

Preface

The present volume gathers contributions by several experts in the theory of isometric immersions between Riemannian manifolds, and focuses on the geometry of CR structures on submanifolds in Hermitian manifolds. CR structures are a bundle theoretic recast of the tangential Cauchy–Riemann equations in complex analysis in several complex variables. Let $\Omega \subset \mathbb{C}^n$ ($n \geq 2$) be an open set and let

$$\bar{\partial}f \equiv \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j = 0 \quad (1)$$

be the ordinary Cauchy–Riemann equations in \mathbb{C}^n . A function $f \in C^1(\Omega, \mathbb{C})$ is *holomorphic* in Ω if f satisfies (1) everywhere in Ω . Let M be an embedded real hypersurface in \mathbb{C}^n such that $U = M \cap \Omega \neq \emptyset$ and let us set

$$T_{1,0}(M)_x = [T_x(M) \otimes_{\mathbb{R}} \mathbb{C}] \cap T^{1,0}(\mathbb{C}^n)_x, \quad x \in M, \quad (2)$$

where $T^{1,0}(\mathbb{C}^n)$ is the holomorphic tangent bundle over \mathbb{C}^n (the span of $\{\partial/\partial z^j : 1 \leq j \leq n\}$). Then $T_{1,0}(M)$ is a rank $n - 1$ complex vector bundle over M , referred to as the *CR structure* of M (induced on M by the complex structure of the ambient space \mathbb{C}^n) and one may consider the first order differential operator

$$\bar{\partial}_b : C^1(M, \mathbb{C}) \rightarrow C(T_{0,1}(M)^*), \quad (3)$$

$$(\bar{\partial}_b u)\bar{Z} = \bar{Z}(u), \quad u \in C^1(M, \mathbb{C}), \quad Z \in T_{1,0}(M),$$

(the *tangential Cauchy–Riemann operator*) where $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (overbars denote complex conjugates). A function $u \in C^1(M, \mathbb{C})$ is a *CR function* on M if u satisfies

$$\bar{\partial}_b u = 0 \quad (4)$$

(the *tangential Cauchy–Riemann equations*) everywhere on M . Let $\text{CR}^1(M)$ be the space of all CR functions on M . The trace on U of any holomorphic function $f \in \mathcal{O}(\Omega)$ is a CR function $u \in \text{CR}^1(U)$. In other words, the Cauchy–Riemann equations (1) induce on M the first order partial differential system (4). A sufficiently small open piece U of M may be described by a smooth defining function $\rho : \Omega \rightarrow \mathbb{R}$ i.e.,

$$U = \{x \in \Omega : \rho(x) = 0\}$$

such that $D\rho(x) \neq 0$ for any $x \in U$. By eventually restricting the open set U we may assume that $\rho_{z^n}(x) \neq 0$ for any $x \in U$. Here $\rho_{z^j} \equiv \partial\rho/\partial z^j$ for $1 \leq j \leq n$. The portion of $T_{1,0}(M)$ over U is then the span of

$$Z_\alpha \equiv \rho_{z^n} \frac{\partial}{\partial z^\alpha} - \rho_{z^\alpha} \frac{\partial}{\partial z^n}, \quad 1 \leq \alpha \leq n-1,$$

and the tangential Cauchy–Riemann equations (4) on U may be written as

$$Z_{\bar{\alpha}}(u) = 0, \quad 1 \leq \alpha \leq n-1, \quad (5)$$

where $Z_{\bar{\alpha}} \equiv \overline{Z_\alpha}$. As such the tangential Cauchy–Riemann equations may be seen to be a first order overdetermined PDEs system with smooth complex valued coefficients. While constant coefficient equations are nowadays fairly well understood, there is still much work to do on variable coefficient PDEs such as (5).

