Chapter 2
Kinematics

In this chapter we introduce the main concepts of kinematics of continua. These concepts are universal and they do not depend on the choice of specific materials.

2.1 Deformation Gradient

In *continuum mechanics* the atomistic or molecular structure of material is approximated by a continuously distributed set of the so-called *material points* (*material particles*). A continuum material point is an abstraction that is used to designate a small representative volume of real material including many *physical particles* (e.g., atoms, molecules).

Material point that occupied position $\mathbf{x}$ in the *reference configuration* moves to position $\mathbf{y}(\mathbf{x}, t)$ in the *current configuration* of the continuum—Fig. 2.1. It is usually convenient, yet not necessary, to assume that the reference state is the initial one: $\mathbf{x} = \mathbf{y}(\mathbf{x}, 0)$. In accordance with the motion of its material points a body that occupied region Ω_0 with boundary $\partial\Omega_0$ in the initial state moves to region Ω with boundary $\partial\Omega$ in the current state.

If we consider $\mathbf{x}$ as an independent variable then we follow motion of a material point that occupied position $\mathbf{x}$ in the reference configuration. Such description is called *referential* or *material* or *Lagrangean*. If, alternatively, we consider $\mathbf{y}$ as an independent variable then we follow motion of *various* material points passing through the fixed spatial point $\mathbf{y}$ in the current configuration. The latter description is called *spatial* or *Eulerian*. The Eulerian description is often preferable when the evolution of the body boundaries is known in advance like in many problems of fluid mechanics while the Lagrangean description is often preferable when the evolution of the body boundaries is not known in advance like in many problems of solid mechanics. Such a division is conditional, of course, and we will use both Lagrangean and Eulerian descriptions in this book.

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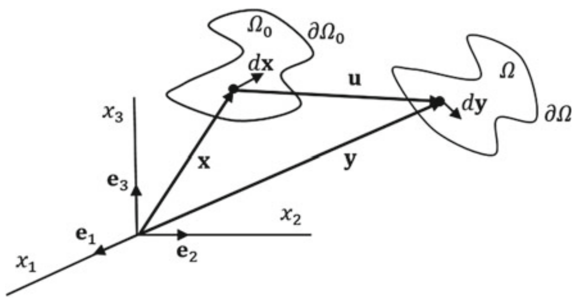


Fig. 2.1 Deformation

An infinitesimal material fiber at points $\mathbf{x}$ and $\mathbf{y}$ before and after deformation accordingly are related by the linear mapping

$$d\mathbf{y} = \mathbf{F}d\mathbf{x}, \tag{2.1}$$

where[1]

$$\mathbf{F} = \mathrm{Grad}\mathbf{y} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \frac{\partial y_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j \tag{2.2}$$

is the *deformation gradient*. This tensor is related to two configurations simultaneously and because of that it is called *two-point*.

We can also use the displacement vector, $\mathbf{u} = \mathbf{y} - \mathbf{x}$, to get

$$\mathbf{F} = \mathrm{Grad}(\mathbf{x} + \mathbf{u}) = \mathbf{1} + \mathbf{H}, \tag{2.3}$$

where

$$\mathbf{H} = \mathrm{Grad}\mathbf{u} = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j \tag{2.4}$$

is the *displacement gradient*.

It is possible to calculate any deformation in the vicinity of a given point when the deformation gradient is known there. We consider deformations of volume, area, and fiber.

We start with the volume deformation—Fig. 2.2

In this case we have

$$d\mathbf{y}^{(m)} = \mathbf{F}d\mathbf{x}^{(m)}, \tag{2.5}$$

[1] We capitalize the first character in differential operators: “Grad”, “Div”, “Curl”, when differentiation is with respect to $\mathbf{x}$. The operators are written as usual: “grad”, “div”, “curl”, when differentiation is with respect to $\mathbf{y}$.

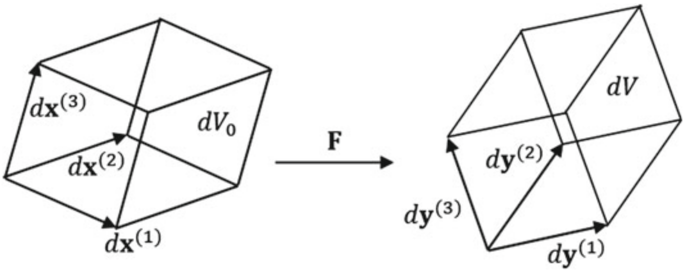


Fig. 2.2 Volume mapping

and, by a direct calculation,

$$\begin{aligned} dV &= \det \begin{bmatrix} dy_1^{(1)} & dy_2^{(1)} & dy_3^{(1)} \\ dy_1^{(2)} & dy_2^{(2)} & dy_3^{(2)} \\ dy_1^{(3)} & dy_2^{(3)} & dy_3^{(3)} \end{bmatrix} \\ &= \det \begin{bmatrix} F_{1j}dx_j^{(1)} & F_{2j}dx_j^{(1)} & F_{3j}dx_j^{(1)} \\ F_{1j}dx_j^{(2)} & F_{2j}dx_j^{(2)} & F_{3j}dx_j^{(2)} \\ F_{1j}dx_j^{(3)} & F_{2j}dx_j^{(3)} & F_{3j}dx_j^{(3)} \end{bmatrix} \\ &= \det \begin{bmatrix} dx_1^{(1)} & dx_2^{(1)} & dx_3^{(1)} \\ dx_1^{(2)} & dx_2^{(2)} & dx_3^{(2)} \\ dx_1^{(3)} & dx_2^{(3)} & dx_3^{(3)} \end{bmatrix} \det \begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix} \\ &= J dV_0, \end{aligned} \tag{2.6}$$

where

$$J = \det \mathbf{F} > 0. \tag{2.7}$$

The physical meaning of the latter restriction is simple—material cannot disappear during deformation.

