Chapter 2
CR-Submanifolds and δ-Invariants

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2.1 Introduction

In 1956, John F. Nash proved in [34] the following famous embedding theorem.

Theorem 2.1 *Every Riemannian n-manifold can be isometrically embedded in a Euclidean m-space* $\mathbb{E}^m$ *with* $m = \frac{n}{2}(n+1)(3n+11)$.

For example, Nash's theorem implies that every Riemannian 3-manifold can be isometrically embedded in $\mathbb{E}^{120}$ with codimension 117.

The Nash embedding theorem was aimed for in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. Till when observed in [32] as such by M.L. Gromov, this hope had not been materialized however.

There were several reasons why it is so difficult to apply Nash's theorem. One reason is that it requires very large codimension for a Riemannian manifold to admit an isometric embedding in Euclidean spaces in general. On the other hand, submanifolds of higher codimension are very difficult to understand, e.g., there are no general results for arbitrary Riemannian submanifolds, except the three fundamental equations of Gauss, Codazzi, and Ricci. Another reason for this is lack of controls of the extrinsic invariants of the submanifold by the known classical intrinsic invariants.

In order to overcome such difficulties as well as to provide answers to an open question on minimal immersions proposed by S.S. Chern in the 1960s, the author

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introduced in the early 1990s a new type of Riemannian invariants; namely, the δ-invariants. At the same time, the author was able to establish universal optimal inequalities for Riemannian submanifolds involving the intrinsic δ-invariants and the most important extrinsic invariant; namely, the squared mean curvature.

The δ-invariants are very different in nature from the "classical" Ricci and scalar curvatures; simply due to the fact that both scalar and Ricci curvatures are "total sum" of sectional curvatures on a Riemannian manifold. In contrast, the author's δ-invariants are obtained from the scalar curvature by throwing away a certain amount of sectional curvatures. After δ-invariants were introduced and the corresponding inequalities were established, δ-invariants were investigated by many mathematicians in the past two decades (see [21, 22] for recent surveys on δ-invariants and their applications).

Let N be a Riemannian manifold isometrically immersed in a Kähler manifold $\tilde{M}$ with complex structure J. For each point $x \in N$, let $\mathcal{D}_x$ denote the maximal complex subspace $T_xN \cap J(T_xN)$ of the tangent space T_xN, $x \in N$. If the dimension of $\mathcal{D}_x$ is the same for all $x \in N$, then $\{\mathcal{D}_x, x \in N\}$ defines a complex distribution $\mathcal{D}$ on N. A submanifold N in a Kähler manifold $\tilde{M}$ is called a *CR-submanifold* if there exists on N a totally real distribution $\mathcal{D}^{\perp}$ whose orthogonal complement is $\mathcal{D}$, i.e., $TN = \mathcal{D} \oplus \mathcal{D}^{\perp}$ and $J\mathcal{D}_x^{\perp} \subset T_x^{\perp}N$, $x \in N$ (cf. [3]).

A Riemannian submersion $\pi : M \to B$ is an everywhere surjective map from a Riemannian manifold M onto another Riemannian manifold B such that the differential π_* preserves lengths of horizontal vectors.

The main purpose of this article is to present recent results on CR-submanifolds in complex space forms which are closely related to δ-invariants and Riemannian submersions.

2.2 Preliminaries

Let M be an n-dimensional submanifold of a Riemannian m-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\tilde{M}^m$ such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and hence $e_{n+1}, \ldots, e_m$ are normal to M.

For the submanifold M in $\tilde{M}^m$, we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given, respectively, by (see, for instance, [22])

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for any vector fields X, Y tangent to M and vector field ξ normal to M, where σ denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold.

Let $\{\sigma_{ij}^r\}$, $i, j = 1, \ldots, n;\ r = n+1, \ldots, m$, denote the coefficients of the second fundamental form h with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$. Then, we have

$$\sigma_{ij}^r = \left\langle \sigma(e_i, e_j), e_r \right\rangle = \left\langle A_{e_r} e_i, e_j \right\rangle,$$

where $\langle\ ,\ \rangle$ denotes the inner product.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n} \operatorname{trace} \sigma = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i). \tag{2.3}$$

The squared mean curvature is then given by

$$H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle.$$

The submanifold M is called minimal in $\tilde{M}^m$ if its mean curvature vector vanishes identically. It is called totally geodesic if its second fundamental form σ vanishes identically.

Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\tilde{M}^m$, respectively. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) &+ \langle \sigma(X, W), \sigma(Y, Z) \rangle \\ &- \langle \sigma(X, Z), \sigma(Y, W) \rangle \end{aligned} \tag{2.4}$$

for vectors X, Y, Z, W tangent to M. In particular, for a submanifold of a Riemannian manifold of constant sectional curvature c, we have

$$\begin{aligned} R(X, Y; Z, W) = c\{&\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle\} \\ &+ \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle\,. \end{aligned} \tag{2.5}$$

Let M be a Riemannian p-manifold and $e_1, \ldots, e_p$ be an orthonormal frame fields on M. For differentiable function φ on M, the Laplacian $\Delta\varphi$ of φ is defined by

$$\Delta\varphi = \sum_{j=1}^{p} \{(\nabla_{e_j} e_j)\varphi - e_j e_j \varphi\}. \tag{2.6}$$

We simply call a Kähler manifold of constant holomorphic sectional curvature a *complex space form*. In this article, we denote a complete simply connected complex m-dimensional complex space form of constant holomorphic sectional curvature $4c$ by $\tilde{M}^m(4c)$.

The curvature tensor $\tilde{R}$ of a complex space form $\tilde{M}^m(4c)$ satisfies

$$\tilde{R}(U, V, W) = c\{\langle V, W\rangle U - \langle X, W\rangle V + \langle JV, W\rangle JU \\ - \langle JU, W\rangle JV + 2\langle U, JV\rangle JW\}. \tag{2.7}$$

It is well-known that $\tilde{M}^m(4c)$ is holomorphically isometric to the complex projective m-space $CP^m(4c)$, the complex Euclidean m-space $\mathbf{C}^m$, or the complex hyperbolic m-space $CH^m(4c)$ according to $c > 0$, $c = 0$, or $c < 0$, respectively.

2.3 CR-Submanifolds and CR-Warped Products

A submanifold N of a Kähler manifold $\tilde{M}$ is called *totally real* if the complex structure J of $\tilde{M}$ carries each tangent space T_xN of N into the normal space $T_x^{\perp}N$ for each $x \in N$ (cf. [25]). A totally real submanifold N is called *Lagrangian* if $\dim_{\mathbf{C}} \tilde{M} = \dim N$. For a point $x \in N$, we denote by $\mathcal{D}_x$ the maximal complex subspace $T_xN \cap J(T_xN)$ of the tangent space T_xN. If the dimension of $\mathcal{D}_x$ is the same for all $x \in N$, then $\{\mathcal{D}_x, x \in N\}$ defines a complex distribution $\mathcal{D}$ on N.

A submanifold N in a Kähler manifold $\tilde{M}$ is called a *CR*-submanifold if there exists on N a totally real distribution $\mathcal{D}^{\perp}$ whose orthogonal complement is $\mathcal{D}$, i.e., $TN = \mathcal{D} \oplus \mathcal{D}^{\perp}$ and $J\mathcal{D}_x^{\perp} \subset T_x^{\perp}N, x \in N$ (cf. [3]).

For a *CR*-submanifold N with the complex distribution $\mathcal{D}$ and the totally real distribution $\mathcal{D}^{\perp}$, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^{\perp}$ in the normal bundle $T^{\perp}N$ of N. Then, we have the following orthogonal direct sum decomposition:

$$T^{\perp}N = J\mathcal{D}^{\perp} \oplus \nu, \quad J\mathcal{D}^{\perp} \perp \nu. \tag{2.8}$$

A *CR*-submanifold is called *proper* if it is neither holomorphic nor totally real. Throughout this article, let h denote the complex rank of the complex distribution $\mathcal{D}$ and p the real rank of the totally real distribution $\mathcal{D}^{\perp}$.

A *CR*-submanifold N is called a *CR-product* if it is a Riemannian product of a holomorphic submanifold N^T and a totally real submanifold $N^{\perp}$ of $\tilde{M}$. It was proved in [8, Theorem 4.6] that every *CR*-product in a complex Euclidean space is a direct product of a holomorphic submanifold of a linear complex subspace and a totally real submanifold of another linear complex subspace. Also, it was proved in [8, Theorem 4.4] that there do not exist proper *CR*-products in complex hyperbolic spaces. Furthermore, it was known in [8, Theorem 5.3] that *CR*-products in complex projective space CP^{h+p+hp} are obtained from the Segre imbedding in a natural way.

Let B and F be two Riemannian manifolds endowed with Riemannian metrics g_B and g_F, respectively, and let f be a positive differentiable function on B. The warped product $B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian metric

$$g = g_B + f^2 g_F.$$

The function f is called the *warping function* of the warped product (cf. [36]).

It was shown in [16, Theorem 3.1] that there do not exist warped products of the form: $N^{\perp} \times_f N^T$ in any Kähler manifold besides CR-products, where $N^{\perp}$ is a totally real submanifold and N^T is a holomorphic submanifold.