The geometric approach to the study of (local and global properties of) solutions to (4) or (5) is to study the complex vector bundle $T_{1,0}(M)$. This is commonly accomplished by introducing additional geometric objects, familiar within differential geometry. For instance, should one need to compute the Chern classes of $T_{1,0}(M)$, one would need a connection in $T_{1,0}(M)$. Indeed it is rather well known (cf. e.g., [10]) that Tanaka and Webster built (cf. [14] and [15]) a linear connection ∇ on any nondegenerate real hypersurface $M \subset \mathbb{C}^n$, uniquely determined by a fixed contact form θ on M [the *Tanaka–Webster connection* of (M, θ)] and such that ∇ descends to a connection in $T_{1,0}(M)$ as a vector bundle. Chern classes of $T_{1,0}(M)$ may then be computed in terms of the curvature of the Tanaka–Webster connection, in the presence of a fixed contact form on M .

CR structures induced on real hypersurfaces of \mathbb{C}^n are but a particular instance of a more general notion, that of an abstract CR structure on a $(2n+k)$ -dimensional manifold M . A complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$, of complex rank n , of the complexified tangent bundle, is said to be an (*abstract*) *CR structure* on M if

$$T_{1,0}(M)_x \cap T_{0,1}(M)_x = (0), \quad x \in M, \quad (6)$$

$$Z, W \in C^\infty(U, T_{1,0}(M)) \implies [Z, W] \in C^\infty(U, T_{1,0}(M)), \quad (7)$$

for any open subset $U \subset M$. The integers n and k are respectively the *CR dimension* and *CR codimension* of $T_{1,0}(M)$ while the pair (n, k) is its type. A complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ satisfying only axiom (6) is an *almost* CR structure on M . Axiom (7) is often referred to as the *formal* (or *Frobenius*) *integrability* property. An almost CR structure satisfying the formal integrability property (7) is a CR structure. CR structures on real hypersurfaces $M \subset \mathbb{C}^n$ have type $(n-1, 1)$.

A large portion of the mathematical literature devoted to the study of CR structures is confined to the case of CR codimension 1 in the presence of additional nondegeneracy assumptions (cf. [16]). Let $(M, T_{1,0}(M))$ be a CR manifold of type (n, k) . The *Levi distribution* is the real rank $2n$ distribution

$$H(M) = \operatorname{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

It carries the complex structure

$$J : H(M) \rightarrow H(M), \quad J(Z + \bar{Z}) = i(Z - \bar{Z}), \quad Z \in T_{1,0}(M),$$

($i = \sqrt{-1}$). The *Levi form* of $(M, T_{1,0}(M))$ is

$$L_x : T_{1,0}(M)_x \times T_{0,1}(M)_x \rightarrow \frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{H(M)_x \otimes_{\mathbb{R}} \mathbb{C}}, \quad x \in M,$$

$$L_x(z, w) = i\pi_x[Z, \bar{W}]_x, \quad z, w \in T_{1,0}(M)_x,$$

where $Z, W \in C^\infty(T_{1,0}(M))$ are arbitrary globally defined smooth sections such that $Z_x = z$ and $W_x = w$. Also $\pi : T(M) \otimes \mathbb{C} \rightarrow [T(M) \otimes \mathbb{C}]/[H(M) \otimes \mathbb{C}]$ is the natural projection. The CR structure $T_{1,0}(M)$ [or the CR manifold $(M, T_{1,0}(M))$] is *nondegenerate* if L_x is nondegenerate for any $x \in M$. Assuming that $k = 1$ there is yet another customary description of the Levi form and of nondegeneracy, as understood in complex analysis. Let

$$H(M)_x^\perp = \{\omega \in T_x^*(M) : \operatorname{Ker}(\omega) \supset H(M)_x\}, \quad x \in M,$$

be the conormal bundle associated to $H(M)$. Assume that M is oriented, so that $T(M)$ is oriented as a vector bundle. The Levi distribution $H(M)$ is oriented by its complex structure J . Hence the quotient $T(M)/H(M)$ is oriented. There are (non-canonical) bundle isomorphisms $H(M)^\perp \approx T(M)/H(M)$, hence $H(M)^\perp$ is oriented, as well. Any oriented real line bundle over a connected manifold is trivial, hence $H(M)^\perp \approx M \times \mathbb{R}$ (a vector bundle isomorphism). Hence globally defined nowhere zero smooth sections $\theta \in C^\infty(H(M)^\perp)$ [referred to as *pseudohermitian structures* on M] do exist. Let \mathcal{P} be the set of all pseudohermitian structures on M . Given $\theta \in \mathcal{P}$ one may set

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M),$$

and one may easily check that L_θ and L agree. As it turns out, if $T_{1,0}(M)$ is nondegenerate then each $\theta \in \mathcal{P}$ is a contact form, i.e., $\theta \wedge (d\theta)^n$ is a volume form on M .

