# An Alternative Proof of the Theorem of Woodford on the Existence of a Sunspot Equilibrium in a Continuous-Time Model

**Kazuo Nishimura and Tadashi Shigoka**

**Abstract** Nishimura and Shigoka, Int J Econ Theory 2:199–216, (2006) has proved a continuous-time version of the theorem of Woodford, Stationary sunspot equilibria: the case of small fluctuations around a deterministic steady state, mimeo, (1986) to the effect that there exists a stationary sunspot equilibrium for a continuous-time model with a predetermined variable and with an unstable root, if equilibrium is indeterminate near either a steady state or a closed orbit, and if a stable manifold is well-located in an ambient space. The present study provides this theorem with an alternative proof that is due to Murakami et al. Homoclinic orbit and stationary sunspot equilibrium in a three-dimensional continuous-time model with a predetermined variable forthcoming in: Nishimura K, Venditti A, Yannelis NC (eds) Sunspots and non-linear dynamics. Springer, (2016) and simpler than that of Nishimura and Shigoka, Int J Econ Theory 2:199–216, (2006).

## 1 Introduction

If for a given deterministic model, there exists a continuum of perfect foresight equilibria, equilibrium is said to be indeterminate. Suppose that fundamental characteristics of an economy are deterministic, but that economic agents believe nevertheless that equilibrium dynamics is affected by random factors apparently irrelevant to the fundamental characteristics (sunspots). This prophecy could be self-fulfilling, and one will get a sunspot equilibrium, if the resulting equilibrium dynamics is subject to a nontrivial stochastic process. A sunspot equilibrium is called a stationary sunspot equilibrium, if the equilibrium stochastic process is stationary. See Shell (1977), Azariadis (1981), and Cass and Shell (1983) for the concept of a sunspot

K. Nishimura (✉)
Research Institute for Economics and Business Administration,
Kobe University, Kobe, Japan
e-mail: nishimura@rieb.kobe-u.ac.jp

T. Shigoka
Institute of Economic Research, Kyoto University, Kyoto, Japan
e-mail: sigoka@kier.kyoto-u.ac.jp



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equilibrium. For a large class of models whose fundamental characteristics are deterministic, if equilibrium is indeterminate, there exists a sunspot equilibrium. See Chiappori and Guesnerie (1991) and Guesnerie and Woodford (1992) for thorough surveys on the sunspot literature. Woodford (1986) has proved that there exists a stationary sunspot equilibrium for a discrete-time model with a predetermined variable and with an unstable root, if equilibrium is indeterminate near a steady state, and if a stable manifold is well-located in an ambient space. Nishimura and Shigoka (2006) treats a three-dimensional continuous-time deterministic model that includes one predetermined variable and two non-predetermined variables and that includes a well-located two-dimensional invariant manifold that might be a stable manifold of either a steady state or a closed orbit, and has constructed a stationary sunspot equilibrium in this model by means of extending the method of Shigoka (1994). This is a continuous-time version of the theorem due to Woodford (1986).

Murakami et al. (2016) treats a three-dimensional continuous-time deterministic model that includes one predetermined variable and two non-predetermined variables and that is amenable to the existence of a homoclinic orbit with multiple steady states, and has constructed a stationary sunspot equilibrium in this model by means of extending the method of Benhabib et al. (2008). The underlying deterministic dynamics in Murakami et al. (2016) is more complex than that in Nishimura and Shigoka (2006). On the other hand, as discussed in Sect. 2.4, the structure of a stochastic differential equation in Benhabib et al. (2008) the extension of which is Murakami et al. (2016) is simpler than that in Shigoka (1994) the extension of which is Nishimura and Shigoka (2006). The present study applies the method of Murakami et al. (2016) to the simpler underlying deterministic dynamics treated by Nishimura and Shigoka (2006), and provides an alternative proof of the existence of a stationary sunspot equilibrium for this model. The alternative proof in this study is simpler than that of Nishimura and Shigoka (2006), because the structure of a stochastic differential equation in the present study is simpler than that in Nishimura and Shigoka (2006).

In Sect. 2.1, we specify an underlying deterministic model. In Sect. 2.2, we specify a stochastic process that generates sunspot variables. In Sect. 2.3, we define a stationary sunspot equilibrium, and state a main theorem that is a continuous-time version of the theorem of Woodford (1986). In Sect. 2.4, we relate our result to those of Shigoka (1994), Nishimura and Shigoka (2006), Benhabib et al. (2008), and Murakami et al. (2016). Section 3 provides the main theorem with a proof the method of which is due to Murakami et al. (2016).

