

Chapter 2

Preliminary—Quantum Mechanics

This chapter is a preliminary chapter for forthcoming discussions. First, we briefly review representation theories for boson and fermion. Second, we turn to consider partition function

$$Z = \text{Tr}(e^{-\beta\hat{H}}). \quad (2.1)$$

Third, we generalize it by turning on an insertion of $(-1)^{\hat{F}}$ into the trace:

$$I = \text{Tr}\left((-1)^{\hat{F}} e^{-\beta\hat{H}}\right). \quad (2.2)$$

This quantity is called *Witten index*, a prototype of the superconformal index in Chaps. 3–5. \hat{F} is fermion number operator which counts the number of fermionic excitations. In the last section, we generalize it and the generalized index gives the basis for Chap. 3.

2.1 Representation Theory

We briefly review the basics of boson and fermion in QM. We emphasize the relationship between operator formalism and path integral formalism for later use.

2.1.1 Boson

Classical prescription Bosonic Lagrangian typically takes the following form:

$$L_b = \frac{1}{2} \dot{x}^2 - V(x). \quad (2.3)$$

As a next step, we define the conjugate momentum of x by

$$p = \frac{\partial L_b}{\partial \dot{x}}. \quad (2.4)$$

Then, the Hamiltonian is defined by the Legendre transformation of L_b :

$$\begin{aligned} H_b &= p\dot{x} - L_b \\ &= \frac{1}{2} p^2 + V(x). \end{aligned} \quad (2.5)$$

Canonical quantization We start with the representation of the bosonic algebra, i.e. Heisenberg algebra:

$$[\hat{p}, \hat{x}] = -i, \quad (2.6)$$

where \hat{p} and \hat{x} are momentum and position operators respectively. In principle, we do not need to stick on the definition of \pm sign in (2.6) if we treat it in self consistent way [1]. As a basis of the Hilbert space, we can take

$$|x\rangle \quad \text{or} \quad |p\rangle. \quad (2.7)$$

These states are defined by

$$\hat{x}|x\rangle = x|x\rangle, \quad \int_{-\infty}^{+\infty} dx |x\rangle \langle x| = 1, \quad (2.8)$$

$$\hat{p}|p\rangle = p|p\rangle, \quad \int_{-\infty}^{+\infty} dp |p\rangle \langle p| = 1. \quad (2.9)$$

There are two important facts. First fact is that $e^{-i\hat{p}a}$ generates translation of $|x\rangle$:

$$e^{-i\hat{p}a}|x\rangle = |x+a\rangle. \quad (2.10)$$

Second fact is that the explicit form of the inner product becomes as follows.¹

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi}} e^{-ipx}. \quad (2.11)$$

¹The simplest way to derive this relation is to use the differential equation. For example,

The constant of integration, $\frac{1}{\sqrt{2\pi}}$, is determined by requiring the orthonormality condition $\langle x'|x\rangle = \delta(x - x')$.

2.1.2 Fermion

Classical prescription Fermionic Lagrangian typically takes the following form:

$$L_f = i\psi_+\dot{\psi}_- - V(\psi_\pm). \quad (2.12)$$

Here we treat ψ_+ , ψ_- as independent Grassmann numbers:

$$\psi_+^2 = 0, \quad \psi_-^2 = 0, \quad \psi_+\psi_- = -\psi_-\psi_+. \quad (2.13)$$

The left² conjugate momentum of ψ_- is defined by

$$\Pi_- = \frac{\partial}{\partial \dot{\psi}_-} L_f. \quad (2.14)$$

The Hamiltonian is defined by the Legendre transformation of L_f :

$$\begin{aligned} H_f &= \Pi_- \dot{\psi}_- - L_f \\ &= V(\psi_\pm). \end{aligned} \quad (2.15)$$

Canonical quantization We start with the representation of the fermionic algebra, i.e. Clifford algebra³:

$$\{\hat{\psi}_+, \hat{\psi}_-\} = +1. \quad (2.16)$$

(Footnote 1 continued)

$$\begin{aligned} \frac{\partial}{\partial x} \langle p|x\rangle &= \lim_{a \rightarrow 0} \frac{\langle p|x+a\rangle - \langle p|x\rangle}{a} \\ &= \lim_{a \rightarrow 0} \frac{\langle p|e^{-i\hat{p}a}|x\rangle - \langle p|x\rangle}{a} \\ &= \lim_{a \rightarrow 0} \frac{e^{-ipa} \langle p|x\rangle - \langle p|x\rangle}{a} \\ &= -ip \langle p|x\rangle. \end{aligned}$$

²Because of the fermionic natures in (2.13), we have to be careful with the order of ψ_+ and ψ_- .