In the case of the area deformation—Fig. 2.3—we have for a cylinder built on the infinitesimal base area

$$\begin{aligned} dV_0 &= dA_0\mathbf{n}_0 \cdot d\mathbf{x}, \\ dV &= dA\mathbf{n} \cdot d\mathbf{y} = dA\mathbf{n} \cdot \mathbf{F}d\mathbf{x}. \end{aligned} \tag{2.8}$$

Using (2.6) we derive

$$dA\mathbf{n} \cdot \mathbf{F}d\mathbf{x} = JdA_0\mathbf{n}_0 \cdot d\mathbf{x}, \tag{2.9}$$

and, consequently,

$$(dA\mathbf{F}^{\mathrm{T}}\mathbf{n} - JdA_0\mathbf{n}_0) \cdot d\mathbf{x} = 0. \tag{2.10}$$

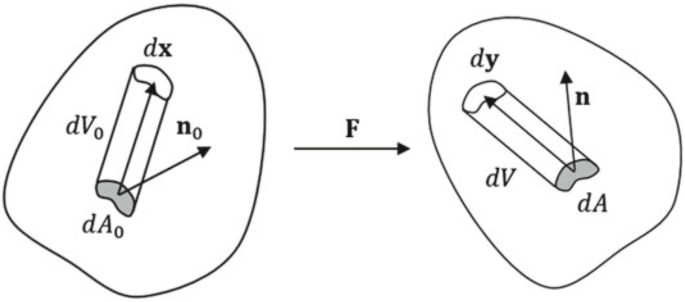


Fig. 2.3 Area mapping

Since $d\mathbf{x}$ is arbitrary we can write down the *Nanson formula*

$$\mathbf{n}dA = J\mathbf{F}^{-\mathrm{T}}\mathbf{n}_0 dA_0. \tag{2.11}$$

Now, we define the fiber stretch—Fig. 2.4—in direction $\mathbf{m}$, $|\mathbf{m}| = 1$,

$$\lambda(\mathbf{m}) = \frac{|d\mathbf{y}|}{|d\mathbf{x}|} = \frac{|\mathbf{F}d\mathbf{x}|}{|d\mathbf{x}|} = |\mathbf{F}\mathbf{m}|\,. \tag{2.12}$$

We can also define the change of the angle between two fibers—Fig. 2.5—by using stretches as follows, for example,

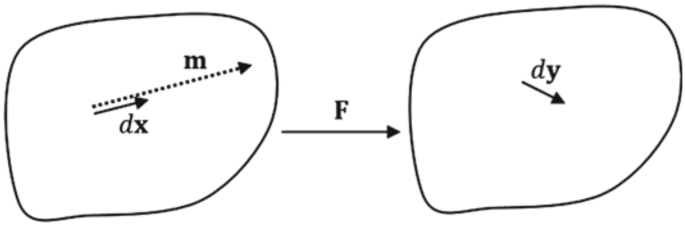


Fig. 2.4 Fiber mapping

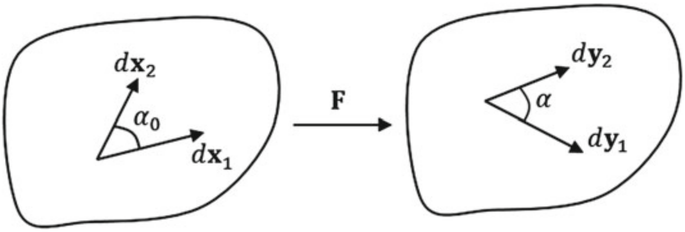
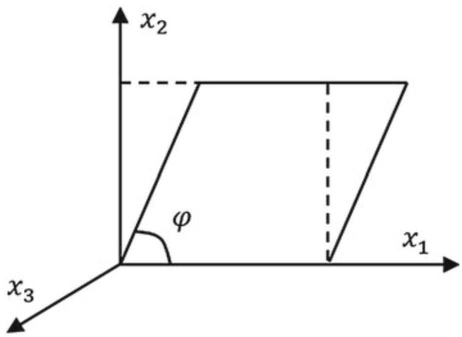


Fig. 2.5 Angle mapping

Fig. 2.6 Simple shear



$$
\begin{aligned}
\cos\alpha &= \frac{d\mathbf{y}_1 \cdot d\mathbf{y}_2}{|d\mathbf{y}_1|\,|d\mathbf{y}_2|} = \frac{(\mathbf{Fm}_1)\cdot(\mathbf{Fm}_2)}{\lambda(\mathbf{m}_1)\lambda(\mathbf{m}_2)},\\
\cos\alpha_0 &= \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{|d\mathbf{x}_1|\,|d\mathbf{x}_2|} = \mathbf{m}_1\cdot\mathbf{m}_2.
\end{aligned}
\tag{2.13}
$$

To illustrate the formulas above we consider the *simple shear* deformation—Fig. 2.6.

