta, \theta+1)$, $EX = \theta + \frac{1}{2}$, then $(X - \frac{1}{2})$ is an unbiased estimator of θ and its variance is $\frac{1}{12}$. For this pdf, unbiased estimators of zero are periodic functions with period 1.

If $h(x)$ satisfies $\int_\theta^{\theta+1} h(x) = 0$

$$\frac{d}{d\theta}\int\limits_\theta^{\theta+1} h(x) = 0$$

$$h(\theta+1) - h(\theta) = 0 \quad \forall \quad \theta$$

Such a function is $h(x) = \sin 2\pi x$.
Now,

$$\mathrm{Cov}\left[X - \frac{1}{2}, \sin 2\pi X\right] = \mathrm{Cov}[X, \sin 2\pi X] = \int\limits_\theta^{\theta+1} x \sin 2\pi x dx$$

$$= -\frac{(\theta+1)\cos 2\pi(\theta+1)}{2\pi} + \theta\frac{\cos 2\pi\theta}{2\pi} + \frac{\sin 2\pi(\theta+1)}{4\pi^2} - \frac{\sin 2\pi\theta}{4\pi^2}$$

Since $\sin 2\pi(\theta+1) = \sin 2\pi\theta$

$$\cos 2\pi(\theta+1) = \cos 2\pi\theta \cos 2\pi - \sin 2\pi\theta \sin 2\pi$$

$$= \cos 2\pi\theta \quad (\cos 2\pi = 1, \sin 2\pi = 0)$$

$$\mathrm{Cov}[X, \sin 2\pi X] = -\frac{\cos 2\pi\theta}{2\pi}$$

Hence $\left(X-\frac{1}{2}\right)$ is correlated with an unbiased estimator of zero. Therefore, $\left(X-\frac{1}{2}\right)$ cannot be the best unbiased estimator of θ.

Example 2.2.3 Sometimes UMVUE is not sensible.

Let $X_1, X_2, \ldots, X_n$ be $N(\mu, 1)$. Now X_1 is unbiased estimator for μ and $\bar{X}$ is complete sufficient statistic for μ then $E(X_1|\bar{X})$ is UMVUE. We will show that $E(X_1|\bar{X}) = \bar{X}$. See (ii) of Example 2.2.11
Note that $\bar{X}$ is $N(\mu, \frac{1}{n})$

$$E(X_1\bar{X}) = \frac{1}{n}EX_1[X_1 + X_2 + \cdots + X_n]$$

$$= \frac{1}{n}[E(X_1^2) + E(X_1X_2) + \cdots + E(X_1X_n)]$$

$$= \frac{1}{n}[1 + \mu^2 + \mu^2 + \cdots + \mu^2]$$

$$\text{Cov}(X_1, \bar{X}) = \frac{1 + n\mu^2}{n} - \mu^2 = \frac{1}{n}$$

$$E(X_1|\bar{X}) = EX_1 + \frac{\text{Cov}(X_1, \bar{X})}{V(\bar{X})}[\bar{X} - E\bar{X}]$$

$$= \mu + \frac{1}{n}n[\bar{X} - \mu]$$

$$= \mu + [\bar{X} - \mu] = \bar{X}$$

$(X_1, \bar{X})$ is a bivariate rv with mean

$$\begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

and covariance matrix

$$\begin{pmatrix} 1 & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} \end{pmatrix}$$

In this example, we want to estimate $d(\mu) = \mu^2$ then $\left(\bar{X}^2 - \frac{1}{n}\right)$ is UMVUE for μ^2. One can easily see that $E\bar{X}^2 = \frac{1}{n} + \mu^2$.

Hence $\mathrm{E}\left(\bar{X}^2 - \frac{1}{n}\right) = \mu^2$ and $\bar{X}^2$ is sufficient and complete for μ^2.
Now μ^2 is always positive but sometimes $\left(\bar{X}^2 - \frac{1}{n}\right)$ may be negative. Therefore, UMVUE for μ^2 is not sensible, see (2.2.56).
Now, we will find UMVUE for different estimators for different distributions.

Example 2.2.4 Let $X_1, X_2, \ldots, X_n$ are iid rvs with $B(n, p)$, $0 < p < 1$. In this case, we have to find the UMVUE of $p^r q^s$, $q = 1 - p$, r , $s \neq 0$ and $P[X \leq c]$. Assume n is known.

Binomial distribution belongs to exponential family. So that $\sum_{i=1}^{n} X_i$ is sufficient and complete for p.
(i) The distribution of T is $B(mn, p)$.
Let $U(t)$ be unbiased estimator for $p^r q^s$.

$$\sum_{t=0}^{nm} u(t) \binom{nm}{t} p^t q^{nm-t} = p^r q^s \tag{2.2.13}$$

$$\sum_{t=0}^{nm} u(t) \binom{nm}{t} p^{t-r} q^{nm-t-s} = 1$$

$$\sum_{t=r}^{nm-s} u(t) \frac{\binom{nm}{t}}{\binom{nm-s-r}{t-r}} \binom{nm-s-r}{t-r} p^{t-r} q^{nm-t-s} = 1$$

Then

$$u(t) \frac{\binom{nm}{t}}{\binom{nm-s-r}{t-r}} = 1$$

Hence

$$u(t) = \begin{cases} \frac{\binom{nm-s-r}{t-r}}{\binom{nm}{t}} & ;\ t = r, r+1, r+2, \ldots, nm-s \\ 0 & ;\ \text{otherwise} \end{cases} \tag{2.2.14}$$

Note: For $m = n = 1$, $r = 2$, and $s = 0$, the unbiased estimator of p^2 does not exist, see Example 2.1.9
(ii) To find UMVUE of $P[X \leq c]$
Now

$$P[X \leq c] = \sum_{x=0}^{c} \binom{n}{x} p^x q^{n-x}$$

Then UMVUE of

$$p^x q^{n-x} = \frac{\binom{nm-n}{t-x}}{\binom{nm}{t}}$$

Hence UMVUE of $P[X \le c]$

$$= \begin{cases} \sum_{x=0}^{c} \binom{n}{x} \frac{\binom{nm-n}{t-x}}{\binom{nm}{t}} \ ; \ t = x, x+1, x+2, \ldots, nm-n+x, \quad c \le \min(t, n) \\ 1 \qquad ; \text{otherwise} \end{cases} \tag{2.2.15}$$

Note: UMVUE of $P[X = x] = \binom{n}{x} p^x q^{n-x}$ is $\frac{\binom{n}{x}\binom{nm-n}{t-x}}{\binom{nm}{t}}$; $\quad x = 0, 1, 2, \ldots, t$

Particular cases:

(a) $r = 1, s = 0$. From (2.2.14), we will get UMVUE of p,

$$u(t) = \frac{\binom{nm-1}{t-1}}{\binom{nm}{t}} = \frac{t}{nm} \tag{2.2.16}$$

(b) $r = 0, s = 1$. From (2.2.14), we will get UMVUE of q,

$$u(t) = \frac{\binom{nm-1}{t}}{\binom{nm}{t}} = \frac{nm-t}{nm} = 1 - \frac{t}{nm} \tag{2.2.17}$$

(c) $r = 1, s = 1$. From (2.2.14), we will get UMVUE of pq,

$$u(t) = \left(\frac{t}{nm}\right)\left(\frac{nm-t}{nm-1}\right) \tag{2.2.18}$$

Remark We have seen that in (2.2.16), (2.2.17), and (2.2.18),

$$\hat{p} = \frac{t}{nm}; \hat{q} = 1 - \frac{t}{nm} \text{ and } \widehat{pq} = \left(\frac{t}{nm}\right)\left(\frac{nm-t}{nm-1}\right)$$

Hence, UMVUE of $pq \ne$ (UMVUE of p) (UMVUE of q).

Example 2.2.5 Let $X_1, X_2, \ldots, X_m$ are iid rvs with $P(\lambda)$. In this case we have to find UMVUE of (i) $\lambda^r e^{-s\lambda}$ (ii) $P[X \le c]$

Poisson distribution belongs to exponential family. So that $T = \sum_{i=1}^{n} X_i$ is sufficient and complete for λ.

(i) The distribution of T is $P(m\lambda)$.

Let $U(t)$ be unbiased estimator for $\lambda^r e^{-s\lambda}$

$$\sum_{t=0}^{\infty} u(t) \frac{e^{-m\lambda}(m\lambda)^t}{t!} = e^{-s\lambda}\lambda^r \tag{2.2.19}$$

$$\sum_{t=0}^{\infty} u(t) \frac{e^{-(m-s)\lambda} m^t \lambda^{t-r}}{t!} = 1$$

$$\sum_{t=r}^{\infty} u(t) \frac{m^t}{(m-s)^{t-r}} \frac{(t-r)!}{t!} \frac{e^{-(m-s)\lambda}[(m-s)\lambda]^{t-r}}{(t-r)!} = 1$$

Then

$$u(t) \frac{m^t}{(m-s)^{t-r}} \frac{(t-r)!}{t!} = 1$$

$$u(t) = \begin{cases} \frac{(m-s)^{t-r}}{m^t} \frac{t!}{(t-r)!} & ;\ t = r, r+1, \ldots, s \le m \\ 0 & ; \text{otherwise} \end{cases} \tag{2.2.20}$$

(ii) To find UMVUE of $P[X \le c]$

$$P[X \le c] = \sum_{x=0}^{c} \frac{e^{-\lambda} \lambda^x}{x!}$$

Now, UMVUE of $e^{-\lambda}\lambda^x$ is $\frac{(m-1)^{(t-x)}}{m^t} \frac{t!}{(t-x)!}$
UMVUE of $P[X \le c]$

$$= \sum_{x=0}^{c} \frac{t!}{(t-x)!x!} \left(\frac{m-1}{m}\right)^t \left(\frac{1}{m-1}\right)^x$$

$$= \begin{cases} \sum_{x=0}^{c} \binom{t}{x} \left(\frac{1}{m}\right)^x \left(\frac{m-1}{m}\right)^{t-x} & ; c \le t \\ 1 & ;\ \text{otherwise} \end{cases} \tag{2.2.21}$$

Remark UMVUE of $P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!}$ is $\binom{t}{x} \left(\frac{1}{m}\right)^x \left(\frac{m-1}{m}\right)^{t-x}$; $x = 0, 1, \ldots, t$
Particular cases:
(a) $s = 0, r = 1$
From (2.2.20), we will get the UMVUE of λ,

$$u(t) = \frac{m^{t-1} t!}{m^t (t-1)!} = \frac{t}{m} \tag{2.2.22}$$

(b) $s = 1, r = 0$
From (2.2.20), we will get the UMVUE of $e^{-\lambda}$,

$$u(t) = \left(\frac{m-1}{m}\right)^t \tag{2.2.23}$$

(c) $s = 1, r = 1$
From (2.2.20), we will get the UMVUE of $\lambda e^{-\lambda}$

$$u(t) = \frac{(m-1)^{t-1} t!}{m^t (t-1)!} = \left(\frac{m-1}{m}\right)^t \frac{t}{m-1} \tag{2.2.24}$$

Remark UMVUE of $\lambda e^{-\lambda} \neq$ (UMVUE of λ)(UMVUE of $e^{-\lambda}$)

Example 2.2.6 Let $X_1, X_2, \ldots, X_m$ are iid rvs with $NB(k, p)$. In this case we have to find UMVUE of

1. $p^r q^s (r, s \neq 0)$
2. $P[X \leq c]$

$P[X = x] =$ Probability of getting kth successes at the xth trial

$$= \binom{k+x-1}{x} p^k q^x; \quad x = 0, 1, 2, \ldots, 0 < p < 1, \; q = 1 - p \tag{2.2.25}$$

Negative Binomial distribution belongs to exponential family.
Therefore, $T = \sum_{i=1}^m X_i$ is complete and sufficient for p. Distribution of T is $NB(mk, p)$.
Let $U(t)$ be unbiased estimator for $p^r q^s$

$$\sum_{t=0}^{\infty} u(t) \binom{mk+t-1}{t} p^{mk} q^t = p^r q^s$$

$$\sum_{t=0}^{\infty} u(t) \binom{mk+t-1}{t} p^{mk-r} q^{t-s} = 1$$

$$\sum_{s=0}^{\infty} u(t) \frac{\binom{mk+t-1}{t}}{\binom{mk-r-s+t-1}{t-s}} \binom{mk-r-s+t-1}{t-s} p^{mk-r} q^{t-s} = 1$$

Then

$$u(t) \frac{\binom{mk+t-1}{t}}{\binom{mk-r-s+t-1}{t-s}} = 1$$

Hence,

$$u(t) = \frac{\binom{mk-r-s+t-1}{t-s}}{\binom{mk+t-1}{t}}$$

$$u(t) = \begin{cases} \frac{\binom{mk-r-s+t-1}{t-s}}{\binom{mk+t-1}{t}} & ;\ t = s, s+1, \ldots, r \le mk \\ 0 & ;\ \text{otherwise} \end{cases} \tag{2.2.26}$$

(ii) To find UMVUE of $P[X \le c]$

$$P[X \le c] = \sum_{x=0}^{c} \binom{k+x-1}{x} p^k q^x$$

Now UMVUE of $p^k q^x = \frac{\binom{mk-k-x+t}{t-x}}{\binom{mk+t-1}{t}}$
UMVUE of $P[X \le c]$

$$= \begin{cases} \sum_{x=0}^{c} \frac{\binom{k+x-1}{x}\binom{mk-k-x+t}{t-x}}{\binom{mk+t-1}{t}} & ;\ t = x, x+1, \ldots \\ 1 & ;\ \text{otherwise.} \end{cases} \tag{2.2.27}$$

Remark UMVUE of $P[X = x] = \binom{k+x-1}{x} p^k q^x$ is $\frac{\binom{k+x-1}{x}\binom{mk-k-x+t}{t-x}}{\binom{mk+t-1}{t}}$
Particular cases:
(a) $r = 1, s = 0$
From (2.2.26), we will get UMVUE of p,

$$u(t) = \frac{\binom{mk+t-2}{t}}{\binom{mk+t-1}{t}} = \frac{mk-1}{mk+t-1} \tag{2.2.28}$$

(b) $r = 0, s = 1$
From (2.2.26), we will get UMVUE of q,

$$u(t) = \frac{\binom{mk+t-2}{t-1}}{\binom{mk+t-1}{t}} = \frac{t}{mk+t-1} \tag{2.2.29}$$

(c) $r = 1, s = 1$
From (2.2.26), we will get UMVUE of pq,

$$u(t) = \frac{\binom{mk+t-3}{t-1}}{\binom{mk+t-1}{t}} = \frac{t(mk-1)}{(mk+t-1)(mk+t-2)} \tag{2.2.30}$$

Remark UMVUE of $pq \ne$ (UMVUE of p)(UMVUE of q)

Example 2.2.7 Let $X_1, X_2, \ldots, X_m$ be iid discrete uniform rvs with parameter $N (N > 1)$. We have to find UMVUE of $N^s (s \ne 0)$.