By contrast, it was also shown in [16] that there exist many CR-submanifolds which are warped products of the form $N^T \times_f N^{\perp}$. Such a warped product CR-submanifold is simply called a *CR-warped product* (see [16]). Furthermore, it was proved in [16, Theorem 5.1] that every CR-warped product in an arbitrary Kähler manifold satisfies the following optimal universal inequality:

$$||\sigma||^2 \geq 2p||\nabla(\ln f)||^2, \tag{2.9}$$

where $\nabla(\ln f)$ denotes the gradient of $\ln f$ and σ is the second fundamental form of the CR-warped product.

CR-warped products in complex space forms satisfying the equality case of (2.9) have been completely classified in [16, 17]. Further results on CR-warped products in complex space forms were obtained in [19, 20].

Lemma 1 ([8]) *Let N be a CR-submanifold in a* $CP^m(4)$. *Then, we have*
(a) $\langle \nabla_U Z, X\rangle = \langle JA_{JZ}U, X\rangle$,
(b) $A_{JZ}W = A_{JW}Z$,
(c) $A_{J\eta}X = -A_{\eta}JX$, *and*
(d) $\langle D_U JZ, JW\rangle = \langle \nabla_U Z, W\rangle$, $\langle D_U JZ, J\eta\rangle = \langle h(U, Z), \eta\rangle$,
for any vector fields $U \in TN$; $X, Y \in \mathcal{D}$; $Z, W \in \mathcal{D}^{\perp}$ *and* $\eta \in \nu$, *where* ν *is defined by (2.8).*

Lemma 2 *The totally real distribution* $\mathcal{D}^{\perp}$ *of a CR-submanifold of a Kähler manifold is an integrable distribution.*

Lemma 3 *Let N be a CR-submanifold of a Kähler manifold* $\tilde{M}$. *Then, we have:*

(1) the complex distribution $\mathcal{D}$ *is integrable if and only if*

$$\langle \sigma(X, JY), JZ\rangle = \langle \sigma(JX, Y), JZ\rangle \tag{2.10}$$

holds for any $X, Y \in \mathcal{D}$ *and* $Z \in \mathcal{D}^{\perp}$,

(2) the leaves of the totally real distribution $\mathcal{D}^{\perp}$ *are totally geodesic in N if and only if* $\langle \sigma(X, Z), JW\rangle = 0$ *holds for any* $X \in \mathcal{D}$ *and* $Z, W \in \mathcal{D}^{\perp}$.

A CR-submanifold N of a Kähler manifold $\tilde{M}$ is called *anti-holomorphic* if we have $J\mathcal{D}^{\perp}_x = T^{\perp}_x N,\ x \in N$. And it is called *mixed totally geodesic* if its second fundamental form σ satisfies $\sigma(X, Z) = 0$ for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. A mixed totally geodesic CR-submanifold is called *mixed foliate* if its complex distribution $\mathcal{D}$ is also integrable. Obviously, real hypersurfaces of a Kähler manifold are anti-holomorphic submanifolds with $p = \text{rank}\ \mathcal{D}^{\perp} = 1$.

Lemma 4 *A complex space form* $\tilde{M}^m(4c)$ *with* $c \neq 0$ *admits no mixed foliate proper CR-submanifolds.*

Lemma 4 is due to [4] for $c > 0$ and due to [29] for $c < 0$.

For mixed foliate *CR*-submanifolds in a complex Euclidean space, we have the following results from [8].

Lemma 5 *Let N be a CR-submanifold of $\mathbf{C}^m$. Then N is mixed foliate if and only if N is a CR-product.*

Lemma 6 *Every CR-product in a complex Euclidean m-space $\mathbf{C}^m$ is a direct product of a holomorphic submanifold of a linear complex subspace and a totally real submanifold of another linear complex subspace.*

Lemma 7 *Let N be a mixed foliate CR-submanifold of a Kähler manifold $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature $4c$. Then, for any unit vectors $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, we have $||A_{JZ}X||^2 = -c$.*

Lemma 8 *Let N be a CR-submanifold of a Kähler manifold $\tilde{M}$ with totally real distribution $\mathcal{D}^\perp$. Then at each point $x \in N$ there exists an orthonormal basis $\{e_1, \dots, e_p\}$ of $\mathcal{D}_x^\perp$ such that the second fundamental form h of N in $\tilde{M}$ satisfies*

$$\langle A_{Je_1} e_1, e_i \rangle = 0, \quad i = 2, \dots, p. \tag{2.11}$$

Lemma 8 extends a result of [31].

For *CR*-warped products, we have [16]

Lemma 9 *For a CR-warped product $N^T \times_f N^\perp$ in a Kähler manifold $\tilde{M}$, we have*

(1) $\left\langle \sigma(\mathcal{D}, \mathcal{D}), J\mathcal{D}^\perp \right\rangle = 0$;
(2) $\nabla_X Z = \nabla_Z X = (X \ln f) Z$;
(3) $\langle \sigma(JX, Z), JW \rangle = (X \ln f) \langle Z, W \rangle$;
(4) $D_X(JZ) = J\nabla_X Z$*, whenever* $\sigma(\mathcal{D}, \mathcal{D}^\perp) \subset J\mathcal{D}^\perp$;
(5) $\left\langle \sigma(\mathcal{D}, \mathcal{D}^\perp), J\mathcal{D}^\perp \right\rangle = 0$ *if and only if* $N^T \times_f N^\perp$ *is a trivial CR-warped product in* $\tilde{M}$,

where $X, Y \in \mathcal{D}$ *and* $Z, W \in \mathcal{D}^\perp$.

2.4 Riemannian Submersions

A *Riemannian submersion* $\pi : M \to B$ is a map from a Riemannian manifold M onto another Riemannian manifold B such that π has maximal rank and the differential π_* preserves lengths of horizontal vectors. Throughout this article, we only consider Riemannian submersions $\pi : M \to B$ with $m > b > 0$, where $m = \dim M$ and $b = \dim B$.

For each $x \in B$, $\pi^{-1}(x)$ is an $(m - b)$-dimensional submanifold of M. The submanifolds $\pi^{-1}(x)$, $x \in B$, are called *fibers*. A vector field on M is called *vertical* if it is always tangent to fibers; and *horizontal* if it is always orthogonal to fibers. We use corresponding terminology for individual tangent vectors as well. A vector field on M is called *basic* if X is horizontal and π-related to a vector field X_* on B, i.e., $\pi_* X_u = X_{*\pi(u)}$, for all $u \in M$.

Let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of tangent spaces of M onto the subspaces of horizontal and vertical vectors, respectively. We use the same letters to denote the horizontal and vertical distributions.

Let g and g_B be the metric tensors of M and B, respectively, and g_F the induced metric on fibers. Denote by R_M, R_B and R_F the Riemann curvature tensors of the metrics g, g_B and g_F, respectively.

Associated with a Riemannian submersion $\pi : M \to B$, there exists a natural $(1, 2)$-tensor $\mathcal{A}$ on M, known as the O'Neill's integrability tensor, defined by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) \tag{2.12}$$

for vector fields E, F tangent to M. In particular, for a horizontal vector field X and a vertical vector field V, we have

$$\mathcal{A}_X V = \mathcal{H}\nabla_X V. \tag{2.13}$$

For horizontal vector fields X, Y, the tensor $\mathcal{A}$ has the alternation property

$$\mathcal{A}_X Y = -\mathcal{A}_Y X. \tag{2.14}$$

Associated with $\pi : M \to B$, the invariants $\breve{A}_\pi$ and $\mathring{A}_\pi$ on M are defined by (cf. [22])

$$\breve{A}_\pi = \sum_{i=1}^{b} \sum_{s=b+1}^{m} ||\mathcal{A}_{X_i} V_s||^2, \quad \mathring{A}_\pi = \sum_{1 \le i < j \le b} ||\mathcal{A}_{X_i} X_j||^2, \tag{2.15}$$

where $X_1, \ldots, X_b$ are orthonormal basic horizontal vector fields and $V_{b+1}, \ldots, V_m$ are orthonormal vertical vector fields.

The following lemma can be found in [22, 36].

Lemma 10 *For vector fields X, Y tangent to B, we have*
(1) $\langle \bar{X}, \bar{Y} \rangle = \langle X, Y \rangle \circ \pi$,
(2) $\mathcal{H}[\bar{X}, \bar{Y}] = [X, Y]^-$,
(3) $\mathcal{H}\nabla_{\bar{X}} \bar{Y} = (\nabla'_X Y)^-$, *where ∇' is the Levi-Civita connection of B,*
where $\bar{X}, \bar{Y}$ and $[X, Y]^-$ are the horizontal lifts of X, Y and $[X, Y]$, respectively.

Lemma 11 *Let X, Y be horizontal vector fields and E, F be vector fields on M. Then, each of the following holds:*
(a) $\mathcal{A}_X Y = -\mathcal{A}_Y X$, *or equivalently,* $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$,
(b) $\mathcal{A}_{\mathcal{H}E} F = \mathcal{A}_E F$,
(c) $\mathcal{A}_E$ *maps the horizontal subspace into the vertical one and the vertical subspace into the horizontal one.*

Lemma 12 *Let $\pi : M \to B$ be a pseudo-Riemannian submersion. Then*

$$R_B(\pi_* X, \pi_* Y; \pi_* Y, \pi_* X) = R_M(X, Y; Y, X) + 3||\mathcal{A}_X Y||^2. \tag{2.16}$$

Moreover, if π has totally geodesic fibers, then we also have
(1) $R_M(U, V; V, U) = R_F(U, V; V, U)$,
(2) $R_M(X, U; U, X) = ||\mathcal{A}_X U||^2$,
for horizontal vector filed X, Y and vertical vector field U.