We designate the amount of shear by $\gamma = \tan(\pi/2 - \varphi) = \cot\varphi$. The law of motion (deformation) takes form: $y_1 = x_1 + \gamma x_2$, $y_2 = x_2$, $y_3 = x_3$. The deformation gradient is

$$\mathbf{F} = \frac{\partial y_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{1} + \gamma\mathbf{e}_1 \otimes \mathbf{e}_2,$$

and we obtain the following stretches for the axial directions

$$
\begin{aligned}
\lambda(\mathbf{e}_1) &= \sqrt{(\mathbf{Fe}_1)\cdot(\mathbf{Fe}_1)} = \sqrt{\mathbf{e}_1\cdot\mathbf{e}_1} = 1,\\
\lambda(\mathbf{e}_2) &= \sqrt{(\mathbf{Fe}_2)\cdot(\mathbf{Fe}_2)} = \sqrt{(\mathbf{e}_2+\gamma\mathbf{e}_1)\cdot(\mathbf{e}_2+\gamma\mathbf{e}_1)} = \sqrt{1+\gamma^2},
\end{aligned}
$$

and the right angle between the directions becomes

$$\alpha = \arccos\frac{(\mathbf{Fe}_1)\cdot(\mathbf{Fe}_2)}{|\mathbf{Fe}_1|\,|\mathbf{Fe}_2|} = \arccos\frac{\mathbf{e}_1\cdot(\mathbf{e}_2+\gamma\mathbf{e}_1)}{\sqrt{1+\gamma^2}} = \arccos\frac{\gamma}{\sqrt{1+\gamma^2}}.$$

2.2 Deformation Gradient in Curvilinear Coordinates

In this section we consider the deformation gradient in curvilinear coordinates. To be specific we choose the deformation law in cylindrical coordinates R, Φ, Z before and r, φ, z after the deformation respectively

$$r = r(R,\Phi,Z), \quad \varphi = \varphi(R,\Phi,Z), \quad z = z(R,\Phi,Z). \tag{2.14}$$

We further introduce the natural curvilinear basis vectors for the reference

$$\mathbf{G}_R = \cos\Phi\mathbf{e}_1 + \sin\Phi\mathbf{e}_2, \quad \mathbf{G}_\Phi = -\sin\Phi\mathbf{e}_1 + \cos\Phi\mathbf{e}_2, \quad \mathbf{G}_Z = \mathbf{e}_3, \tag{2.15}$$

and current configurations

$$\mathbf{g}_r = \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2, \quad \mathbf{g}_\varphi = -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2, \quad \mathbf{g}_z = \mathbf{e}_3, \tag{2.16}$$

accordingly.

Now, the deformation gradient can be written as follows

$$\mathbf{F} = \operatorname{Grad}\mathbf{y} = \frac{\partial\mathbf{y}}{\partial R}\otimes\mathbf{G}_R + \frac{1}{R}\frac{\partial\mathbf{y}}{\partial\Phi}\otimes\mathbf{G}_\Phi + \frac{\partial\mathbf{y}}{\partial Z}\otimes\mathbf{G}_Z, \tag{2.17}$$

where

$$\begin{aligned}\mathbf{y} &= y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3\\ &= r\cos\varphi(\cos\varphi\mathbf{g}_r - \sin\varphi\mathbf{g}_\varphi) + r\sin\varphi(\sin\varphi\mathbf{g}_r + \cos\varphi\mathbf{g}_\varphi) + z\mathbf{g}_z\\ &= r\mathbf{g}_r + z\mathbf{g}_z.\end{aligned} \tag{2.18}$$

Substituting (2.18) in (2.17) we obtain

$$\begin{aligned}\mathbf{F} = {} & \frac{\partial r}{\partial R}\mathbf{g}_r\otimes\mathbf{G}_R + r\frac{\partial\mathbf{g}_r}{\partial R}\otimes\mathbf{G}_R + \frac{\partial z}{\partial R}\mathbf{g}_z\otimes\mathbf{G}_R\\ & + \frac{1}{R}\frac{\partial r}{\partial\Phi}\mathbf{g}_r\otimes\mathbf{G}_\Phi + \frac{r}{R}\frac{\partial\mathbf{g}_r}{\partial\Phi}\otimes\mathbf{G}_\Phi + \frac{1}{R}\frac{\partial z}{\partial\Phi}\mathbf{g}_z\otimes\mathbf{G}_\Phi\\ & + \frac{\partial r}{\partial Z}\mathbf{g}_r\otimes\mathbf{G}_Z + r\frac{\partial\mathbf{g}_r}{\partial Z}\otimes\mathbf{G}_Z + \frac{\partial z}{\partial Z}\mathbf{g}_z\otimes\mathbf{G}_Z,\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}\frac{\partial\mathbf{g}_r}{\partial R} &= \frac{\partial\mathbf{g}_r}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial\mathbf{g}_r}{\partial\varphi}\frac{\partial\varphi}{\partial R} + \frac{\partial\mathbf{g}_r}{\partial z}\frac{\partial z}{\partial R} = \frac{\partial\varphi}{\partial R}\mathbf{g}_\varphi,\\ \frac{\partial\mathbf{g}_r}{\partial\Phi} &= \frac{\partial\mathbf{g}_r}{\partial r}\frac{\partial r}{\partial\Phi} + \frac{\partial\mathbf{g}_r}{\partial\varphi}\frac{\partial\varphi}{\partial\Phi} + \frac{\partial\mathbf{g}_r}{\partial z}\frac{\partial z}{\partial\Phi} = \frac{\partial\varphi}{\partial\Phi}\mathbf{g}_\varphi,\\ \frac{\partial\mathbf{g}_r}{\partial Z} &= \frac{\partial\mathbf{g}_r}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial\mathbf{g}_r}{\partial\varphi}\frac{\partial\varphi}{\partial Z} + \frac{\partial\mathbf{g}_r}{\partial z}\frac{\partial z}{\partial Z} = \frac{\partial\varphi}{\partial Z}\mathbf{g}_\varphi.\end{aligned} \tag{2.20}$$