## 2 Main Result

### *2.1 Deterministic Equilibrium Dynamics*

In the present section, we specify a three-dimensional continuous-time deterministic model that includes one predetermined variable and two non-predetermined variables. We will assume that the model has a well-located two-dimensional invariant manifold that might be a stable manifold of either a steady state or a closed orbit. Since the number of a predetermined variable is one, and since the dimension of the invariant manifold is two, equilibrium is indeterminate near either the steady state or the closed orbit. Let $V$ be a nonempty open subset of $\mathbb{R}^2$ homeomorphic to some convex set. Let $I$ be a nonempty open connected subset of $\mathbb{R}$. Let $W$ be defined as $W := V \times I$. Let $f_i : W \to \mathbb{R}$, $i = 1, 2, 3$, be a continuously differentiable function, i.e., a $C^1$-function, and let $F : W \to \mathbb{R}^3$ be a $C^1$-function defined as

$$F(X, u, Q) := \begin{bmatrix} f_1(X, u, Q) \\ f_2(X, u, Q) \\ f_3(X, u, Q) \end{bmatrix},$$

where $(X, u, Q) \in W$. We assume that $X$ is a predetermined variable, whereas $u$ and $Q$ are non-predetermined variables, and that a perfect foresight equilibrium is a solution of an ordinary differential equation $[\dot{X}, \dot{u}, \dot{Q}]^{\mathbf{T}} = F(X, u, Q)$, where $^{\mathbf{T}}$ denotes the transpose of a given vector. We assume:

**Assumption 1** There exists a $C^1$-function $\varphi : V \to I$ such that, for $(X, u) \in V$,

$$f_3(X, u, \varphi(X, u)) = \frac{\partial \varphi}{\partial X}(X, u) f_1(X, u, \varphi(X, u)) + \frac{\partial \varphi}{\partial u}(X, u) f_2(X, u, \varphi(X, u)).$$

Under Assumption 1, $\{(X, u, Q) \in W : (X, u) \in V \wedge Q = \varphi(X, u)\}$ constitutes a two-dimensional manifold invariant under the action of $[\dot{X}, \dot{u}, \dot{Q}]^{\mathbf{T}} = F(X, u, Q)$. Let $G : V \to \mathbb{R}^2$ be a $C^1$-function defined as

$$G(X, u) := \begin{bmatrix} f_1(X, u, \varphi(X, u)) \\ f_2(X, u, \varphi(X, u)) \end{bmatrix},$$

for $(X, u) \in V$. Under Assumption 1, we further assume that either of the following two assumptions is satisfied.

**Assumption 2** There exists a closed subset $D$ of $V$ homeomorphic to the two-dimensional closed unit disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ such that the vector field $[\dot{X}, \dot{u}]^{\mathbf{T}} = G(X, u)$ points inward on the boundary $\partial D$ of $D$, where $\partial D$ is homeomorphic to $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

**Assumption 3** There exists a closed subset $D$ of $V$ homeomorphic to the two-dimensional closed doughnut $\{(x, y) \in \mathbb{R}^2 : \frac{1}{2} \leq x^2 + y^2 \leq 1\}$ such that the vector field $[\dot{X}, \dot{u}]^{\mathbf{T}} = G(X, u)$ points inward on the boundary $\partial D$ of $D$, where $\partial D$ is homeomorphic to $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = \frac{1}{2} \vee x^2 + y^2 = 1\}$.

For some $(\bar{X}, \bar{u}) \in V$, if $(\bar{X}, \bar{u}, \varphi(\bar{X}, \bar{u})) \in W$ is a hyperbolic steady state, and if $\{(X, u, Q) \in W : (X, u) \in V \wedge Q = \varphi(X, u)\}$ constitutes a two-dimensional stable manifold of the steady state, then Assumptions 1 and 2 are satisfied, and equilibrium is indeterminate near the steady state. See Nishimura and Shigoka (2006, pp. 204–205) for the method of assuring that Assumptions 1 and 2 are satisfied, and see

Sect. 3 in Nishimura and Shigoka (2006) for concrete economic models that satisfy Assumptions 1 and 2.

For some closed curve $\gamma$ in $V$, if $\hat{\gamma} := \{(X, u, Q) \in W : (X, u) \in \gamma \wedge Q = \varphi(X, u)\}$ is a closed orbit of $[\dot{X}, \dot{u}, \dot{Q}]^{\mathbf{T}} = F(X, u, Q)$, and if $\{(X, u, Q) \in W : (X, u) \in V \wedge Q = \varphi(X, u)\}$ includes a two-dimensional invariant manifold each point of which asymptotically converges to this closed orbit $\hat{\gamma}$, then Assumptions 1 and 3 are satisfied, and equilibrium is indeterminate near the closed orbit. See Sect. 2.4 in Nishimura and Shigoka (2006) for the method of assuring that Assumptions 1 and 3 are satisfied, and see Sect. 3 in Nishimura and Shigoka (2006) for concrete examples that satisfy Assumptions 1 and 3.