Then joint distribution of $(X_1, X_2, \ldots, X_m)$ is

$$f(x_1, x_2, \ldots, x_m) = \frac{1}{N^m} I(N - x_{(m)}) I(x_{(1)} - 1)$$

$$I(y) = \begin{cases} 1 \; ; \; y > 0 \\ 0 \; ; \text{otherwise} \end{cases}$$

By factorization theorem, $X_{(m)}$ is sufficient for N.
Now, we will find the distribution of $X_{(m)}$.

$$P[X_{(m)} \le z] = \prod_{i=1}^{m} P[X_i \le z] = \frac{z^m}{N^m}$$

$$P[X_{(m)} = z] = P[X_{(m)} \le z] - P[X_{(m)} \le z - 1]$$

$$= \frac{z^m}{N^m} - \frac{(z-1)^m}{N^m}; \quad z = 1, 2, \ldots, N \tag{2.2.31}$$

We have to show that this distribution is complete, i.e., we have to show if $\mathrm{E}h(z) = 0$ then $h(z) = 0$ with probability 1.

$$\mathrm{E}h(z) = \sum_{z=1}^{N} h(z) \left[\frac{z^m}{N^m} - \frac{(z-1)^m}{N^m} \right] = 0$$

Now $\left(\frac{z^m - (z-1)^m}{N^m} \right)$ is always positive then $h(z) = 0$ with probability 1.
Therefore, $X_{(m)}$ is sufficient and complete for N.
Let $u(z)$ be unbiased estimator of N^s
Then

$$\sum_{z=1}^{N} u(z) \left[\frac{z^m - (z-1)^m}{N^m} \right] = N^s$$

$$\sum_{z=1}^{N} u(z) \left[\frac{z^m - (z-1)^m}{N^{m+s}} \right] = 1$$

$$\sum_{z=1}^{N} u(z) \left[\frac{z^m - (z-1)^m}{z^{m+s} - (z-1)^{m+s}} \right] \left[\frac{z^{m+s} - (z-1)^{m+s}}{N^{m+s}} \right] = 1$$

Hence,

$$u(z) \left[\frac{z^m - (z-1)^m}{z^{m+s} - (z-1)^{m+s}} \right] = 1$$

$$u(z) = \left[\frac{z^{m+s} - (z-1)^{m+s}}{z^m - (z-1)^m}\right]$$

Therefore,

$$u(X_{(m)}) = \left[\frac{X_{(m)}^{m+s} - (X_{(m)}-1)^{m+s}}{X_{(m)}^m - (X_{(m)}-1)^m}\right] \tag{2.2.32}$$

Then $u(X_{(m)})$ in (2.2.32) is UMVUE of N^s.
Particular cases:
(a) $s = 1$
From (2.2.32), we get UMVUE of N,

$$\hat{N} = \left[\frac{X_{(m)}^{m+1} - (X_{(m)}-1)^{m+1}}{X_{(m)}^m - (X_{(m)}-1)^m}\right] \tag{2.2.33}$$

(b) $s = 5$
From (2.2.33), we get UMVUE of N^5

$$\hat{N^5} = \left[\frac{X_{(m)}^{m+5} - (X_{(m)}-1)^{m+5}}{X_{(m)}^m - (X_{(m)}-1)^m}\right] \tag{2.2.34}$$

(c) To find UMVUE of e^N
Now

$$e^N = \sum_{j=0}^{\infty} \frac{N^j}{j!} \tag{2.2.35}$$

Using (2.2.32), UMVUE of e^N is

$$e^{\hat{N}} = \sum_{j=0}^{\infty} \frac{1}{j!}\left[\frac{X_{(m)}^{m+j} - (X_{(m)}-1)^{m+j}}{X_{(m)}^m - (X_{(m)}-1)^m}\right]$$

Remark UMVUE of $e^N \neq e^{\hat{N}}$

Example 2.2.8 Let $X_1, X_2, \ldots, X_m$ be iid rvs with power series distribution.

$$P(X = x) = \frac{a(x)\theta^x}{c(\theta)}; x = 0, 1, 2, \ldots \tag{2.2.36}$$

where $c(\theta) = \sum_{x=0}^{\infty} a(x)\theta^x$.
This distribution belongs to exponential family.

Therefore, $T = \sum X_i$ is sufficient and complete for θ. In this case, we will find UMVUE of $\frac{\theta^r}{[c(\theta)]^s}(r,\ s \neq 0)$.
This distribution of T is again a power series distribution, see Roy and Mitra (1957), and Patil (1962)

$$P(T = t) = \frac{A(t, m)\theta^t}{[c(\theta)]^m}, \tag{2.2.37}$$

where $A(t, m) = \sum_{(x_1, x_2, \ldots, x_m)} \prod_{i=1}^{m} a(x_i)$
Let $U(t)$ be an unbiased estimator of $\frac{\theta^r}{[c(\theta)]^s}$

$$\sum_{t=0}^{\infty} u(t) \frac{A(t, m)\theta^t}{[c(\theta)]^m} = \frac{\theta^r}{[c(\theta)]^s} \tag{2.2.38}$$

$$\sum_{t=0}^{\infty} u(t) \frac{A(t, m)\theta^{t-r}}{[c(\theta)]^{m-s}} = 1$$

$$\sum_{t=0}^{\infty} u(t) \frac{A(t, m)}{A(t-r, m-s)} \frac{A(t-r, m-s)\theta^{t-r}}{[c(\theta)]^{m-s}} = 1$$

Now

$$u(t) \frac{A(t, m)}{A(t-r, m-s)} = 1$$

This implies

$$U(t) = \begin{cases} \frac{A(t-r, m-s)}{A(t, m)} & ;\ t \geq r,\quad m \geq s \\ 0 & ;\ \text{otherwise} \end{cases} \tag{2.2.39}$$

Example 2.2.9 Let $X_1, X_2, \ldots, X_m$ be iid rvs with $G(p, \frac{1}{\theta})$.
Let

$$f(x, \theta) = \frac{e^{-\frac{x}{\theta}} x^{p-1}}{\theta^p \Gamma(p)};\ x > 0,\ p > 0,\ \theta > 0 \tag{2.2.40}$$

Now gamma distribution belongs to an exponential family. $T = \sum X_i$ is sufficient and complete for θ.

The distribution of T is

$$f(t) = \frac{e^{-\frac{t}{\theta}} t^{mp-1}}{\theta^{mp} \Gamma(mp)};\ t > 0,\ p > 0,\ \theta > 0 \tag{2.2.41}$$

We have to find UMVUE of (i) $e^{-\frac{k}{\theta}}\theta^r$ (ii) $P(X \geq k)$
(i) Let $u(t)$ be an unbiased estimator of $e^{-\frac{k}{\theta}}\theta^r$

$$\int_0^\infty u(t)\frac{e^{-\frac{t}{\theta}}t^{mp-1}}{\theta^{mp}\Gamma(mp)} = e^{-\frac{k}{\theta}}\theta^r$$

$$\int_0^\infty u(t)\frac{e^{-\frac{t-k}{\theta}}t^{mp-1}}{\theta^{mp+r}\Gamma(mp)} = 1$$

$$\int_k^\infty \left(u(t)\frac{t^{mp-1}\Gamma(mp+r)}{(t-k)^{mp+r-1}\Gamma(mp)}\right)\left(\frac{e^{-\frac{t-k}{\theta}}(t-k)^{mp+r-1}}{\theta^{mp+r}\Gamma(mp+r)}\right)dt = 1$$

Then,

$$u(t)\frac{t^{mp-1}\Gamma(mp+r)}{(t-k)^{mp+r-1}\Gamma(mp)} = 1$$

$$u(t) = \begin{cases} \frac{(t-k)^{mp+r-1}\Gamma(mp)}{t^{mp-1}\Gamma(mp+r)} & ;\ t > k,\ mp > -r \\ 0 & ;\ \text{otherwise} \end{cases} \tag{2.2.42}$$

(ii) We have to find UMVUE of $P[X \ge k]$. Note that

$$P[X \ge k] = \int_k^\infty \frac{e^{-\frac{x}{\theta}}x^{p-1}}{\theta^p\Gamma(p)}dx$$

Let

$$Y = \begin{cases} 1\ ;\ X_1 \ge k \\ 0\ ;\ \text{otherwise} \end{cases}$$

$$\mathrm{E}(Y) = P[X_1 \ge k]$$

Hence Y is unbiased estimator for $P[X_1 \ge k]$. We have seen in Sect. 2.2 that $[\mathrm{E}Y|T = t]$ is an estimator and has minimum variance.
So $\mathrm{E}[Y|T = t] = P[X_1 \ge k|T = t]$. Now we will require the distribution of $X_1|T = t$

$$P[X_1|T = t] = \frac{f(x_1)f(t_1)}{f(t)}, \quad \text{where } T_1 = \sum_{i=2}^m X_i$$

Distribution of $(T_1 = t_1) = f(t_1)$

$$f(t_1) = \frac{e^{-\frac{t_1}{\theta}}t_1^{(m-1)p-1}}{\Gamma((m-1)p)\theta^{(m-1)p}}; \quad t_1 \ge 0$$

$$P[X_1|T=t] = \frac{e^{-\frac{x_1}{\theta}} x_1^{p-1}}{\Gamma(p)\theta^p} \frac{e^{-\frac{t_1}{\theta}} t_1^{(m-1)p-1}}{\Gamma((m-1)p)\theta^{(m-1)p}} \frac{\Gamma(mp)\theta^{mp}}{e^{-\frac{t}{\theta}} t^{mp-1}}$$

$$= \frac{\left(\frac{x_1}{t}\right)^{p-1}\left(1-\frac{x_1}{t}\right)^{(m-1)p-1}}{t\beta(p,(m-1)p)}; \quad 0 \le \frac{x_1}{t} \le 1 \tag{2.2.43}$$

$$\mathrm{E}[Y|T=t] = P[X_1 \ge k|T=t] = \int_k^t \frac{\left(\frac{x_1}{t}\right)^{p-1}\left(1-\frac{x_1}{t}\right)^{(m-1)p-1}}{t\beta(p,(m-1)p)} dx_1$$

Let $\frac{x_1}{t} = w$

$$= \int_{\frac{k}{t}}^{1} \frac{w^{p-1}(1-w)^{(m-1)p-1}}{\beta(p,(m-1)p)} dw$$

$$= 1 - \int_0^{\frac{k}{t}} \frac{w^{p-1}(1-w)^{(m-1)p-1}}{\beta(p,(m-1)p)} dw \tag{2.2.44}$$

$$P[X_1 \ge k|T=t] = \begin{cases} 1 - I_{\frac{k}{t}}(p, mp-p) \ ; & 0 < k < t \\ 0 & ; \ k \ge t \end{cases}$$

Now

$$P[X \ge k] = \int_k^\infty \frac{e^{-\frac{x}{\theta}} x^{p-1}}{\theta^p \Gamma(p)} dx$$

$$1 - I_{\frac{k}{\theta}}(p) = \text{Incomplete Gamma function.} \tag{2.2.45}$$

Hence UMVUE of $1 - I_{\frac{k}{\theta}}(p)$ is given by incomplete Beta function $1 - I_{\frac{k}{t}}(p, mp-p)$.
Note: Student should use R or Minitab software to calculate UMVUE.

Example 2.2.10 Let $X_1, X_2, \ldots, X_m$ be iid rvs with the following pdfs.

1. $f(x|\lambda) = \frac{\lambda}{(1+x)^{\lambda+1}}; \quad x > 0$
2. $f(x|\lambda) = \lambda x^{\lambda-1}; \quad 0 < x < 1, \quad \lambda > 0$
3. $f(x|\lambda) = \frac{1}{2\lambda} e^{\frac{|x|}{\lambda}}; \quad x > 0, \quad \lambda > 0$
4. $f(x|\lambda) = \frac{\alpha}{\lambda} x^{\alpha-1} e^{-\frac{x^\alpha}{\lambda}}; \quad x > 0, \quad \lambda > 0, \quad \alpha > 0$
5. $f(x|\lambda) = \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{x^2}{2\lambda}}; \quad x > 0, \quad \lambda > 0$

(i) Let $Y = \log(1+x)$ then $f(y|\lambda) = \lambda e^{-\lambda y}; \quad y > 0, \quad \lambda > 0$

The UMVUE of λ^r is given as

$$u(t) = \frac{t^{-r}\Gamma(m)}{\Gamma(m-r)}; \quad m > r \tag{2.2.46}$$

Consider $r = 1$ then we will get UMVUE of λ,

$$\hat{\lambda} = \frac{m-1}{T} \tag{2.2.47}$$

(ii) Let $Y = -\log X$ then $f(y|\lambda) = \lambda e^{-\lambda y}; \quad y > 0, \quad \lambda > 0$
We will get the UMVUE of λ^r in (2.2.46) and for $r = 1$, UMVUE of λ is given in (2.2.47)
(iii) Let $|x| = y$ then $f(y|\lambda) = \lambda e^{-\lambda y}; \quad y > 0, \quad \lambda > 0$
In the same way as (i) and (ii) we can obtain the UMVUE of λ^{-r}.
(iv) Let $x^\alpha = y$ then $f(y|\lambda) = \frac{1}{\lambda}e^{-\frac{y}{\lambda}}; \quad y > 0, \quad \lambda > 0$
In the same way as (i) and (ii), we can obtain the UMVUE of λ^r (here $\theta = \lambda$).
(v) Let $\frac{x^2}{2} = y$ then $f(y|\lambda) = \frac{e^{-\frac{y}{\lambda}}y^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\lambda^{\frac{1}{2}}}; \quad y > 0, \quad \lambda > 0$
In this case $p = \frac{1}{2}$ and $\theta = \lambda$.
Similarly, we can obtain the UMVUE of λ^r.

Example 2.2.11 Let $X_1, X_2, \ldots, X_m$ be iid rvs with $N(\mu, \sigma^2)$. We will consider three cases

$$(i)\ \mu \text{ known, } \sigma^2 \text{ unknown}$$

$$(ii)\ \mu \text{ unknown, } \sigma^2 \text{ known}$$

$$(iii)\ \mu \text{ and } \sigma^2 \text{ both unknown}$$

(i) Normal distribution belongs to exponential family.
$T = \sum_{i=1}^{m}(X_i - \mu)^2$ is complete and sufficient for σ^2.

$$\frac{\sum_{i=1}^{m}(X_i - \mu)^2}{\sigma^2} \text{ has } \chi^2 \text{ with } m\ df \tag{2.2.48}$$

Hence, $\mathrm{E}\frac{T}{\sigma^2} = m$. This implies that UMVUE of σ^2 is $\hat{\sigma^2} = \frac{\sum(X_i-\mu)^2}{m}$
Let $\sigma^2 = \theta$ and $Y = \frac{T}{\theta}$
Then

$$f(y) = \frac{e^{-\frac{y}{2}}y^{\frac{m}{2}-1}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}; y > 0$$

To find the unbiased estimator of θ^r. Let $u(y)$ be an unbiased estimator of θ^r.

$$\int_0^\infty u(y)\frac{e^{-\frac{y}{2}}y^{\frac{m}{2}-1}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}dy=\theta^r=\frac{t^r}{y^r}$$

$$\int_0^\infty \frac{u(y)}{t^r}\frac{e^{-\frac{y}{2}}y^{\frac{m}{2}+r-1}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}dy=1$$

$$\int_0^\infty \left(\frac{u(y)}{t^r}\frac{\Gamma\left(\frac{m}{2}+r\right)2^{\frac{m}{2}+r}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}\right)\frac{e^{-\frac{y}{2}}y^{\frac{m}{2}+r-1}}{2^{\frac{m}{2}+r}\Gamma\left(\frac{m}{2}+r\right)}dy=1$$

Now

$$\left(\frac{u(y)}{t^r}\frac{\Gamma\left(\frac{m}{2}+r\right)2^{\frac{m}{2}+r}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}\right)=1$$

$$u(y)=\frac{t^r\Gamma\left(\frac{m}{2}\right)}{2^r\Gamma\left(\frac{m}{2}+r\right)};\quad r=1,2,\ldots \tag{2.2.49}$$

Particular cases: $r=1$

$$u(y)=\frac{t\Gamma\left(\frac{m}{2}\right)}{2\Gamma\left(\frac{m}{2}+1\right)}=\frac{t}{(2)\left(\frac{m}{2}\right)}=\frac{t}{m}$$

$$=\frac{\sum(X_i-\mu)^2}{m} \tag{2.2.50}$$

Therefore, $\frac{\sum(X_i-\mu)^2}{m}$ is the UMVUE of σ^2.
Next, we will find the UMVUE of $P[X_1\ge k]$

$$\begin{aligned}P[X_1\ge k]&=P\left[\frac{X_1-\mu}{\sigma}\ge\frac{k-\mu}{\sigma}\right]\\&=1-P\left[\frac{X_1-\mu}{\sigma}<\frac{k-\mu}{\sigma}\right]\\&=1-\Phi\left[\frac{k-\mu}{\sigma}\right]\end{aligned} \tag{2.2.51}$$

Define

$$Y_1=\begin{cases}1\ ;\ X_1\ge k\\0\ ;\ \text{otherwise}\end{cases}$$

$$\mathrm{E}Y_1=P[X_1\ge k]$$

According to Rao–Blackwell theorem, we have to find $P[X_1 \geq k|T = t]$.
For this we will have to find the distribution of X_1 given $T = t$. Then it is necessary to find the joint distribution of X_1 and $T = t$
Let $T = (X_1 - \mu)^2 + Z$ and $\frac{T}{\sigma^2}$ has χ^2_m. So $Z = T - (X_1 - \mu)^2$ then $\frac{Z}{\sigma^2}$ has χ^2_{m-1}. Let $y = \frac{z}{\sigma^2}$. Then

$$f(y) = \frac{e^{-\frac{y}{2}} y^{\frac{m-1}{2}-1}}{2^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right)}$$

$$f(z) = \frac{e^{-\frac{z}{2\sigma^2}} z^{\frac{m-1}{2}-1}}{2^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right) \sigma^{m-1}}; \quad z > 0 \tag{2.2.52}$$

$$\begin{aligned} f(x_1, t) &= f(x_1) f(z) \\ &= \frac{e^{-\frac{(x_1-\mu)^2}{2\sigma^2}} e^{-\frac{[z-(x_1-\mu)^2]}{2\sigma^2}} [t - (x_1 - \mu)^2]^{\frac{m-1}{2}-1}}{(\sigma\sqrt{2\pi}) 2^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right) \sigma^{m-1}} \\ &= \frac{e^{-\frac{t}{2\sigma^2}} \left[t - (x_1 - \mu)^2\right]^{\frac{m-1}{2}-1}}{2^{\frac{m}{2}} \Gamma\left(\frac{m-1}{2}\right) \sigma^m \sqrt{\pi}} \\ f(t) &= \frac{e^{-\frac{t}{2\sigma^2}} t^{\frac{m}{2}-1}}{2^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right) \sigma^m} \end{aligned} \tag{2.2.53}$$

$$f(x_1|T = t) = \begin{cases} \frac{\Gamma\left(\frac{m}{2}\right)\left[t-(x_1-\mu)^2\right]^{\frac{m-1}{2}-1}}{\Gamma\left(\frac{1}{2}\right) t^{\frac{m}{2}-1} \Gamma\left(\frac{m-1}{2}\right)} & ; \ \mu - \sqrt{t} < x_1 < \mu + \sqrt{t} \\ 0 & ; \text{otherwise} \end{cases} \tag{2.2.54}$$