³In order to derive this relation from the usual canonical quantization method, considering Poisson bracket is not enough. Instead of it, Dirac bracket is necessary.

In contrast to the bosonic case, the sign in (2.16) is important to get the unitary representation [1]. As an orthonormal basis of the Hilbert space, we can take

$$\{|0\rangle, |1\rangle\}. \quad (2.17)$$

The states $|0\rangle, |1\rangle$ are defined by

$$\begin{aligned} \hat{\psi}_-|0\rangle &= 0, & \hat{\psi}_+|0\rangle &= |1\rangle, \\ \hat{\psi}_-|1\rangle &= |0\rangle, & \hat{\psi}_+|1\rangle &= 0, \\ |0\rangle\langle 0| + |1\rangle\langle 1| &= 1. \end{aligned} \quad (2.18)$$

One can regard $|0\rangle$ as a hole-state, and $|1\rangle$ as an occupied state. We cannot make $|2\rangle := \hat{\psi}_+|1\rangle$ because it is automatically zero. This is an algebraic representation of the famous Pauli exclusion principle.

Coherent state basis In later discussions, we consider the path integral formalism. In order to derive it, there is a more useful basis than the basis in (2.17), the coherent state basis [2]:

$$|\Psi\rangle = e^{-\Psi\hat{\psi}_+}|0\rangle, \quad \langle\Psi| = \langle 0|e^{\Psi\hat{\psi}_-}. \quad (2.19)$$

We should take Ψ as a Grassmann valuable, therefore $\Psi^2 = 0$ and we get

$$|\Psi\rangle = (1 - \Psi\hat{\psi}_+)|0\rangle. \quad (2.20)$$

These states satisfy the following relations.

$$\hat{\psi}_-|\Psi\rangle = \Psi|\Psi\rangle, \quad \langle\Psi|\hat{\psi}_+ = \langle\Psi|. \quad (2.21)$$

After a direct calculation, one can get the inner product formula

$$\langle\Psi_+|\Psi_-\rangle = e^{\Psi_+\Psi_-} \quad (2.22)$$

and the complete relation

$$\int d\Psi_+ d\Psi_- |\Psi_-\rangle e^{-\Psi_+\Psi_-} \langle\Psi_+| = 1. \quad (2.23)$$

2.2 Partition Function

One of the most important objects in QM is the partition function:

$$Z = \text{Tr}(e^{-\beta \hat{H}}). \quad (2.24)$$

It contains all informations of the energy spectra because we can extract each energy eigenvalue by taking following procedure [3, 4]⁴:

1. Taking $\beta \rightarrow \infty$ of Z , then Z behaves $e^{-\beta E_0}$ where E_0 is the ground state energy.
2. Subtracting $e^{-\beta E_0}$ from Z , and rename it Z_1 , and
taking $\beta \rightarrow \infty$ of Z_1 , then Z_1 , behaves $e^{-\beta E_1}$ where E_1 is the 1st excited state energy.
3. Repeating this procedure.

2.2.1 Boson Sector

Partition function of the bosonic degrees of freedom is described by the Hamiltonian operator defined from (2.5):

$$\hat{H} = \hat{H}_b, \quad \hat{H}_b = \frac{1}{2} \hat{p}^2 + V(\hat{x}). \quad (2.25)$$

Harmonic oscillator The simplest example is

$$V(\hat{x}) = \frac{1}{2} \omega^2 \hat{x}^2. \quad (2.26)$$

In this case, as well known, once we define \hat{a} and \hat{a}^\dagger [5] so that

$$\hat{H}_b = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad (2.27)$$

and by constructing a basis

$$\left\{ |0\rangle, |1\rangle, |2\rangle, \dots \right\}, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad (2.28)$$

⁴This is valid if there is no degeneracy.