## 2.2 Sunspot Variables

In the present section, we specify a continuous-time stochastic process that generates sunspot variables. We assume that the stochastic process is subject to a *separable two-state Markov process with stationary transition matrices.* A sample function of a random variable subject to this process generates a sequence of discontinuous points such that each discontinuous point is, of itself, a random variable. We will utilize a sequence of discontinuous points in the sample path of a sunspot variable in order to construct a sunspot equilibrium.

Let $\mathbb{T}$ denote the set of all nonnegative real numbers, i.e., $\mathbb{T} := \mathbb{R}_+$. We denote the set of all function from $\mathbb{T}$ to $\{1, 2\}$ by $\{1, 2\}^{\mathbb{T}}$. Let $B$ be some subset of $\{1, 2\}^{\mathbb{T}}$. Let $\varepsilon(t, b)$ denote the $t$th coordinate of $b \in B$. Let $\mathcal{B}(B)$ be some $\sigma$-field on $B$ such that $\varepsilon(t, b)$ is a measurable function of $b$ for each $t \in \mathbb{T}$. And let $\hat{\mathbf{P}}_1 : \mathcal{B}(B) \to [0, 1]$ be some probability measure defined on $\mathcal{B}(B)$. $\varepsilon(t, b)$ that is considered as a function of $t \in \mathbb{T}$ will be called a sample function of $b \in B$. Let $\mathcal{L} = \mathcal{L}(\mathbb{R}^2)$ be the set of all $2 \times 2$ real square matrices. Let $\lambda > 0$ be a given positive constant, and let $\mathbf{\Lambda} \in \mathcal{L}(\mathbb{R}^2)$ be given by

$$\mathbf{\Lambda} := \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}.$$

Let $\hat{\mathbf{Q}} : \mathbb{T} \to \mathcal{L}(\mathbb{R}^2)$ be defined as

$$\hat{\mathbf{Q}}(h) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{h^k}{k!} \mathbf{\Lambda}^k,$$

for each $h \in \mathbb{T}$. Let $\hat{B}(\{1, 2\}^{\mathbb{T}})$ be the set of all functions $b$ in $\{1, 2\}^{\mathbb{T}}$ such that $b$ is a piecewise-continuous function of $t \in \mathbb{T}$ and such that $b$ is continuous on the right at each discontinuous point in $\mathbb{T}$. Let $\mathbb{N}$ be the set of all positive integers, i.e., $\mathbb{N} := \{1, 2, \ldots\}$. For $b \in B \cap \hat{B}(\{1, 2\}^{\mathbb{T}})$ and for $m \in \mathbb{N}$, let $\hat{t}(m, b) \in \mathbb{T}$ be the $m$th discontinuous point of the sample function $\varepsilon(t, b)$ of $b$ if the $m$th discontinuous point exists. Then, there exists a stochastic process $(B, \mathcal{B}(B), \hat{\mathbf{P}}_1)$ that satisfies the following conditions. See Proposition 1 in Murakami et al. (2016) for the proof and for the reference to the relevant parts of Doob (1953).

**Proposition 1** *There exists a continuous-time two-state Markov process* $(B, \mathcal{B}(B), \hat{\mathbf{P}}_1)$ *with stationary transition probabilities that satisfies the following conditions:*

*(1) The initial probability is given by*

$$\hat{\mathbf{P}}_1\{b \in B : \varepsilon(0, b) = 1\} = \frac{1}{2} \wedge \hat{\mathbf{P}}_1\{b \in B : \varepsilon(0, b) = 2\} = \frac{1}{2}.$$

*(2) A family of the stationary transition probabilities is given by a family of the matrices* $\hat{\mathbf{Q}}(h)$ *with* $h \in \mathbb{T}$.

*(3)* $B \subset \hat{B}(\{1, 2\}^{\mathbb{T}})$. *[The separability.]*

*(4) For each* $b \in B$ *and for each* $m \in \mathbb{N}$, *there exists the* $m$*th discontinuous point* $\hat{t}(m, b)$ *in the sample function* $\varepsilon(t, b)$ *of* $b$, *and* $\lim_{m\to\infty} \hat{t}(m, b) = \infty$.