Lemma 13 *For a Riemannian submersion $\pi : M \to B$, we have*

$$K_M(\bar{X}, \bar{Y}) = K_B(X, Y) - 3\,||\mathcal{A}_{\bar{X}} \bar{Y}||^2$$

for orthonormal vector fields X, Y on B.

Let $\mathbb{C}^{m+1}$ denote the complex Euclidean $(m+1)$-space and let

$$S^{2m+1} = \{z = (z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1} : \langle z, z\rangle = 1\}$$

be the unit hypersphere of $\mathbb{C}^{m+1}$. Consider the Hopf fibration

$$\pi : S^{2m+1}(c) \to CP^m(4c). \tag{2.17}$$

Then, π is a Riemannian submersion with totally geodesic fibers.

Given $z \in S^{2m+1}$, the vector $\xi = \mathrm{i}z$ is tangent to the fibers and the horizontal space at z is the orthogonal complement of $\mathrm{i}z$ with respect to the induced metric on S^{2m+1} from the standard metric on $\mathbb{C}^{m+1}$. Moreover, given a horizontal vector X, then $\mathrm{i}X$ is again horizontal and $\pi_*(\mathrm{i}X) = J(\pi_*(X))$, where J is the complex structure on $CP^m(4)$. It is well-known that S^{2m+1} is a Sasakian manifold with characteristic vector field ξ and with the contact structure obtained from the projection of the complex structure J of $\mathbb{C}^{m+1}$.

Let $\phi : N \to CP^m(4)$ be an isometric immersion of a Riemannian n-manifold N into $CP^m(4)$. Then, $\tilde{N} = \pi^{-1}(N)$ is a principal circle bundle over N with totally geodesic fibers and the lift $\tilde{\phi} : \tilde{N} \to S^{2m+1}$ of ϕ is an isometric immersion such that

$$\begin{array}{ccc}
\tilde{N} & \xrightarrow{\tilde{\phi}} & S^{2m+1} \\
{\scriptstyle \tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
N & \xrightarrow{\phi} & CP^m(4)
\end{array}$$

commutes. Since ξ generate the vertical subspaces of $\pi : S^{2m+1}(c) \to CP^m(4c)$, we have the orthogonal decomposition

$$T_z\tilde{N} = \overline{T_{\pi(z)}N} \oplus Span\{\xi\}.$$

2.5 Warped Product CR Submanifolds and δ-Invariants

Let M be a Riemannian n-manifold. Let $K(\pi)$ be the sectional curvature associated with a plane section $\pi \subset T_pM,\ p \in M$. For an orthonormal basis $e_1, \dots, e_n$ of T_pM, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

Let N be a CR-submanifold of a Kähler manifold. Denote by $\mathcal{D}$ and $\mathcal{D}^\perp$ the complex and the totally real distributions of N as before. The *CR δ-invariant* $\delta(\mathcal{D})$ is then defined by

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x), \tag{2.18}$$

where $\tau(x)$ and $\tau(\mathcal{D}_x)$ denote the scalar curvature of N at $x \in N$ and the scalar curvature of $\mathcal{D}_x \subset T_xN$, respectively (see [23] for details).

For a CR-warped product $N^T \times_f N^\perp$ in the complex space form $\tilde{M}^{h+p}(4c)$ with $h = \dim_C N^T$ and $p = \dim N^\perp$, let us choose a local orthonormal frame $\{e_1, \dots, e_{2h+p}\}$ on N such that $e_1, \dots, e_h, e_{h+1} = Je_1, \dots, e_{2h} = Je_h$ are in $\mathcal{D}$ and $e_{2h+1}, \dots, e_{2h+p}$ are in $\mathcal{D}^\perp$.

In the following, we shall use the following convention on the range of indices *unless mentioned otherwise*:

$$i, j, k = 1, \dots, 2h; \quad \alpha, \beta, \gamma = 1, \dots, h,$$
$$r, s, t = 2h+1, \dots, 2h+p; \quad A, B, C = 1, \dots, 2h+p.$$

Let us put $\sigma^r_{AB} = \langle \sigma(e_A, e_B), Je_r \rangle$.

It follows from Lemma 9(2) that we have

$$\frac{\Delta f}{f} = \sum_{j=1}^{2h} K(e_j \wedge e_r) \tag{2.19}$$

for each $r \in \{2h+1, \dots, 2h+p\}$.

The next theorem provides an optimal inequality for CR-warped submanifolds in complex space forms involving the CR δ-invariant.

Theorem 2.2 ([23]) *Let $N = N^T \times_f N^\perp$ be a CR-warped product in a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \dim_C N^T \geq 1$ and $p = \dim N^\perp \geq 2$. Then, we have*

$$H^2 \geq \frac{2(p+2)}{(2h+p)^2(p-1)}\left\{\delta(\mathcal{D}) - \frac{p\Delta f}{f} - \frac{p(p-1)c}{2}\right\}, \tag{2.20}$$

where Δf is the Laplacian of the warping function f and H^2 is the squared mean curvature.

The equality sign of (2.20) holds at a point $x \in N$ if and only if there exists an orthonormal basis $\{e_{2h+1}, \ldots, e_n\}$ of $\mathcal{D}_x^\perp$ such that the coefficients of the second fundamental σ with respect to $\{e_{2h+1}, \ldots, e_n\}$ satisfy

$$\begin{aligned} &\sigma_{rr}^r = 3\sigma_{ss}^r, && \text{for } 2h+1 \leq r \neq s \leq 2h+p, \\ &\sigma_{st}^r = 0, && \text{for distinct } r, s, t \in \{2h+1, \ldots, 2h+p\}. \end{aligned} \tag{2.21}$$

Proof Let $N = N^T \times_f N^\perp$ be a CR-warped product in a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \dim_C N^T \geq 1$ and $p = \dim N^\perp \geq 2$. Let us choose an orthonormal frame $\{e_1, \ldots, e_{2h+p}\}$ on N as above.

It follows from Gauss' equation, (2.18), and (2.19) that $\delta(\mathcal{D})$ satisfies

$$\begin{aligned} \delta(\mathcal{D}) &= \sum_{i,r} K(e_i, e_r) + \sum_{r<s} K(e_r, e_s) \\ &= \frac{p\Delta f}{f} + \sum_{r<s} \langle \sigma(e_r, e_r), \sigma(e_s, e_s)\rangle - \sum_{r<s} ||\sigma(e_r, e_s)||^2 + \frac{p(p-1)}{2}c. \end{aligned} \tag{2.22}$$

On the other hand, it follows from Lemma 9(1) and $J\mathcal{D}^\perp = T^\perp N$ that

$$\sum_{r<s} \langle \sigma(e_r, e_r), \sigma(e_s, e_s)\rangle - \sum_{r<s} ||\sigma(e_r, e_s)||^2 = \frac{n^2}{2}||H||^2 - \frac{1}{2}||\sigma_\perp||^2, \tag{2.23}$$

where $n = 2h + p$ and $||\sigma_\perp||^2$ is the squared norm of σ restricted to $\mathcal{D}^\perp$, i.e.

$$||\sigma_\perp||^2 = \sum_{r,s} ||\sigma(e_r, e_s)||^2. \tag{2.24}$$

By combining (2.22) and (2.23) we find

$$\delta(\mathcal{D}) = \frac{p\Delta f}{f} + \frac{p(p-1)}{2}c + \frac{n^2}{2}||H||^2 - \frac{1}{2}||\sigma_\perp||^2. \tag{2.25}$$

Thus we obtain

$$n^2||H||^2 + \frac{2(p+2)}{p-1}\left(\frac{p\Delta f}{f} - \delta(\mathcal{D})\right) + p(p+2)c$$
$$= \frac{3n^2}{1-p}||H||^2 + \frac{p+2}{p-1}||\sigma_\perp||^2. \tag{2.26}$$

From Lemma 1(a) we find $\sigma^r_{st} = \sigma^s_{rt} = \sigma^t_{rs}$. Now, we derive from (2.26) and Lemma 9(1) that

$$\begin{aligned}
&n^2||H||^2 + \frac{2(p+2)}{p-1}\left(\frac{p\Delta f}{f} - \delta(\mathcal{D})\right) + p(p+2)c \\
&= \sum_r \Big(\sum_s \sigma^r_{ss}\Big)^2 + \frac{3(p+1)}{p-1}\sum_{r\neq s}(\sigma^r_{ss})^2 + \frac{6(p+2)}{p-1}\sum_{r<s<t}(\sigma^r_{st})^2 \\
&\quad + \frac{2(p+2)}{p-1}\sum_r\sum_{s<t}\sigma^r_{ss}\sigma^r_{tt} \\
&= \sum_r(\sigma^r_{rr})^2 + \frac{3(p+1)}{p-1}\sum_{r\neq s}(\sigma^r_{ss})^2 + \frac{6(p+2)}{p-1}\sum_{r<s<t}(\sigma^r_{st})^2 \\
&\quad - \frac{6}{p-1}\sum_r\sum_{s<t}\sigma^r_{ss}\sigma^r_{tt} \\
&= \frac{6(p+2)}{p-1}\sum_{r<s<t}(\sigma^r_{st})^2 + \frac{3}{p-1}\sum_{r\neq s,t}\sum_{s<t}(\sigma^r_{ss} - \sigma^r_{tt})^2 \\
&\quad + \frac{1}{p-1}\sum_{s\neq r}(\sigma^r_{rr} - 3\sigma^r_{ss})^2 \\
&\geq 0.
\end{aligned} \tag{2.27}$$

Consequently, inequality (2.20) follows from (2.27). Moreover, it is easy to verify that the equality sign of (2.20) holds if and only if (2.21) holds.