then, we can diagonalize the Hamiltonian: $\hat{H}_b|n\rangle = \omega(n + \frac{1}{2})|n\rangle$. By using this basis, the partition function can be computed by utilizing the formula of power series

$$\begin{aligned} \text{Tr}(e^{-\beta\hat{H}_b}) &= \sum_{n=0}^{\infty} e^{-\beta\omega(n+\frac{1}{2})} \\ &= \frac{e^{-\frac{\beta\omega}{2}}}{1 - e^{-\beta\omega}} \\ &= \frac{1}{2 \sinh \frac{\beta\omega}{2}}. \end{aligned} \quad (2.29)$$

The zero energy which corresponds to $n = 0$ is often called Casimir energy.

Path integral formalism By Inserting the complete set (2.8) and (2.9) into the trace in (2.24), we can re-express it as

$$\begin{aligned} Z_b &= \int_{x(0)=x(\beta)} \left(\prod_{t \in [0, \beta]} dx(t) \frac{dp(t)}{2\pi} \right) e^{-\int_0^\beta dt \left(ip\dot{x} + \frac{1}{2}p^2 + V(x) \right)} \\ &= \int_{x(0)=x(\beta)} \left(\prod_{t \in [0, \beta]} \frac{dx(t)}{\sqrt{2\pi}} \right) e^{-\int_0^\beta dt \left(\frac{1}{2}x\partial_t^2 x + V(x) \right)}. \end{aligned} \quad (2.30)$$

Path integral description of harmonic oscillator We have the following action

$$-\int_0^\beta dt \left(\frac{1}{2}x\partial_t^2 x + V(x) \right) = -\frac{1}{2} \int_0^\beta dt x(-\partial_t^2 + \omega^2)x. \quad (2.31)$$

Thanks to the Gaussian integral formula in (A.13), we get

$$Z_b = \frac{1}{\sqrt{\det_{x(0)=x(\beta)}(-\partial_t^2 + \omega^2)}}. \quad (2.32)$$

The “matrix” ∂_t ’s eigenvectors are $x_n(t) = e^{\frac{2\pi i}{\beta}nt}$, $n \in \mathbb{Z}$ because

$$\partial_t x_n(t) = \frac{2\pi i}{\beta} n x_n(t). \quad (2.33)$$

Therefore, we get the following representation of the determinant.

$$\det_{x(0)=x(\beta)}(-\partial_t^2 + \omega^2) = \prod_{n=-\infty}^{\infty} \left(\frac{(2\pi)^2}{\beta^2} n^2 + \omega^2 \right)$$

$$\begin{aligned}
&= \omega^2 \left[\prod_{n=1}^{\infty} \left(\frac{(2\pi)^2}{\beta^2} n^2 + \omega^2 \right) \right]^2 \\
&= \left[\prod_{n=1}^{\infty} \frac{2\pi}{\beta} n \right]^4 \times \omega^2 \prod_{n=1}^{\infty} \left(1 + \frac{(\beta\omega)^2}{(2\pi n)^2} \right)^2. \tag{2.34}
\end{aligned}$$

Obviously, the first factor diverges. We regularize it by using zeta-function regularization. (See Appendix A for $\zeta(0)$, $\zeta'(0)$ values' derivation.):

$$\begin{aligned}
\left[\prod_{n=1}^{\infty} \frac{2\pi}{\beta} n \right]^4 &= \exp \left(4 \sum_{n=1}^{\infty} \log \frac{2\pi}{\beta} n \right) \rightarrow \exp \left(4 \left[-\zeta'(0) - \zeta(0) \log \frac{\beta}{2\pi} \right] \right) \\
&= \exp \left(4 \left[-\left(-\frac{1}{2}\right) \log 2\pi - \left(-\frac{1}{2}\right) \log \frac{\beta}{2\pi} \right] \right) \\
&= \beta^2. \tag{2.35}
\end{aligned}$$

Then, by using the infinite product formula (A.1), we get

$$(2.34) = \left[(\beta\omega) \prod_{n=1}^{\infty} \left(1 + \frac{(\beta\omega)^2}{(2\pi n)^2} \right) \right]^2 = \left[2 \sinh \frac{\beta\omega}{2} \right]^2. \tag{2.36}$$