Let $\mathbb{N}_0$ denote the set of all nonnegative integers, i.e., $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Let $\tau : \mathbb{N}_0 \times B \to \mathbb{T}$ be a function constructed in the following way. For each $b \in B$, set $\tau(0, b) = 0$. For each $m \in \mathbb{N}$ and for each $b \in B$, set $\tau(m, b) = \hat{t}(m, b)$. Then, by Proposition 1, $\tau(m, b) \in \mathbb{T}$ is well-defined for all $(m, b) \in \mathbb{N}_0 \times B$. By construction, and since $B \subset \hat{B}(\{1, 2\}^{\mathbb{T}})$, $\tau(0, b) = 0$ for each $b \in B$, and $\tau(m + 1, b) - \tau(m, b) > 0$ for each $(m, b) \in \mathbb{N}_0 \times B$. Let $\mathcal{B}_t(B)$ be *the smallest* $\sigma$*-field* with respect to which $(\varepsilon(s, b), 0 \leq s \leq t)$ is a family of measurable functions of $b \in B$. For each $m \in \mathbb{N}_0$, if $t \geq \tau(m, b)$, $\tau(m, b)$ is a $\mathcal{B}_t(B)$-measurable function of $b$, and if $\tau(m, b) > s \geq 0$, $\tau(m, b)$ is not $\mathcal{B}_s(B)$-measurable function of $b$. Proposition 2 in Murakami et al. (2016) shows that each element of the set of random variables $\{\tau(m + 1, b) - \tau(m, b)\}_{m\in\mathbb{N}_0}$ is independently and identically subject to a exponential distribution with a parameter $\lambda > 0$.

## 2.3 Stationary Sunspot Equilibrium

In the present section, we assume that the underlying deterministic dynamics $[\dot{X}, \dot{u}, \dot{Q}]^{\mathbf{T}} = F(X, u, Q)$ satisfies Assumption 1 and either of Assumptions 2 and 3. Under these assumptions, we define a stationary sunspot equilibrium *formally*, and state that there exists a stationary sunspot equilibrium thus defined. The proof of the statement will be given in Sect. 3. Let $\hat{D} \subset W$ be defined as $\hat{D} := \{(X, u, Q) \in W : (X, u) \in D \wedge Q = \varphi(X, u)\}$, where $\varphi$ and $D$ are specified as in Assumption 1 and either of Assumptions 2 and 3, respectively. Let $\mathcal{B}(\hat{D})$ be the Borel $\sigma$-field on $\hat{D}$, and let $\hat{\mathbf{P}}_0 : \mathcal{B}(\hat{D}) \to [0, 1]$ be some probability measure on $\mathcal{B}(\hat{D})$. Let $(B, \mathcal{B}(B), \hat{\mathbf{P}}_1)$ be the probability space the existence of which is assured by Proposition 1, and let $\Omega := \hat{D} \times B$ and let $\mathcal{B}_\Omega = \mathcal{B}(\Omega)$ be the product $\sigma$-field of $\mathcal{B}(\hat{D})$ and $\mathcal{B}(B)$, i.e., $\mathcal{B}(\Omega) := \mathcal{B}(\hat{D}) \times \mathcal{B}(B)$. Let $\pi_0 : \Omega \to \hat{D}$ be the projection of $\hat{D} \times B$ onto $\hat{D}$. Let $\pi_1 : \Omega \to B$ be the projection of $\hat{D} \times B$ onto $B$. We have denoted the $t$th coordinate of $b \in B$ by $\varepsilon(t, b)$. Let $\mathcal{B}_t(\Omega)$ be *the smallest* $\sigma$*-field* with respect to which

$(\pi_0(\omega), \varepsilon(s, \pi_1(\omega)), 0 \le s \le t)$ is a family of measurable functions of $\omega \in \Omega$. Let $\mathbf{P} : \mathcal{B}(\Omega) \to [0, 1]$ be defined as the product measure of $\hat{\mathbf{P}}_0$ and $\hat{\mathbf{P}}_1$, and let $\mathbf{E}_t[\cdot]$ be the conditional expectation operator relative to $\mathcal{B}_t(\Omega)$. We denote a set of all functions from $\mathbb{T}$ to $\hat{D}$ by $\hat{D}^{\mathbb{T}}$. Let $\hat{l} : \Omega \to \hat{D}^{\mathbb{T}}$ be a function such that the $t$th coordinate of $\hat{l}(\omega) \in \hat{D}^{\mathbb{T}}$ is $\mathcal{B}_\Omega$-measurable function of $\omega \in \Omega$ for each $t \in \mathbb{T}$. Let $(X(t, \omega), u(t, \omega), Q(t, \omega))$ denote the $t$th coordinate of $\hat{l}(\omega) \in \hat{D}^{\mathbb{T}}$. Let $\mathcal{B}(\hat{D}^{\mathbb{T}})$ be some $\sigma$-field on $\hat{D}^{\mathbb{T}}$, and let $\hat{\mathbf{P}} : \mathcal{B}(\hat{D}^{\mathbb{T}}) \to [0, 1]$ be some probability measure defined on $\mathcal{B}(\hat{D}^{\mathbb{T}})$. We define a stationary sunspot equilibrium in the following way.