All *CR*-warped products in the complex Euclidean $(h+p)$-space $\mathbb{C}^{h+p}$ satisfying the equality case of inequality (2.20) identically have been completely classified in [23] as follows.

Theorem 2.3 *Let $\psi : N^T \times_f N^\perp \to \mathbb{C}^{h+p}$ be a CR-warped product in $\mathbb{C}^{h+p}$ with $h = \dim_C N^T \geq 1$ and $p = \dim N^\perp \geq 2$. Then*

$$H^2 \geq \frac{2(p+2)}{(2h+p)^2(p-1)}\left\{\delta(\mathcal{D}) - \frac{p\Delta f}{f}\right\}. \tag{2.28}$$

The equality sign of (2.28) holds identically if and only if, up to dilations and rigid motions of $\mathbb{C}^{h+p}$, one of the following three cases occurs:

(a) *The CR-warped product is an open part of the CR-product* $\mathbb{C}^h \times W^p \subset \mathbb{C}^h \times \mathbb{C}^p$, *where* W^p *is the Whitney p-sphere in* $\mathbb{C}^p$*;*

(b) N^T *is an open part of* $\mathbb{C}^h$, $N^\perp$ *is an open part of the unit p-sphere* S^p, $f = |z_1|$ *and* ψ *is the minimal immersion defined by*

$$\big(z_1 w_0,\ \cdots, z_1 w_p, z_2, \ldots, z_h\big),$$

where $z = (z_1, \ldots, z_h) \in \mathbb{C}^h$ *and* $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}$*;*

(c) N^T *is an open part of* $\mathbb{C}^h$, $N^\perp$ *is the warped product of a curve and an open part of* S^{p-1} *with warping function* $\varphi = (\sqrt{c^2-1}/\sqrt{2})\mathrm{cn}\big(c\,t, \sqrt{c^2-1}/\sqrt{2}c\big)$, $c > 1, f = |z_1|$, *and* ψ *is the non-minimal immersion defined by*

$$\left(z_1 e^{\int \frac{\varphi(\varphi' + \mathrm{i}k\varphi^2)}{\varphi^2-1}dt}, z_1\varphi e^{\mathrm{i}k\int\varphi dt} w_1, \cdots z_1\varphi e^{\mathrm{i}k\int\varphi dt} w_p, z_2, \ldots, z_h\right),$$

with $z = (z_1, \ldots, z_h) \in \mathbb{C}^h$, $(w_1, \ldots, w_p) \in S^{p-1}(1) \subset \mathbb{E}^p$, *and* $k = \sqrt{c^4-1}/2$, *where* cn *is a Jacobi's elliptic function.*

For the proof of Theorem 2.3, see [23].

2.6 Anti-holomorphic Submanifolds with $p \geq 2$

For a *CR*-submanifold N of a Kähler manifold, the two partial mean curvature vectors $\overrightarrow{H}_{\mathcal{D}}$ and $\overrightarrow{H}_{\mathcal{D}^\perp}$ of N are defined by

$$\overrightarrow{H}_{\mathcal{D}} = \frac{1}{2h}\sum_{i=1}^{2h}\sigma(e_i, e_i), \qquad \overrightarrow{H}_{\mathcal{D}^\perp} = \frac{1}{p}\sum_{r=2h+1}^{2h+p}\sigma(e_r, e_r). \tag{2.29}$$

An anti-holomorphic submanifold N of a Kähler manifold $\tilde{M}$ is called *minimal* (resp., $\mathcal{D}$-*minimal* or $\mathcal{D}^\perp$-*minimal*) if $H = 0$ holds identical (resp., $\overrightarrow{H}_{\mathcal{D}} = 0$ or $\overrightarrow{H}_{\mathcal{D}^\perp} = 0$ hold identically).

For anti-holomorphic submanifolds with $p = \mathrm{rank}\,\mathcal{D}^\perp \geq 2$, we have the following optimal inequality.

Theorem 2.4 ([2]) *Let N be an anti-holomorphic submanifold of a complex space form* $\tilde{M}^{h+p}(4c)$ *with* $h = \mathrm{rank}_{\mathbf{C}}\,\mathcal{D} \geq 1$ *and* $p = \mathrm{rank}\,\mathcal{D}^\perp \geq 2$. *Then we have*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(2h+p)^2}{2(p+2)}H^2 + \frac{p}{2}(4h+p-1)c. \tag{2.30}$$

The equality sign of inequality (2.30) holds identically if and only if the following three conditions are satisfied:

(a) *N is $\mathcal{D}$-minimal, i.e., $\overrightarrow{H}_{\mathcal{D}} = 0$,*
(b) *N is mixed totally geodesic, and*
(c) *there exist an orthonormal frame $\{e_{2h+1}, \ldots, e_n\}$ of $\mathcal{D}^{\perp}$ such that the second fundamental σ of N satisfies*

$$\begin{cases} \sigma^r_{rr} = 3\sigma^r_{ss}, & \text{for } 2h+1 \le r \ne s \le 2h+p, \\ \sigma^r_{st} = 0, & \text{for distinct } r, s, t \in \{2h+1, \ldots, 2h+p\}. \end{cases} \tag{2.31}$$

Proof Let N be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{h+p}(4c)$. Let us choose an orthonormal frame $\{e_1, \ldots, e_{2h+p}\}$ on N as above.

It follows from the equation of Gauss and the definition of *CR* δ-invariant that $\delta(\mathcal{D})$ satisfies

$$\begin{aligned} \delta(\mathcal{D}) &= \sum_{i=1}^{2h} \sum_{r=2h+1}^{2h+p} K(e_i, e_r) + \sum_{2h+1 \le r \ne s \le 2h+p} \frac{1}{2} K(e_r, e_s) \\ &= \sum_{i=1}^{2h} \sum_{r=2h+1}^{2h+p} \langle \sigma(e_i, e_i), \sigma(e_r, e_r) \rangle + \sum_{r,s=2h+1}^{2h+p} \frac{1}{2} \langle \sigma(e_r, e_r), \sigma(e_s, e_s) \rangle \\ &\quad - \sum_{i=1}^{2h} \sum_{r=2h+1}^{2h+p} ||\sigma(e_i, e_r)||^2 - \sum_{r,s=2h+1}^{2h+p} \frac{1}{2} ||\sigma(e_r, e_s)||^2 + \frac{p}{2}(4h+p-1)c. \end{aligned} \tag{2.32}$$

On the other hand, we have

$$\begin{aligned} &\sum_{i=1}^{2h} \sum_{r=2h+1}^{2h+p} \langle \sigma(e_i, e_i), \sigma(e_r, e_r) \rangle + \sum_{r,s=2h+1}^{2h+p} \frac{1}{2} \langle \sigma(e_r, e_r), \sigma(e_s, e_s) \rangle \\ &\quad - \sum_{r,s=2h+1}^{2h+p} \frac{1}{2} ||\sigma(e_r, e_s)||^2 \\ &\quad = \frac{(2h+p)^2}{2} H^2 - 2h^2 |\overrightarrow{H}_{\mathcal{D}}|^2 - \frac{1}{2} ||\sigma_{\mathcal{D}^{\perp}}||^2, \end{aligned} \tag{2.33}$$

where $||\sigma_{\mathcal{D}^{\perp}}||^2$ is defined by

$$||\sigma_{\perp}||^2 = \sum_{r,s=2h+1}^{2h+p} ||\sigma(e_r, e_s)||^2. \tag{2.34}$$

By combining (2.32) and (2.33) we find

$$\delta(\mathcal{D}) = \frac{(2h+p)^2}{2}H^2 + \frac{p}{2}(4h+p-1)c - 2h^2|\overrightarrow{H}_{\mathcal{D}}|^2 \\ - \sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2 - \frac{1}{2}||\sigma_{\mathcal{D}^\perp}||^2. \tag{2.35}$$