After simplifications, we obtain

$$\begin{aligned}\mathbf{F} = {} & \frac{\partial r}{\partial R}\mathbf{g}_r\otimes\mathbf{G}_R + \frac{1}{R}\frac{\partial r}{\partial\Phi}\mathbf{g}_r\otimes\mathbf{G}_\Phi + \frac{\partial r}{\partial Z}\mathbf{g}_r\otimes\mathbf{G}_Z\\ & + r\frac{\partial\varphi}{\partial R}\mathbf{g}_\varphi\otimes\mathbf{G}_R + \frac{r}{R}\frac{\partial\varphi}{\partial\Phi}\mathbf{g}_\varphi\otimes\mathbf{G}_\Phi + r\frac{\partial\varphi}{\partial Z}\mathbf{g}_\varphi\otimes\mathbf{G}_Z\\ & + \frac{\partial z}{\partial R}\mathbf{g}_z\otimes\mathbf{G}_R + \frac{1}{R}\frac{\partial z}{\partial\Phi}\mathbf{g}_z\otimes\mathbf{G}_\Phi + \frac{\partial z}{\partial Z}\mathbf{g}_z\otimes\mathbf{G}_Z.\end{aligned} \tag{2.21}$$

2.3 Polar Decomposition of Deformation Gradient

Let us square the expression for stretch (2.12) and rewrite it as follows

$$\lambda^2(\mathbf{m}) = (\mathbf{Fm}) \cdot (\mathbf{Fm}) = \mathbf{m} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{Fm} = \mathbf{m} \cdot \mathbf{Cm}, \tag{2.22}$$

where

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{2.23}$$

is the *right* Cauchy–Green tensor.

Choosing $\mathbf{m} = \mathbf{m}^{(i)}$ as an eigenvector of tensor $\mathbf{C}$ we have

$$\lambda^2(\mathbf{m}^{(i)}) = \mathbf{m}^{(i)} \cdot \mathbf{Cm}^{(i)} = \mathbf{m}^{(i)} \cdot \zeta_i \mathbf{m}^{(i)} = \zeta_i, \tag{2.24}$$

where ζ_i is the corresponding eigenvalue of $\mathbf{C}$.

The latter equation means that eigenvalues of the right Cauchy–Green tensor are equal to the squared stretches in principal directions. Thus, we can write the spectral decomposition of $\mathbf{C}$ in the form

$$\mathbf{C} = \lambda_1^2 \mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2^2 \mathbf{m}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3^2 \mathbf{m}^{(3)} \otimes \mathbf{m}^{(3)}. \tag{2.25}$$

Now, we define the *right stretch* tensor as a square root of the right Cauchy–Green tensor

$$\mathbf{U} = \sqrt{\mathbf{C}} = \lambda_1 \mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2 \mathbf{m}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3 \mathbf{m}^{(3)} \otimes \mathbf{m}^{(3)}, \tag{2.26}$$

where all principal stretches are positive.

We assume then that any deformation gradient can be *multiplicatively* decomposed as

$$\mathbf{F} = \mathbf{RU}. \tag{2.27}$$

This is called the *polar decomposition* of the deformation gradient and, consequently, we have

$$\mathbf{R} = \mathbf{FU}^{-1}. \tag{2.28}$$

Let us analyze properties of $\mathbf{R}$. First, we observe that it is *orthogonal*

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = (\mathbf{FU}^{-1})^{\mathrm{T}}\mathbf{FU}^{-1} = \mathbf{U}^{-\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{FU}^{-1} = \mathbf{U}^{-\mathrm{T}}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{1}. \tag{2.29}$$

Orthogonal tensors do not change lengths of vectors that they map

$$|d\mathbf{y}| = \sqrt{d\mathbf{y} \cdot d\mathbf{y}} = \sqrt{(\mathbf{R}d\mathbf{x}) \cdot (\mathbf{R}d\mathbf{x})} = \sqrt{d\mathbf{x} \cdot \mathbf{R}^{\mathrm{T}}\mathbf{R}d\mathbf{x}} = \sqrt{d\mathbf{x} \cdot d\mathbf{x}} = |d\mathbf{x}|\,. \tag{2.30}$$

In addition, we observe (for $\det \mathbf{F} > 0$)

$$\det \mathbf{R} = \frac{\det \mathbf{F}}{\det \mathbf{U}} = \frac{\sqrt{\det \mathbf{F}^{\mathrm{T}} \det \mathbf{F}}}{\det \mathbf{U}} = \frac{\sqrt{\det \mathbf{C}}}{\det \mathbf{U}} = \frac{\sqrt{\det \mathbf{U}^2}}{\det \mathbf{U}} = \frac{\det \mathbf{U}}{\det \mathbf{U}} = 1. \tag{2.31}$$

The latter is the property of the *rotation* tensor. Thus, $\mathbf{R}$ is the *proper orthogonal* or rotation tensor.