**Definition 1** If the probability measure $\mathbf{P} : \mathcal{B}(\Omega) \to [0, 1]$ satisfies the following conditions, then function $\hat{l} : \Omega \to \hat{D}^{\mathbb{T}}$ constitutes a stationary sunspot equilibrium.

(1) For each $t \in \mathbb{T}$, $(X(t, \omega), u(t, \omega), Q(t, \omega)) \in \hat{D}$ is a $\mathcal{B}_t(\Omega)$-measurable function of $\omega \in \Omega$.
(2) The distribution of $(X(0, \omega), u(0, \omega), Q(0, \omega))$ is given by $(\hat{D}, \mathcal{B}(\hat{D}), \hat{\mathbf{P}}_0)$.
(3) There exists a stochastic process $(\hat{D}^{\mathbb{T}}, \mathcal{B}(\hat{D}^{\mathbb{T}}), \hat{\mathbf{P}})$ on $\hat{D}^{\mathbb{T}}$ such that if $\{t_i\}_{i=1}^N$ is a given set of points in $\mathbb{T}$ with $N \ge 1$, and if $\hat{Y}$ is a given Borel subset of $\hat{D}^N$, then

$$\begin{aligned}&\mathbf{P}\{\omega \in \Omega : (\hat{l}(t_1, \omega), \dots, \hat{l}(t_N, \omega)) \in \hat{Y}\}\\ &\quad = \hat{\mathbf{P}}\{\hat{d} \in \hat{D}^{\mathbb{T}} : (\hat{d}(t_1), \dots, \hat{d}(t_N)) \in \hat{Y}\},\end{aligned}$$

where $\hat{l}(t, \omega) := (X(t, \omega), u(t, \omega), Q(t, \omega)) \in \hat{D}$, and $\hat{d}(t)$ denotes the $t$th coordinate of $\hat{d} \in \hat{D}^{\mathbb{T}}$. [The existence of a stochastic process.]
(4) For each $\omega \in \Omega$, $X(t, \omega)$ is a continuous function of $t \in \mathbb{T}$, and for each $t > 0$,

$$\mathbf{P}\{\omega \in \Omega : \lim_{h \to 0} \frac{X(t+h, \omega) - X(t, \omega)}{h} = f_1(X(t, \omega), u(t, \omega), Q(t, \omega))\} = 1.$$

(5) For each $\omega \in \Omega$, $(u(t, \omega), Q(t, \omega))$ is a piecewise-continuous function of $t \in \mathbb{T}$ and continuous on the right at each discontinuous point in $\mathbb{T}$, and for each $t \in \mathbb{T}$,

$$\begin{bmatrix} \lim_{h\to+0} \frac{X(t+h,\omega)-X(t,\omega)}{h} \\ \mathbf{E}_t[\lim_{h\to+0} \frac{u(t+h,\omega)-u(t,\omega)}{h}] \\ \mathbf{E}_t[\lim_{h\to+0} \frac{Q(t+h,\omega)-Q(t,\omega)}{h}] \end{bmatrix} = \begin{bmatrix} f_1(X(t, \omega), u(t, \omega), Q(t, \omega)) \\ f_2(X(t, \omega), u(t, \omega), Q(t, \omega)) \\ f_3(X(t, \omega), u(t, \omega), Q(t, \omega)) \end{bmatrix}.$$

(6) For each $t > s \ge 0$, $(X(t, \omega), u(t, \omega), Q(t, \omega))$ is not $\mathcal{B}_s$-measurable function of $\omega \in \Omega$.
(7) If $\{t_i\}_{i=1}^N$ is a given set of points in $\mathbb{T}$ with $N \ge 1$, and if $\hat{Y}$ is a given Borel subset of $\hat{D}^N$, for any $h \in \mathbb{R}$ such that $\{t_i + h\}_{i=1}^N \subset \mathbb{T}$,

$$\begin{aligned}&\mathbf{P}\{\omega \in \Omega : (\hat{l}(t_1, \omega), \dots, \hat{l}(t_N, \omega)) \in \hat{Y}\}\\ &\quad = \mathbf{P}\{\omega \in \Omega : (\hat{l}(t_1 + h, \omega), \dots, \hat{l}(t_N + h, \omega)) \in \hat{Y}\},\end{aligned}$$

where $\hat{l}(t, \omega) := (X(t, \omega), u(t, \omega), Q(t, \omega)) \in \hat{D}$. [The stationarity.]