It follows from statement (2) of Lemma 1 the coefficients of the second fundamental form satisfy

$$\sigma^r_{st} = \sigma^s_{rt} = \sigma^t_{rs}. \tag{2.36}$$

We find from (2.29), (2.34), and (2.36) that

$$\begin{aligned}
&(p+2)||\sigma_{\mathcal{D}^\perp}||^2 - 3p^2|H_{\mathcal{D}^\perp}|^2 \\
&= (p-1)\sum_{r=2h+1}^{2h+p}\Big(\sum_{s=2h+1}^{2h+p}\sigma^r_{ss}\Big)^2 \\
&\quad + \sum_{2h+1\le r\neq s\le 2h+p} 3(p+1)(\sigma^r_{ss})^2 + \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 \\
&\quad + \sum_{r=2h+1}^{2h+p}\sum_{2h+1\le s<t\le 2h+p} 2(p+2)\sigma^r_{ss}\sigma^r_{tt} \\
&= \sum_{r=2h+1}^{2h+p}(p-1)(\sigma^r_{rr})^2 + \sum_{2h+1\le r\neq s\le 2h+p} 3(p+1)(\sigma^r_{ss})^2 \\
&\quad + \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 - \sum_{r=2h+1}^{2h+p}\sum_{2h+1\le s<t\le 2h+p} 6\sigma^r_{ss}\sigma^r_{tt} \\
&= \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 + \sum_{2h+1\le s\neq r\le 2h+p}(\sigma^r_{rr} - 3\sigma^r_{ss})^2 \\
&\quad + \sum_{r\neq s,t}\sum_{2h+1\le s<t\le 2h+p} 3(\sigma^r_{ss} - \sigma^r_{tt})^2 \\
&\ge 0.
\end{aligned} \tag{2.37}$$

Thus, we get

$$||\sigma_{\mathcal{D}^\perp}||^2 \ge \frac{3p^2}{p+2}|H_{\mathcal{D}^\perp}|^2, \tag{2.38}$$

with equality holding if and only if

$$\begin{aligned}
&\sigma^r_{rr} = 3\sigma^r_{ss}, \qquad \text{for } 2h+1 \le r \neq s \le 2h+p, \\
&\sigma^r_{st} = 0, \qquad \text{for distinct } r,s,t \in \{2h+1,\dots,2h+p\}.
\end{aligned} \tag{2.39}$$

Now, by combining (2.35) and (2.38), we obtain

$$
\begin{aligned}
&\frac{(2h+p)^2}{2}H^2 + \frac{p}{2}(4h+p-1)c - \delta(\mathcal{D}) \\
&\geq 2h^2|\overrightarrow{H}_{\mathcal{D}}|^2 + \sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p} ||\sigma(e_i, e_r)||^2 + \frac{3p^2}{2(p+2)}|H_{\mathcal{D}^\perp}|^2 \\
&= \frac{3}{2(p+2)}\left\{(2h+p)^2H^2 - 4h^2|\overrightarrow{H}_{\mathcal{D}}|^2 - 2\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p} ||\sigma(e_i, e_r)||^2\right\} \\
&\quad + 2h^2|\overrightarrow{H}_{\mathcal{D}}|^2 + \sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p} ||\sigma(e_i, e_r)||^2. \\
&= \frac{3(2h+p)^2}{2(p+2)}H^2 + \frac{2h^2(p-1)}{p+2}|\overrightarrow{H}_{\mathcal{D}}|^2 + \frac{p-1}{p+2}\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p} ||\sigma(e_i, e_r)||^2 \\
&\geq \frac{3(2h+p)^2}{2(p+2)}H^2. \qquad (2.40)
\end{aligned}
$$

It is obvious that the equality of the last inequality in (2.40) holds if and only if N is $\mathcal{D}$-minimal and mixed totally geodesic. Consequently, we may obtain inequality (2.30) from (2.40).

It is straightforward to verify that the equality sign of (2.30) holds identically if and only if conditions (a), (b) and (c) of Theorem 2.4 are satisfied.

2.7 Anti-holomorphic Submanifolds with Equality in (2.30)

The notion of H-umbilical Lagrangian submanifolds was introduced in [12, 13].

Definition 1 A Lagrangian submanifold is said to be *H-umbilical* if its second fundamental form satisfies the following simple form:

$$
\begin{aligned}
&\sigma(e_1, e_1) = \lambda J e_1, \quad \sigma(e_1, e_j) = \mu J e_j, \\
&\sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = \mu J e_1, \\
&\sigma(e_j, e_k) = 0, \quad j \neq k, \quad j, k = 2, \ldots, n,
\end{aligned}
\qquad (2.41)
$$

for some suitable functions φ and ψ with respect to some suitable orthonormal local frame field $\{e_1, \ldots, e_n\}$.

Since there do not exist umbilical Lagrangian submanifold in Kähler manifolds other than totally geodesic ones (cf. [26]), H-umbilical Lagrangian submanifolds are the simplest Lagrangian submanifolds next to totally geodesic one (cf. [12, 13]).

Let $G : N^{p-1} \to \mathbb{E}^p$ be an isometric immersion of a Riemannian $(p-1)$-manifold into the Euclidean p-space $\mathbb{E}^p$ and let $F : I \to \mathbb{C}^*$ be a unit speed curve in $\mathbb{C}^* = \mathbb{C} - \{0\}$. We extend $G : N^{p-1} \to \mathbb{E}^p$ to an immersion of $I \times N^{p-1}$ into $\mathbb{C}^p$ as

$$F \otimes G : I \times N^{p-1} \to \mathbb{C} \otimes \mathbb{E}^p = \mathbb{C}^p, \tag{2.42}$$

where $(F \otimes G)(s, q) = F(s) \otimes G(q)$ for $s \in I,\ q \in N^{p-1}$. This extension $F \otimes G$ of G via tensor product is called the *complex extensor* of G via F.

Example 1 (*Whitney sphere*) Let $w : S^p(1) \to \mathbb{C}^p$ be the map of the unit p-sphere into $\mathbb{C}^p$ defined by

$$w(y_0, y_1, \dots, y_p) = \frac{1 + \mathrm{i}y_0}{1 + y_0^2}(y_1, \dots, y_p), \quad y_0^2 + y_1^2 + \cdots + y_p^2 = 1.$$

The map w is a (non-isometric) Lagrangian immersion with one self-intersection point which is called the *Whitney p-sphere.* The Whitney p-sphere is a complex extensor $\phi = F \otimes \iota$ of $\iota : S^{p-1}(1) \subset \mathbb{E}^p$ via F, where $F = F(s)$ is an arclength reparametrization of the curve $f : I \to \mathbb{C}$ defined by

$$f(t) = \frac{\sin t + \mathrm{i} \sin t \cos t}{1 + \cos^2 t}.$$

Up to rigid motions and dilations, the Whitney sphere is the only Lagrangian H-umbilical submanifold in $\mathbb{C}^p$ which satisfies (2.41) with $\lambda = 3\mu$ (see [12]).

Consider the product immersion

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^h \oplus \mathbf{C}^p = \mathbf{C}^{h+p}$$

defined by

$$\phi(z, x) = (z, w(x)), \quad \forall z \in \mathbf{C}^h, \quad \forall x \in S^p(1). \tag{2.43}$$

It is straightforward to verify that ϕ is an anti-holomorphic isometric immersion which satisfies the equality sign of (2.30) identically.

Anti-holomorphic submanifolds satisfying the equality case of inequality (2.30) were classified by the following two theorems.

Theorem 2.5 ([2]) *Let N be an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \mathrm{rank}_{\mathbf{C}}\, \mathcal{D} \geq 1$ and $p = \mathrm{rank}\, \mathcal{D}^{\perp} \geq 2$. If N satisfies the equality case of (2.30) identically and if the complex distribution $\mathcal{D}$ is integrable, then $c = 0$ so that $\tilde{M}^{h+p}(4c) = \mathbf{C}^{h+p}$. Moreover, we have either*

(i) *N is a totally geodesic anti-holomorphic submanifold of* $\mathbf{C}^{h+p}$ *or,*
(ii) *up to dilations and rigid motions of* $\mathbf{C}^{h+p}$*, N is given by an open portion of the following product immersion:*

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^{h+p}; \quad (z, x) \mapsto (z, w(x)), \quad z \in \mathbf{C}^h, \ x \in S^p(1),$$

where $w : S^p(1) \to \mathbf{C}^p$ *is the Whitney p-sphere.*

Proof Assume that N is an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \operatorname{rank}_{\mathbf{C}} \mathcal{D} \geq 1$ and $p = \operatorname{rank} \mathcal{D}^{\perp} \geq 2$. If N satisfies the equality case of (2.30) and if the complex distribution $\mathcal{D}$ is integrable, then it follows from Theorem 2.4 that N is mixed foliate. Hence Lemma 4 implies that $c = 0$. Thus, according to Lemma 5, N is a *CR*-product. Therefore, N is locally a *CR*-product given by

$$\mathbf{C}^h \times N^{\perp} \subset \mathbf{C}^h \times \mathbf{C}^p,$$

where $\mathbf{C}^h$ is a complex Euclidean h-subspace and $N^{\perp}$ is a Lagrangian submanifold of $\mathbf{C}^p$. Consequently, condition (c) of Theorem 2.4 implies that $N^{\perp}$ is a Lagrangian H-umbilical submanifold in $\mathbf{C}^p$ whose second fundamental form satisfying

$$\begin{aligned}
&\sigma(e_{2h+1}, e_{2h+1}) = 3\lambda J e_{2h+1}, \\
&\sigma(e_{2h+1}, e_s) = \lambda J e_s, \\
&\sigma(e_{2h+2}, e_{2h+2}) = \cdots = \sigma(e_{2h+p}, e_{2h+p}) = \lambda J e_{2h+1}, \\
&\sigma(e_r, e_s) = 0, \quad 2h + 2 \leq r \neq s \leq 2h + p,
\end{aligned} \tag{2.44}$$

for some suitable function λ with respect to some suitable orthonormal local frame field $\{e_{2h+1}, \ldots, e_{2h+p}\}$ of $TN^{\perp}$.