Note that $\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$
Consider

$$\frac{\left[t - (x_1 - \mu)^2\right]^{\frac{m-1}{2}-1}}{t^{\frac{m}{2}-1}}$$

$$= \frac{t^{\frac{m-1}{2}-1} \left[1 - \left(\frac{x_1-\mu}{\sqrt{t}}\right)^2\right]^{\frac{m-1}{2}-1}}{t^{\frac{m}{2}-1}}$$

$$= t^{-\frac{1}{2}} \left[1 - \left(\frac{x_1 - \mu}{\sqrt{t}}\right)^2\right]^{\frac{m-1}{2}-1}$$

$$P[X_1 \geq k|T = t] = \int_{k}^{\mu+\sqrt{t}} \frac{t^{-\frac{1}{2}}\left[1 - \left(\frac{x_1-\mu}{\sqrt{t}}\right)^2\right]^{\frac{m-1}{2}-1}}{\beta\left(\frac{1}{2}, \frac{m-1}{2}\right)} dx_1$$

Let $\left(\frac{x_1-\mu}{\sqrt{t}}\right)^2 = v \Rightarrow \frac{2}{\sqrt{t}}\left(\frac{x_1-\mu}{\sqrt{t}}\right) dx_1 = dv$
$\Rightarrow dx_1 = \frac{\sqrt{t}}{2} v^{-\frac{1}{2}} dv$
when $X_1 = k \Rightarrow v = \left(\frac{k-\mu}{\sqrt{t}}\right)^2$ and $X_1 = \mu + \sqrt{t} \Rightarrow v = 1$

$$= \frac{1}{2} \int_{\left(\frac{k-\mu}{\sqrt{t}}\right)^2}^{1} \frac{v^{-\frac{1}{2}}(1-v)^{\frac{m-1}{2}-1}}{\beta\left(\frac{1}{2}, \frac{m-1}{2}\right)} dv$$

Hence

$$2P[X_1 \geq k|T = t] = 1 - \int_{0}^{\left(\frac{k-\mu}{\sqrt{t}}\right)^2} \frac{v^{-\frac{1}{2}}(1-v)^{\frac{m-1}{2}-1}}{\beta\left(\frac{1}{2}, \frac{m-1}{2}\right)} dv$$

$$P[X_1 \geq k|T = t] = \frac{1}{2} - \frac{1}{2}\mathbf{I}_{\left(\frac{k-\mu}{\sqrt{t}}\right)^2}\left(\frac{1}{2}, \frac{m-1}{2}\right)$$

UMVUE of $P[X_1 \geq k] = 1 - \Phi\left(\frac{k-\mu}{\sigma}\right)$ is $P[X_1 \geq k|T = t]$

$$P[X_1 \geq k|T = t] = \begin{cases} \frac{1}{2} - \frac{1}{2}\mathbf{I}_{\left(\frac{k-\mu}{\sqrt{t}}\right)^2}\left(\frac{1}{2}, \frac{m-1}{2}\right) & ; \ \mu - \sqrt{t} < k < \mu + \sqrt{t} \\ 1 & ; \ k < \mu - \sqrt{t} \\ 0 & ; \ k > \mu + \sqrt{t} \end{cases} \tag{2.2.55}$$

(ii) For σ known, $\sum X_i$ or $\bar{X}$ is complete and sufficient for μ. The distribution of $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{m}\right)$.
Now, $E\bar{X} = \mu$ and $E\bar{X}^2 = \mu^2 + \frac{\sigma^2}{n}$

$$E\left(\bar{X}^2 - \frac{\sigma^2}{m}\right) = \mu^2$$

Hence,

$$\left(\bar{X}^2 - \frac{\sigma^2}{m}\right) \text{ is UMVUE for } \mu^2 \tag{2.2.56}$$

For (2.2.56), see Example 2.2.3.

$$\mathrm{E}(\bar{X}^r) = \int_{-\infty}^{\infty} \bar{x}^r \frac{\sqrt{m}}{\sigma\sqrt{2\pi}} e^{-\frac{m}{2\sigma^2}(\bar{x}-\mu)^2} d\bar{x}$$

Let $w = \frac{(\bar{x}-\mu)\sqrt{m}}{\sigma} \Rightarrow \bar{x} = \mu + \frac{w\sigma}{\sqrt{m}}$

$$= \int_{-\infty}^{\infty} \left(\mu + \frac{w\sigma}{\sqrt{m}}\right)^r \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$$

Since odd moments of w are 0

$$= \int_{-\infty}^{\infty} \left[\left(\frac{w\sigma}{\sqrt{m}}\right)^r + \binom{r}{1}\left(\frac{w\sigma}{\sqrt{m}}\right)^{r-1} \mu + \binom{r}{2}\left(\frac{w\sigma}{\sqrt{n}}\right)^{r-2} \mu^2 + \cdots + \mu^r\right] \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw$$

$$= \frac{\sigma^r}{m^{\frac{r}{2}}}\mu_r + \binom{r}{1}\frac{\sigma^{r-1}}{m^{\frac{r-1}{2}}}\mu_{r-1}\mu + \binom{r}{2}\frac{\sigma^{r-2}}{m^{\frac{r-2}{2}}}\mu_{r-2}\mu^2 \cdots + \mu^r$$

$$\mu_r = \begin{cases} 0 & ;\ r \text{ is odd} \\ (r-1)(r-3)\ldots 1 & ;\ r \text{ is even} \end{cases}$$

Particular cases: (a) $r = 3$ (b) $r = 4$
(a) $r = 3$

$$\mathrm{E}(\bar{X}^3) = \frac{\sigma^3}{m^{\frac{3}{2}}}\mu_3 + \binom{3}{1}\frac{\sigma^2\mu_2\mu}{m} + \binom{3}{2}\frac{\sigma\mu_1\mu^2}{m^{\frac{1}{2}}} + \mu^3$$

$$= 3\frac{\sigma^2\mu}{m} + \mu^3$$

$$\text{UMVUE of } \mu^3 = \bar{X}^3 - 3\frac{\sigma^2\mu}{m} \tag{2.2.57}$$

(b) $r = 4$

$$\mathrm{E}(\bar{X}^4) = \frac{\sigma^4}{m^2}\mu_4 + \binom{4}{1}\frac{\sigma^3}{m^{\frac{3}{2}}}\mu_3(\mu) + \binom{4}{2}\frac{\sigma^2}{m}\mu_2(\mu)^2 + \binom{4}{3}\frac{\sigma}{m^{\frac{1}{2}}}\mu_1(\mu)^3 + \mu^4$$

$$\mu_4 = (4-1)(4-3) = 3,\ \ \mu_3 = 0,\ \ \mu_2 = 1$$

$$\mathrm{E}(\bar{X}^4) = \frac{3\sigma^4}{m^2} + \frac{6\sigma^2}{m}(\mu)^2 + \mu^4$$

UMVUE of μ^4 is

$$\bar{X}^4 - \frac{3\sigma^4}{m^2} - 6\frac{\sigma^2(\mu)^2}{m}$$

$$= \bar{X}^4 - \frac{3\sigma^4}{m^2} - \frac{6\sigma^2}{m}\left(\bar{x}^2 - \frac{\sigma^2}{m}\right) \tag{2.2.58}$$

Similarly, we can find UMVUE of $\mu^r (r \geq 1)$
Next, find the UMVUE of $P[X_1 \geq k]$
Again define

$$Y_1 = \begin{cases} 1 \ ; \ X_1 \geq k \\ 0 \ ; \text{otherwise} \end{cases}$$

$$\mathrm{E}Y_1 = P[X_1 \geq k]$$

According to Rao–Blackwell theorem, we have to find $P[X_1 \geq k | T = t]$ where $T = \sum_{i=1}^{m} X_i$ and $T_1 = \sum_{i=2}^{m} X_i$. $T \sim N(m\mu, m\sigma^2)$ and $T_1 \sim N\left((m-1)\mu, (m-1)\sigma^2\right)$

$$f(x_1, t) = f(x_1)f(t_1)$$

$$f(x_1, t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_1-\mu)^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi(m-1)}} e^{-\frac{[t_1-(m-1)\mu]^2}{2(m-1)\sigma^2}}$$

$$f(t) = \frac{1}{\sigma\sqrt{2\pi m}} \exp\left[-\frac{1}{2m\sigma^2}(t-\mu)^2\right]$$

$$f(x_1|T = t) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{\frac{m-1}{m}}} e^{-\frac{m}{2(m-1)\sigma^2}(x_1 - \frac{t}{m})^2} \tag{2.2.59}$$

Therefore, $(X_1|T = t)$ has $N\left(\frac{t}{m}, \frac{(m-1)\sigma^2}{m}\right)$
To find $P[X_1 \geq k|T = t]$

$$= 1 - \Phi\left(\frac{k - \frac{t}{m}}{\sigma\sqrt{\frac{m-1}{m}}}\right)$$

$$= 1 - \Phi\left(\frac{k - \bar{x}}{\sigma\sqrt{\frac{m-1}{m}}}\right) \tag{2.2.60}$$

We conclude that $\Phi\left(\frac{k-\bar{x}}{\sigma\sqrt{\frac{m-1}{m}}}\right)$ is UMVUE of $\Phi\left(\frac{k-\mu}{\sigma}\right)$.

(iii) Both μ and σ are unknown
$(\bar{x}, S^2)$ is jointly sufficient and complete for (μ, σ^2) because normal distribution belongs to exponential family where $S^2 = \sum(x_i - \bar{x})^2$.
Now, $\frac{S^2}{\sigma^2}$ has χ^2 distribution with $m-1$ df.
Let $\frac{S^2}{\sigma^2} = y$ then $\mathrm{E}Y^r = \frac{\Gamma\left(\frac{m-1}{2}+r\right)}{\Gamma\left(\frac{m-1}{2}\right)} 2^r$
Hence

$$\mathrm{E}(S^2)^r = \frac{\Gamma\left(\frac{m-1}{2}+r\right)}{\Gamma\left(\frac{m-1}{2}\right)} (2\sigma^2)^r \tag{2.2.61}$$

Therefore, $\frac{\Gamma\left(\frac{m-1}{2}\right)S^2}{\Gamma\left(\frac{m-1}{2}+r\right)2^r}$ is UMVUE of σ^{2r}
Particular case: (a) $r = \frac{1}{2}$ (b)$r = 1$
(a)

$$\hat{\sigma} = \frac{\Gamma\left(\frac{m-1}{2}\right) S}{\Gamma\left(\frac{m-1}{2}+\frac{1}{2}\right) 2^{\frac{1}{2}}}$$

(b)

$$\hat{\sigma^2} = \frac{\Gamma\left(\frac{m-1}{2}\right) S^2}{\Gamma\left(\frac{m-1}{2}+\frac{1}{2}\right) 2} = \frac{S^2}{m-1}$$

$$\mathrm{E}(\bar{X}^2) = \mu^2 + \frac{\sigma^2}{m}$$

Then

$$\mathrm{E}\left[\bar{X}^2 - \frac{S^2}{m(m-1)}\right] = \mu^2$$

So that

$$\text{UMVUE of } \mu^2 \text{ is } \bar{X}^2 - \frac{S^2}{m(m-1)} \tag{2.2.62}$$

Next,

$$\mathrm{E}(\bar{X}^3) = \mu^3 + \frac{3\sigma^2}{n}\mu$$

$$\mathrm{E}\left[\bar{X}^3 - \frac{3\bar{x}S^2}{m(m-1)}\right] = \mu^3$$

$$[\bar{X}^3 - \frac{3\bar{x}S^2}{m(m-1)}] \text{ is UMVUE of } \mu^3 \tag{2.2.63}$$

Similarly, one can obtain UMVUE of $\mu^r (r \geq 1)$
(c) $r = -1$

$$\text{UMVUE of } \frac{1}{\sigma^2} = \widehat{\sigma^{-2}}$$

$$= \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m-1}{2}-1\right)} \frac{S^{-2}}{2^{-1}}$$

$$= \frac{m-3}{S^2}; \quad m > 3 \tag{2.2.64}$$

Next, we will find the UMVUE of $P[X_1 \geq k]$

$$P[X_1 \geq k] = 1 - \Phi\left(\frac{k-\mu}{\sigma}\right)$$

As usual

$$Y = \begin{cases} 1 \; ; \; X_1 \geq k \\ 0 \; ; \text{otherwise} \end{cases} \tag{2.2.65}$$

$$\mathrm{E}Y = P[X_1 \geq k] = 1 - \Phi\left(\frac{k-\mu}{\sigma}\right)$$

As we have done earlier,

$$\mathrm{E}(Y|\bar{X}, S^2) = P[X_1 \geq k|\bar{X}, S^2]$$

We need to find the distribution of $(X_1, \bar{X}, S^2)$.
Consider the following orthogonal transformation:

$$z_1 = \frac{1}{\sqrt{m}}(x_1 + x_2 + \cdots + x_m) = \sqrt{m}\bar{x}$$

$$z_2 = \left[\left(1 - \frac{1}{m}\right)x_1 - \frac{x_2}{m} - \cdots - \frac{x_m}{m}\right]\sqrt{\frac{m}{m-1}}$$

$$z_i = c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{im}x_m \;\; i = 3, 4, \ldots, m$$

where $\sum_{j=1}^{m} c_{ij} = 0, i = 3, 4, \ldots, m$ and $\sum_{j=1}^{m} {c_{jj}}^2 = 1$

$$z_1 \sim N(\sqrt{m}\mu, \sigma^2) \tag{2.2.66}$$

$$z_r \sim N(0, \sigma^2) \;\; r = 2, 3, \ldots, n$$

Let $Z = PX$, where P is an orthogonal matrix

$$Z'Z = X'P'PX = X'X$$

Hence,

$$\sum_{i=1}^{m} {z_i}^2 = \sum_{i=1}^{m} {x_i}^2 \tag{2.2.67}$$

$$\sum_{i=3}^{m} {z_i}^2 = \sum_{i=1}^{m} {x_i}^2 - {z_1}^2 - {z_2}^2$$

$$= \sum_{i=1}^{m} {x_i}^2 - m\bar{x}^2 - {z_2}^2 = S^2 - {z_2}^2$$

$$\text{Let } \ v = S^2 - {z_2}^2,$$

where $v = \sum_{i=3}^{m} {z_i}^2$
Let $z_1 = \sqrt{m}\bar{x}$, $z_2 = \sqrt{\frac{m}{m-1}}(x_1 - \bar{x})$, $v = S^2 - {z_2}^2$

$$J = \frac{\partial(z_1, z_2, v)}{\partial(x_1, \bar{x}, S^2)}$$

$$= \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial \bar{x}} & \frac{\partial z_1}{\partial S^2} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial \bar{x}} & \frac{\partial z_2}{\partial S^2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial \bar{x}} & \frac{\partial v}{\partial S^2} \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & \sqrt{m} & 0 \\ \sqrt{\frac{m}{m-1}} & -\sqrt{\frac{m}{m-1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\frac{m}{\sqrt{m-1}}$$

Therefore,

$$|J| = \frac{m}{\sqrt{m-1}}$$

$$f(z_1, z_2, v) = \frac{e^{-\frac{(z_1 - \sqrt{m}\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \frac{e^{-\frac{(z_2)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \frac{e^{-\frac{v}{2\sigma^2}} v^{\frac{m-2}{2}-1}}{\Gamma\left(\frac{m-2}{2}\right) 2^{\frac{m-2}{2}} \sigma^{m-2}} |J| \tag{2.2.68}$$