Finally we note that the polar decomposition can be interpreted as the successive stretch and rotation—Fig. 2.7.

It is possible, of course, to change the order of stretch and rotation

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \tag{2.32}$$

where $\mathbf{V}$ is called the *left stretch* tensor.

By a direct calculation we have

$$\mathbf{V} = \mathbf{F}\mathbf{R}^{-1} = \mathbf{F}\mathbf{R}^{\mathrm{T}} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}} = \mathbf{V}^{\mathrm{T}}, \tag{2.33}$$

which means that the left stretch tensor is the rotated right stretch tensor and, consequently, they have the same eigenvalues—principal stretches, while their eigenvectors (principal directions) are different.

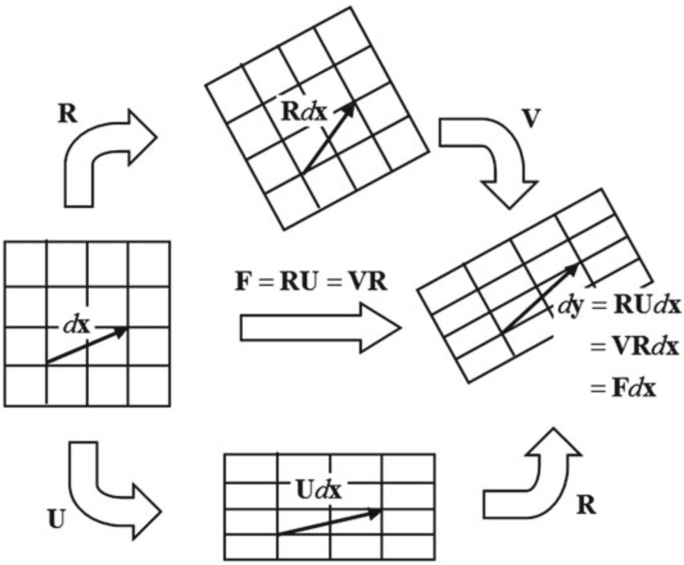


Fig. 2.7 Polar decomposition of deformation gradient

With account of the spectral decomposition of $\mathbf{U}$ we have

$$\mathbf{V} = \lambda_1 \mathbf{n}^{(1)} \otimes \mathbf{n}^{(1)} + \lambda_2 \mathbf{n}^{(2)} \otimes \mathbf{n}^{(2)} + \lambda_3 \mathbf{n}^{(3)} \otimes \mathbf{n}^{(3)}, \tag{2.34}$$

where (no sum on i)

$$\mathbf{n}^{(i)} \otimes \mathbf{n}^{(i)} = \mathbf{R}(\mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)})\mathbf{R}^{\mathrm{T}} = \mathbf{R}\mathbf{m}^{(i)} \otimes \mathbf{R}\mathbf{m}^{(i)}. \tag{2.35}$$

To clarify the meaning of the principal directions of $\mathbf{V}$ we square the tensor as follows

$$\mathbf{V}^2 = (\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}})(\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}) = \mathbf{R}\mathbf{U}\mathbf{U}\mathbf{R}^{\mathrm{T}} = \mathbf{R}\mathbf{U}(\mathbf{R}\mathbf{U})^{\mathrm{T}} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{B}, \tag{2.36}$$

and, consequently,

$$\mathbf{B} = \lambda_1^2 \mathbf{n}^{(1)} \otimes \mathbf{n}^{(1)} + \lambda_2^2 \mathbf{n}^{(2)} \otimes \mathbf{n}^{(2)} + \lambda_3^2 \mathbf{n}^{(3)} \otimes \mathbf{n}^{(3)}, \tag{2.37}$$

where $\mathbf{B}$ is the *left* Cauchy–Green tensor (also called Finger strain tensor), which principal directions coincide with the principal directions of $\mathbf{V}$ while the principal values of $\mathbf{B}$ are squared principal stretches.

Unfortunately, we cannot directly write the relations between the directions of eigenvectors $\mathbf{m}^{(i)}$ and $\mathbf{n}^{(i)}$ in the reference and current configurations because these directions are not defined uniquely and can always be reversed. However, we can *define* the principal directions uniquely by the following procedure. Assume, for example, that the principal directions in the reference configuration, $\mathbf{m}^{(i)}$, are uniquely chosen then we calculate the principal directions in the current configuration as follows

$$\mathbf{n}^{(i)} = \mathbf{R}\mathbf{m}^{(i)}. \tag{2.38}$$

Of course, we could start with the current configuration otherwise.