In Sect. 3, we will show the following theorem by means of the method due to Murakami et al. (2016).

**Theorem 1** *Suppose that the underlying deterministic dynamics* $[\dot{X}, \dot{u}, \dot{Q}]^T = F(X, u, Q)$ *satisfies Assumption 1 and either of Assumptions 2 and 3. Then, there exists a stationary sunspot equilibrium.*

## *2.4 On Relations of the Present Result to Other Results*

Before leaving Sect. 2, we relate our result to those of Shigoka (1994), Nishimura and Shigoka (2006), Benhabib et al. (2008), and Murakami et al. (2016) here. Let $g_i : V \to \mathbb{R}, i = 1, 2,$ be defined as $g_i(X, u) := f_i(X, u, \varphi(X, u)), i = 1, 2$. Suppose that the vector field $[\dot{X}, \dot{u}]^{\mathbf{T}} = G(X, u)$ satisfies either of Assumptions 2 and 3. Let $\{(X(t, \omega), u(t, \omega))\}_{t\in\mathbb{T}}$ be a set of random variables that will have been constructed in Sect. 3. Then, we have the following:

(1) For each $\omega \in \Omega$, $X(t, \omega)$ is a continuous function of $t \in \mathbb{T}$, and for each $t > 0$,

$$\mathbf{P}\{\omega \in \Omega : \lim_{h\to 0} \frac{X(t+h, \omega) - X(t, \omega)}{h} = g_1(X(t, \omega), u(t, \omega))\} = 1.$$

(2) For each $\omega \in \Omega$, $u(t, \omega)$ is a piecewise-continuous function of $t \in \mathbb{T}$ and continuous on the right at each discontinuous point in $\mathbb{T}$, and for each $t \in \mathbb{T}$,

$$\begin{bmatrix} \lim_{h\to +0} \frac{X(t+h,\omega)-X(t,\omega)}{h} \\ \mathbf{E}_t[\lim_{h\to +0} \frac{u(t+h,\omega)-u(t,\omega)}{h}] \end{bmatrix} = \begin{bmatrix} g_1(X(t, \omega), u(t, \omega)) \\ g_2(X(t, \omega), u(t, \omega)) \end{bmatrix}.$$

(3) For each $t > s \geq 0$, $(X(t, \omega), u(t, \omega))$ is not $\mathcal{B}_s$-measurable function of $\omega \in \Omega$.

(4) If $\{t_i\}_{i=1}^N$ is a given set of points in $\mathbb{T}$ with $N \geq 1$, and if $Y$ is a given Borel subset of $D^N$, for any $h \in \mathbb{R}$ such that $\{t_i + h\}_{i=1}^N \subset \mathbb{T}$,

$$\begin{aligned} &\mathbf{P}\{\omega \in \Omega : (l(t_1, \omega), \ldots, l(t_N, \omega)) \in Y\} \\ &\quad = \mathbf{P}\{\omega \in \Omega : (l(t_1 + h, \omega), \ldots, l(t_N + h, \omega)) \in Y\}, \end{aligned}$$

where $l(t, \omega) := (X(t, \omega), u(t, \omega)) \in D$. [The stationarity.]

The existence of such a set of random variables $\{(X(t, \omega), u(t, \omega))\}_{t\in\mathbb{T}}$ is also the assertion of Theorem 1 in Shigoka (1994).

The present study provides Theorem 1 in Nishimura and Shigoka (2006) with an alternative proof that is due to Murakami et al. (2016). The proof due to Nishimura and Shigoka (2006) is an extension of that due to Shigoka (1994), whereas the proof due to Murakami et al. (2016) is an extension of that due to Benhabib et al. (2008). Shigoka (1994) treats a deterministic model that includes either a steady state or a closed orbit with a two-dimensional stable manifold, whereas Benhabib et al.