Note that $\frac{v}{\sigma^2} \sim \chi^2_{m-3}$

$$f(x_1|\bar{x}, S^2) = \frac{f(z_1, z_2, v)}{f(\bar{x}, S^2)}$$

$$= |J| \frac{\exp\left[\frac{(z_1-\sqrt{m}\mu)^2}{2\sigma^2} - \frac{z_2^2}{2\sigma^2} - \frac{v}{2\sigma^2}\right]}{\exp\left[-\frac{m}{2\sigma^2}(\bar{x}-\mu)^2 - \frac{s^2}{2\sigma^2}\right]} \frac{v^{\frac{m-2}{2}-1} \frac{\sigma}{\sqrt{n}} \sqrt{2\pi} 2^{\frac{m-1}{2}} \sigma^{m-1} \Gamma\left(\frac{m-1}{2}\right)}{\sigma^m (2\pi) \Gamma\left(\frac{m-2}{2}\right) 2^{\frac{m-2}{2}} (s^2)^{\frac{m-1}{2}-1}}$$

Consider

$$\exp\left[-\frac{m}{2\sigma^2}(\bar{x}-\mu)^2 - \frac{m}{m-1}\frac{(x_1-\bar{x})^2}{2\sigma^2} - \frac{S^2}{2\sigma^2} + \frac{m}{m-1}\frac{(x_1-\bar{x})^2}{2\sigma^2} + \frac{m}{2\sigma^2}(\bar{x}-\mu)^2 + \frac{S^2}{2\sigma^2}\right] = 1$$

$$f(x_1|\bar{x}, S^2) = \frac{m}{\sqrt{m-1}} \frac{2^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right)}{\sqrt{m}\sqrt{2\pi}} \frac{v^{\frac{m-2}{2}-1}}{\Gamma\left(\frac{m-2}{2}\right) 2^{\frac{m-1}{2}} (S^2)^{\frac{m-1}{2}-1}}$$

$$= \frac{m 2^{\frac{1}{2}}}{2^{\frac{1}{2}} \sqrt{m-1} \sqrt{\pi}} \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m-2}{2}\right)} \frac{[S^2 - \frac{m}{m-1}(x_1-\bar{x})^2]^{\frac{m-2}{2}-1}}{(S^2)^{\frac{m-1}{2}-1}\sqrt{m}}$$

$$= \frac{m}{\sqrt{m-1}} \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m-2}{2}\right)} \frac{[S^2 - \frac{m}{m-1}(x_1-\bar{x})^2]^{\frac{m-2}{2}-1}}{(S^2)^{\frac{m-1}{2}-1}}$$

$$= \frac{\sqrt{m}}{\sqrt{m-1}} \frac{[S^2 - \frac{m}{m-1}(x_1-\bar{x})^2]^{\frac{m-2}{2}-1}}{(S^2)^{\frac{m-1}{2}-1} \beta\left(\frac{1}{2}, \frac{m-2}{2}\right)} \tag{2.2.69}$$

$$= \sqrt{\frac{m}{m-1}} \frac{\sqrt{\frac{m-1}{2}}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m-2}{2}\right)} \frac{[S^2 - \frac{m}{m-1}(x_1-\bar{x})^2]^{\frac{m-2}{2}-1}}{(S^2)^{\frac{m-1}{2}-1}}$$

$$= \sqrt{\frac{m}{m-1}} \frac{1}{\beta\left(\frac{1}{2}, \frac{m-2}{2}\right)} \left(\frac{1}{S^2}\right)^{\frac{m-1}{2}-1} [S^2 - m(x_1-\bar{x})^2]^{\frac{m-2}{2}-1}$$

$$= \sqrt{\frac{m}{m-1}} \frac{1}{\beta\left(\frac{1}{2}, \frac{m-2}{2}\right)} (S^2)^{-\frac{1}{2}} \left[1 - \frac{m}{m-1}\left(\frac{x_1-\bar{x}}{S}\right)^2\right]^{\frac{m-2}{2}-1} \tag{2.2.70}$$

Now

$$S^2 > \frac{m}{m-1}(x_1-\bar{x})^2 \Rightarrow \frac{(m-1)S^2}{m} > (x_1-\bar{x})^2$$

This implies that $|x_1 - \bar{x}| < S\sqrt{\frac{m-1}{m}}$
Hence,

$$\bar{x} - S\sqrt{\frac{m-1}{m}} \le x_1 \le \bar{x} + S\sqrt{\frac{m-1}{m}} \tag{2.2.71}$$

$$P[X_1 \ge k|T = t] = \int_k^{\bar{x}+S\sqrt{\frac{m-1}{m}}} \frac{1}{\beta\left(\frac{1}{2}, \frac{m-2}{2}\right)}\sqrt{\frac{m}{m-1}}(S^2)^{-\frac{1}{2}}\left[1 - \frac{m}{m-1}\left(\frac{x_1 - \bar{x}}{S}\right)^2\right]^{\frac{m-2}{2}-1} dx_1$$

Let $\frac{m}{m-1}\frac{(x_1-\bar{x})^2}{S^2} = t$, $\frac{2m}{m-1}\frac{(x_1-\bar{x})}{S^2}dx_1 = dt$, and $dx_1 = \frac{m-1}{2m}\frac{S^2}{(x_1-\bar{x})}dt$

$$= \int_{\frac{m}{m-1}\left(\frac{k-\bar{x}}{S}\right)^2}^{1} \frac{1}{2\beta\left(\frac{1}{2}, \frac{m-2}{2}\right)}[1-t]^{\frac{m-2}{2}-1}t^{-\frac{1}{2}}dt \tag{2.2.72}$$

UMVUE of $P[X_1 \ge k]$ is

$$P[X_1 \ge k|\bar{x}, S^2] = \begin{cases} 0 & ;\ k > \bar{x} + S\sqrt{\frac{m-1}{m}} \\ \int_k^{\bar{x}+S\sqrt{\frac{m-1}{m}}} f(x_1|\bar{x}, S^2)dx_1 & ;\ \bar{x} - S\sqrt{\frac{m-1}{m}} \le x_1 \le \bar{x} + S\sqrt{\frac{m-1}{m}} \\ 1 & ;\ k > \bar{x} - S\sqrt{\frac{m-1}{m}} \end{cases} \tag{2.2.73}$$

Further, if $\bar{x} - S\sqrt{\frac{m-1}{m}} \le x_1 \le \bar{x} + S\sqrt{\frac{m-1}{m}}$

$$\int_k^{\bar{x}+S\sqrt{\frac{m-1}{m}}} f(x_1|\bar{x}, s^2)dx_1 = \frac{1}{2}\left[1 - \mathbf{I}_{\frac{m}{m-1}\left(\frac{k-\bar{x}}{s}\right)^2}\left(\frac{1}{2}, \frac{m-2}{2}\right)\right] \tag{2.2.74}$$

where $\mathbf{I}$ is an incomplete Beta distribution.

2.3 UMVUE in Nonexponential Families

This section is devoted to find UMVUE from right, left, and both truncation families. One can see Tate (1959), Guenther (1978), and Jadhav (1996).

Example 2.3.1 Let $X_1, X_2, \ldots, X_m$ be iid rvs from the following pdf:

$$f(x|\theta) = \begin{cases} Q_1(\theta)M_1(x) & ;\ a < x < \theta \\ 0 & ;\ \text{otherwise} \end{cases} \tag{2.3.1}$$

where $M_1(x)$ is nonnegative and absolutely continuous over (a, θ) and $Q_1(\theta) = \left[\int_a^\theta M_1(x)dx\right]^{-1}$, $Q_1(\theta)$ is differentiable everywhere.

The joint pdf of $X_1, X_2, \ldots, X_m$ is

$$f(x_1, x_2, \ldots, x_m|\theta) = [Q_1(\theta)]^m \prod_{i=1}^{m} M_1(x_i)\mathbf{I}(\theta - x_{(m)})\mathbf{I}(x_{(1)} - a)$$

where

$$\mathbf{I}(y) = \begin{cases} 1\ ;\ y > 0 \\ 0\ ;\ y \le 0 \end{cases}$$

By factorization theorem, $X_{(m)}$ is sufficient for θ. The distribution of $X_{(m)}$ is $w(x|\theta)$, where

$$w(x|\theta) = m[F(x)]^{m-1}f(x) \tag{2.3.2}$$

Now

$$\int_a^\theta Q_1(\theta)M_1(x)dx = 1$$

This implies

$$\int_a^\theta M_1(x)dx = \frac{1}{Q_1(\theta)}$$

Then

$$\int_a^x M_1(x)dx = \frac{1}{Q_1(x)} \tag{2.3.3}$$

This implies $F(x) = \frac{Q_1(\theta)}{Q_1(x)}$
From (2.3.2)

$$w(x|\theta) = \frac{m[Q_1(\theta)]^m M_1(x)}{[Q_1(x)]^{m-1}},\quad a < x < \theta \tag{2.3.4}$$

Let $h(x)$ be a function $X_{(m)}$. Now, we will show that $X_{(m)}$ is complete.

$$\mathrm{E}[h(x)] = \int_a^\theta h(x)\frac{[Q_1(\theta)]^m M_1(x)}{[Q_1(x)]^{m-1}}dx = 0 \tag{2.3.5}$$

Consider the following result
Let $f = f(x|\theta)$, $a = a(\theta)$, $b = b(\theta)$

$$\frac{d}{d\theta}\left[\int_a^b f dx\right] = \int_a^b \frac{df}{d\theta}dx + f(b|\theta)\frac{db}{d\theta} - f(a|\theta)\frac{da}{d\theta} \tag{2.3.6}$$

Now,

$$\int_a^\theta h(x)\frac{M_1(x)}{[Q_1(x)]^{m-1}}dx = 0 \tag{2.3.7}$$

Using (2.3.6),

$$\frac{dh(x)\frac{M_1(x)}{[Q_1(x)]^{m-1}}}{d\theta} = 0 \tag{2.3.8}$$

Differentiating (2.3.7) with respect to θ

$$\frac{h(\theta)M_1(\theta)}{[Q_1(\theta)]^{m-1}} = 0 \ \text{ and, } \ M_1(\theta) \text{ and } Q_1(\theta) \neq 0$$

Hence $h(\theta) = 0$ for $a < x < \theta$.
This implies $h(x) = 0$ for $a < x < \theta$.
We will find UMVUE of $g(\theta)$. Let $U(x)$ be an unbiased estimator of $g(\theta)$.

$$\int_a^\theta u(x)\frac{m[Q_1(\theta)]^m M_1(x)}{[Q_1(x)]^{m-1}}dx = g(\theta)$$

$$\int_a^\theta u(x)\frac{M_1(x)}{[Q_1(x)]^{m-1}}dx = \frac{g(\theta)}{m[Q_1(\theta)]^m} \tag{2.3.9}$$

Differentiating (2.3.9) with respect to θ

$$\frac{u(\theta)M_1(\theta)}{[Q_1(\theta)]^{m-1}} = \frac{1}{m}\left[\frac{g^{(1)}(\theta)}{[Q_1(\theta)]^m} + \frac{g(\theta)[Q_1^{(1)}(\theta)](-m)}{[Q_1(\theta)]^{m+1}}\right]$$

$$= \frac{1}{m}\left[\frac{g^{(1)}(\theta)}{[Q_1(\theta)]^m} - \frac{mg(\theta)Q_1^{(1)}(\theta)}{[Q_1(\theta)]^{m+1}}\right] \tag{2.3.10}$$

where $g^{(1)}(\theta)$ = First derivative of $g(\theta)$
$Q_1^{(1)}(\theta)$ = First derivative of $Q_1(\theta)$
Now

$$\int_a^\theta M_1(x)dx = \frac{1}{Q_1(\theta)} \tag{2.3.11}$$

Differentiating (2.3.11) with respect to θ

$$M_1(\theta) = -\frac{Q_1^{(1)}(\theta)}{Q_1^2(\theta)} \tag{2.3.12}$$

Substitute (2.3.12) in (2.3.10),

$$\frac{u(\theta)M_1(\theta)}{[Q_1(\theta)]^{m-1}} = \frac{1}{m}\left[\frac{g^{(1)}(\theta)}{[Q_1^m(\theta)]} + \frac{mg(\theta)M_1(\theta)}{[Q_1(\theta)]^{m-1}}\right]$$

$$u(\theta) = \frac{g^{(1)}(\theta)}{m[Q_1^m(\theta)]}\frac{[Q_1(\theta)]^{m-1}}{M_1(\theta)} + \frac{g(\theta)M_1(\theta)}{[Q_1(\theta)]^{m-1}}\frac{[Q_1(\theta)]^{m-1}}{M_1(\theta)}$$

$$= \frac{g^{(1)}(\theta)}{mQ_1(\theta)M_1(\theta)} + g(\theta) \quad \forall \ \theta$$

Therefore,

$$u(x_{(m)}) = \frac{g^{(1)}(x_{(m)})}{mQ_1(x_{(m)})M_1(x_{(m)})} + g(x_{(m)}) \tag{2.3.13}$$

We can conclude that $U(x_{(m)})$ is UMVUE of $g(\theta)$.
Particular cases:
(a)

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} \ ; \ 0 < x < \theta \\ 0 \ ; \text{otherwise} \end{cases} \tag{2.3.14}$$

Comparing (2.3.14) with (2.3.1), $Q_1(\theta) = \frac{1}{\theta}$ and $M_1(x) = 1$
In this case we will find UMVUE of $\theta^r (r > 0)$.
Then $g(\theta) = \theta^r$. Using (2.3.13), $g(x_{(m)}) = [x_{(m)}]^r$, $g^{(1)}(x_{(m)}) = r[x_{(m)}]^{r-1}$, $Q_1(x_{(m)}) = \frac{1}{x_{(m)}}$, $M_1(x_{(m)}) = 1$

$$u(x_{(m)}) = \frac{r(x_{(m)})^{r-1}}{m\frac{1}{x_{(m)}}(1)} + (x_{(m)})^r$$

$$= x_{(m)}^r\left[\frac{r}{m} + 1\right] \tag{2.3.15}$$

If $r = 1$, then

$$u(x_{(m)}) = \frac{m+1}{m} x_{(m)} \tag{2.3.16}$$

is UMVUE of θ.

(b)

$$f(x|\theta) = \begin{cases} \frac{a\theta}{(\theta-a)x^2} & ; \ a < x < \theta \\ 0 & ; \text{otherwise} \end{cases} \tag{2.3.17}$$

In comparing (2.3.17) with (2.3.1), $Q_1(\theta) = \frac{a\theta}{(\theta-a)}$ and $M_1(x) = \frac{1}{x^2}$

Let $g(\theta) = \theta^r \ \ (r > 0), \ \ g^{(1)}(\theta) = r\theta^{r-1}$

Using (2.3.13),

$$u(x_{(m)}) = \frac{r x_{(m)}^{r-1}}{m\left(\frac{a x_{(m)}}{x_{(m)}-a}\right)\left(\frac{1}{x_{(m)}^2}\right)} + x_{(m)}^r \tag{2.3.18}$$

$$= x_{(m)}^r \left[\frac{r(x_{(m)} - a)}{am} + 1\right] \tag{2.3.19}$$

Put $r = 1$ in (2.3.19)

$$u(x_{(m)}) = x_{(m)} \left[\frac{x_{(m)} - a}{am} + 1\right] \tag{2.3.20}$$

is UMVUE of θ

(c)

$$f(x|\theta) = \frac{3x^2}{\theta^3}; \quad 0 < x < \theta \tag{2.3.21}$$

In this case $M_1(x) = 3x^2$, $Q_1(\theta) = \frac{1}{\theta^3}$, $g(\theta) = \theta^r$

$$u(x_{(m)}) = \frac{r x_{(m)}^{r-1}}{m \frac{1}{x_{(m)}^3} 3x_{(m)}^2} + x_{(m)}^r$$

$$= x_{(m)}^r \left[\frac{r + 3m}{3m}\right] \tag{2.3.22}$$

Put $r = 1$ in (2.3.22) then $U(x_{(m)}) = x_{(m)} \left(\frac{3m+1}{3m}\right)$ is UMVUE of θ.

(d)

$$f(x|\theta) = \frac{1}{\theta}; \quad -\theta < x < 0 \tag{2.3.23}$$

Let

$$Y_i = |X_i|, i = 1, 2, \ldots, m \tag{2.3.24}$$

then $Y_1, Y_2, \ldots, Y_m$ are iid rvs with $\cup(0, \theta)$.
From (2.3.15), UMVUE of θ^r is $u(y_{(m)})$, hence

$$u(y_{(m)}) = y_{(m)}^r \left[\frac{r}{m} + 1\right] \tag{2.3.25}$$

Example 2.3.2 Let $X_1, X_2, \ ldots, X_m$ be iid rvs from the following pdf:

$$f(x|\theta) = \begin{cases} Q_2(\theta)M_2(x) & ; \theta < x < b \\ 0 & ; \text{otherwise} \end{cases} \tag{2.3.26}$$

where $M_2(x)$ is nonnegative and absolutely continuous over (θ, b) and $Q_2(\theta) = \left[\int_\theta^b M_2(x)dx\right]^{-1}$, $Q_2(\theta)$ is differentiable everywhere.

