Chapter 2
Exponential Stability for Nonlinear Thermoelastic Equations with Second Sound

2.1 Introduction

This chapter is concerned with the global existence and asymptotic behavior of solutions to the equations of one-dimensional nonlinear thermoelasticity with thermal memory and second sound. We adopt the results in this chapter from [223].

The reference configuration under consideration is the unit interval $\Omega = (0, 1)$. The equations under consideration read as follows

$$\begin{cases} u_{tt} - S(u_x, \theta)_x = 0, & (2.1.1)\\ \theta_t + \gamma q_x + k_1 * q_x + \beta u_{xt} = 0, & (2.1.2)\\ q_t + q + k\theta_x = 0 & (2.1.3) \end{cases}$$

subject to the boundary conditions

$$u(0, t) = u(1, t) = q(0, t) = q(1, t) = 0 \tag{2.1.4}$$

and initial conditions

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ \theta(x, 0) = \theta_0(x),\ q(x, 0) = q_0(x). \tag{2.1.5}$$

Here by $u = u(x, t)$, $\theta = \theta(x, t)$ and $q = q(x, t)$, we denote the displacement, absolute temperature and heat flux respectively, $S(u_x, \theta)$ is the Piola-Kirchhoff stress tensor, $k_1 = k_1(t)$ is the relaxation kernel. The sign $*$ denotes the convolution product, i.e., $k_1 * y(\cdot, t) = \int_0^t k_1(t - \tau)y(\cdot, \tau)d\tau$. Finally, $\alpha,\ \beta, \gamma$ are positive constants.

When the problem (2.1.1)–(2.1.3) has no the relaxation kernel k_1 (i.e., $k_1 = 0$), Messaoud and Said-Houari [169] considered a one-dimensional homogeneous body occupying, in its reference configuration, an interval I, the laws of balance of momentum, balance of energy, and growth of entropy with the forms

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$$\begin{cases} \rho u_{tt} = \sigma_x + b, \\ e_t + q_x = \sigma \varepsilon_t + r, \\ \eta_t \geq \frac{r}{\theta} - \left(\frac{q}{\theta}\right)_x, \end{cases}$$

where the displacement u, the strain $\varepsilon = u_x$, the stress σ, the 'absolute' temperature θ, the heat flux q, the internal energy e, the body force b, and the external heat supply r are all functions of (x, t) $(t \geq 0, x \in I = (0, 1))$. Moreover, the strain and the temperature are required to satisfy $\varepsilon > -1$ and $\theta > 0$, in the absence of the body force b and the external heat supply r, assuming that the material density ρ is equal to one, and taking in some considerations, the equations together with Cattaneo's law take the form

$$\begin{cases} u_{tt} - au_{xx} + b\theta_x = \alpha_1 qq_x, \\ \theta_t + gq_x + du_{tx} = \alpha_2 qq_t, \\ \tau q_t + q_x + k\theta_x = 0 \end{cases}$$

where

$$\begin{aligned} &a = a(u_x, \theta, q),\ b = b(u_x, \theta, q),\ g = g(u_x, \theta, q),\ d = d(u_x, \theta, q), \\ &\tau = \tau(u_x, \theta),\ \alpha_1 = \alpha_1(u_x, \theta),\ k = k(u_x, \theta),\ \alpha_2 = \alpha_2(u_x, \theta), \end{aligned}$$

the authors established an exponential decay result for solutions with sufficiently small initial data and proved that the dissipation given by the heat conduction is strong enough to stabilize the system exponentially. This work has extended the result of Racke [235] to a more general situation.

We assume that S are C^3-functions satisfying

$$\frac{\partial S}{\partial u_x}(0, 0) = 1 > 0, \quad \frac{\partial S}{\partial \theta}(0, 0) \neq 0. \tag{2.1.6}$$

Concerning the kernel, we assume that $k_1(t) \in C^1(\mathbb{R}^+)$ and that $k_1(t)$ is a strongly positive definite kernel. Additionally, we assume that there exist positive constants $c_0 \leq c_1$, such that for all $t \geq 0$,

$$k_1(t) > 0, \quad k_1'(t) + c_0 k_1(t) \leq 0 \leq k_1'(t) + c_1 k_1(t). \tag{2.1.7}$$

To simplify notations, we shall introduce $-\frac{\partial S}{\partial \theta}(0, 0) = \alpha$ satisfying the product $\alpha\beta > 0$.