If $\lambda = 0$, then $N^{\perp}$ is an open portion of a totally geodesic totally real p-plane in $\mathbf{C}^p$. Hence, in this case, N is a totally geodesic anti-holomorphic submanifold.

If $\lambda \neq 0$, it follows from (2.44) that, up to dilations and rigid motions, $N^{\perp}$ is an open part of the Whitney p-sphere in $\mathbf{C}^p$ (cf. [7, 22]). Consequently, up to dilations and rigid motions of $\mathbf{C}^{h+p}$, the anti-holomorphic submanifold is locally given by the product immersion

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^{h+p}; \qquad (z, x) \mapsto (z, w(x)), \tag{2.45}$$

for $z \in \mathbf{C}^h$ and $x \in S^p(1)$, where $w : S^p(1) \to \mathbf{C}^p$ is the Whitney p-sphere.

The converse is easy to verify.

Theorem 2.6 ([2]) *Let N be an anti-holomorphic submanifold in a complex space form* $\tilde{M}^{1+p}(4c)$ *with* $h = \operatorname{rank}_{\mathbf{C}} \mathcal{D} = 1$ *and* $p = \operatorname{rank} \mathcal{D}^{\perp} \geq 2$*. Then, we have*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(p+2)^2}{2(p+2)} H^2 + \frac{p}{2}(p+3)c. \tag{2.46}$$

The equality case of (2.46) holds identically if and only if $c = 0$ *and either*

(i) *N is a totally geodesic anti-holomorphic submanifold of* $\mathbf{C}^{h+p}$ *or,*
(ii) *up to dilations and rigid motions, N is given by an open portion of the following product immersion:*

$$\phi : \mathbf{C} \times S^p(1) \to \mathbf{C}^{1+p}; \qquad (z, x) \mapsto (z, w(x)), \quad z \in \mathbf{C}, \; x \in S^p(1),$$

where $w : S^p(1) \to \mathbf{C}^p$ *is the Whitney p-sphere.*

Proof Let N be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{1+p}(4c)$. Then we have inequality (2.44) from inequality (2.30).

Assume that N satisfies the equality case of (2.46) identically. Then, Theorem 2.4 implies that N satisfies conditions (a), (b), and (c) of Theorem 2.4.

By condition (a), N is $\mathcal{D}$-minimal. Thus, we find

$$\sigma(Je_1, Je_1) = -\sigma(e_1, e_1) \tag{2.47}$$

for any unit vector $e_1 \in \mathcal{D}$. It is direct to verify from (2.47) and polarization that the second fundamental form satisfies the following condition:

$$\sigma(X, JY) = \sigma(JX, Y), \quad \forall X, Y \in \mathcal{D}.$$

Therefore, according to Lemma 2(1), we may conclude that $\mathcal{D}$ is integrable. Consequently, we obtain Theorem 2.6 from Theorem 2.5.

2.8 An Optimal Inequality for Real Hypersurfaces

Obviously, anti-holomorphic submanifolds with rank $\mathcal{D}^\perp = 1$ are nothing but real hypersurfaces. A real hypersurface N of a Kähler manifold $\tilde{M}$ is called a *Hopf hypersurface* if $J\xi$ is a principal curvature vector, i.e., an eigenvector of the shape operator A_ξ, where ξ is a unit normal vector of N. In the following, we call a Hopf hypersurface N *special* if the vector field $J\xi$ is an eigenvector field of A_ξ with eigenvalue 0, i.e., $A_\xi(J\xi) = 0$.

For real hypersurfaces, we have the following:

Theorem 2.7 ([2]) *If N is a real hypersurface of a complex space form* $\tilde{M}^{h+1}(4c)$, *then the Ricci tensor Ric of N satisfies*

$$Ric(J\xi, J\xi) \leq \frac{(2h+1)^2}{2} H^2 + 2hc. \tag{2.48}$$

where ξ *is a unit normal vector field of N in* $\tilde{M}^{h+1}(4c)$.

The equality sign of inequality (2.48) holds identically if and only if N is a minimal special Hopf hypersurface.

Proof Let N be a real hypersurface of a complex space form $\tilde{M}^{h+1}(4c)$. Then it follows from the definition of $\delta(\mathcal{D})$ that

$$\delta(\mathcal{D}) = Ric(J\xi, J\xi). \tag{2.49}$$

Let us choose an orthonormal frame

$$\{e_1, \dots, e_h, e_{h+1} = Je_1, \dots, e_{2h} = Je_h\}$$

for the complex distribution $\mathcal{D}$ and let e_{2h+1} be a unit vector field in $\mathcal{D}^\perp$.

We put

$$\sigma_{a,b} = \langle \sigma(e_a, e_b), Je_{2h+1} \rangle\,, \quad a, b = 1, \dots, 2h+1. \tag{2.50}$$

Let us define the connection forms by

$$\begin{aligned}\nabla_X e_i &= \sum_{j=1}^{2h} \omega_i^j(X) e_j + \omega_i^{2h+1}(X) e_{2h+1}, \\ \nabla_X e_{2h+1} &= \sum_{j=1}^{2h} \omega_{2h+1}^j(X) e_j,\end{aligned} \tag{2.51}$$

for $i = 1, \dots, 2h$. It follows from (2.18) and the equation of Gauss that

$$\delta(\mathcal{D}) = \sum_{i=1}^{2h} \sigma_{i,i}\sigma_{2h+1,2h+1} - \sum_{i=1}^{2h} (\sigma_{i,2h+1})^2 + 2hc. \tag{2.52}$$

On the other hand, we have

$$\sum_{i=1}^{2h} \sigma_{i,i}\sigma_{2h+1,2h+1} = \frac{(2h+1)^2}{2} H^2 - \frac{1}{2}(\sigma_{2h+1,2h+1})^2 - 2h^2 |\overrightarrow{H}_{\mathcal{D}}|^2. \tag{2.53}$$

By combining (2.52) and (2.53) we obtain

$$\begin{aligned}\delta(\mathcal{D}) &= \frac{(2h+1)^2}{2} H^2 + 2hc - 2h^2 |\overrightarrow{H}_{\mathcal{D}}|^2 - \frac{1}{2}(\sigma_{2h+1,2h+1})^2 \\ &\quad - \sum_{i=1}^{2h} (\sigma_{i,2h+1})^2 \\ &\le \frac{(2h+1)^2}{2} H^2 + 2hc.\end{aligned} \tag{2.54}$$

It follows from (2.54) and Lemma 3(2) that the equality sign of inequality (2.48) holds identically if and only if the following two statements hold:

(i) N is a special Hopf hypersurface and
(ii) N is $\mathcal{D}$-minimal in $\tilde{M}^{h+1}(4c)$.

Obviously, conditions (i) and (ii) imply that N is a minimal real hypersurface of $\tilde{M}^{h+1}(4c)$.

The converse is easy to verify.

The following corollary follows from Theorem 2.7.

Corollary 6 ([2]) *Let N be a real hypersurface of a complex space form $\tilde{M}^{h+1}(4c)$. If N satisfies the equality case of (2.48) identically, then the complex distribution of N is non-integrable, unless $c = 0$ and N is totally geodesic.*

Proof Under the hypothesis, if N satisfies the equality case of (2.48) identically and if the complex distribution $\mathcal{D}$ is integrable, then Theorem 2.7 implies that N is mixed foliate. So, it follows from Lemmas 4 and 5 that $c = 0$ and N is a CR-product of a complex h-subspace in $\mathbf{C}^h$ and an open portion of line in $\mathbf{C}$. Consequently, N must be totally geodesic.

The following results are some further applications of Theorem 2.7

Theorem 2.8 ([2]) *If N is a real hypersurface of $\tilde{M}^2(4c)$, then we have*

$$Ric(J\xi, J\xi) \leq \frac{9}{2}H^2 + 2c. \tag{2.55}$$

The equality sign of inequality (2.55) holds identically if and only if $c = 0$ and N is totally geodesic.

Theorem 2.9 ([2]) *Let N be a real hypersurface of $\mathbf{C}^3$. We have*

$$Ric(J\xi, J\xi) \leq \frac{25}{2}H^2. \tag{2.56}$$

If the equality case of inequality (2.56) holds identically, then N is a totally real 3-ruled minimal submanifold of $\mathbf{C}^3$.

Theorem 2.10 ([2]) *If N is a real hypersurface of $CP^3(4)$, then we have*

$$Ric(J\xi, J\xi) \leq \frac{25}{2}H^2 + 4. \tag{2.57}$$

The equality sign of inequality (2.57) holds identically if and only if locally there exists an orthonormal frame $\{e_1, e_2, e_3 = Je_1, e_4 = Je_2, e_5\}$ such that

$$\sigma(e_1,e_1)=\lambda\xi,\ \sigma(e_2,e_2)=-\lambda\xi,$$
$$\sigma(e_3,e_3)=\frac{1}{\lambda}\xi,\ \sigma(e_4,e_4)=-\frac{1}{\lambda}\xi,$$
$$\sigma(e_a,e_b)=0\ \textit{otherwise},$$

where λ is a nowhere zero function.