The joint pdf of $X_1, X_2, \ldots, X_m$ is

$$f(x_1, x_2, \ldots, x_m|\theta) = [Q_2(\theta)]^m \prod_{i=1}^{m} M_2(x_i)\mathbf{I}(\theta - x_{(1)})\mathbf{I}(x_{(m)} - b)$$

By factorization theorem, $X_{(1)}$ is sufficient for θ. The distribution of $X_{(1)}$ is $w(x|\theta)$, where

$$w(x|\theta) = m[1 - F(x)]^{m-1}f(x) \tag{2.3.27}$$

Now

$$\int_\theta^b M_2(x)dx = \frac{1}{Q_2(\theta)}$$

This implies then

$$\int_x^b M_2(x)dx = \frac{1}{Q_2(x)} \tag{2.3.28}$$

$$1 - F(x) = P[x \geq x] = \int_x^b Q_2(\theta)M_2(x)dx$$

$$= \frac{Q_2(\theta)}{Q_2(x)} \tag{2.3.29}$$

$$w(x|\theta) = m\frac{[Q_2(\theta)]^m M_2(x)}{[Q_2(x)]^{m-1}}, \theta < x < b \tag{2.3.30}$$

Using (2.3.6), we can get

$$\frac{h(\theta)M_2(\theta)}{[Q_2(\theta)]^{m-1}} = 0 \text{ and } M_2(\theta), Q_2(\theta) \neq 0$$

Hence $h(\theta) = 0$ for $\theta < x < b$
This implies $h(x) = 0$ for $\theta < x < b$
We conclude that $X_{(1)}$ is complete.
Let $U(x)$ be an unbiased estimator of $g(\theta)$.

$$\int_\theta^b u(x)\frac{m[Q_2(\theta)]^m M_2(x)}{[Q_2(x)]^{m-1}}dx = g(\theta)$$

Using (2.3.6)

$$-\frac{u(\theta)M_2(\theta)}{[Q_2(\theta)]^{m-1}} = \frac{1}{m}\left[\frac{g^{(1)}(\theta)}{[Q_2(\theta)]^m} - \frac{mg(\theta)[Q_2^{(1)}(\theta)]}{[Q_2(\theta)]^{m+1}}\right] \tag{2.3.31}$$

Now,

$$\int_\theta^b M_2(x)dx = \frac{1}{Q_2(\theta)} \tag{2.3.32}$$

Differentiating (2.3.32) with respect to θ

$$M_2(\theta) = \frac{Q_2^{(1)}(\theta)}{Q_2(\theta)} \tag{2.3.33}$$

Substituting (2.3.33) into (2.3.31)

$$u(\theta) = g(\theta) - \frac{1}{m}\frac{g^{(1)}(\theta)}{Q_2(\theta)M_2(\theta)}$$

Hence

$$u(x_{(1)}) = g(x_{(1)}) - \frac{1}{m}\frac{g^{(1)}(x_{(1)})}{Q_2(x_{(1)})M_2(x_{(1)})} \tag{2.3.34}$$

Particular cases:
(a)

$$f(x|\theta) = \frac{\left(\frac{1}{\theta}\right)\left(\frac{\theta}{x}\right)^2}{1-\frac{\theta}{b}}; \quad \theta < x < b \tag{2.3.35}$$

Here $Q_2(\theta) = \frac{\theta}{1-\frac{\theta}{b}}$ and $M_2(x) = x^{-2}$
We wish to find UMVUE of $g(\theta) = \theta^r$ using (2.3.31),

$$u(x_{(1)}) = x_{(1)}^r - \frac{1}{m}\frac{r x_{(1)}^{r-1}}{\left(\frac{x_{(1)}}{1-\frac{x_{(1)}}{b}} x_{(1)}^{-2}\right)}$$

$$= x_{(1)}^r \left[1 - \frac{1}{m}\frac{r(b - x_{(1)})}{b}\right]$$

For $r = 1$

$$u(x_{(1)}) = x_{(1)}\left[1 - \frac{b - x_{(1)}}{mb}\right] \tag{2.3.36}$$

(b)

$$f(x, \theta) = \begin{cases} \frac{e^{-x}}{e^{-\theta}-e^{-b}} & ; \ \theta < x < b \\ 0 & ; \text{otherwise} \end{cases} \tag{2.3.37}$$

Comparing (2.3.37) and (2.3.23)
$Q_2(\theta) = (e^{-\theta} - e^{-b})^{-1}$ and $M_2(x) = e^{-x}$
To find UMVUE of $g(\theta) = \theta^r$ using (2.3.31),

$$u(x_{(1)}) = x_{(1)}^r - \frac{1}{m}\frac{r x_{(1)}^{r-1}(e^{-x_{(1)}} - e^{-b})}{e^{-x_{(1)}}}$$

Put $r = 1$, then UMVUE of θ

$$u(x_{(1)}) = x_{(1)} - \frac{1}{m}e^{x_{(1)}}(e^{-x_{(1)}} - e^{-b}) \tag{2.3.38}$$

In the following example, we will find UMVUE from two-point truncation parameter families. This technique was introduced by Hogg and Craig (1972) and developed by Karakostas (1985).

Example 2.3.3 Let $X_1, X_2, \ldots, X_m$ be iid rvs from the following pdf:

$$f(x|\theta_1, \theta_2) = \begin{cases} Q(\theta_1, \theta_2)M(x) & ; \theta_1 < x < \theta_2 \\ 0 & ; \text{otherwise} \end{cases} \tag{2.3.39}$$

where $M(x)$ is an absolutely continuous function and $Q(\theta_1, \theta_2)$ is differentiable everywhere.

The joint pdf of $X_1, X_2, \ldots, X_m$ is

$$f(x_1, x_2, \ldots, x_m|\theta_1, \theta_2) = [Q(\theta_1, \theta_2)]^m \prod_{i=1}^{m} M(x_i)\mathbf{I}(x_{(1)} - \theta_1)\mathbf{I}(\theta_2 - x_{(m)}) \quad (2.3.40)$$

By factorization theorem, $(x_{(1)}, x_{(m)})$ is jointly sufficient for (θ_1, θ_2). Suppose we are looking for UMVUE of $g(\theta_1, \theta_2)$ is such that $\frac{dg(x_{(1)},x_{(m)})}{dx_{(1)}}$ and $\frac{dg(x_{(1)},x_{(m)})}{dx_{(m)}}$ both exists.
The joint pdf of $(x_{(1)}, x_{(m)})$ is

$$f_{(x_{(1)},x_{(m)})}(x, y) = \begin{cases} m(m-1)[F(y) - F(x)]^{m-2}f(x)f(y) & ;\ \theta_1 < x < y < \theta_2 \\ 0 & ;\ \text{otherwise} \end{cases} \quad (2.3.41)$$

Now,

$$\int_{\theta_1}^{\theta_2} M(x)dx = \frac{1}{Q(\theta_1, \theta_2)} \quad (2.3.42)$$

Hence

$$\int_{x}^{y} M(t)dt = \frac{1}{Q(x, y)} \quad (2.3.43)$$

$$F(y) - F(x) = \int_{x}^{y} Q(\theta_1, \theta_2)M(t)dt$$

$$= \frac{Q(\theta_1, \theta_2)}{Q(x, y)} \quad (2.3.44)$$

$$f(x, y|\theta_1, \theta_2) = \begin{cases} m(m-1)\frac{[Q(\theta_1,\theta_2)]^m}{[Q(x,y)]^{m-2}}M(x)M(y) & ;\ \theta_1 < x < y < \theta_2 \\ 0 & ;\ \text{otherwise} \end{cases} \quad (2.3.45)$$

Assume that $\frac{df(x,y)}{dx}$ and $\frac{df(x,y)}{dy}$ both exists.
To prove the completeness of $f(x, y|\theta_1, \theta_2)$, let

$$R(y, \theta_1) = \int_{\theta_1}^{y} h(x, y)[Q(x, y)]^{-(m-2)}M(x)dx$$

where $h(x, y)$ is any continuous function of (x, y) and

$$R(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} M(y) R(y, \theta_1) dy$$

$$R(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \int_{\theta_1}^{y} h(x, y)[Q(x, y)]^{-(m-2)} M(x) M(y) dx dy = 0 \tag{2.3.46}$$

Hence to prove $h(x, y) = 0$, i.e., to prove $h(\theta_1, \theta_2) = 0$

$$\frac{\partial R(\theta_1, \theta_2)}{\partial \theta_1} = \int_{\theta_1}^{\theta_2} -h(\theta_1, y)[Q(\theta_1, y)]^{-(m-2)} M(\theta_1) M(y) dy \tag{2.3.47}$$

$$\frac{\partial^2 R(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} = -h(\theta_1, \theta_2)[Q(\theta_1, \theta_2)]^{-(m-2)} M(\theta_1) M(\theta_2) = 0 \tag{2.3.48}$$

which implies that $h(\theta_1, \theta_2) = 0$. Hence $h(x, y) = 0$.
Completeness of $f(x, y|\theta_1, \theta_2)$ implies that a UMVUE $u(x, y)$ for some function of θ's, $g(\theta_1, \theta_2)$, say, will be found by solving the integral equation.

$$g(\theta_1, \theta_2) = \mathrm{E}[u(x, y)]$$

That is,

$$g(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \int_{x}^{\theta_2} u(x, y) m(m-1) M(x) M(y) \frac{[Q(\theta_1, \theta_2)]^m}{[Q(x, y)]^{m-2}} dx dy$$

$$= [Q(\theta_1, \theta_2)]^m \int_{\theta_1}^{\theta_2} m(m-1) M(x) \left\{ \int_{x}^{\theta_2} \frac{u(x, y) M(y)}{[Q(x, y)]^{m-2}} dy \right\} dx \tag{2.3.49}$$

Now, we will have to find the solution of the integral equation (2.3.49).
Since

$$\frac{1}{Q(\theta_1, \theta_2)} = \int_{\theta_1}^{\theta_2} M(x) dx$$

$$-\frac{\frac{\partial Q(\theta_1, \theta_2)}{\partial \theta_1}}{[Q(\theta_1, \theta_2)]^2} = -M(\theta_1)$$

$$\frac{\partial Q(\theta_1,\theta_2)}{\partial\theta_1} = [Q(\theta_1,\theta_2)]^2 M(\theta_1) \tag{2.3.50}$$

Let

$$Q_1(\theta_1,\theta_2) = Q^2(\theta_1,\theta_2)M(\theta_1), \tag{2.3.51}$$

where $Q_1(\theta_1,\theta_2) = \frac{\partial Q(\theta_1,\theta_2)}{\partial\theta_1}$
Next,

$$\frac{-\frac{\partial Q(\theta_1,\theta_2)}{\partial\theta_2}}{Q^2(\theta_1,\theta_2)} = M(\theta_2)$$

$$-\frac{\partial Q(\theta_1,\theta_2)}{\partial\theta_2} = Q^2(\theta_1,\theta_2)M(\theta_2)$$

Let

$$Q_2(\theta_1,\theta_2) = -Q^2(\theta_1,\theta_2)M(\theta_2) \tag{2.3.52}$$

where $Q_2(\theta_1,\theta_2) = \frac{\partial Q(\theta_1,\theta_2)}{\partial\theta_2}$

$$\frac{\partial^2 Q(\theta_1,\theta_2)}{\partial\theta_1\theta_2} = Q_{12}(\theta_1,\theta_2) = -2Q^3(\theta_1,\theta_2)M(\theta_1)M(\theta_2) \tag{2.3.53}$$

Differentiating (2.3.49) with respect to θ_1,

$$\begin{aligned} g_1(\theta_1,\theta_2) &= [Q(\theta_1,\theta_2)]^m[-m(m-1)M(\theta_1)]\left\{\int_{\theta_1}^{\theta_2}\frac{u(\theta_1,y)M(y)}{[Q(\theta_1,y)]^{m-2}}dy\right\} \\ &\quad + mQ^{m-1}(\theta_1,\theta_2)Q_1(\theta_1,\theta_2)\int_{\theta_1}^{\theta_2} m(m-1)M(x)\left\{\int_x^{\theta_2}\frac{u(x,y)M(y)}{[Q(x,y)]^{m-2}}dy\right\}dx \end{aligned}$$

where $g_1(\theta_1,\theta_2) = \frac{\partial g}{\partial\theta_1}$
Using (2.3.51)

$$\begin{aligned} &= mQ^{m+1}(\theta_1,\theta_2)M(\theta_1)\int_{\theta_1}^{\theta_2} m(m-1)M(x)\left\{\int_x^{\theta_2}\frac{u(x,y)M(y)}{[Q(x,y)]^{m-2}}dy\right\}dx \\ &\quad - Q^m(\theta_1,\theta_2)[m(m-1)M(\theta_1)]\left\{\int_{\theta_1}^{\theta_2}\frac{u(\theta_1,y)M(y)}{[Q(\theta_1,y)]^{m-2}}dy\right\}, \end{aligned} \tag{2.3.54}$$

Using (2.3.49)

$$g_1(\theta_1,\theta_2) = mQ(\theta_1,\theta_2)M(\theta_1)g(\theta_1,\theta_2) - m(m-1)Q^m(\theta_1,\theta_2)M(\theta_1)\left[\int_{\theta_1}^{\theta_2} \frac{u(\theta_1,y)M(y)}{[Q(\theta_1,y)]^{m-2}}dy\right] \tag{2.3.55}$$

This equation can be written as

$$\int_{\theta_1}^{\theta_2} \frac{u(\theta_1,y)M(y)}{[Q(\theta_1,y)]^{m-2}}dy = \frac{g_1(\theta_1,\theta_2) - mQ(\theta_1,\theta_2)M(\theta_1)g(\theta_1,\theta_2)}{-m(m-1)[Q(\theta_1,\theta_2)]^m M(\theta_1)}$$

$$= \frac{g(\theta_1,\theta_2)}{(m-1)[Q(\theta_1,\theta_2)]^{m-1}} - \frac{g_1(\theta_1,\theta_2)}{m(m-1)M(\theta_1)[Q(\theta_1,\theta_2)]^m} \tag{2.3.56}$$

Differentiating with respect to θ_2,

$$\begin{aligned}\frac{u(\theta_1,\theta_2)M(\theta_2)}{[Q(\theta_1,\theta_2)]^{m-2}(\theta_1,\theta_2)} &= \frac{g(\theta_1,\theta_2)[-(m-1)]Q[(\theta_1,\theta_2)]^{-(m-1)-1}Q_2(\theta_1,\theta_2)}{m-1} \\ &+ \frac{g_2(\theta_1,\theta_2)}{(m-1)[Q(\theta_1,\theta_2)]^{m-1}} \\ &- \left[\frac{g_1(\theta_1,\theta_2)(-m)[Q(\theta_1,\theta_2)]^{-(m+1)}Q_2(\theta_1,\theta_2)}{m(m-1)M(\theta_1)}\right. \\ &\left. + \frac{g_{12}(\theta_1,\theta_2)}{m(m-1)[Q(\theta_1,\theta_2)]^m M(\theta_1)}\right]\end{aligned} \tag{2.3.57}$$

$$\begin{aligned}\frac{u(\theta_1,\theta_2)M(\theta_2)}{[Q(\theta_1,\theta_2)]^{m-2}} &= \frac{g(\theta_1,\theta_2)[-(m-1)]Q_2(\theta_1,\theta_2)}{(m-1)[Q(\theta_1,\theta_2)]^m} \\ &+ \frac{g_2(\theta_1,\theta_2)}{(m-1)[Q(\theta_1,\theta_2)]^{m-1}} \\ &- \left[\frac{g_1(\theta_1,\theta_2)(-m)Q_2(\theta_1,\theta_2)}{m(m-1)Q^{m+1}(\theta_1,\theta_2)M(\theta_1)}\right] \\ &- \frac{g_{12}(\theta_1,\theta_2)}{m(m-1)Q^m(\theta_1,\theta_2)M(\theta_1)}\end{aligned} \tag{2.3.58}$$

$$\begin{aligned}\frac{u(\theta_1,\theta_2)M(\theta_2)}{Q^{m-2}(\theta_1,\theta_2)} &= \frac{g(\theta_1,\theta_2)M(\theta_2)}{[Q(\theta_1,\theta_2)]^{m-2}}\\ &\quad - \frac{g_1(\theta_1,\theta_2)M(\theta_2)}{(m-1)[Q(\theta_1,\theta_2)]^{m-1}M(\theta_1)}\\ &\quad + \frac{g_2(\theta_1,\theta_2)}{(m-1)[Q(\theta_1,\theta_2)]^{m-1}}\\ &\quad - \frac{g_{12}(\theta_1,\theta_2)}{m(m-1)[Q(\theta_1,\theta_2)]^{m}M(\theta_1)}\end{aligned} \tag{2.3.59}$$

$$\begin{aligned}u(\theta_1,\theta_2) &= g(\theta_1,\theta_2) - \frac{g_1(\theta_1,\theta_2)}{(m-1)[Q(\theta_1,\theta_2)]^{m}M(\theta_1)}\\ &\quad + \frac{g_2(\theta_1,\theta_2)}{(m-1)M(\theta_2)Q(\theta_1,\theta_2)}\\ &\quad - \frac{g_{12}(\theta_1,\theta_2)}{m(m-1)M(\theta_1)M(\theta_2)[Q(\theta_1,\theta_2)]^2}\end{aligned} \tag{2.3.60}$$

Replacing θ_1 by $X_{(1)}$ and θ_2 by $X_{(m)}$,

$$\begin{aligned}u(X_{(1)},X_{(m)}) &= g(X_{(1)},X_{(m)}) - \frac{g_1(X_{(1)},X_{(m)})}{(m-1)Q(X_{(1)},X_{(m)})M(X_{(1)})}\\ &\quad + \frac{g_2(X_{(1)},X_{(m)})}{(m-1)M(X_{(m)})Q(X_{(1)},X_{(m)})}\\ &\quad - \frac{g_{12}(X_{(1)},X_{(m)})}{m(m-1)M(X_{(1)})M(X_{(m)})[Q(X_{(1)},X_{(m)})]^2}\end{aligned} \tag{2.3.61}$$

is UMVUE of $g(\theta_1,\theta_2)$.