For the initial data, we assume that

$$\begin{cases} (u_0, u_1) \in \left(H^3(0, 1) \bigcap (H_0^1)\right) \times \left(H^2(0, 1) \bigcap H_0^1(0, 1)\right) \times H_0^1(0, 1), \\ (\theta_0, \theta_1) \in H^2(0, 1) \times H^1(0, 1), \\ (q_0, q_1) \in H^2(0, 1) \times H^1(0, 1), \end{cases} \tag{2.1.8}$$

with

$$\int_0^1 \theta_0(x)dx = 0, \quad u_2 = u_{tt}\mid_{t=0}= \frac{d}{dx}S(u_{0x}(x), \theta_0(x)). \tag{2.1.9}$$

We put $\|\cdot\|=\|\cdot\|_{L^2(0,1)}$, and use C (sometimes $C_1, C_2, \ldots$) to denote the generic positive constant independent of time $t > 0$.

Our main result of this chapter reads as follows.

Theorem 2.1.1 *Under assumptions (2.1.6)–(2.1.9), there exists a small constant* $0 < \epsilon_0 < 1$ *such that for any* $\epsilon \in (0, \epsilon_0)$ *and for any initial data* $(u_0, u_1, \theta_0, q_0)$ *satisfying*

$$\|u_0\|_{H^3}^2 + \| u_1 \|_{H^2}^2 + \| \theta_0 \|_{H^2}^2 + \| q_0 \|_{H^2}^2 < \epsilon, \tag{2.1.10}$$

problem (2.1.1)–(2.1.5) admits a unique global solution $(u(t), \theta(t), q(t))$ *satisfying*

$$u(t) \in \bigcap_{m=0}^{2} C^m([0, +\infty), H^{3-m}(0, 1) \cap H_0^1(0, 1)),\ (\partial_t^3 u)(t) \in C([0, +\infty), L^2(0, 1)), \tag{2.1.11}$$

$$(k_1 * \theta)(t), \theta(t) \in \bigcap_{m=0}^{1} C^m([0, +\infty), H^{2-m}(0, 1)), \tag{2.1.12}$$

$$(k_1 * \theta)(t), \theta(t) \in C^2([0, +\infty), L^2(0, 1)), \tag{2.1.13}$$

$$q(t) \in C^1([0, +\infty), L^2(0, 1)) \cap C([0, +\infty), H^1(0, 1)), \tag{2.1.14}$$

$$(k_1 * \partial_t^i \theta_x)(t),\ (k_1 * \partial_t^j \theta_{xx})(t) \in L^2([0, +\infty), L^2(0, 1)),\ i = 0, 1, 2;\ j = 0, 1, \tag{2.1.15}$$

$$\partial_t^i q(t),\ \partial_t^j q_x(t) \in L^2([0, +\infty), L^2(0, 1)),\ i = 0, 1, 2;\ j = 0, 1. \tag{2.1.16}$$

Moreover, the solution $(u(t), \theta(t), q(t))$ *decays exponentially as* $t \to +\infty$*, i.e., there exists a large time* $t_0 > 0$ *such that as* $t \geq t_0$*,*

$$\| u(t) \|_{H^3}^2 + \| u_t(t) \|_{H^2}^2 + \| \theta(t) \|_{H^2}^2 + \| q(t) \|_{H^2}^2 \leq Ce^{-C't}, \tag{2.1.17}$$

where C *and* C' *are positive constants.*

2.2 Global Existence and Exponential Stability

In this section, we shall prove Theorem 2.1.1. The main idea of the proof is that we first shall prove the local solutions, by a standard contraction mapping argument, we then can show that problem (2.1.1)–(2.1.5) admits a unique local solution $(u(t), \theta(t), q(t))$ such that

$$u(t) \in \bigcap_{m=0}^{2} C^m([0,T), H^{3-m}(0,1) \cap H_0^1(0,1)), \quad (\partial_t^3 u)(t) \in C([0,T), L^2(0,1)), \tag{2.2.1}$$

$$(k_1 * \theta)(t), \theta(t) \in \bigcap_{m=0}^{1} C^m([0,T), H^{2-m}(0,1)), \tag{2.2.2}$$