(2008) treats a two-dimensional deterministic model that includes one predetermined variable and one non-predetermined variable and that is amenable to the existence of a homoclinic orbit with multiple steady states. The former deterministic model is simpler than the latter deterministic model. On the other hand, the specification of a stochastic differential equation in Shigoka (1994) is more complex than that in Benhabib et al. (2008). According to the specification due to Shigoka (1994),

$$du(t,\omega) = g_2(X(t,\omega), u(t,\omega)) + \bar{s}d\varepsilon(t, \pi_1(\omega)),$$

where $\bar{s}$ is some constant with $\bar{s} \neq 0$, and where $du(t,\omega)$ and $d\varepsilon(t, \pi_1(\omega))$ denote Lebesgue-Stieljes signed measures relative to $t \in \mathbb{T}$, respectively. According to the specification due to Benhabib et al. (2008),

$$\lim_{h \to +0} \frac{u(t+h,\omega) - u(t,\omega)}{h} = g_2(X(t,\omega), u(t,\omega)).$$

Although Benhabib et al. (2008) does not include the proof of the stationarity, Murakami et al. (2016) includes the proof of this. The proof of the present study is simpler than that of Nishimura and Shigoka (2006), because the stochastic differential equation in Benhabib et al. (2008) is simpler than that in Shigoka (1994).

## 3 Proof of Theorem 1

In the present section, we will prove Theorem 1. We assume that Assumption 1 and either of Assumptions 2 and 3 are satisfied. Let $U \subset \mathbb{R}^2$ be a set of all interior points of $D$ that is the closed subset specified in either of Assumptions 2 and 3. Then there exists an open subset $N$ of $\mathbb{R} \times V$ such that $\mathbb{T} \times D \subset N \subset \mathbb{R} \times V$ and there exists a continuous function $\phi : N \to V$ that satisfies following conditions:

(1) For each $(t, X, u) \in N$, $\phi(t, X, u)$ is $C^1$-function of $t$, with $\phi(0, X, u) = (X, u) \in V$.
(2) For each $(X, u) \in D$,

$$\lim_{h \to 0} \frac{\phi(t+h, X, u) - \phi(t, X, u)}{h} = G(\phi(t, X, u)).$$

(3) $\phi(\mathbb{T} \times D) \subset D$, and $\phi(\mathbb{T} \times U) \subset U$.

Let $d : D \to \mathbb{R}_+$ be defined as

$$d(X, u) := \min \sqrt{(X - x_1)^2 + (u - x_2)^2} \text{ subject to } (x_1, x_2) \in \partial D.$$

Since $\partial D$ is a compact set, $d = d(X, u)$ is well-defined, and since $d = d(X, u)$ is a distance between a point in $D$ and the set $\partial D$, $d = d(X, u)$ is a continuous function of $(X, u) \in D$. Since $U$ is the interior region of $D$, and since $\partial D$ is the boundary of $U$, for any $(X, u) \in U$, $d = d(X, u) > 0$, and $(X, u + \frac{1}{2}d(X, u)) \in U$.

Let $\hat{\mathbf{P}}_0 : \mathcal{B}(\hat{D}) \to [0, 1]$ be a probability measure that satisfies

$$\hat{\mathbf{P}}_0\{(X, u, Q) \in \hat{D} : (X, u) \in U\} = 1.$$

We have defined $\mathbf{P} : \mathcal{B}(\Omega) \to [0, 1]$ as the product measure of $\hat{\mathbf{P}}_0$ and $\hat{\mathbf{P}}_1$, where $\hat{\mathbf{P}}_1 : \mathcal{B}(B) \to [0, 1]$ is the probability measure in Proposition 1. Let $\hat{\pi} : \hat{D} \to D$ be the projection of $\hat{D}$ onto $D$ so that $\hat{\pi}(X, u, Q) = (X, u)$ for $(X, u, Q) \in \hat{D}$. Then, we have

$$\mathbf{P}\{\omega \in \Omega : (X, u) \in U\} = 1.$$

Since $\pi_1(\omega) = b$, $\varepsilon(t, \pi_1(\omega))$ and $\tau(m, \pi_1(\omega))$ are measurable functions of $\omega \in \Omega$. We have defined $\mathcal{B}_t(\Omega)$ as the smallest $\sigma$-field with respect to which $(\pi_0(\omega), \varepsilon(s, \pi_1(\omega)), 0 \le s \le t)$ is a family of measurable functions of $\omega \in \Omega$. For each $m \in \mathbb{N}_0$, if $t \ge \tau(m, \pi_1(\omega))$, then $\tau(m, \pi_1(\omega))$ is a $\mathcal{B}_t(\Omega)$-measurable function of $\omega \in \Omega$, because $t \ge \tau(m, b)$ so that $\tau(m, b)$ is a $\mathcal{B}_t(B)$-measurable function of $b \in B$.