Particular cases:

(a)

$$f(x|\theta_1,\theta_2) = \begin{cases}\frac{1}{\theta_2-\theta_1} & ;\ \theta_1 < x < \theta_2\\ 0 & ;\ \text{otherwise}\end{cases} \tag{2.3.62}$$

Comparing (2.3.62) and (2.3.39), $Q(\theta_1,\theta_2) = \frac{1}{\theta_2-\theta_1}$, $M(x) = 1$

To find UMVUE of (i) θ_1, (ii) θ_2, (iii) $\frac{\theta_1-\theta_2}{2}$ and (iv) $\frac{\theta_1+\theta_2}{2}$

(i) $g(\theta_1,\theta_2) = \theta_1$, $g(X_{(1)},X_{(m)}) = X_{(1)}$, $g_1(X_{(1)},X_{(m)}) = 1$,
$g_2(X_{(1)},X_{(m)}) = 0$ and $g_{12}(X_{(1)},X_{(m)}) = 0$
$M(X_{(1)}) = M(X_{(m)}) = 1$, $Q(X_{(1)},X_{(m)}) = \frac{1}{X_{(m)}-X_{(1)}}$. Using (2.3.61),

$$u(X_{(1)},X_{(m)}) = X_{(1)} - \frac{X_{(m)}-X_{(1)}}{(m-1)}$$

$$= \frac{mX_{(1)} - X_{(1)} - X_{(m)} + X_{(1)}}{(m-1)}$$

$$= \frac{mX_{(1)} - X_{(m)}}{(m-1)} \tag{2.3.63}$$

Hence, $\frac{mX_{(1)} - X_{(m)}}{(m-1)}$ is UMVUE of θ_1
(ii) $g(\theta_1, \theta_2) = \theta_2$, $g(X_{(1)}, X_{(m)}) = X_{(m)}$, $g_1(X_{(1)}, X_{(m)}) = 0$, $g_2(X_{(1)}, X_{(m)}) = 1$ and $g_{12}(X_{(1)}, X_{(m)}) = 0$
$M(X_{(1)}) = M(X_{(m)}) = 1$, $Q(X_{(1)}, X_{(m)}) = \frac{1}{X_{(m)} - X_{(1)}}$

$$u(X_{(1)}, X_{(m)}) = X_{(m)} + \frac{X_{(m)} - X_{(1)}}{(m-1)}$$

$$= \frac{mX_{(m)} - X_{(m)} + X_{(n)} - X_{(1)}}{(m-1)}$$

$$= \frac{mX_{(m)} - X_{(1)}}{(m-1)} \tag{2.3.64}$$

Hence, $\frac{mX_{(m)} - X_{(1)}}{(m-1)}$ is UMVUE of θ_2
(iii) UMVUE of $\frac{\theta_1 - \theta_2}{2}$

$$= \frac{mX_{(1)} - X_{(m)} - mX_{(m)} + X_{(1)}}{2(m-1)}$$
$$= \frac{(m+1)}{2(m-1)}[X_{(m)} - X_{(1)}] \tag{2.3.65}$$

(iv) UMVUE of $\frac{\theta_1 + \theta_2}{2}$

$$= \frac{1}{2}\left[\frac{mX_{(1)} - X_{(m)}}{(m-1)} + \frac{mX_{(m)} - X_{(1)}}{(m-1)}\right]$$
$$= \frac{1}{2(m-1)}\left[(m-1)X_{(1)} + (m-1)X_{(m)}\right]$$
$$= \frac{X_{(m)} + X_{(1)}}{2} \tag{2.3.66}$$

(b)

$$f(x, \theta_1, \theta_2) = \begin{cases} \frac{\theta_1\theta_2}{\theta_2 - \theta_1} x^{-2} & ; \theta_1 < x < \theta_2 \\ 0 & ; \text{otherwise} \end{cases} \tag{2.3.67}$$

Comparing (2.3.67) to (2.3.39)

$$Q(\theta_1, \theta_2) = \frac{\theta_1\theta_2}{\theta_2 - \theta_1}, M(x) = x^{-2}$$

To find UMVUE of $(\theta_1\theta_2)^m$

$$g(X_{(1)}, X_{(m)}) = [(X_{(1)}X_{(m)}]^m, g_1(X_{(1)}, X_{(m)}) = m[(X_{(1)}X_{(m)}]^{m-1}X_{(m)}$$

$$g_2(X_{(1)}, X_{(m)}) = m[(X_{(1)}X_{(m)}]^{m-1}X_{(1)},$$

$$g_{12}(X_{(1)}, X_{(m)}) = m(m-1)[X_{(1)}X_{(m)}]^{m-2}X_{(1)}X_{(m)} + m[X_{(1)}X_{(m)}]^{m-1}$$

$$M(X_{(1)}) = X_{(1)}^{-2}, M(X_{(m)}) = X_{(m)}^{-2}, Q(X_{(1)}, X_{(m)}) = \frac{X_{(1)}X_{(m)}}{X_{(m)} - X_{(1)}}$$

$$\begin{aligned}
U(X_{(1)}, X_{(m)}) = {} & (X_{(1)}X_{(m)})^m - \frac{m[X_{(1)}X_{(m)}]^{m-1}X_{(m)}[X_{(m)} - X_{(1)}]}{(m-1)X_{(1)}X_{(m)}X_{(1)}^{-2}} \\
& + \frac{m[X_{(1)}X_{(m)}]^{m-1}X_{(1)}[X_{(m)} - X_{(1)}]}{(m-1)X_{(1)}X_{(m)}X_{(m)}^{-2}} \\
& - \frac{m(m-1)[X_{(1)}X_{(m)}]^{m-2}X_{(1)}X_{(m)} + m[X_{(m)}X_{(1)}]^{m-1}}{m(m-1)X_{(1)}^{-2}X_{(m)}^{-2}X_{(1)}^2X_{(m)}^2}[X_{(m)} - X_{(1)}]^2
\end{aligned}$$

$$\begin{aligned}
&= (X_{(1)}X_{(m)})^m - \frac{m[X_{(m)} - X_{(1)}][X_{(1)}X_{(m)}]^{m-1}}{(m-1)X_{(1)}^{-1}} \\
&+ \frac{m[X_{(1)}X_{(m)}]^{m-1}[X_{(m)} - X_{(1)}]}{(m-1)X_{(m)}^{-1}} - \frac{[X_{(1)}X_{(m)}]^{m-1}[X_{(m)} - X_{(1)}]^2}{1} \\
&- \frac{m[X_{(1)}X_{(m)}]^{m-1}[X_{(m)} - X_{(1)}]^2}{m(m-1)X_{(1)}^{-2}X_{(m)}^{-2}X_{(1)}^2X_{(m)}^2}
\end{aligned}$$

$$\begin{aligned}
&= (X_{(1)}X_{(m)})^m - \frac{m}{m-1}X_{(1)}[X_{(m)} - X_{(1)}][X_{(1)}X_{(m)}]^{m-1} \\
&+ \frac{m}{m-1}X_{(m)}[X_{(m)} - X_{(1)}][X_{(1)}X_{(m)}]^{m-1} - [X_{(1)}X_{(m)}]^{m-1}[X_{(m)} - X_{(1)}]^2 \\
&- \frac{[X_{(1)}X_{(m)}]^{m-1}[X_{(m)} - X_{(1)}]^2}{(m-1)}
\end{aligned}$$

$$= (X_{(1)}X_{(m)})^m + \frac{m}{m-1}[X_{(m)} - X_{(1)}]^2[X_{(1)}X_{(m)}]^{m-1}$$

$$- [X_{(1)}X_{(m)}]^{m-1}[X_{(m)} - X_{(1)}]^2 - \frac{[X_{(1)}X_{(m)}]^{m-1}[X_{(m)} - X_{(1)}]^2}{m-1}$$

$$= (X_{(1)}X_{(m)})^m + [X_{(m)} - X_{(1)}]^2[X_{(1)}X_{(m)}]^{m-1}\left[\frac{m}{m-1} - 1 - \frac{1}{m-1}\right]$$

$$= (X_{(1)}X_{(m)})^m \tag{2.3.68}$$

Hence, $(X_{(1)}X_{(m)})^m$ is UMVUE of $(\theta_1\theta_2)^m$. One should note that MLE of $(\theta_1\theta_2)^m$ is again the same.
Stigler (1972) had obtained an UMVUE for an incomplete family.

Example 2.3.4 Consider the Example 1.5.5.
Further, consider a single observation $X \sim P_N$.

$$P[X = k] = \begin{cases} \frac{1}{N} \; ; \; k = 1, 2, \ldots, N \\ 0 \; ; \text{otherwise} \end{cases}$$

Now X is sufficient and complete.

$$\text{E}X = \frac{N+1}{2} \;\text{ and }\; \text{E}[2X - 1] = N$$

Then, $\Phi_1(X) = (2X - 1)$ is UMVUE of N.

$$\text{V}[\Phi_1(X)] = \frac{N^2 - 1}{3} \tag{2.3.69}$$

Now the family $\wp - P_n$ is not complete, see Example 1.5.5.
We will show that for this family the UMVUE of N is

$$\Phi_2(k) = \begin{cases} 2k - 1 \; ; \; k \neq n, k \neq n+1 \\ 2n \qquad ; \; k = n, n+1 \end{cases} \tag{2.3.70}$$

According to Theorem 2.2.3, we have to show that $\Phi_2(k)$ is UMVUE iff it is uncorrelated with all unbiased estimates of zero.
In Example 1.5.5, we have shown that $g(X)$ is an unbiased estimator of zero, where

$$g(x) = \begin{cases} 0 \;\; ; \;\; x = 1, 2, \ldots, n-1, n+2, n+3 \ldots \\ a \;\; ; \;\; x = n \\ -a \; ; \;\; x = n+1 \end{cases} \tag{2.3.71}$$

where a is nonzero constant.

Case (i) $N < n$

$$\mathrm{E}g(X) = \sum_{k=1}^{N} g(x)\frac{1}{N} = 0$$

Case (ii) $N > n$

$$\mathrm{E}g(X) = \sum_{k=1}^{N} \frac{1}{N} g(x)$$

$$= \frac{1}{N}\left[0 + \cdots + 0 + (-a) + (a) + 0\right] = 0$$

Case (iii) $N = n$

$$\mathrm{E}g(X) = \sum_{k=1}^{N} g(x)\frac{1}{N}$$

$$= \frac{1}{N}\left[0 + \cdots + 0 + (a)\right] = \frac{a}{N}$$

$$\mathrm{E}g(X) = \begin{cases} 0 \;;\; N = n \\ \frac{a}{N} \;;\quad N = n \end{cases}$$

Thus we see that $g(x)$ is an unbiased estimate of zero for the family $\wp - P_n$ and therefore the family is not complete.

Remark: Completeness is a property of a family of distribution rather than the random variable or the parametric form, that the statistical definition of "complete" is related to every day usage, and that removing even one point from a parameter set may alter the completeness of the family, see Stigler (1972).

Now, we know that the family $\wp - \{P_n\}$ is not complete. Hence $\Phi_1(X)$ is not UMVUE of N for the family $\wp - \{P_n\}$. For this family consider the UMVUE of N as $\Phi_2(X)$, where

$$\Phi_2(X) = \begin{cases} 2x - 1 \;;\; x \neq n, x \neq n+1 \\ 2n \qquad ;\; x = n, n+1 \end{cases} \tag{2.3.72}$$

According to Theorem 2.2.3, $\Phi_2(X)$ is UMVUE iff it is uncorrelated with all unbiased estimates of zero.

Already, we have shown that $g(x)$ is an unbiased estimator of zero for the family $\wp - \{P_n\}$.

Since $\mathrm{E}g(x) = 0$ for $N \neq n$

Now, we have to show that $\mathrm{Cov}[g(x), \Phi_2(X)] = 0$.

$$\operatorname{Cov}[g(x), \Phi_2(X)] = \mathrm{E}[g(x)\Phi_2(X)]$$

Case (i) $N > n$

$$\begin{aligned}\mathrm{E}[g(x)\Phi_2(X)] &= \frac{1}{N}\sum_{k=1}^{N} g(x)\Phi_2(k) \\ &= \frac{1}{N}\left[(0)(2k-1) + (a)(2n) + (-a)(2n)\right] = 0\end{aligned}$$

Case (ii) $N < n$

$$\mathrm{E}[g(x)\Phi_2(X)] = \frac{1}{N}\left[(0)(2k-1)\right] = 0$$

Thus, $\Phi_2(X)$ is UMVUE of N for the family $\wp - \{P_n\}$.
Note that $\mathrm{E}\Phi_2(X) = N$. We can compute the variance of $\Phi_2(X)$
Case (i) $N < n$

$$\begin{aligned}\mathrm{E}\Phi_2(x) &= \sum_{x=1}^{N}(2x-1)\frac{1}{N} \\ &= \frac{1}{N}\left[\frac{2N(N+1)}{2} - N\right] = N\end{aligned}$$

$$\begin{aligned}\mathrm{E}{\Phi_2}^2(x) &= \frac{1}{N}\sum_{x=1}^{N}(2x-1)^2\frac{1}{N} \\ &= \frac{1}{N}\left[\sum_{k=1}^{N}(4x^2 - 4x + 1)\right] \\ &= \frac{1}{N}\left[\frac{4N(N+1)(2N+1)}{6} - \frac{4N(N+1)}{2} + N\right] \\ &= \frac{2(N+1)(2N+1)}{3} - 2(N+1) + 1 \\ &= \frac{4N^2-1}{3}\end{aligned}$$

$$\begin{aligned}\operatorname{Var}[\Phi_2(X)] &= \frac{4N^2-1}{3} - N^2 \\ &= \frac{N^2-1}{3}\end{aligned}$$

Case (ii) $N > n$

$$
\begin{aligned}
E[\Phi_2(x)] &= \frac{1}{N}\left[\sum_{x=1}^{N}\Phi_2(x)\right] \\
&= \frac{1}{N}[\Phi_2(1)+\Phi_2(2)+\cdots+\Phi_2(n-1)+\Phi_2(n)+\Phi_2(n+1) \\
&+ \Phi_2(n+2)+\cdots+\Phi_2(N)] \\
&= \frac{1}{N}[1+3+\cdots+2n-3+2n+2n+2n+3+2n+5+\cdots+2N-1] \\
&= \frac{1}{N}[1+3+\cdots+2n-3+(2n-1+2n+1)+2n+3+\cdots+2N-1 \\
&+ 2n+2n-(2n-1+2n+1)] \\
&= \frac{1}{N}\left[\frac{N}{2}(1+2N-1)+0\right] = N
\end{aligned}
$$

$$
\begin{aligned}
E{\Phi_2}^2(x) &= \frac{1}{N}[{\Phi_2}^2(1)+{\Phi_2}^2(2)+\cdots+{\Phi_2}^2(n-1)+{\Phi_2}^2(n) \\
&+ {\Phi_2}^2(n+1)+{\Phi_2}^2(n+2)+\cdots+{\Phi_2}^2(N)] \\
&= \frac{1}{N}[1^2+3^2+5^2\cdots+(2n-3)^2+\{(2n-1)^2+(2n+1)^2\} \\
&+ (2n+3)^2+(2n+5)^2+\cdots+(2N-1)^2+(2n)^2+(2n)^2-\{(2n-1)^+(2n+1)^2\}] \\
&= \frac{1}{N}\left[\sum_{k=1}^{N}(2k-1)^2+4n^2+4n^2-4n^2+4n-1-4n^2-4n-1\right] \\
&= \frac{4N^2}{3}-\frac{1}{3}-\frac{2}{N}
\end{aligned}
$$

$$
\mathrm{Var}[\Phi_2(X)] = \frac{4N^2}{3}-\frac{1}{3}-\frac{2}{N}-N^2 = \frac{N^2-1}{3}-\frac{2}{N}
$$

$$
\mathrm{Var}[\Phi_2(X)] = \begin{cases} \frac{N^2-1}{3} & ;\ N < n \\ \frac{N^2-1}{3}-\frac{2}{N} & ;\ N > n \end{cases} \tag{2.3.73}
$$