It reproduces the result (2.29):

$$Z_b = \frac{1}{\sqrt{\det_{x(0)=x(\beta)} (-\partial_t^2 + \omega^2)}} = \frac{1}{2 \sinh \frac{\beta\omega}{2}}. \tag{2.37}$$

2.2.2 Fermion Sector

Partition function of the fermionic degrees of freedom is described by the Hamiltonian operator defined from (2.15):

$$\hat{H} = \hat{H}_f, \quad \hat{H}_f = V(\hat{\psi}_{\pm}). \tag{2.38}$$

Harmonic oscillator The simplest example is

$$V(\hat{\psi}_{\pm}) = \omega \left(\hat{\psi}_+ \hat{\psi}_- - \frac{1}{2} \right). \tag{2.39}$$

Then, the basis (2.17) diagonalizes this Hamiltonian:

$$\hat{H}_f |n\rangle = \omega \left(n - \frac{1}{2} \right) |n\rangle, \quad n = 0, 1. \quad (2.40)$$

The partition function is, therefore,

$$\begin{aligned} \text{Tr}(e^{-\beta \hat{H}_f}) &= \sum_{n=0}^1 e^{-\beta \omega (n - \frac{1}{2})} \\ &= e^{\frac{\beta \omega}{2}} + e^{-\frac{\beta \omega}{2}} \\ &= 2 \cosh \frac{\beta \omega}{2}. \end{aligned} \quad (2.41)$$

There are two important differences compared with the bosonic harmonic oscillator.

- The absolute value of Casimir energy is same but the sign is different.
- cosh function appears unlike the sinh in bosonic case.

As we will see later, if we insert $(-1)^{\hat{\psi}_+ \hat{\psi}_-}$ into the trace, we get sinh not cosh.

Path integral formalism When we derive fermion's path integral representation of the partition function, we have to be careful with the periodicity as described below. First, we re-express Tr in the partition function with coherent basis in (2.19):

$$\begin{aligned} Z_f &= \text{Tr}(e^{-\beta \hat{H}_f}) \\ &= \int d\Psi_+ d\Psi_- e^{\Psi_+ \Psi_-} \langle \Psi_+ | e^{-\beta V(\hat{\psi}_+, \hat{\psi}_-)} | \Psi_- \rangle. \end{aligned} \quad (2.42)$$

Second, we divide β into N pieces: $\epsilon = \frac{\beta}{N}$, say $N=2$,

$$\begin{aligned} (2.42) &= \int d\Psi_+ d\Psi_- \int d\Lambda_+ d\Lambda_- e^{\Psi_+ \Psi_-} \langle \Psi_+ | e^{-\epsilon V(\hat{\psi}_+, \hat{\psi}_-)} | \Lambda_- \rangle e^{-\Lambda_+ \Lambda_-} \langle \Lambda_+ | e^{-\epsilon V(\hat{\psi}_+, \hat{\psi}_-)} | \Psi_- \rangle \\ &= \int d\Psi_+ d\Psi_- \int d\Lambda_+ d\Lambda_- e^{\Psi_+ \Psi_-} e^{-\epsilon V(\Psi_+, \Lambda_-)} \langle \Psi_+ | \Lambda_- \rangle e^{-\Lambda_+ \Lambda_-} \langle \Lambda_+ | \Psi_- \rangle e^{-\epsilon V(\Lambda_+, \Psi_-)} \\ &= \int d\Psi_+ d\Psi_- \int d\Lambda_+ d\Lambda_- e^{\Psi_+ \Psi_- + \Psi_+ \Lambda_- - \Lambda_+ \Lambda_- + \Lambda_+ \Psi_-} e^{-\epsilon V(\Psi_+, \Lambda_-) - \epsilon V(\Lambda_+, \Psi_-)}. \end{aligned} \quad (2.43)$$

We rename fermionic variables:

$$\Psi_+ = \Psi_+^2, \quad \Lambda_- = \Psi_-^2, \quad \Lambda_+ = \Psi_+^1, \quad \Psi_- = \Psi_-^1, \quad (2.44)$$

then we get

$$(2.43) = \int d\Psi_+^2 d\Psi_-^2 d\Psi_+^1 d\Psi_-^1 e^{\Psi_+^2 \Psi_-^1 + \Psi_+^2 \Psi_-^2 - \Psi_+^1 \Psi_-^2 + \Psi_+^1 \Psi_-^1 - \epsilon V(\Psi_+^2, \Psi_-^2) - \epsilon V(\Psi_+^1, \Psi_-^1)}. \quad (2.45)$$