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold, of type $(n, 1)$, and let $\theta \in \mathcal{P}$. The *Reeb vector* of (M, θ) is the unique globally defined nowhere zero tangent vector field $\xi \in \mathfrak{X}(M)$ determined by

$$\theta(\xi) = 1, \quad \xi \lrcorner d\theta = 0.$$

The *Webster metric* is the semi-Riemannian metric g_θ on M given by

$$g_\theta(X, Y) = (d\theta)(X, JY), \quad g_\theta(X, \xi) = 0, \quad g_\theta(\xi, \xi) = 1, \quad (8)$$

for any $X, Y \in H(M)$. Axioms (8) uniquely determine g_θ because of $T(M) = H(M) \oplus \mathbb{R}\xi$. For any nondegenerate CR manifold $(M, T_{1,0}(M))$, on which a contact form $\theta \in \mathcal{P}$ has been specified, there is a unique linear connection ∇ on M [the *Tanaka–Webster connection* of (M, θ)] obeying to the following axioms (i) $H(M)$ is parallel with respect to ∇ , (ii) $\nabla J = 0$ and $\nabla g_\theta = 0$, and (iii) the torsion tensor field T_∇ is pure, i.e.,

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2iL_\theta(Z, \overline{W})\xi,$$

$$\tau \circ J + J \circ \tau = 0,$$

for any $Z, W \in T_{1,0}(M)$, where $\tau(X) = T_\nabla(\xi, X)$ for any $X \in \mathfrak{X}(M)$. One should notice that the existence of ∇ is tied to that of ξ , which in turn is a direct consequence of nondegeneracy and orientability.

We say $(M, T_{1,0}(M))$ is *strictly pseudoconvex* if L_θ is positive definite for some $\theta \in \mathcal{P}$. To emphasize on the role play by Chern classes $c_j(T_{1,0}(M))$, let us recall (cf. [13, 14]) the *Lee conjecture* according to which any abstract strictly pseudoconvex CR manifold M with $c_1(T_{1,0}(M)) = 0$ should admit a contact form θ such that (M, θ) is pseudo-Einstein [i.e., the pseudohermitian Ricci tensor (of the Tanaka–Webster connection of (M, θ)) is proportional to the Levi form]. The sphere $S^{2n-1} \subset \mathbb{C}^n$ is pseudo-Einstein with the canonical contact form $\theta = \frac{i}{2}(\overline{\partial} - \partial)|z|^2$ [and of course $c_1(T_{1,0}(S^{2n-1})) = 0$].

From the definition of the notion of an (abstract) CR structure, it is manifest that the prospective study of the CR structure $T_{1,0}(M)$ of a real hypersurface $M \subset \mathbb{C}^n$ ignores the metric structure (the canonical Euclidean structure of $\mathbb{C}^n \approx \mathbb{R}^{2n}$) and only takes into consideration the complex structure on \mathbb{C}^n . Whatever metric structure M is seen to possess *a posteriori*, such as the Levi form L_θ , springs from the CR structure (from the complex structure J along $H(M)$) and is determined by it only up to a “conformal factor”, very much as in the theory of Riemann surfaces.

Indeed if $\theta, \hat{\theta} \in \mathcal{P}$ then $\hat{\theta} = \lambda\theta$ for some C^∞ function $\lambda : M \rightarrow \mathbb{R} \setminus \{0\}$, implying that $L_{\hat{\theta}} = \lambda L_\theta$. However, the fact that the Webster metric g_θ is semi-Riemannian is tied to nondegeneracy (g_θ is actually Riemannian when $(M, T_{1,0}(M))$ is strictly pseudoconvex) and none of these objects, including of course the Tanaka–Webster connection, is available on a CR manifold whose Levi form has a degeneracy locus (for instance in the extreme case where $(M, T_{1,0}(M))$ is *Levi flat*, i.e., $L_\theta = 0$ for some $\theta \in \mathcal{P}$, and thus for all). We also underline that everything said and done in pseudohermitian geometry is confined to the starting assumption that the given CR manifold has CR codimension $k = 1$.