Thus $\Phi_2(X)$ is UMVUE for $\wp - \{P_n\}$ but $\Phi_2(X)$ is not unbiased for the family $\wp$. Note that for $N = n$,

$$
\begin{aligned}
E[\Phi_2(X)] &= \frac{1}{n}\sum_{x=1}^{N}\Phi_2(X) \\
&= \frac{1}{n}[\Phi_2(1)+\cdots+\Phi_2(n-1)+\Phi_2(n)] \\
&= \frac{1}{n}[1+3+\cdots+2n-3+2n] \\
&= \frac{1}{n}\left[\sum_{x=1}^{N}(2x-1)^2+2n-(2n-1)\right] = \frac{n^2+1}{n}
\end{aligned} \tag{2.3.74}
$$

$$E[\Phi_2{}^2(X)] = \frac{1}{n}\left[\sum_{x=1}^{N}(2x-1)^2 + (2n)^2 - (2n-1)^2\right]$$
$$= \frac{4n^2-1}{3} + \frac{4n-1}{n}$$

$$\text{Var}[\Phi_2(X)] = \frac{4n^2-1}{3} + \frac{4n-1}{n} - \left(\frac{n^2+1}{n}\right)^2$$

Example 2.3.5 Let $X_1, X_2, \ldots, X_m$ be iid discrete rvs with following pmf $f(x|N)$. Find the UMVUE of $g(N)$.

$$f(x|N) = \begin{cases} \phi(N)M(x) \ ; \ a \le X \le N \\ 0 \qquad\qquad\quad ; \text{otherwise} \end{cases} \tag{2.3.75}$$

where $\sum_{x=a}^{N} M(x) = \frac{1}{\phi(N)}$.
According to Example 2.2.7, we can show that $X_{(m)}$ is sufficient and complete for N.

$$P[X_{(m)} \le z] = \left[\frac{\phi(N)}{\phi(z)}\right]^m$$

$$P[X_{(m)} \le z-1] = \left[\frac{\phi(N)}{\phi(z-1)}\right]^m$$

$$P[X_{(m)} = z] = \phi^m(N)[\phi^{-m}(z) - \phi^{-m}(z-1)]$$

Let $u(X_{(m)})$ is UMVUE of $g(N)$

$$\sum_{z=a}^{N} u(z)\phi^m(N)[\phi^{-m}(z) - \phi^{-m}(z-1)] = g(N)$$

$$\sum_{z=a}^{N} u(z)\frac{\phi^m(N)}{g(N)}[\phi^{-m}(z) - \phi^{-m}(z-1)] = 1$$

Let $\psi(N) = \frac{\phi(N)}{g^{\frac{1}{m}}(N)}$

$$\sum_{z=a}^{N} u(z)\psi^m(N)[\phi^{-m}(z) - \phi^{-m}(z-1)] = 1$$

$$\sum_{z=a}^{N} \frac{u(z)[\phi^{-m}(z) - \phi^{-m}(z-1)]}{[\psi^{-m}(z) - \psi^{-m}(z-1)]} \psi^m(N)[\psi^{-m}(z) - \psi^{-m}(z-1)] = 1,$$

Hence

$$\frac{u(z)[\phi^{-m}(z) - \phi^{-m}(z-1)]}{[\psi^{-m}(z) - \psi^{-m}(z-1)]} = 1,$$

This implies

$$u(z) = \frac{[\psi^{-m}(z) - \psi^{-m}(z-1)]}{[\phi^{-m}(z) - \phi^{-m}(z-1)]},$$

Therefore,

$$u(X_{(m)}) = \frac{[\psi^{-m}(X_{(m)}) - \psi^{-m}(X_{(m)} - 1)]}{[\phi^{-m}(X_{(m)}) - \phi^{-m}(X_{(m)} - 1)]},$$

We conclude that $U(X_{(m)})$ is UMVUE of $g(N)$.
Particular cases:
(a) $g(N) = N^s$, s is a real number.
According to (2.3.75), $\phi(N) = N^{-1}$, $M(x) = 1$,
$\psi(N) = N^{-\frac{(s+m)}{m}}$, $\psi(X_{(m)}) = X_{(m)}^{-\frac{(s+m)}{m}}$, $\phi(X_{(m)}) = X_{(m)}^{-1}$.

$$u(X_{(m)}) = \frac{X_{(m)}^{m+s} - (X_{(m)} - 1)^{m+s}}{X_{(m)}^m - (X_{(m)} - 1)^m},$$

which is same as (2.2.32).
(b) $g(N) = e^N$

$$\psi(N) = N^{-1} e^{-\frac{N}{m}} \Rightarrow \psi(X_{(m)}) = X_{(m)}^{-1} e^{-\frac{X_{(m)}}{m}}$$

Hence $u(X_{(m)})$ is UMVUE of e^N.
Hence,

$$u(X_{(m)}) = \frac{X_{(m)}^m e^{X_{(m)}} - (X_{(m)} - 1)^m e^{X_{(m)} - 1}}{X_{(m)}^m - (X_{(m)} - 1)^m},$$

Reader should show that the above UMVUE of e^N is same as in Example 2.2.7.
Now, we will consider some examples which can be solved using R software.

Example 2.3.6 2, 5, 7, 3, 4, 2, 5, 4 is a sample of size 8 drawn from binomial distribution B(10,p). Obtain UMVUE of p, p^2, p^2q, $p(x \leq 2)$, $p(x > 6)$.

```
a=function (r,s)
{
m<-8
n<-10
x<-c(2,5,7,3,4,2,5,4)
t<-sum(x)
umvue=(choose(m*n-r-s,t-r)/choose(m*n,t))
print(umvue)
}
a(1,0) #UMVUE of p
a(2,0) #UMVUE of p^2
a(2,1) #UMVUE of p^2*q
b=function(c)
{
m<-8
n<-10
x<-c(2,5,7,3,4,2,5,4)
t<-sum(x)
g<-array(,c(1,c+1))
for (i in 1:c)
{
g[i]=((choose(n,i)*choose(m*n-n,t-i))/choose(m*n,t))
}
g[c+1]=((choose(n,0)*choose(m*n-n,t))/choose(m*n,t))
umvue=sum(g)
print (umvue)
}
b(2)#UMVUE of P(X<=2)
1-b(6)#UMVUE of P(X<=6) & P(X>6)
```

Example 2.3.7 0, 3, 1, 5, 5, 3, 2, 4, 5, 4 is a sample of size 10 from the Poisson distribution $P(\lambda)$. Obtain UMVUE of λ, λ^2, $\lambda e^{-\lambda}$, and $P(x \geq 4)$.

```
 d=function (s,r) {
m<-10
x<-c(0,3,1,5,5,3,2,4,5,4)
t<-sum(x)
umvue=((m-s)^(t-r)*factorial(t))/(m^t*factorial(t-r))
print (umvue) } d(0,1) #UMVUE of lamda d(0,2) #UMVUE of
lamda^2 d(1,1) #UMVUE of lamda*e^(-lamda) f=function (c) {
m<-10
x<-c(0,3,1,5,5,3,2,4,5,4)
t<-sum(x)
g<-array(,c(1,c+1))
for (i in 1:c)
```

```
{
g[i]<-(choose(t,i)*(1/m)^i*(1-(1/m))^(t-i))
}
g[c+1]=choose(t,0)*(1-(1/m))^t
umvue=sum(g)
print (umvue) } 1-f(3) #UMVUE of P(X<4) & P(X>=4)
```

Example 2.3.8 8, 4, 6, 2, 9, 10, 5, 8, 10, 8, 3, 10, 1, 6, 2 is a sample of size 15 from the following distribution:

$$P[X = k] = \begin{cases} \frac{1}{N} & ;\ k = 1, 2, \ldots, N \\ 0 & ; \text{otherwise} \end{cases}$$

Obtain UMVUE of N^5.

```
 h<-function (s) {
n<-15
x<-c(8,4,6,2,9,10,5,8,10,8,3,10,1,6,2)
z<-max(x)
umvue=(z^(n+s)-(z-1)^(n+s))/((z^n)-(z-1)^n)
print (umvue) } h(5) #UMVUE of N^5
```

Example 2.3.9 Lots of manufactured articles are made up of items each of which is an independent trial with probability p of it being defective. Suppose that four such lots are sent to a consumer, who inspects a sample of size 50 from each lot. If the observed number of defectives in the ith lot is 0, 1, or 2, the consumer accepts this lot. The observed numbers of defectives are 0, 0, 0, 3. Obtain UMVUE of the probability that a given lot will be accepted.

```
j=function (c) {
m<-4
n<-50
t<-3
g<-array(,c(1,c+1))
for (i in 1:c)
{

g[i]<-(choose(50,i)*choose((m*n)-n,t-i))/(choose(m*n,t))
}
g[c+1]<-(choose(m*n-n,t))/(choose(m*n,t))
umvue=sum(g)
print (umvue) } j(2) #UMVUE of P(X<=2)
```

Example 2.3.10 Let $X_1, X_2, \ldots X_n$ be a sample from $NB(1, \theta)$.
Find the UMVUE of $d(\theta) = P(X = 0)$, for the data 3, 4, 3, 1, 6, 2, 1, 8

```
k=function (r,s) {
m<-8
k<-1
x<-c(3,4,3,1,6,2,1,8)
t=sum(x)
umvue=choose(t-s+m*k-r-1,m*k-r-1)/choose(t+m*k-1,t)
print(umvue) } k(1,0) #UMVUE of P(X=0), i.e., p
```

Example 2.3.11 The following observations were recorded on a random variable X having pdf:

$$f(x) = \begin{cases} \frac{x^{p-1} e^{-\frac{x}{\sigma}}}{\sigma^p \Gamma(p)} & ;\ x > 0,\ \sigma > 0,\ p = 4 \\ 0 & ;\text{ otherwise} \end{cases}$$

7.89, 10.88, 17.09, 16.17, 11.32, 18.44, 3.32, 19.51, 6.45, 6.22.
Find UMVUE of σ^3

```
x1<-function (k,r) {
p<-4
n<-10
y<-c(7.89,10.88,17.09,16.17,11.32,18.44,3.32,19.51,6.45,6.22)
t<-sum(y)
umvue=((gamma(n*p))*(t-k)^(n*p-r-1))/((gamma(n*p-r))*t^(n*p-1))
print (umvue) } x1(0,-3) #UMVUE of sigma^3
```

Example 2.3.12 A random sample of size 10 is drawn from the following pdf:

1.

$$f(x, \theta) = \begin{cases} \frac{\theta}{(1+x)^{\theta+1}} & ;\ x > 0,\ \theta > 0 \\ 0 & ;\text{ otherwise} \end{cases}$$

Data: 0.10, 0.34, 0.35, 0.08, 0.03, 2.88, 0.45, 0.49, 0.86, 3.88

2.

$$f(x, \theta) = \begin{cases} \theta x^{\theta-1} & ;\quad 0 < x < 1 \\ 0 & ;\text{ otherwise} \end{cases}$$

Data: 0.52, 0.79, 0.77, 0.76, 0.71, 0.76, 0.47, 0.35, 0.55, 0.63

3.

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{|x|}{\theta}} & ;\quad -\infty < x < \infty \\ 0 & ;\text{ otherwise} \end{cases}$$

Data: 9.97, 0.64, 3.17, 1.48, 0.81, 0.61, 0.62, 0.72, 3.14, 2.99
Find UMVUE of θ in (i), (ii), and (iii).

(i)

```
x2<-function (k,r) {
n<-10
y<-c(0.10,0.34,0.35,0.08,0.03,2.88,0.45,0.49,0.86,3.88)
x<-array(,c(1,10))
for (i in 1:10)
{
x[i]=log(1+y[i])
}
t<-sum(x)
umvue=(((t-k)^(n-r-1))*gamma(n))/((t^(n-1))*gamma(n-r))
print (umvue) } x2(0,1) #UMVUE of theta
```

(ii)

```
 x3<-function (k,r) {
n<-10
y<-c(0.52,0.79,0.77,0.76,0.71,0.76,0.47,0.35,0.55,0.63)
x<-array(,c(1,10))
for (i in 1:10)
{
x[i]=-log(y[i])
}
t<-sum(x)
umvue=(((t-k)^(n-r-1))*gamma(n))/((t^(n-1))*gamma(n-r))
print (umvue) } x3(0,1) #UMVUE of theta
```

(iii)

```
 x4<-function (k,r) {
n<-10
y<-c(9.97,0.64,3.17,1.48,0.81,0.61,0.62,0.72,3.14,2.99)
t<-sum(y)
umvue=(((t-k)^(n-r-1))*gamma(n))/((t^(n-1))*gamma(n-r))
print (umvue) } x4(0,-1) #UMVUE of theta
```

Example 2.3.13 The following observations were obtained on an rv X following:

1. $N(\theta, \sigma^2)$
   Data: 5.77, 3.81, 5.24, 8.81, 0.98, 8.44, 3.16, 11.27, 4.40, 4.87, 7.28, 8.48, 6.43, −0.00, 9.67, 12.04, −5.06, 13.71, 6.12, 4.76
   Find UMVUE of θ, θ^2, ϑ^3 and $P(x \le 2)$

2. $N(6, \sigma^2)$
   Data: 7.26, −0.23, 7.55, 3.09, 7.62, 16.79, 5.27, 8.46, 5.16, −0.66.
   Find UMVUE of $\frac{1}{\sigma}$, σ, σ^2, $P(X \geq 2)$
3. $N(\theta, \sigma^2)$
   Data: 10.59, −1.50, 6.40, 7.55, 4.70, 1.63, 0.04, 2.96, 6.47, 6.42
   Find UMVUE of θ, θ^2, $\theta + 2\sigma$,

(i)

```
x5<-function (sigsq,n,k)
{x<-c(5.77,3.81,5.24,8.81,0.98,8.44,3.16,11.27,4.4,4.87,7.28,
8.48,6.43,0,9.67,12.04,-5.06,13.71,6.12,4.76)
umvue1=mean(x)
umvue2=umvue1^2-(sigsq/n)
umvue3=umvue1^3-(3*sigsq*umvue1/n)
umvue4=pnorm((k-(mean(x)))/(sqrt((sigsq*((n-1)/n)))))
print (umvue1) #UMVUE of theta
print (umvue2) #UMVUE of theta^2
print (umvue3) #UMVUE of theta^3
print (umvue4) #UMVUE of P(X<=2) } x5(4,20,2)
```

(ii)

```
x6<-function (n,r) {
x<-c(7.26,-0.23,7.55,3.09,7.62,16.79,5.27,8.46,5.16,-0.66)
t<-sum((x-6)^2)
umvue=(((t^r)*gamma(n/2))/((2^r)*gamma((n/2)+r)))
print (umvue) } x6 (10,-0.5) #UMVUE of 1/sigma x6 (10,0.5)
#UMVUE of sigma x6 (10,1) #UMVUE of sigma^2

x7<-function (n,k) {
x<-c(7.26,-0.23,7.55,3.09,7.62,16.79,5.27,8.46,5.16,-0.66)
t<-sum((x-6)^2)
umvue<-(1-pbeta(((k-6)/sqrt(t))^2,0.5, ((n-1)/2)))*0.5
print (umvue) } x7(10,2) #UMVUE of P(X>=2)
```

(iii)

```
x8<-function(n,r) {
x<-c(10.59,-1.5,6.4,7.55,4.7,1.63,0.04,2.96,6.47,6.42)
s<-sum((x-mean(x))^2)
umvue1<-mean(x) #UMVUE of theta
umvue2<-((s^(r))*gamma((n-1)/2))/(gamma(((n-1)/2)+r)*(2^r))
#UMVUE of sigma^2
print (umvue1)
```

```
print (umvue2)
print ((umvue1^2)-(umvue2/n))#UMVUE of theta^2
print (umvue1+2*sqrt(umvue2))#UMVUE of theta+2*sigma }
x8(10,1)
```