$$(k_1 * \theta)(t), \theta(t) \in C^2([0,T), L^2(0,1)), \tag{2.2.3}$$

$$(k_1 * \partial_t^i \theta_x)(t),\ (k_1 * \partial_t^j \theta_{xx})(t) \in L^2([0,T), L^2(0,1)),\ i = 0,1,2;\ j = 0,1, \tag{2.2.4}$$

$$\partial_t^i q(t),\ \partial_t^j q_x(t) \in L^2([0,T), L^2(0,1)),\ i = 0,1,2;\ j = 0,1, \tag{2.2.5}$$

$$q(t) \in C^1([0,T), L^2(0,1)) \cap C([0,T), H^1(0,1)), \tag{2.2.6}$$

where the constant $T > 0$ is the maximal existence interval of solutions, and second we shall use the contradiction argument to continue the local solutions in time.

The next lemma concerns the property of a strongly positive definite kernel.

Lemma 2.2.1 *Assume that $\hat{k}(t) \in L^1(\mathbb{R}^+)$ is a strongly positive definite kernel satisfying $\hat{k}'(t) \in L^1(\mathbb{R}^+)$, then for any $y(t) \in L^1_{loc}(\mathbb{R}^+)$, it follows that*

$$\int_0^t \left|\hat{k} * y(\tau)\right|^2 d\tau \le \beta_0 k_2 \int_0^t y(\tau)\ \ \hat{k} * y(\tau) d\tau \tag{2.2.7}$$

where $k_2 = \left(\int_0^{+\infty} \left|\hat{k}(t)\right| dt\right)^2 + 4\left(\int_0^{+\infty} \left|\hat{k}'(t)\right| dt\right)^2$ and $\beta_0 > 0$ is a constant such that the function $\hat{k}(t) - \beta_0 e^{-t}$ is a positively definite kernel.

Proof Define

$$y_t(\tau) = \begin{cases} y(\tau), & 0 \le \tau \le t, \\ 0, & \text{otherwise.} \end{cases}$$

By the Plancherel identity and the fact that convolution is mapped into pointwise multiplication by the Fourier transform,

$$\begin{aligned} \int_0^t \left|\hat{k} * y(\tau)\right|^2 d\tau &\le \int_0^{+\infty} \left|\int_0^\tau \hat{k}(\tau - s) y_t(s) ds\right|^2 d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\tilde{\hat{k}}(w)\right|^2 |\tilde{y}_t(w)|^2 dw \end{aligned} \tag{2.2.8}$$

where

$$\left|\tilde{\hat{k}}(w)\right| = \left|\int_0^{+\infty} e^{-iwt}\hat{k}(t)dt\right| \le \int_0^{+\infty} |\hat{k}(t)|dt, \tag{2.2.9}$$

$$\left|w\tilde{\hat{k}}(w)\right| = \left|\int_0^{+\infty} (e^{-iwt}-1)\hat{k}'(t)dt\right| \le 2\int_0^{+\infty} \left|\hat{k}'(t)\right| dt \tag{2.2.10}$$

and $\tilde{f}$ denotes the Fourier transform of f.
Square these two inequalities (2.2.9)–(2.2.10), and add (2.2.10) to (2.2.9) to get

$$\left|\tilde{\hat{k}}(w)\right|^2 \le \frac{k_2}{1+w^2}$$

which, combined with (2.2.8), yields

$$\begin{aligned}
&\int_0^t \left|\int_0^\tau \hat{k}(\tau-s)y(s)ds\right|^2 d\tau \\
&\le \frac{k_2}{2\pi}\int_{-\infty}^{+\infty} \frac{|\tilde{y}_t(w)|^2}{1+w^2}dw \\
&= \frac{k_2}{2}\int_{-\infty}^{+\infty} y_t(\tau)\int_{-\infty}^{+\infty} e^{-|\tau-s|}y_t(s)dsd\tau = k_2\int_0^t y(\tau)\int_0^\tau e^{-(\tau-s)}y(s)dsd\tau \\
&= k_2\int_0^t y(\tau)(e^{-\tau} * y)(\tau)d\tau \le k_2\beta_0\int_0^t y(\tau)(\hat{k} * y)(\tau)d\tau.
\end{aligned}$$