Examples of real hypersurfaces $M \subset \mathbb{C}^n$ which are not nondegenerate, abound (for instance, boundaries of worm domains, cf. [12]). Also, CR manifolds of higher CR codimension ($k \geq 2$) are frequently met, as submanifolds of certain Hermitian manifolds. The absence of an analog to pseudohermitian geometry in these cases may be compensated by making full use of the additional metric structure on M , as the first fundamental form of a given immersion $M \hookrightarrow \tilde{M}$, where \tilde{M} is a Hermitian manifold. This became apparent, and came as a surprise to the CR community, with the work of A. Bejancu at the end of the 1970s (cf. [2–3]).

Let \tilde{M} be a Hermitian manifold, of complex dimension N , with the complex structure J and the Hermitian metric \tilde{g} . Let M be a real m -dimensional submanifold, i.e., the inclusion $\iota : M \hookrightarrow \tilde{M}$ is an embedding. Let us assume that $N = m + p$ and $m = 2n + k$ with $p \geq 1$, $n \geq 1$ and $k \geq 1$. Let \mathcal{D} be a smooth real rank $2n$ distribution on M such that (i) $J_x(\mathcal{D}_x) = \mathcal{D}_x$ and (ii) $J_x(\mathcal{D}_x^\perp) \subset T(M)_x^\perp$, for any $x \in M$. Here \mathcal{D}_x^\perp is the g_x -orthogonal complement of \mathcal{D}_x in $T_x(M)$ and $T(M)^\perp \rightarrow M$ is the normal bundle of the given immersion ι [so that $T(M)_x^\perp$ is the \tilde{g}_x -orthogonal complement of $T_x(M)$ in $T_x(\tilde{M})$]. Also $g = \iota^* \tilde{g}$ is the induced metric (the first fundamental form of ι). A pair (M, \mathcal{D}) , consisting of a $(2n + k)$ -dimensional submanifold of \tilde{M} and of a distribution \mathcal{D} as above, is called a *CR submanifold* of the Hermitian manifold $(\tilde{M}, J, \tilde{g})$. This is the notion of a CR submanifold as introduced by Bejancu (cf. [2]) except for Bejancu's original request that the ambient space be a Kählerian manifold, i.e., that \tilde{g} be a Kähler metric. Orientable real hypersurfaces $M \subset \mathbb{C}^n$ fit into this category, for one may choose a unit normal vector field N on M , take a rotation of N of angle $\pi/2$ so that to get $\xi = J_0 N \in \mathfrak{X}(M)$, and set $\eta(X) = g_0(X, \xi)$ for any $X \in \mathfrak{X}(M)$, where J_0 and g_0 are respectively the canonical complex structure and (flat) Kählerian metric on \mathbb{C}^n . Then $\mathcal{D} = \text{Ker}(\eta)$ organizes M as a CR submanifold of (\mathbb{C}^n, J_0, g_0) .

A CR submanifold (M, \mathcal{D}) with $\mathcal{D} = T(M)$ is a complex submanifold of \tilde{M} , while one with $\mathcal{D} = (0)$ is totally real (such submanifolds are also referred to as anti-invariant). It has been argued by a number of authors that A. Bejancu has introduced the notion of a CR submanifold in an attempt to unify the notions of complex (or invariant) and totally real submanifolds, and of course that of a generic submanifold (where $J(\mathcal{D}^\perp) = T(M)^\perp$). Whether or not A. Bejancu had the insight that his notion was tied to the theory of tangential Cauchy–Riemann equations (as understood in complex analysis) became soon irrelevant, with the nice and

elementary discovery by Blair and Chen (cf. [5]) that each CR submanifold (M, \mathcal{D}) of a Hermitian manifold may be organized as a CR manifold with the CR structure

$$T_{1,0}(M) = \{X - iJX : X \in \mathcal{D}\}.$$

With respect to this CR structure the inclusion is a CR immersion [i.e., $(d_x \iota)T_{1,0}(M)_x \subset T^{1,0}(\tilde{M})_{\iota(x)}$ for any $x \in M$, where $T^{1,0}(\tilde{M})$ is the holomorphic tangent bundle over the complex manifold (\tilde{M}, J)] so that Bejancu's CR submanifolds appear as *embedded* CR manifolds.