Since $\phi(\mathbb{T}, U) \subset U$ and since $\tau(0, \pi_1(\omega)) = 0 \wedge \tau(m+1, \pi_1(\omega)) - \tau(m, \pi_1(\omega)) > 0 \wedge \lim_{m\to\infty} \tau(m, \pi_1(\omega)) = \infty$ for each $(m, \omega) \in \mathbb{N}_0 \times \Omega$, the following constructions are well-defined. Let $f : \mathbb{N}_0 \times \Omega \to U$ and $g : \mathbb{N}_0 \times \Omega \to U$ be defined as follows. For $\omega \in \Omega$, if $\hat{\pi}(\pi_0(\omega)) = (X, u) \in U$, let $f(0, \omega)$ and $g(0, \omega)$ be given by

$$\begin{aligned} f(0, \omega) &:= (\hat{\pi}(\pi_0(\omega))), \\ g(0, \omega) &:= f(0, \omega). \end{aligned}$$

Choose some specific point $(X', u')$ from $U$ in advance, and for $\omega \in \Omega$, if $\hat{\pi}(\pi_0(\omega)) = (X, u) \in \partial D$, let $f(0, \omega)$ and $g(0, \omega)$ be given by

$$\begin{aligned} f(0, \omega) &:= (X', u'), \\ g(0, \omega) &:= f(0, \omega). \end{aligned}$$

For $(m, \omega) \in \mathbb{N}_0 \times \Omega$, and for given $f(m, \omega)$ and $g(m, \omega)$, let $f(m+1, \omega)$ and $g(m+1, \omega)$ be given by

$$\begin{aligned} f(m+1, \omega) &:= \phi(\tau(m+1, \pi_1(\omega)) - \tau(m, \pi_1(\omega)), g(m, \omega)), \\ g(m+1, \omega) &:= f(m+1, \omega) + (0, \frac{1}{2}d(f(m+1, \omega))). \end{aligned}$$

Then, $f(m, \omega) \subset U \wedge g(m, \omega) \subset U$ for each $(m, \omega) \in \mathbb{N}_0 \times \Omega$. For each $m \in \mathbb{N}_0$, if $t \ge \tau(m, \pi_1(\omega))$, $f(m, \omega)$ and $g(m, \omega)$ are $\mathcal{B}_t(\Omega)$-measurable functions of $\omega \in \Omega$.

Note that for each $(t, \omega) \in \mathbb{T} \times \Omega$, there exists a unique element $m$ in $\mathbb{N}_0$ such that $\tau(m, \pi_1(\omega)) \le t < \tau(m+1, \pi_1(\omega))$. Let $\theta : \mathbb{T} \times \Omega \to U$ be defined as follows. For each $(t, \omega) \in \mathbb{T} \times \Omega$, if $\tau(m, \pi_1(\omega)) \le t < \tau(m+1, \pi_1(\omega))$, let $\theta(t, \omega)$ be given by

$$\theta(t,\omega) := \phi(t - \tau(m, \pi_1(\omega)), g(m,\omega)).$$

Then, for each $(t,\omega) \in \mathbb{T} \times \Omega$, $\theta(t,\omega) \in U$, and for each $t \in \mathbb{T}$, $\theta(t,\omega)$ is a $\mathcal{B}_t(\Omega)$-measurable function of $\omega \in \Omega$. For each $(t,\omega) \in \mathbb{T} \times \Omega$, let $(X(t,\omega), u(t,\omega), Q(t,\omega))$ be defined as

$$(X(t,\omega), u(t,\omega), Q(t,\omega)) := (\theta(t,\omega), \varphi(\theta(t,\omega)).$$

Then, we can use the same argument as that of Sect. 5.2 in Murakami et al. (2016) to show that a set of random variables $\{(X(t,\omega), u(t,\omega), Q(t,\omega))\}_{t\in\mathbb{T}}$ thus constructed satisfies the conditions (1)–(6) in Definition 1. We can use the same arguments as that of Sect. 5.3 in Murakami et al. (2016) to show that a set of random variables $\{(X(t,\omega), u(t,\omega), Q(t,\omega), \varepsilon(t, \pi_1(\omega)))\}_{t\in\mathbb{T}}$ is subject to a Markov process with stationary transition probabilities and that there is an invariant measure on $\hat{D} \times \{1,2\}$ such that if we assign this measure as an initial probability measure to $\hat{D} \times \{1,2\}$, then the resulting stochastic process is stationary, which implies that the condition (7) in Definition 1 is satisfied.

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