Finally, we can calculate the spectral decomposition, which is the *singular value decomposition*, of the deformation gradient as follows

$$\begin{aligned}\mathbf{F} = \mathbf{R}\mathbf{U} &= \lambda_1 \mathbf{R}\mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2 \mathbf{R}\mathbf{m}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3 \mathbf{R}\mathbf{m}^{(3)} \otimes \mathbf{m}^{(3)} \\ &= \lambda_1 \mathbf{n}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2 \mathbf{n}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3 \mathbf{n}^{(3)} \otimes \mathbf{m}^{(3)}.\end{aligned} \tag{2.39}$$

A nice analytical example on the spectral and polar decompositions of the deformation gradient was found by Marsden and Hughes (1983). They considered the following law of deformation: $y_1 = \sqrt{3}x_1 + x_2$, $y_2 = 2x_2$, $y_3 = x_3$; and they calculated the corresponding quantities in Cartesian coordinates

$$[\mathbf{F}] = \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\mathbf{C}] = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\lambda_1 = \sqrt{6},\ \left[\mathbf{m}^{(1)}\right] = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \end{bmatrix},\quad \lambda_2 = \sqrt{2},\ \left[\mathbf{m}^{(2)}\right] = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ -1 \\ 0 \end{bmatrix},\quad \lambda_3 = 1,\ \left[\mathbf{m}^{(3)}\right] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$[\mathbf{U}] = \frac{1}{2\sqrt{2}}\begin{bmatrix} 3+\sqrt{3} & 3-\sqrt{3} & 0 \\ 3-\sqrt{3} & 1+3\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix},\quad \left[\mathbf{U}^{-1}\right] = \frac{1}{4\sqrt{6}}\begin{bmatrix} 1+3\sqrt{3} & \sqrt{3}-3 & 0 \\ \sqrt{3}-3 & 3+\sqrt{3} & 0 \\ 0 & 0 & 4\sqrt{6} \end{bmatrix},$$

$$[\mathbf{R}] = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1+\sqrt{3} & \sqrt{3}-1 & 0 \\ 1-\sqrt{3} & 1+\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix},\quad [\mathbf{V}] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1+\sqrt{3} & \sqrt{3}-1 & 0 \\ \sqrt{3}-1 & 1+\sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

2.4 Strain

Strain is a geometric measure of deformation and it can be introduced in various ways. We start with one-dimensional measures for the change of the length of a material fiber—Fig. 2.8.

We can introduce the *engineering strain*, *logarithmic strain*, or the *Green strain* accordingly

$$\begin{aligned} E_\mathrm{E} &= \frac{L - L_0}{L_0} = \lambda - 1, \\ E_\mathrm{L} &= \int_{L_0}^{L} \frac{dL}{L_0} = \ln\frac{L}{L_0} = \ln\lambda, \\ E_\mathrm{G} &= \frac{L^2 - L_0^2}{2L_0^2} = \frac{1}{2}(\lambda^2 - 1). \end{aligned} \tag{2.40}$$

In order to generalize one-dimensional strains to the three-dimensional ones we assume that the previous formulas are valid in the principal directions of the reference configuration. In this case, the three-dimensional strain tensors take forms

$$\begin{aligned} \mathbf{E}_\mathrm{E} &= \sum_{i=1}^{3}(\lambda_i - 1)\mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)} = \mathbf{U} - \mathbf{1}, \\ \mathbf{E}_\mathrm{L} &= \sum_{i=1}^{3}(\ln\lambda_i)\mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)} = \ln\mathbf{U}, \\ \mathbf{E}_\mathrm{G} &= \sum_{i=1}^{3}\frac{1}{2}(\lambda_i^2 - 1)\mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}). \end{aligned} \tag{2.41}$$

Fig. 2.8 Strain

The Green strain tensor is very popular and it can be written without the suffix

$$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}) = \frac{1}{2}(\mathbf{C} - \mathbf{1}) = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}). \tag{2.42}$$

2.5 Motion

Velocity and acceleration vectors are defined as material time derivatives of the placement vector $\mathbf{y}(\mathbf{x}, t)$ as follows

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{y}(\mathbf{x}, t)}{dt} = \dot{\mathbf{y}} = \dot{\mathbf{x}} + \dot{\mathbf{u}} = \dot{\mathbf{u}}, \\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}. \end{aligned} \tag{2.43}$$

When the Eulerian or spatial description is used it is necessary to apply the chain rule for differentiation of function $f(\mathbf{y}(t), t)$

$$\frac{df}{dt} = \dot{f}(\mathbf{y}(t), t) = \frac{\partial f}{\partial t} + \mathrm{grad} f \cdot \frac{\partial \mathbf{y}}{\partial t} = \frac{\partial f}{\partial t} + \mathrm{grad} f \cdot \mathbf{v}. \tag{2.44}$$

For example, we have for the acceleration vector

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{L}\mathbf{v}, \tag{2.45}$$

in which another important quantity—*velocity gradient* —is introduced

$$\mathbf{L} = \mathrm{grad}\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}}. \tag{2.46}$$

We emphasize that the partial time derivatives are taken for $\mathbf{y}$ fixed

$$\frac{\partial f}{\partial t} \equiv \left(\frac{\partial f(\mathbf{y}, t)}{\partial t} \right)_{\mathbf{y}\ \text{fixed}}. \tag{2.47}$$

Another way to calculate the velocity gradient comes from identity

$$\dot{\mathbf{F}} = \frac{d}{dt}\mathrm{Grad}\mathbf{y} = \mathrm{Grad}\dot{\mathbf{y}} = \mathrm{Grad}\mathbf{v} = (\mathrm{grad}\mathbf{v})\mathbf{F} = \mathbf{L}\mathbf{F}. \tag{2.48}$$

From the latter equation we get

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{2.49}$$

The velocity gradient can be decomposed into symmetric and skew parts

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}), \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}}), \tag{2.50}$$

where $\mathbf{D}$ and $\mathbf{W}$ are the *deformation rate* and the *spin (vorticity)* tensors accordingly.