A CR manifold $(M, T_{1,0}(M))$ is *locally embeddable* if there is $N > \dim(M)$ such that for any point $x_0 \in M$ there is an open neighborhood $U \subset M$ and a C^∞ immersion $\Psi : U \rightarrow \mathbb{C}^N$ such that

$$(d_x \Psi)T_{1,0}(M)_x = [T_{\Psi(x)}(\Psi(U)) \otimes_{\mathbb{R}} \mathbb{C}] \cap T^{1,0}(\mathbb{C}^N)_{\Psi(x)}, \quad x \in U.$$

At the time A. Bejancu introduced the notion of a CR submanifold L. Nirenberg's problem (i.e., whether a given abstract CR manifold may embed, even if just locally, cf. e.g., [6]) was far less popular than nowadays, and perhaps unknown to Riemannian geometers, who embraced early Bejancu's notion and produced a significant amount of work (cf. e.g., [16]). In the meanwhile it became classical mathematics that real analytic CR manifolds are always locally embeddable (cf. [1]) while in the C^∞ category all strictly pseudoconvex CR manifolds of dimension $\dim(M) \geq 7$ embed locally (and there are known counterexamples in dimension 3, while the case $\dim(M) = 5$ is open). The positive embeddability results close a circle of ideas and incorporating the Hermitian manifold \tilde{M} in the definition of a CR (sub) manifold may not any longer be seen as a limitation of sorts. It should also be noticed that \tilde{M} in Bejancu's definition is an arbitrary Hermitian manifold, and not necessarily $\tilde{M} = \mathbb{C}^N$ for some N (e.g., \tilde{M} may be the complex projective space $\mathbb{C}P^N$, or the complex hyperbolic space CH^N).

A comment is due on Bejancu's motivation¹ for fixing a complement to the holomorphic, or invariant, distribution \mathcal{D} (its orthogonal complement \mathcal{D}^\perp , with respect to the induced metric $g = \iota^* \tilde{g}$). As it appears, inspiration was drawn from the work by S. Greenfield (cf. [11]) where a complement to the Levi distribution $H(M)$ is fixed to start with. S. Greenfield's choice may be criticized as non-canonical. Indeed, to make such a choice within pseudohermitian geometry ($k = 1$) one first requires nondegeneracy of the given CR structure and then chooses $\mathbb{R}\xi$ as a complement to $H(M)$, where ξ is the Reeb vector associated to a fixed contact form θ . This choice is perhaps natural enough, yet certainly has one leave the realm of CR geometry (and confines oneself to pseudohermitian geometry). In turn

¹Of course one may interview Professor Bejancu on the argument. Here we adopt the historian perspective giving preference to historical reconstructions based on documents rather than testimonies.

Bejancu's choice of \mathcal{D}^\perp as a complement to \mathcal{D} uses the metric structure [say $g = \iota^* g_0$ for a given real hypersurface $\iota : M \hookrightarrow \mathbb{C}^n$] and, be it canonical or not, it fits nicely into the adopted philosophy [which is to exploit the additional metric structure to compensate for the eventual lack of nondegeneracy]. We end this remark by recalling that $\mathbb{R}\xi$ is certainly orthogonal to $H(M)$ with respect to the Webster metric g_θ , but in general it cannot be expected to be orthogonal to $H(M)$ with respect to the induced metric. For only in rare occasions [e.g., for the sphere $S^{2n-1} \hookrightarrow \mathbb{C}^n$] does the Webster metric (associated to some contact form) coincide with the induced metric [in fact, all the (infinitely many) Webster metrics of the boundary of the Siegel domain $\Omega = \{(z, w) \in \mathbb{C}^2 : \text{Im}(w) > |z|^2\}$ are distinct from the first fundamental form of the immersion $\partial\Omega \hookrightarrow \mathbb{C}^2$]. This is a point of divergence between Bejancu's theory and pseudohermitian geometry (Webster's theory) yet hasn't proved to be counterproductive so far. On the contrary, one was led to study the geometry of the foliation tangent to \mathcal{D}^\perp [\mathcal{D}^\perp is always Frobenius integrable, provided that the ambient space is Kählerian or locally conformal Kähler] resulting into a deeper understanding of the geometry of the CR submanifold (M, \mathcal{D}) itself.