Theorem 2.11 ([2]) *If N is a real hypersurface of $CH^3(-4)$, then we have*

$$Ric(J\xi,J\xi)\leq\frac{25}{2}H^2-4. \tag{2.58}$$

The equality sign of inequality (2.58) holds identically if and only if locally there exists an orthonormal frame $\{e_1,e_2,e_3=Je_1,e_4=Je_2,e_5\}$ on N such that

$$\sigma(e_1,e_1)=\lambda\xi,\ \sigma(e_2,e_2)=-\lambda\xi,$$
$$\sigma(e_3,e_3)=-\frac{1}{\lambda}\xi,\ \sigma(e_4,e_4)=\frac{1}{\lambda}\xi,$$
$$\sigma(e_a,e_b)=0\ \textit{otherwise},$$

where λ is a nowhere zero function.

Corollary 7 ([2]) *Every real hypersurface of $CP^3(4)$ (resp., of $CH^3(-4)$) satisfying the equality case of (2.57) (resp., the equality case of (2.58)) is $\delta(2,2)$-ideal in the sense of [15, 22].*

For the proofs of the above, see [2].

2.9 An Inequality Involving a Submersion δ-Invariant

Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers and let N be a Riemannian n-manifold isometrically immersed in B. Denote the pre-image $\pi^{-1}(N)$ of N in M by $\tilde{N}$. Then $\tilde{\pi} : \tilde{N} \to N$ is also a Riemannian submersion with totally geodesic fibers, where $\tilde{\pi}$ is the restriction $\pi|_{\tilde{N}}$.

For a horizontal 2-plane $P_x \subset T_x\tilde{N}$ we denote the $(m-b+2)$–subspace spanned by P_x and the vertical $\mathcal{V}_x$ by $\bar{P}_x$. The *submersion δ-invariant* δ^H on $\widetilde{N}$ is defined by (cf. [1]):

$$\delta^H(x)=\tau_{\tilde{N}}(x)-\inf_{\bar{P}_x}\tau_{\tilde{N}}(\bar{P}_x), \tag{2.59}$$

where $\bar{P}_x$ runs over $(m-b+2)$–subspaces associated with all horizontal 2-planes P_x at $x\in\tilde{N}$. Obviously, we have $\delta_{\tilde{N}}(r)\geq\delta^H$ with $r=2+m-b$.

Lemma 14 ([1]) *Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. Then the scalar curvature τ_M of M and the scalar curvature τ_B of B satisfy*

$$\tau_M = \tau_B + \breve{A}_\pi - 3\mathring{A}_\pi + \tau_F, \tag{2.60}$$

where τ_F is the scalar curvature of fibers.

Proof Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. For orthonormal basic horizontal vector fields $X_1, \dots, X_b$ and orthonormal vertical vector fields $V_{b+1}, \dots, V_m$ on M, it follows from Lemma 12(2) that

$$K_M(X_i \wedge V_\alpha) = ||\mathcal{A}_{X_i} V_\alpha||^2. \tag{2.61}$$

Also, it follows from Lemma 4 and (2.15) that the scalar curvature $\tau(\mathcal{H})$ of the horizontal space satisfies

$$\tau(\mathcal{H}) = \tau_B - 3\mathring{A}_\pi. \tag{2.62}$$

Moreover, since π has totally geodesic fibers, the scalar curvature τ_F equals the scalar curvature $\tau(\mathcal{V})$ of the vertical distribution. Consequently, we obtain (2.60) from (2.7), (2.61) and (2.62).

Lemma 15 *Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers and N be a submanifold of B. Then, for orthonormal vectors e_1, e_2 at $\pi(x) \in N$, $x \in \tilde{N}$, we have*

$$\begin{aligned} \tau_{\tilde{N}}(x) - \tau_{\tilde{N}}(\bar{P}_x) &= \tau_N - K_N(e_1, e_2) - 3(\mathring{A}_{\tilde{\pi}} - ||\mathcal{A}_{\bar{e}_1}\bar{e}_2||^2) \\ &\quad + \breve{A}_{\tilde{\pi}} - \sum_{i=1}^{2}\sum_{\alpha=1}^{m-b} K_{\tilde{N}}(\bar{e}_i, v_\alpha), \end{aligned} \tag{2.63}$$

where $\bar{e}_1, \bar{e}_2$ are horizontal vectors at x, $\{v_1, \dots, v_{m-b}\}$ is an orthonormal basis of the vertical space $\mathcal{V}_x$, and $\bar{P}_x$ is the subspace spanned by $\bar{e}_1, \bar{e}_2$ and $\mathcal{V}_x$.

Proof Under the hypothesis, $\tilde{\pi} : \tilde{N} = \pi^{-1}(N) \to N$ is a Riemannian submersion with totally geodesic fibers. Thus it follows from Lemma 14 that

$$\tau_{\tilde{N}} = \tau_N + \breve{A}_{\tilde{\pi}} - 3\mathring{A}_{\tilde{\pi}} + \tau_F. \tag{2.64}$$

Let $x \in \tilde{N}$ and e_1, e_2 orthonormal vectors at $\pi(x) \in N$. Denote by $\bar{e}_1, \bar{e}_2$ the horizontal lifts of e_1, e_2 at $x \in \tilde{N}$. As before let $\bar{P}_x$ denote the subspace of $T_x\tilde{N}$ spanned by $\bar{e}_1, \bar{e}_2$ and $\mathcal{V}_x$. Then we have

$$\tau_{\tilde{N}}(\bar{P}_x) = \tau_F + K_{\tilde{N}}(\bar{e}_1, \bar{e}_2) + \sum_{i=1}^{2}\sum_{\alpha=1}^{m-b} K_{\tilde{N}}(\bar{e}_i, v_\alpha). \tag{2.65}$$

From Lemma 4 we find

$$K_{\tilde{N}}(\bar{e}_1, \bar{e}_2) = K_N(e_1, e_2) - 3\,||\mathcal{A}_{\bar{e}_1}\bar{e}_2||^2. \tag{2.66}$$

By combining (2.64)–(2.66) and Lemma 10(2), we obtain (2.63).

Let N be a Riemannian submanifold of a Kähler manifold. For $X \in TN$ we put

$$JX = PX + FX, \tag{2.67}$$

where PX and FX are the tangential and normal components of JX, respectively. It follows from $J^2 = -I$ and (2.67) that

$$\langle PX, Y\rangle = -\langle X, PY\rangle \tag{2.68}$$

for X, Y tangent to N.

Let ψ be a 2-plane section of $T_{\bar{x}}N$, $\bar{x} \in N$, spanned by two orthonormal vectors $e_1, e_2 \in T_{\bar{x}}N$. We put

$$\Theta(\psi) = \langle Pe_1, e_2\rangle^2\,. \tag{2.69}$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal frame of N, then the squared norms $||P||^2$ and $||F||^2$ of P and F are defined, respectively, by

$$||P||^2 = \sum_{i=1}^{n} ||Pe_i||^2, \quad ||F||^2 = \sum_{i=1}^{n} ||Fe_i||^2. \tag{2.70}$$

Lemma 16 ([11]) *Let $\phi : N \to CP^m(4)$ be an isometric immersion from a Riemannian n-manifold N into the complex projective m-space $CP^m(4)$. Then, for any 2-plane section $\psi \subset T_yN$, $y \in N$, we have*

$$\tau_N - K_N(\psi) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{(n+1)(n-2)}{2} + \frac{3}{2}||P||^2 - 3\Theta(\psi). \tag{2.71}$$

The equality of inequality (2.71) holds at a point $y \in N$ if and only if there is an orthonormal basis $e_1, \ldots, e_m$ at y such that
(i) $\psi = \mathrm{Span}\{e_1, e_2\}$ *and*
(ii) *the shape operator A at y satisfies*

$$A_{e_s} = \begin{pmatrix} B_s & 0 \\ 0 & \mu_s I \end{pmatrix}, \quad s = n+1, \ldots, 2m, \tag{2.72}$$

where I is an identity $(n-2) \times (n-2)$-submatrix and B_s are symmetric 2×2 submatrices with $\mu_s = \mathrm{trace}\, B_s$, $s = n+1, \ldots, 2m$.

An important application of Lemma 15 is the following.

Theorem 2.12 ([1]) *Let $\pi : S^{2m+1} \to CP^m(4)$ be the Hopf fibration and let N be an n-dimensional submanifold of $CP^m(4)$. Then we have*

$$\delta^H \leq \frac{n^2(n-2)}{2(n-1)}||H||^2 + ||P||^2 + \frac{1}{2}(n^2 - n - 2), \tag{2.73}$$

where $||H||^2$ is the squared mean curvature of N in $CP^m(4)$.

The equality sign of (2.73) holds identically if and only if there is an orthonormal frame $e_1, \ldots, e_m$ such that
(a) *the shape operator A of N in $CP^m(4)$ satisfies*

$$A_{e_s} = \begin{pmatrix} B_s & 0 \\ 0 & \mu_s I \end{pmatrix}, \quad s = n+1, \ldots, 2m, \tag{2.74}$$

where I is an identity $(n-2) \times (n-2)$ matrix and B_s are symmetric 2×2 submatrices satisfying $\mu_s = \mathrm{trace}\, B_s$, $s = n+1, \ldots, 2m$, and
(b) $Pe_1 = Pe_2 = 0$.