2.6 Rigid Body Motion

The *rigid body motion* (RBM) superimposed on the current configuration—Fig. 2.9—is of importance for constitutive modeling, which will be discussed in the coming chapters. It is generally required that the constitutive laws should not be affected by the superimposed rigid body motion—they should be *objective*.

The superimposed RBM, designated with asterisk, can be described as follows

$$\mathbf{y}^* = \mathbf{Q}(t)\mathbf{y} + \mathbf{c}(t), \tag{2.51}$$

where

$$\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}, \quad \det \mathbf{Q} = 1 \tag{2.52}$$

is a proper-orthogonal tensor.

We remind the reader that since $\det \mathbf{Q} = 1 > 0$ then material does not disappear and tensor $\mathbf{Q}$ describes rotation.

The transformation law for a material fiber takes form

$$\mathbf{s}^* = \mathbf{y}_2^* - \mathbf{y}_1^* = \mathbf{Q}(\mathbf{y}_2 - \mathbf{y}_1) = \mathbf{Q}\mathbf{s}. \tag{2.53}$$

This motion preserves length

$$\left|\mathbf{s}^*\right| = \sqrt{\mathbf{s}^* \cdot \mathbf{s}^*} = \sqrt{\mathbf{s} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{s}} = \sqrt{\mathbf{s} \cdot \mathbf{1}\mathbf{s}} = |\mathbf{s}|\,. \tag{2.54}$$

Besides, it preserves angles between fibers. Check it.

All quantities related to the reference configuration at $t = 0$ are unaffected by RBM and only quantities related to the current configuration are affected by RBM.

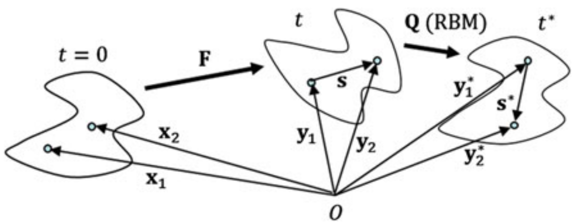


Fig. 2.9 Superimposed rigid body motion

Vector quantities that transform under superimposed RBM following the rule

$$\mathbf{s}^* = \mathbf{Q}\mathbf{s} \tag{2.55}$$

are called objective, i.e. unaffected by RBM.

Not all vectors are objective. For example, velocity and acceleration vectors are not objective

$$\begin{aligned}\mathbf{v}^* &= \dot{\mathbf{y}}^* = \frac{d}{dt}(\mathbf{Q}\mathbf{y} + \mathbf{c}) = \mathbf{Q}\dot{\mathbf{y}} + \dot{\mathbf{Q}}\mathbf{y} + \dot{\mathbf{c}} = \mathbf{Q}\mathbf{v} + \boldsymbol{\Omega}(\mathbf{y}^* - \mathbf{c}) + \dot{\mathbf{c}},\\ \mathbf{a}^* &= \dot{\mathbf{v}}^* = \mathbf{Q}\mathbf{a} + \ddot{\mathbf{c}} + (\dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega}^2)(\mathbf{y}^* - \mathbf{c}) + 2\boldsymbol{\Omega}(\mathbf{v}^* - \dot{\mathbf{c}}),\end{aligned} \tag{2.56}$$

where

$$\begin{aligned}\mathbf{y} &= \mathbf{Q}(\mathbf{y}^* - \mathbf{c}),\\ \boldsymbol{\Omega} &= \dot{\mathbf{Q}}\mathbf{Q}^{-1} = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = -\boldsymbol{\Omega}^{\mathrm{T}},\\ \frac{d}{dt}(\mathbf{Q}\mathbf{Q}^{\mathrm{T}}) &= \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} = \mathbf{0},\end{aligned} \tag{2.57}$$

and tensor $\boldsymbol{\Omega}$ is the spin of RBM.

Second-order tensors defined in the current configuration are called objective if they preserve objectivity of the vectors that they map. Let tensor $\mathbf{A}$ map objective vector $\mathbf{s}$ into objective vector $\mathbf{r}$:

$$\mathbf{r} = \mathbf{A}\mathbf{s}. \tag{2.58}$$

Then, tensor $\mathbf{A}^*$ maps proper transformations of the objective vectors

$$\mathbf{r}^* = \mathbf{A}^*\mathbf{s}^*, \tag{2.59}$$

or

$$\mathbf{Q}\mathbf{r} = \mathbf{A}^*\mathbf{Q}\mathbf{s}, \tag{2.60}$$

and, consequently, we have

$$\mathbf{r} = \mathbf{Q}^{\mathrm{T}}\mathbf{A}^*\mathbf{Q}\mathbf{s}. \tag{2.61}$$

Comparing (2.58) and (2.61) and assuming that the choice of vectors is arbitrary, we obtain the transformation rule for the objective second-order tensor

$$\mathbf{A}^* = \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}}. \tag{2.62}$$