Now, we regard each Ψ_{\pm}^n as $\Psi_{\pm}(t_n) = \Psi_{\pm}^n$, where $t_n = \epsilon n$. In this $N=2$ case,

$$\begin{aligned}
 & \Psi_+^2 \Psi_-^1 + \Psi_+^2 \Psi_-^2 - \Psi_+^1 \Psi_-^2 + \Psi_+^1 \Psi_-^1 \\
 &= \Psi_+(t_2) \left(\underbrace{\Psi_-(t_1)}_{\Psi_-(0) + \epsilon \dot{\Psi}_-(0)} + \Psi_-(t_2) \right) - \Psi_+(t_1) \left(\underbrace{\Psi_-(t_2)}_{\Psi_-(t_1) + \epsilon \dot{\Psi}_-(t_1)} - \Psi_-(t_1) \right) \\
 &= \Psi_+(t_2) \left(\epsilon \dot{\Psi}_-(0) + \underbrace{[\Psi_-(0) + \Psi_-(t_2)]}_{\text{we have to make it zero.}} \right) - \Psi_+(t_1) \left(\epsilon \dot{\Psi}_-(t_1) + \underbrace{[\Psi_-(t_1) - \Psi_-(t_1)]}_0 \right)
 \end{aligned}$$

As we can see above, in order to drop the $\mathcal{O}(\epsilon^0)$ term, we have to take

$$\Psi_-(t_2) = \Psi_-(\beta) = -\Psi_-(0). \quad (2.46)$$

Therefore, corresponding fermionic fields $\Psi_{\pm}(t)$ are anti-periodic⁵ under the translation $t \rightarrow t + \beta$. Then, by using

$$\dot{\Psi}(0) = \frac{d}{dt} \Big|_{t=0} \Psi(t) = -\frac{d}{dt} \Big|_{t=0} \Psi(t + \beta) = -\dot{\Psi}(t_2), \quad (2.47)$$

and taking $N \rightarrow \infty$ limit, we arrive at

$$(2.45) = \int_{\Psi_{\pm}(0) = -\Psi_{\pm}(\beta)} \left(\prod_{t \in [0, \beta]} d\Psi_+(t) d\Psi_-(t) \right) e^{-\int_0^{\beta} dt \left(\Psi_+ \dot{\Psi}_- + V(\Psi_+, \Psi_-) \right)}. \quad (2.48)$$

Path integral description of harmonic oscillator For harmonic oscillator (2.39),

$$\begin{aligned}
 \text{Tr}(e^{-\beta \hat{H}_f}) &= \int_{\Psi_{\pm}(0) = -\Psi_{\pm}(\beta)} \left(\prod_{t \in [0, \beta]} d\Psi_+(t) d\Psi_-(t) \right) e^{-\int_0^{\beta} dt \Psi_+ (\partial_t + \omega) \Psi_-} \\
 &= \det_{\Psi_{\pm}(0) = -\Psi_{\pm}(\beta)} (\partial_t + \omega). \quad (2.49)
 \end{aligned}$$

We used the Gaussian integral formula for fermionic variables (A.16). In this anti-periodic sector, the eigenvectors of ∂_t are $\psi_n(t) = e^{\frac{2\pi i}{\beta}(n - \frac{1}{2})t}$ with $n \in \mathbb{Z}$. Therefore,

$$\begin{aligned}
 \det_{\Psi_{\pm}(0) = -\Psi_{\pm}(\beta)} (\partial_t + \omega) &= \prod_{n=-\infty}^{\infty} \left(\frac{2\pi i}{\beta} \left(n - \frac{1}{2} \right) + \omega \right) \\
 &= \prod_{n=1}^{\infty} \left(\frac{(2\pi)^2}{\beta^2} \left(n - \frac{1}{2} \right)^2 + \omega^2 \right)
 \end{aligned}$$

⁵We have checked it only with Ψ_- , but we can understand the case for Ψ_+ in similar way.