Proof Let $\pi : S^{2m+1} \to CP^m(4)$ be the Hopf fibration and put $\xi = \mathrm{i}z$ as before. Denote by $\hat{\nabla}$ and $\check{\nabla}$ the Levi-Civita connections of S^{2m+1} and $CP^m(4)$, respectively. For vector fields X, Y tangent to $CP^m(4)$, we have

$$\hat{\nabla}_{\bar{X}}\bar{Y} = \overline{\check{\nabla}_X Y} - \langle JX, Y\rangle\, \xi, \tag{2.75}$$

$$\hat{\nabla}_{\bar{X}}\xi = \hat{\nabla}_{\xi}\bar{X} = \overline{JX}. \tag{2.76}$$

Let N be an n-dimensional submanifold of $CP^m(4)$. Denote by $\tilde{N}$ the pre-image of N via the Hopf fibration $\pi : S^{2m+1} \to CP^m(4)$. Let P_y be a 2-plane section of a tangent space T_yN of N spanned by two orthonormal vectors e_1, e_2. As before, we denote by $\bar{P}_x$ the 3-plane spanned by ξ_x and the horizontal lifts $\bar{e}_1, \bar{e}_2$ of e_1, e_2 at a point x with $\pi(x) = y$. Then it follows from Lemma 15 that

$$\begin{aligned} \tau_{\tilde{N}}(x) - \tau_{\tilde{N}}(\bar{P}_x) = \tau_N - K_N(e_1, e_2) - 3\mathring{A}_{\tilde{\pi}} + 3||\mathcal{A}_{\bar{e}_1}\bar{e}_2||^2 \\ + \breve{A}_{\tilde{\pi}} - \sum_{i=1}^{2} K_{\tilde{N}}(\bar{e}_i, \xi), \end{aligned} \tag{2.77}$$

where $\xi = \mathrm{i}z$ is the characteristic vector field of the Sasakian space form S^{2m+1}.

If η is a normal vector field of N in $CP^m(4)$, then by using $\xi = \mathrm{i}z$ we find

$$\hat{\nabla}_{\bar{X}}\bar{\eta} = \overline{\check{\nabla}_X \eta} - \langle FX, \eta\rangle\, \xi. \tag{2.78}$$

Hence, Weingarten's formula yields

$$\hat{A}_{\bar{\xi}}\bar{X} = \overline{A_{\xi}X} + \langle FX, \eta \rangle \, \xi, \quad \hat{D}_{\bar{X}}\bar{\eta} = \overline{D_X \eta}, \tag{2.79}$$

where $A, \hat{A}$ are the shape operators of N in $CP^m(4)$ and $\tilde{N}$ in S^{2m+1}, respectively, and D and $\hat{D}$ are the corresponding normal connections.

From (2.75) we get

$$\hat{h}(\bar{X}, \bar{Y}) = \overline{h(X, Y)}, \tag{2.80}$$

where $\hat{h}$ is the second fundamental form of $\tilde{N}$ in S^{2m+1}.

By using (2.75) we find

$$\tilde{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} - \langle JX, Y \rangle \, \xi, \tag{2.81}$$

where $\tilde{\nabla}$ and ∇ are the Levi-Civita connections of $\tilde{N}$ and N, respectively. Also, it follows from (2.76) that

$$\hat{h}(\bar{X}, \xi) = \overline{FX}, \quad \tilde{\nabla}_{\bar{X}}\xi = \tilde{\nabla}_{\xi}\bar{X} = \overline{FX} \tag{2.82}$$

for $X \in TN$. Moreover, since Hopf's fibration has totally geodesic fibers, we get

$$\hat{h}(\xi, \xi) = 0. \tag{2.83}$$

Now, it follows from (2.82), (2.83) and Gauss' equation that

$$K_{\tilde{N}}(\bar{X}, \xi) = 1 - ||FX||^2 \tag{2.84}$$

for each unit tangent vector X of N.

By applying (2.12), (2.15) and (2.84) we find

$$\breve{A}_{\tilde{\pi}} = n - ||F||^2. \tag{2.85}$$

For an orthonormal frame $\{e_1, \ldots, e_n\}$, we find from $\xi = \mathrm{i}z$ and Lemma 11(a) that

$$\begin{aligned} 2\mathcal{A}_{\bar{e}_i}\bar{e}_j = \mathcal{V}[\bar{e}_i, \bar{e}_j] &= \left\langle \breve{\nabla}_{\bar{e}_i}\bar{e}_j - \breve{\nabla}_{\bar{e}_j}\bar{e}_i, \xi \right\rangle \xi \\ &= \left\langle \bar{e}_i, \mathrm{i}\breve{\nabla}_{\bar{e}_j} z \right\rangle \xi - \left\langle \bar{e}_j, \mathrm{i}\breve{\nabla}_{\bar{e}_i} z \right\rangle \xi \\ &= 2\left\langle \bar{e}_i, \mathrm{i}\bar{e}_j \right\rangle \xi = 2\left\langle e_i, Pe_j \right\rangle \xi, \end{aligned} \tag{2.86}$$

where $\breve{\nabla}$ is the Levi-Civita connection of $\mathbf{C}^{m+1}$. Combining (2.15) and (2.86) gives

$$\mathring{A}_{\tilde{\pi}} = \frac{1}{2}||P||^2. \tag{2.87}$$

By applying (2.77), (2.84), (2.86) and (2.87), we obtain

$$\begin{aligned}\tau_{\tilde{N}}(x) - \tau_{\tilde{N}}(\bar{P}_x) = \tau_N - K_N(e_1, e_2) - \frac{3}{2}||P||^2 + 3\,\langle Pe_1, e_2\rangle^2 \\ + n - 2 - ||F||^2 + \sum_{i=1}^{2} ||Fe_i||^2.\end{aligned} \tag{2.88}$$

Since $||PX||^2 + ||FX||^2 = ||X||^2$, we derive from Lemma 16 and (2.88) that

$$\begin{aligned}\tau_{\tilde{N}}(x) - \tau_{\tilde{N}}(\bar{P}_x) &\le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{n^2+n-6}{2} - ||F||^2 + \sum_{i=1}^{2} ||Fe_i||^2 \\ &= \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{n^2-n-6}{2} + ||P||^2 + \sum_{i=1}^{2} ||Fe_i||^2 \\ &= \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{n^2-n-2}{2} + ||P||^2 - \sum_{i=1}^{2} ||Pe_i||^2 \\ &\le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{n^2-n-2}{2} + ||P||^2,\end{aligned} \tag{2.89}$$

which gives inequality (2.73).

From Lemma 16 and (2.79) we conclude that the equality sign of (2.73) holds identically if and only if there exists an orthonormal frame $e_1, \ldots, e_m$ such that statements (a) and (b) of the theorem hold.

Let N be a CR-submanifold of $CP^m(4)$. The next result from [1] provides the necessary and sufficient condition for the pre-image $\pi^{-1}(N)$ to satisfy the equality case of the inequality (2.73).

Theorem 2.13 ([1]) *Let N be a CR-submanifold of the complex projective m-space $CP^m(4)$. Then, N satisfies the equality case of (2.73) identically if and only if N is a totally real submanifold satisfying*

$$\delta(2) = \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n^2 - n - 2) \tag{2.90}$$

identically.

For the proof of this theorem, see [1].

Finally, we provide many examples of totally real submanifolds of $CP^m(4)$ which satisfy the equality case of inequality (2.73).

Theorem 2.14 *If N is a totally real totally geodesic submanifold of $CP^m(4)$, then the equality sign of (2.73) holds identically.*

Proof Let N be an n-dimensional totally real totally geodesic submanifold of $CP^m(4)$ with $n \geq 3$. In view of (2.84), we have

$$\tau_{\tilde{N}} = \frac{n(n-1)}{2}, \quad \tau_{\tilde{N}}(\bar{P}_x) = 1.$$

Thus, $\delta^H = \frac{1}{2}(n^2 - n - 2)$. Hence, we obtain the equality sign of (2.73) identically due to $||H|| = P = 0$.

Theorem 2.15 *There exist many non-totally geodesic totally real submanifolds of $CP^m(4)$ which satisfy the equality case of inequality (2.73) identically.*

Proof Let N be an n-dimensional submanifold in the unit m-sphere S^m satisfying

$$\delta(2) = \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n^2 - n - 2). \tag{2.91}$$

Then, N can be isometrically immersed as a totally real submanifold of $CP^m(4)$ satisfying the equality case of (2.73) via the following standard isometric immersion:

$$S^m \xrightarrow[\text{covering}]{\text{2 to 1}} RP^m(1) \xrightarrow[\text{totally real}]{\text{totally geodesic}} CP^m(4). \tag{2.92}$$

Since N in S^m satisfies equality (2.91), the shape operator of N in S^m satisfies statement (a) of Theorem 2.12, the shape operator of N in $CP^m(4)$ satisfies (a) as well. Because N is totally real in $CP^m(4)$, it also satisfies statement (b) of Theorem 2.12. It is known that there exist ample submanifolds in spheres which satisfy equality (2.91) identically. Consequently, there exist many non-totally geodesic, totally real submanifolds of $CP^m(4)$ which satisfy the equality case of inequality (2.73) identically according to Theorem 2.12.

Remark 8 For further results on the CR-submanifolds in Kaehler manifolds related to δ-invariants, in particular for CR-submanifolds in complex hyperbolic spaces, see [14, 27, 28, 37, 38].

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