Example 2.3.14 If rv X is drawn from $U(\theta_1, \theta_2)$. Find the UMVUE of θ_1 and θ_2 from the following data:
3.67, 2.65, 4.41, 3.48, 2.07, 2.91, 2.77, 4.82, 2.73, 2.98.

```
x<-c(3.67,2.65,4.41,3.48,2.07,2.91,2.77,4.82,2.73,2.98)
umvue1<-(max(x)-length(x)*min(x))/(1-length(x)) umvue1 #UMVUE
of theta1 umvue2<-(length(x)*max(x)-min(x))/(length(x)-1)
umvue2 #UMVUE of theta1
```

Example 2.3.15 If rv X is drawn from $U(0, \theta)$ Find the UMVUE of θ, θ^2, and $\frac{1}{\theta}$ from the following data:
1.60, 1.91, 3.68, 0.78, 2.52, 4.34, 1.15, 4.69, 1.53, 4.53

```
x9<-function (n,r) {
x<-c(1.6,1.91,3.68,0.78,2.52,4.34,1.15,4.69,1.53,4.53)
umvue<-((max(x)^r)*((n+r)/n))
print (umvue) } x9(10,1) #UMVUE of theta x9(10,2) #UMVUE of
theta^2 x9(10,-1)#UMVUE of (1/theta)
```

2.4 Exercise 2

1. For the geometric distribution,

$$f(x|\theta) = \theta(1-\theta)^{x-1};\, x = 1, 2, 3, \ldots, 0 < \theta < 1$$

Obtain an unbiased estimator of $\frac{1}{\theta}$ for a sample of size n. Calculate it for given data: 6, 1, 1, 14, 1, 1, 6, 5, 2, 2.
2. $X_1, X_2, \ldots, X_n$ is a random sample from an exponential distribution with mean θ. Find an UMVUE of $\exp(-\frac{1}{\theta})$ when $t > 1$, where $T = \sum_{i=1}^{n} X_i$ for the given data: 0.60, 8.71, 15.71, 2.32, 0.02, 6.22, 8.79, 2.05, 2.96, 3.33
3. Let

$$f(x|\mu, \sigma) = \frac{1}{\sigma}\exp\left[-\frac{(x-\mu)}{\sigma}\right];\quad x \geq \mu \in R \text{ and } \sigma > 0$$

For a sample of size n, obtain
(a) an unbiased estimate of μ when σ is known,
(b) an unbiased estimate of σ when μ is known,
(c) Ten unbiased estimators of σ^2 when μ is known.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$, where μ is known and if $T = \frac{1}{n}\sum_{i=1}^{n}|X_i - \mu|$, examine if T is unbiased for σ and if not obtain an unbiased estimator of σ.
5. If $X_1, X_2, \ldots, X_n$ is a random sample from the population

$$f(x|\theta) = (\theta + 1)x^{\theta}; \quad 0 < x < 1, \theta > -1$$

Prove that $\left[-\frac{(n-1)}{\sum \ln X_i} - 1\right]$ is an UMVUE of θ.
6. Suppose X has a truncated Poisson distribution with pmf

$$f(x|\theta) = \begin{cases} \frac{\exp[-\theta]\theta^x}{[1-e^{-\theta}]x!} & ; \quad x = 1, 2 \\ 0 & ; \text{otherwise} \end{cases}$$

Prove that the only unbiased estimator of $[1 - e^{-\theta}]$ based on X is the statistic $T(X)$,

$$T(x) = \begin{cases} 0 \text{ ; when } x \text{ is odd} \\ 2 \text{ ; when } x \text{ is even} \end{cases}$$

[Hint $\sum\limits_{x=1}^{\infty} \dfrac{\theta^{2x}}{(2x)!} = \dfrac{e^{-\theta} + e^{\theta}}{2} - 1$]
7. Let $X_1, X_2, \ldots, X_n$ be iid rvs from $f(x|\theta)$,

$$f(x|\theta) = \begin{cases} \exp[i\theta - x] & ; \ x \geq i\theta \\ 0 & ; \ x < i\theta \end{cases}$$

Prove that

$$T = \min_i\left[\frac{X_i}{i}\right]$$

is minimal sufficient statistic for θ. If possible obtain the distribution of X_1 given T. Can you find an unbiased estimator of θ? If "Yes," find and if "No," explain.
8. Let $X_1, X_2, \ldots, X_n$ be iid rvs with $f(x|\mu)$,

$$f(x|\mu) = \begin{cases} \frac{1}{2i\mu} & ; \quad -i(\mu - 1) < x_i < i(\mu + 1) \\ 0 & ; \text{otherwise} \end{cases}$$

where $\mu > 0$. Find the sufficient statistic for μ. If T is sufficient for μ then find the distribution of X_1, X_2 given T. If possible, find an unbiased estimator of μ.
9. If X_1, X_2, and X_3 are iid rvs with the following pmfs:
(a)

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0, 1, 2, \ldots, \lambda > 0$$

(b)

$$f(x|\lambda) = \binom{n}{x}\lambda^x(1-\lambda)^{n-\lambda};\ \ 0 < \lambda < 1, x = 0, 1, 2, \ldots, n$$

(c)

$$f(x|\lambda) = (1-\lambda)\lambda^x;\ \ x = 0, 1, 2, \ldots \lambda > 0$$

Prove that $X_1 + 2X_2$, $X_2 + 3X_3$, and $X_1 + 2X_2 + X_3$ are not sufficient for λ in (a), (b), and (c). Further, prove that $2(X_1 + X_2 + X_3)$ is sufficient for λ in (a), (b), and (c).
10. Let $X_1, X_2, \ldots, X_n$ be iid rvs having $\cup(\theta, 3\theta)$, $\theta > 0$. Then prove that $(X_{(1)}, X_{(n)})$ is jointly minimal sufficient statistic.
11. Let $\{(X_i, Y_i), i = 1, 2, \ldots, n\}$ be n independent random vectors having a bivariate distribution

$$N = \left(\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right);\ \ -\infty < \theta_1, \theta_2 < \infty,\ \sigma_1, \sigma_2 > 0,\ \ -1 \le \rho \le 1.$$

Prove that

$$\left(\sum X_i, \sum X_i^2, \sum X_iY_i, \sum Y_i \sum Y_i^2\right)$$

is jointly sufficient $(\theta_1, \sigma_1, \rho, \theta_2, \sigma_2)$.
12. Let the rv X_1 is $B(n, \theta)$ and X_2 is $P(\theta)$ where n is known and $0 < \theta < 1$. Obtain four unbiased estimators of θ.
13. Let $X_1, X_2, \ldots, X_n$ are iid rvs with $\cup(\theta, \theta + 1)$.

(i) Find sufficient statistic for θ

(ii) Show that the sufficient statistic is not complete

(iii) Find an unbiased estimator of θ

(iv) Find the distribution of X_1 given T, where T is sufficient for θ

(v) Can you find UMVUE of θ ? If "No," give reasons.

14. Let X be a rv with pmf

$$f(x|p) = \left(\frac{p}{2}\right)^{|x|}(1-p)^{1-|x|};\ x = -1, 0, 1,\ 0 < p < 1$$

(i) Show that X is not complete.

(ii) Show that $|X|$ is sufficient and complete.

15. Let $X_1, X_2, \ldots, X_n$ are iid rvs from the following pdf:
(i)

$$f(x|\alpha) = \frac{\alpha}{(1+x)^{1+\alpha}}; \quad x > 0, \alpha > 0$$

(ii)

$$f(x|\alpha) = \frac{(\ln \alpha)\alpha^x}{\alpha - 1}; \quad 0 < x < \infty, \alpha > 1$$

(iii)

$$f(x|\alpha) = \exp[-(x-\alpha)]\exp[-e^{-(x-\alpha)}]; \quad -\infty < x < \infty, -\infty < \alpha < \infty$$

(iv)

$$f(x|\alpha) = \frac{x^3 e^{-\frac{x}{\alpha}}}{6\alpha^4}; \quad x > 0, \alpha > 0$$

(v)

$$f(x|\alpha) = \frac{kx^{k-1}}{\alpha^k}; \quad 0 < x < \alpha, \alpha > 0$$

Find a complete sufficient statistic or show that it does not exist.
Further if it exists, then find the distribution of X_1 given T, where T is sufficient statistic. Further, find UMVUE of α^r, whenever it exists.
16. Let $X_1, X_2, \ldots, X_N$ are iid rvs with $B(1, p)$, where N is also a random variable taking values 1, 2,…with known probabilities $p_1, p_2, \ldots, \sum p_i = 1$.
(i) Prove that the pair (X, N) is minimal sufficient and N is ancillary for p.
(ii) Prove that the estimator $\frac{X}{N}$ is unbiased for p and has variance $p(1-p)\mathrm{E}\frac{1}{N}$.
17. In a normal distribution $N(\mu, \mu^2)$, prove that $(\sum X_i, \sum X_i^2)$ is not complete in a sample of size n.
18. Let $X_1, X_2, \ldots, X_n$ be iid rvs from the following pdf:
(i)

$$f(x|\theta) = \theta x^{\theta-1}; 0 < x < 1, \theta > 0$$

Find UMVUE of (a) $\theta e^{-\theta}$ (b) $\dfrac{\theta}{\theta+1}$ (c) $\dfrac{1+\theta}{e^{2\theta}}$

(ii)

$$f(x|\theta_1,\theta_2)=\frac{1}{(\theta_2-\theta_1)};\ \theta_1<x<\theta_2,\theta_1,\theta_2>0$$

Find minimal sufficient statistic and show that it is complete, further if possible, find the the distribution of X_1 given T, where T is sufficient statistic. Find UMVUE of $\exp(\theta_2-\theta_1)$, $\dfrac{\theta_1}{\theta_1+\theta_2}$, $\sin(\theta_1-\theta_2)$, and $\cos(\theta_1-\theta_2)$

19. Let T_1, T_2 be two unbiased estimates having common variances $a\sigma^2(a>1)$, where σ^2 is the variance of the UMVUE. Prove that the correlation coefficient between T_1 and T_2 is greater than or equal to $\frac{2-a}{a}$.

20. Let $X_1, X_2, \ldots, X_n$ are iid rvs from discrete uniform distribution

$$f(x|N_1,N_2)=\frac{1}{N_2-N_1};\ x=N_1+1,N_1+2,\ldots,N_2.$$

Find the sufficient statistic for N_1 and N_2.
If exists, find UMVUE for N_1 and N_2.

21. Let $X_1, X_2, \ldots, X_n$ are iid rvs from $P(\lambda)$. Let $g(\lambda)=\sum_{i=0}^{\infty}c_i\lambda^i$ be a parametric function. Find the UMVUE for $g(\lambda)$. In particular, find the UMVUE for (i)$g(\lambda)=(1-\lambda)^{-1}$ (ii) $g(\lambda)=\lambda^r(r>0)$

22. Let $X_1, X_2, \ldots, X_n$ are iid rvs with $N(\theta,1)$. Show that S^2 is ancillary.

23. In scale parameter family, prove that $\left(\frac{X_1}{X_n},\frac{X_2}{X_n},\ldots,\frac{X_{n-1}}{X_n}\right)$ are ancillary.

24. Let X_1, X_2 are iid rvs with $N(0,\sigma^2)$. Prove that $\frac{X_1}{X_2}$ is ancillary.

25. Let $X_1, X_2, \ldots, X_n$ are iid rvs with (i) $N(\mu,\sigma^2)$ (ii)$N(\mu,\mu^2)$. Examine $T=\left(\left(\frac{X_1-\bar{X}}{S}\right),\left(\frac{X_2-\bar{X}}{S}\right),\ldots,\left(\frac{X_n-\bar{X}}{S}\right)\right)$ is ancillary in (i) and (ii).

26. Let $X_1, X_2, \ldots, X_m$ are iid rvs with $B(n,p)$, $0<p<1$ and n is known. Find the UMVUE of $P[X=x]=\binom{n}{x}p^xq^{n-x};\ x=0,1,2,\ldots,n,\ q=1-p$

27. Let $X_1, X_2, \ldots, X_m$ are iid rvs from Poisson (λ). Find the UMVUE of $P[X=x]=\frac{e^{-\lambda}\lambda^x}{x!};\ x=0,1,2,\ldots,\ \lambda>0$

28. Let $X_1, X_2, \ldots, X_m$ are iid rvs from gamma distribution with parameters p and σ. Then find the UMVUE of $\frac{e^{-\frac{x}{\sigma}}x^{p-1}}{\sigma^p\Gamma(p)}$ for p known, $x>0,\ \sigma>0$.

29. Let $X_1, X_2, \ldots, X_n$ are iid rvs from $N(\mu,\sigma^2)$, $\mu\in R$, $\sigma>0$. Find UMVUE of $P[X_1\le k], k>0$.

30. Let $X_1, X_2, \ldots, X_n$ are iid rvs with pdf,

$$f(x|\theta)=\begin{cases}\frac{1}{2\theta}; & -\theta<x<\theta\\ 0 & ;\text{otherwise}\end{cases}$$

Prove that $T(X)=\max\left[-X_{(1)},X_{(n)}\right]$ is a complete sufficient statistic. Find UMVUE of $\theta^r(r>0)$. If $Y=|X|$, then find UMVUE of

1. θ^r
2. $\dfrac{\theta}{1+\theta}$
3. $\sin(\theta)$

based on Y.

31. Let $X_1, X_2, \ldots, X_n$ are iid rvs from the pdf,

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma}\exp\left[-\frac{(x-\mu)}{\sigma}\right]; x \geq \mu, \sigma > 0$$

(i) Prove that $[X_{(1)}, \sum_{j=1}^{n}(X_j - X_{(1)})]$ is a complete sufficient statistic for (μ, σ).
(ii) Prove that UMVUE of μ and σ are given by

$$(\hat{\mu} = X_{(1)}) - \frac{n}{(n-1)}\sum_{j=1}^{n}(X_j - X_{(1)})$$

$$\hat{\sigma} = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - X_{(1)})$$

32. Let $X_1, X_2, \ldots, X_n$ are iid rvs from $\cup(\theta_1, \theta_2)$ or $\cup(\theta_1 + 1, \theta_2 + 1)$. Find the UMVUE of $g(\theta_1, \theta_2)$ without using the general result from Example 2.3.3. Further, find the UMVUE of $\theta_1^r \theta_2^s (r,\ s > 0)$.

33. Let $X_1, X_2, \ldots, X_n$ be iid rvs from $\cup(-k\theta, k\theta),\ k,\ \theta > 0$. Show that the UMVUE of $g(\theta)$ is

$$u(y_{(m)} = g(y_{(m)}) + \frac{y_{(m)}g'(y_{(m)})}{m},$$

where $y_{(m)} = \max_i Y_i,\ Y_i = \frac{|X_i|}{k}\ :\ i = 1, 2, \ldots, n$

34. Let $X_1, X_2, \ldots, X_m$ be iid rvs from discrete uniform distribution where

$$f(x|N) = \begin{cases} \frac{1}{2N}\ ; & x = -N, -N+1, \ldots, -1, 1, 2, \ldots, N \\ 0 & ; \text{otherwise} \end{cases}$$

Find UMVUE of (i) $\sin N$ (ii) $\cos N$ (iii) e^N (iv) $\frac{N}{e^N}$

35. Let $X_1, X_2, \ldots, X_m$ be iid rvs from $f(x|N)$

$$(a)\ f(x|N) = \frac{2x}{N(N+1)}; \quad x = 1, 2, \ldots, N$$

$$(b)\ f(x|N) = \frac{6x^2}{N(N+1)(2N+1)}; \quad x = 1, 2, \ldots, N$$

Find UMVUE of (i) $\sin N$ (ii) $\cos N$ (iii) e^N (iv) $\frac{N}{e^N}$ (v) $\frac{e^N}{\sin N}$ (vi) $\frac{e^N}{\cos N}$

36. Let $X_1, X_2, \ldots, X_m$ be iid rvs from $f(x|N_1, N_2)$

$$f(x|N_1, N_2) = \frac{1}{N_2 - N_1 + 1}; x = N_1, N_1 + 1, \ldots, N_2$$

Find UMVUE of (i) N_1 (ii) N_2 (iii) $(N_1N_2)^2$

37. Let $X_1, X_2, \ldots, X_m$ be iid rvs with $\cup(0, \theta)$.

Then find UMVUE of (i) e^{θ} (ii) $\sin\theta$ (iii) $\dfrac{\theta}{1+\theta}$.

38. Let $X_1, X_2, \ldots, X_m$ be iid rvs with $f(x|\theta)$,

$$f(x|\theta) = \frac{4x^3}{\theta^4}\ ;\ 0 < x < \theta,$$

Find UMVUE of (i) θ^5 (ii) $\dfrac{\theta^2}{1+\theta^3}$ (iii) $\cos\theta$.

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Springer
http://www.springer.com/978-981-10-0888-7

Examples in Parametric Inference with R
Dixit, U.J.
2016, LVIII, 423 p. 26 illus., Hardcover
ISBN: 978-981-10-0888-7