$$= \left[\prod_{n=1}^{\infty} \frac{2\pi}{\beta} \left(n - \frac{1}{2}\right) \right]^2 \times \prod_{n=1}^{\infty} \left(1 + \frac{(\beta\omega)^2}{(2\pi[n - \frac{1}{2}])^2} \right). \quad (2.50)$$

The first factor diverges, so we have to regularize it somehow. One might think that the zeta-function regularization works, however in this case, we should be more careful:

$$\begin{aligned} \left[\prod_{n=1}^{\infty} \frac{2\pi}{\beta} \left(n - \frac{1}{2}\right) \right]^2 &= \left[\prod_{n=1}^{\infty} \frac{2\pi}{\beta} n \right]^2 \times \left[\prod_{n=1}^{\infty} \frac{\frac{2\pi}{\beta} (n - \frac{1}{2})}{\frac{2\pi}{\beta} n} \right]^2 \\ &= \left[\prod_{n=1}^{\infty} \frac{2\pi}{\beta} n \right]^2 \times \left[\prod_{n=1}^{\infty} \frac{\frac{\pi}{\beta} (2n - 1)}{\frac{\pi}{\beta} (2n)} \right]^2 \\ &= \underbrace{\left[\prod_{n=1}^{\infty} \frac{2\pi}{\beta} n \right]^2}_{\rightarrow \beta} \times \frac{\pi}{\beta} \times \underbrace{\left[\prod_{n=1}^{\infty} \frac{\frac{\pi}{\beta} (2n - 1) \times \frac{\pi}{\beta} (2n + 1)}{\frac{\pi^2}{\beta^2} (2n)^2} \right]}_{\frac{2}{\pi}} \\ &\rightarrow 2, \end{aligned} \quad (2.51)$$

where we used Wallis' formula. Another part of (2.50) can be calculated by using infinite product formula for cosh (A.2):

$$\prod_{n=1}^{\infty} \left(1 + \frac{(\beta\omega)^2}{(2\pi[n - \frac{1}{2}])^2} \right) = \cosh \frac{\beta\omega}{2}. \quad (2.52)$$

Gathering all, we recover the result (2.41)

$$\text{Tr}(e^{-\beta\hat{H}_f}) = 2 \cosh \frac{\beta\omega}{2}. \quad (2.53)$$

2.3 Witten Index

As we have reviewed briefly, the partition function of the harmonic oscillator can be calculated easily. However, once we turn on the cubic or more higher interaction, it is difficult to perform the calculation explicitly. In addition to it, the naive zeta-function regularization did not work in the fermionic sector as we have observed in previous page. However, we can overcome such a situation by considering

$$I = \text{Tr}((-1)^{\hat{F}} e^{-\beta\hat{H}}), \quad \text{where } \hat{F} \text{ is a fermion number operator,} \quad (2.54)$$

instead of Z .

Fermion number operator \hat{F} is an operator which counts the number of fermion excitation, 0 or 1. Explicitly, we can write it in our previous notation as

$$\hat{F} = \hat{\psi}_+ \hat{\psi}_-. \quad (2.55)$$

As one can check,

$$(-1)^{\hat{F}} = \begin{cases} +1 & \text{bosonic state} \\ -1 & \text{fermionic state} \end{cases}. \quad (2.56)$$

Therefore, within only bosonic sector, I and Z are identical:

$$I_b = \text{Tr}_b \left((-1)^{\hat{F}} e^{-\beta \hat{H}_b} \right) = \text{Tr}_b (e^{-\beta \hat{H}_b}) = Z_b, \quad (2.57)$$

and nothing different happens compared with the partition function. However, the fermion sector's behavior changes drastically.

2.3.1 Fermion Sector

Let us see what happens in the operator formalism first by using the harmonic oscillator example.