Let (M, \mathcal{D}) be a CR submanifold of the Hermitian manifold $(\tilde{M}, J, \tilde{g})$. When \tilde{g} is a Kählerian, or a locally conformal Kähler, metric, the Levi form L of M as a CR manifold, and the second fundamental form h of the given immersion $M \hookrightarrow \tilde{M}$, are related in a nice computable manner. Studying the geometry of h is then closely related to the study of CR geometry (and pseudohermitian geometry, in the CR codimension case) on M . Only now no nondegeneracy assumptions are needed to start with, and the classical machinery in the theory of isometric immersions of Riemannian manifolds (e.g., the Gauss–Ricci–Codazzi equations) becomes available.

Besides from Nirenberg's CR embedding problem mentioned above, one should recall the equally classical CR extension problem. As already seen at the beginning of this preface, traces of holomorphic functions on real hypersurfaces of \mathbb{C}^n are solutions to the tangential Cauchy–Riemann equations. Conversely, the CR extension problem is whether CR functions (on embedded CR manifolds) extend, at least locally, to holomorphic functions on (some open subset of) \mathbb{C}^n . An interesting feature of the generic CR submanifolds of higher CR codimension is that positive CR extension results depend on the presence of particular cones in the normal space at each point of the submanifold, and the metric structure of the ambient space may not be ignored any longer (cf. [6]).

The past 30 years have seen a great increase in the volume of research devoted to the geometry of CR submanifolds, from the point of view of Riemannian geometry. The results at the level of K. Yano and M. Kon's monograph mentioned above were integrated by the publication of a book by Bejancu himself (cf. [4]) and by the monograph [9] reporting on the case where the ambient metric \tilde{g} is locally conformal Kähler (and the list of monographs devoted to CR submanifolds is by far not complete).

Given the huge amount of work on CR submanifolds produced since the appearance of the last monograph (i.e., [7]) on the argument, the editors thought it appropriate to invite a number of specialists to contribute one or more papers, perhaps of partially expositive nature, illustrating the state of the art in the theory. The following colleagues answered our call (and are listed here in alphabetical order).

Bang-Yen Chen contributes two papers (cf. Chaps. 1 and 2 in this book), one about his theory of δ -invariants as it applies to CR geometry, and another on the geometry of CR-warped products in Kählerian manifolds.

Miroslava Antić and **Luc Vrancken** contribute a study (cf. Chap. 3 in this book) of CR submanifolds of the nearly Kähler 6-sphere.

Elisabetta Barletta and **Sorin Dragomir** contribute new results (cf. Chap. 4 in this book) on the interplay between the geometry of the second fundamental form of a CR submanifold and the tangential Cauchy–Riemann equations.

J.L. Cabrerizo, **A. Carriazo** and **L.M. Fernández** report on their work (cf. Chap. 5 in this book) on CR submanifolds of (locally conformal) Kähler manifolds and normal CR submanifolds of S -manifolds.

Krishan Lal Duggal surveys some of his work (cf. Chap. 6 in this book) relating Lorentzian and Cauchy–Riemann geometry.

Hitoshi Furuhata and **Izumi Hasegawa** contribute their work (cf. Chap. 7 in this book) on CR submanifolds of holomorphic statistical manifolds.

Ion Mihai and **Adela Mihai** contribute their work (cf. Chap. 8 in this book) concerning minimality of warped product CR submanifolds in complex space forms, and estimates on the scalar curvature of such submanifolds.