Not all tensors are objective. Let us examine objectivity of the velocity gradient tensor

$$
\begin{aligned}
\mathbf{L}^* &= \dot{\mathbf{F}}^*\mathbf{F}^{*-1} \\
&= (\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}})\mathbf{F}^{-1}\mathbf{Q}^{-1} \\
&= \dot{\mathbf{Q}}\mathbf{Q}^{-1} + \mathbf{Q}\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{Q}^{-1} \\
&= \boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}}.
\end{aligned}
\tag{2.63}
$$

This tensor is affected by the superimposed RBM and it is not objective. However, its symmetric part—the deformation rate tensor—is objective

$$
\begin{aligned}
\mathbf{D}^* &= \frac{1}{2}(\mathbf{L}^* + \mathbf{L}^{*\mathrm{T}}) \\
&= \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \boldsymbol{\Omega}^{\mathrm{T}} + \mathbf{Q}\mathbf{L}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}) \\
&= \frac{1}{2}\mathbf{Q}(\mathbf{L} + \mathbf{L}^{\mathrm{T}})\mathbf{Q}^{\mathrm{T}} \\
&= \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathrm{T}}.
\end{aligned}
\tag{2.64}
$$

We note that the rate of an objective second-order tensor is not objective

$$
\dot{\mathbf{A}}^* = \frac{d}{dt}(\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}}) = \dot{\mathbf{Q}}\mathbf{A}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{A}\dot{\mathbf{Q}}^{\mathrm{T}}. \tag{2.65}
$$

The latter observation triggered various proposals for an objective rate of an objective tensor. For example, we mention the Jaumann-Zaremba, Truesdell, and Oldroyd objective rates respectively

$$
\begin{aligned}
\overset{\bullet}{\mathbf{A}} &= \dot{\mathbf{A}} - \mathbf{W}\mathbf{A} - \mathbf{A}\mathbf{W}^{\mathrm{T}}, \\
\overset{\circ}{\mathbf{A}} &= \dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^{\mathrm{T}} + (\mathrm{tr}\mathbf{L})\mathbf{A}, \\
\overset{\diamond}{\mathbf{A}} &= \dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^{\mathrm{T}}.
\end{aligned}
\tag{2.66}
$$

The proof of the objectivity of the Oldroyd rate, for example, is by the direct calculation

$$
\begin{aligned}
\overset{\diamond}{\mathbf{A}^*} &= \dot{\mathbf{A}}^* - \mathbf{L}^*\mathbf{A}^* - \mathbf{A}^*\mathbf{L}^{*\mathrm{T}} \\
&= \dot{\mathbf{Q}}\mathbf{A}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{A}\dot{\mathbf{Q}}^{\mathrm{T}} - (\mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}})\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}}(\mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}} \\
&= \dot{\mathbf{Q}}\mathbf{A}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{A}\dot{\mathbf{Q}}^{\mathrm{T}} - \mathbf{Q}\mathbf{L}\mathbf{A}\mathbf{Q}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{A}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{A}\mathbf{L}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{A}\dot{\mathbf{Q}}^{\mathrm{T}} \\
&= \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{L}\mathbf{A}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{A}\mathbf{L}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}} \\
&= \mathbf{Q}(\dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^{\mathrm{T}})\mathbf{Q}^{\mathrm{T}} \\
&= \mathbf{Q}\overset{\diamond}{\mathbf{A}}\mathbf{Q}^{\mathrm{T}}.
\end{aligned}
$$

2.7 Lagrangean, Eulerian and Two-Point Tensors

Tensor fields considered in the previous sections can be classified as Lagrangean, Eulerian, and two-point.

Lagrangean tensors are defined on the initial or referential configuration. For example, the right Cauchy–Green tensor $\mathbf{C}$, right stretch tensor $\mathbf{U}$, strain tensors $\mathbf{E}_{\mathrm{G}}$, $\mathbf{E}_{\mathrm{E}}$, $\mathbf{E}_{\mathrm{L}}$ are Lagrangean.

Eulerian tensors are defined on the current configuration. For example, the left Cauchy–Green tensor $\mathbf{B}$, left stretch tensor $\mathbf{V}$, velocity gradient $\mathbf{L}$, deformation rate $\mathbf{D}$, spin $\mathbf{W}$ are Eulerian.

Two-point tensors belong to both initial and current configurations simultaneously. For example, deformation gradient $\mathbf{F}$ and rotation tensor $\mathbf{R}$ are two-point.

Vectors cannot be two-point—they are Eulerian, like $\mathbf{n}^{(i)}$, or Lagrangean, like $\mathbf{m}^{(j)}$.

It is important to follow the character of the tensor (Lagrangean, Eulerian, and two-point) in order to have physically consistent formulations. It is also important to not confuse Eulerian and Lagrangean tensors with the Eulerian and Lagrangean descriptions of motion. Lagrangean description of motion can be used for Eulerian tensors and Eulerian description of motion can be used for Lagrangean tensors.

2.8 Exercises

1. Find principal directions and stretches for the following deformation law

$$y_1 = (1+\alpha)x_1 + \alpha x_2, \ y_2 = -\alpha x_1 + (1+\alpha)x_2, \ y_3 = x_3, \tag{2.67}$$

   where $\alpha =$ constant.
2. Calculate the polar decomposition of the deformation gradient for the deformation law presented in (2.67).
3. Calculate the Cartesian components of the Green strain for the deformation law presented in (2.67).
4. Derive (2.56).
5. Prove objectivity of the Jaumann-Zaremba and Truesdell rates $(2.66)_{1,2}$.

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