Operator formalism We can get I for fermion sector just by inserting $(-1)^n$ into the previous summation in (2.41) as

$$\begin{aligned} \text{Tr} \left((-1)^{\hat{F}} e^{-\beta \hat{H}_f} \right) &= \sum_{n=0}^1 (-1)^n e^{-\beta \omega (n - \frac{1}{2})} \\ &= e^{\frac{\beta \omega}{2}} - e^{-\frac{\beta \omega}{2}} \\ &= 2 \sinh \frac{\beta \omega}{2}. \end{aligned} \quad (2.58)$$

Path integral formalism After a simple calculation, one can verify that

$$\begin{aligned} I &= \text{Tr} \left((-1)^{\hat{F}} e^{-\beta \hat{H}_f} \right) \\ &= \int d\Psi_+ d\Psi_- e^{-\Psi_+ \Psi_-} \langle \Psi_+ | e^{-\beta \hat{H}_f} | \Psi_- \rangle. \end{aligned} \quad (2.59)$$

Compering with the partition function (2.42), one can see that the sign of the exponential factor is different. This minus sign makes the fermionic fields $\Psi_{\pm}(t)$ in the path integral periodic under $t \rightarrow t + \beta$. In summary,

$$I_f = \int_{\Psi_{\pm}(0)=\Psi_{\pm}(\beta)} \left(d\Psi_+(t) d\Psi_-(t) \right) e^{-\int_0^\beta \left(\Psi_+ \partial_t \Psi_- + V(\Psi_+, \Psi_-) \right)}. \quad (2.60)$$

In this case, we can recover the result (2.58) as follows.

Harminic oscillator

$$\begin{aligned} I_f &= \int_{\Psi_{\pm}(0)=\Psi_{\pm}(\beta)} \left(d\Psi_+(t) d\Psi_-(t) \right) e^{-\int_0^\beta \Psi_+ (\partial_t + \omega) \Psi_-} \\ &= \det_{\Psi_{\pm}(0)=\Psi_{\pm}(\beta)} (\partial_t + \omega) \\ &= \prod_{n=-\infty}^{\infty} \left(\frac{2\pi i}{\beta} n + \omega \right) \\ &= \omega \prod_{n=1}^{\infty} \left(\frac{(2\pi n)^2}{\beta^2} + \omega^2 \right). \end{aligned} \quad (2.61)$$

The same infinite product in the bosonic partition function (2.34) emerges. Therefore, by repeating zeta-function regularization procedure, we arrive at

$$I_f = 2 \sinh \frac{\beta\omega}{2}. \quad (2.62)$$

2.3.2 Supersymmetric Quantum Mechanics

What happens when we consider

$$I = \text{Tr} \left((-1)^{\hat{F}} e^{-\beta \hat{H}} \right), \quad (2.63)$$

with harmonic oscillator Hamiltonian $\hat{H} = \hat{H}_b + \hat{H}_f$? The answer is extremely simple;

$$\begin{aligned} I &= I_b \times I_f \\ &= Z_b \times I_f \\ &= \frac{1}{2 \sinh \frac{\beta\omega}{2}} \times 2 \sinh \frac{\beta\omega}{2} \\ &= 1. \end{aligned} \quad (2.64)$$

Note that if we turn on different frequencies ω_b, ω_f for boson and fermion respectively, we get

$$I = \frac{\sinh \frac{\beta\omega_f}{2}}{\sinh \frac{\beta\omega_b}{2}}, \quad (2.65)$$

and it does depend on β . Therefore, the β independence is equivalent to the condition $\omega_b = \omega_f$. It is strongly related to the concept of *supersymmetry*. In other words, the Hamiltonian

$$\hat{H} = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \omega \left(\hat{\psi}_+ \hat{\psi}_- - \frac{1}{2} \right) = \omega (\hat{a}^\dagger \hat{a} + \hat{\psi}_+ \hat{\psi}_-) \quad (2.66)$$

defines supersymmetric quantum mechanics. The physical meaning is also extremely simple: the state $|0\rangle$ only contributes. This quantity is called Witten index [6]. We can learn other facts of supersymmetry from this extremely simple example by defining

$$\hat{Q} := \sqrt{\omega} \hat{a}^\dagger \hat{\psi}_-, \quad \hat{Q}^\dagger := \sqrt{\omega} \hat{a} \hat{\psi}_+. \quad (2.67)$$

These operators are called *supercharges* which satisfy the following equation.