Andrea Olteanu presents results (cf. Chap. 9 in this book) on geometric inequalities occurring on CR-doubly warped product submanifolds.

Toru Sasahara surveys the known results (cf. Chap. 10 in this book) on δ -ideal submanifolds in complex space forms, the nearly Kähler 6-sphere, and odd dimensional spheres.

Mohammad Hasan Shahid, **Falleh R. Al-Solamy** and **Mohammed Jamali** contribute a survey (cf. Chap. 11 in this book) on the geometry of submersions from a CR submanifold.

Ramesh Sharma's contribution (cf. Chap. 12 in this book) regards the geometry of CR submanifolds in semi-Kählerian manifolds, hinting to possible applications to space–time physics.

Gabriel-Eduard Vilcu presents a generalization (cf. Chap. 13 in this book) of CR submanifold theory to paraquaternionic geometry.

Potenza, Italy
New Delhi, India
Jeddah, Saudi Arabia
October 2015

Sorin Dragomir
Mohammad Hasan Shahid
Falleh R. Al-Solamy

References

1. Andreotti, A., Hill, C.D.: Complex characteristic coordinates and the tangential Cauchy-Riemann equations. *Ann. Scuola Norm. Sup. Pisa* **26**, 299–324 (1972)
2. Bejancu, A.: CR submanifolds of a Kaehler manifold I. *Proc. Amer. Math. Soc.* **69**, 134–142 (1978)
3. Bejancu, A.: CR submanifolds of a Kaehler manifold II. *Trans. Amer. Math. Soc.* **250**, 335–345 (1979)
4. Bejancu, A.: *Geometry of CR-Submanifolds*. D. Reidel Publishing Co., Dordrecht (1986)
5. Blair, D.E., Chen, B.-Y.: On CR submanifolds of Hermitian manifolds. *Israel J. Math.* **34**, 353–363 (1979)
6. Boggess, A.: CR manifolds and the tangential Cauchy-Riemann complex. In: *Studies in Advanced Mathematics*. CRC Press, Boca Raton, Ann Arbor, Boston, London (1991)
7. Djoric, M., Okumura, M.: *CR Submanifolds of Complex Projective Space*. Springer, New York (2010)
8. Dragomir, S.: On a conjecture of J.M. Lee. *Hokkaido Math. J.* **23**(1), 35–49 (1994)
9. Dragomir, S., Ornea, L.: Locally conformal Kähler geometry. In: *Progress in Mathematics*, vol. 155. Birkhäuser, Boston, Basel, Berlin (1998)
10. Dragomir, S., Tomassini, G.: Differential geometry and analysis on CR manifolds. In: *Progress in Mathematics*, vol. 246. Birkhäuser, Boston, Basel, Berlin (2006)
11. Greenfield, S.: Cauchy-Riemann equations in several variables. *Ann. Sc. Norm. Sup. Pisa* **22**, 275–314 (1968).
12. Krantz, S.G., Peloso, M.M.: Analysis and geometry on worm domains. *J. Geom. Anal.* **18**, 478–510 (2008)
13. Lee, J.M.: Pseudo-Einstein structures on CR manifolds. *Am. J. Math.* **110**, 157–178 (1988)
14. Tanaka, N.: *A Differential Geometric Study on Strongly Pseudo-Convex Manifolds*. Kinokuniya Book Store Co., Ltd., Kyoto (1975)
15. Webster, S.: Pseudohermitian structures on a real hypersurface. *J. Diff. Geom.* **13**, 25–41 (1978)
16. Yano, K., Kon, M.: CR submanifolds of Kaehlerian and Sasakian manifolds. In: Coates, J., Helgason, S. (eds.) *Progress in Mathematics*, vol. 30. Birkhäuser, Boston, Basel, Stuttgart (1983)

Geometry of Cauchy-Riemann Submanifolds

Dragomir, S.; Shahid, M.H.; Al-Solamy, F.R. (Eds.)

2016, XVIII, 390 p., Hardcover

ISBN: 978-981-10-0915-0