$$\hat{H} = \{\hat{Q}, \hat{Q}^\dagger\}. \quad (2.68)$$

By using this expression, the reason for β independence of Witten index becomes clear because the differential of the index with respect to β becomes zero:

$$\begin{aligned} \frac{d}{d\beta} \text{Tr}(-1)^{\hat{F}} e^{-\beta \hat{H}} &= \frac{d}{d\beta} \text{Tr}(-1)^{\hat{F}} e^{-\beta \{\hat{Q}, \hat{Q}^\dagger\}} \\ &= -\text{Tr}(-1)^{\hat{F}} (\hat{Q} \hat{Q}^\dagger + \hat{Q}^\dagger \hat{Q}) e^{-\beta \{\hat{Q}, \hat{Q}^\dagger\}} \\ &= -\text{Tr}(-1)^{\hat{F}} (\hat{Q} \hat{Q}^\dagger - \hat{Q} \hat{Q}^\dagger) e^{-\beta \{\hat{Q}, \hat{Q}^\dagger\}} = 0. \end{aligned} \quad (2.69)$$

We can construct a somewhat more non-trivial Hamiltonian (e.g. [6–8]) which contains interaction terms. In such case, Witten index counts the number of degeneracy of ground states, or more technically speaking, it counts the number of *BPS states*.

2.3.3 Generalized Index

In (2.69), we use the following facts:

$$[\hat{H}, \hat{Q}] = [\hat{H}, \hat{Q}^\dagger] = 0. \quad (2.70)$$

It means \hat{Q} and \hat{Q}^\dagger generate symmetry of the system. Suppose there is another generator \hat{J} which commutes with the supercharges:

$$[\hat{Q}, \hat{J}] = 0, \quad [\hat{Q}^\dagger, \hat{J}] = 0, \quad (2.71)$$

then following trace

$$\text{Tr}\left((-1)^{\hat{F}} e^{-\beta\{\hat{Q}, \hat{Q}^\dagger\}} e^{-i\mu\hat{J}}\right) \quad (2.72)$$

also does not depend on β . In Chap. 3, we introduce the concept of *SuperConformal Index* (SCI). SCI can be regarded such a generalized index. $e^{-i\mu\hat{J}}$ insertion makes $x(t)$ and $\Psi_\pm(t)$ not periodic but “twisted”

$$x(t + \beta) = e^{i\mu J_x} x(t), \quad \Psi_\pm(t + \beta) = e^{i\mu J_\psi} \Psi_\pm(t), \quad (2.73)$$

where J_x, J_ψ are eigenvalues of \hat{J} operator. The reason is as follows. For bosonic degrees of freedom, (2.72) can be expressed

$$\begin{aligned} \text{Tr}\left((-1)^{\hat{F}} e^{-\beta\{\hat{Q}, \hat{Q}^\dagger\}} e^{-i\mu\hat{J}}\right) &= \int dx \langle x | (-1)^{\hat{F}} e^{-\beta\{\hat{Q}, \hat{Q}^\dagger\}} e^{-i\mu\hat{J}} | x \rangle \\ &= \int dx \langle x | e^{-\beta\hat{H}} | e^{-i\mu J_x} x \rangle \\ &= \int dx dp dx_1 \underbrace{\langle x | e^{-(\beta-\epsilon)\hat{H}} | x_1 \rangle}_{e^{-\epsilon H(x_1, p) + i p x_1}} \underbrace{\langle x_1 | e^{-\epsilon\hat{H}} | p \rangle}_{e^{-i p e^{-i\mu J_x} x}} \langle p | e^{-i\mu J_x} x \rangle, \end{aligned} \quad (2.74)$$

and at the edge, we have

$$e^{-\epsilon H(x_1, p) + i p x_1 - i p e^{-i\mu J_x} x}. \quad (2.75)$$

In order to get rid of $\mathcal{O}(\epsilon^0)$ term,

$$\begin{aligned} +i p x_1 - i p e^{-i\mu J_x} x &= i p (x_1 - e^{-i\mu J_x} x) \\ &= i p \left(x(t = \epsilon) - e^{-i\mu J_x} x(t = \beta) \right) \\ &= i p \left(\underbrace{\epsilon \dot{x}(0) + x(t = 0) - e^{-i\mu J_x} x(t = \beta)}_{\text{we have to make it zero.}} \right). \end{aligned} \quad (2.76)$$

This is the origin of the twisted boundary condition in (2.73).

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