Thus (2.2.7) follows. □

From (2.1.3), we can deduce that

$$q = q_0e^{-t} - \int_0^t e^{s-t}\theta_x ds. \tag{2.2.11}$$

By inserting (2.2.11) and (2.1.6) into (2.1.1)–(2.1.3), system (2.1.1)–(2.1.3) reduces to the system

$$\begin{cases}
u_{tt} - u_{xx} + \alpha\phi_x = f, & (2.2.12)\\
\theta_t - \tilde{k}_1 * \theta_{xx} + \beta u_{xt} = g, & (2.2.13)\\
q_t + q + k\theta_x = 0, & (2.2.14)
\end{cases}$$

where

$$\begin{cases}
f = \left[\frac{\partial S}{\partial u_x}(u_x,\theta) - 1\right]u_{xx} + \left[\frac{\partial S}{\partial \theta}(u_x,\theta) + \alpha\right]\theta_x, & (2.2.15)\\
g = -\gamma q_{0x}k_2 - k_1 * k_2 q_{0x}, \quad k_2 = e^{-t}, \quad \tilde{k}_1 = \gamma k k_2 + k k_1 * k_2. & (2.2.16)
\end{cases}$$

For simplicity, we put

$$\eta_1 = \frac{\partial S}{\partial u_x}(u_x, \theta) - 1, \quad \eta_2 = \frac{\partial S}{\partial \theta}(u_x, \theta) + \alpha. \tag{2.2.17}$$

From (2.1.7), we can derive that, if $\gamma > k_1(0)$, then there exists positive constants $c_2 \le c_3$, such that for all $t \ge 0$,

$$\tilde{k}_1(t) > 0, \quad \tilde{k}_1'(t) + c_2\tilde{k}_1(t) \le 0 \le \tilde{k}_1'(t) + c_3\tilde{k}_1(t). \tag{2.2.18}$$

If $\gamma \le k_1(0)$, from (2.1.7), there exists some time $t_0 > 0$, as $t \ge t_0$, we have $\gamma > k_1(t_0)$, so we can define $\hat{t} = t - t_0$, then we have for all $\hat{t} \ge 0$,

$$\tilde{k}_1(\hat{t}) > 0, \quad \tilde{k}_1'(\hat{t}) + c_2\tilde{k}_1(\hat{t}) \le 0 \le \tilde{k}_1'(\hat{t}) + c_3\tilde{k}_1(\hat{t}).$$

Thus without loss of generality, we may assume $\gamma > k_1(0)$.
It follows from (2.2.18) that the kernel $\tilde{k}_1(t)$ decays exponentially as time goes to infinity and satisfies

$$\tilde{k}_1(0)e^{-c_3 t} \le \tilde{k}_1(t) \le \tilde{k}_1(0)e^{-c_2 t}.$$

Thus, we can choose $\delta \in \delta_0 \equiv (0, \min(1, c_0/2, c_2/2))$ so small that for any $t \ge 0$,

$$\tilde{k}_1(0)e^{-(c_3/2)t} \le \hat{k}(t) := e^{\delta t}\tilde{k}_1(t) \le \tilde{k}_1(0)e^{-(c_2/2)t} \tag{2.2.19}$$

and for all $t \ge 0$,

$$\hat{k}(t) > 0, \quad \hat{k}'(t) + \frac{c_2}{2}\hat{k}(t) \le 0 \le \hat{k}'(t) + c_3\hat{k}(t). \tag{2.2.20}$$

Let us denote

$$v(x,t) = e^{\delta t}u(x,t), \ \phi(x,t) = e^{\delta t}\theta(x,t), \ p(x,t) = e^{\delta t}q(x,t). \tag{2.2.21}$$

Then the system (2.2.12)–(2.2.14) can be rewritten as

$$\begin{cases} v_{tt} - v_{xx} + \alpha\phi_x = F, & (2.2.22)\\ \phi_t - \hat{k} * \phi_{xx} + \beta v_{xt} = G, & (2.2.23)\\ p_t + (1-\delta)p + k\phi_x = 0, & (2.2.24)\\ v(0,t) = v(1,t) = p(0,t) = p(1,t) = 0, & (2.2.25)\\ v(x,0) = v_0, \ v_t(x,0) = v_1, \ \phi(x,0) = \phi_0, \ p(x,0) = p_0, & (2.2.26) \end{cases}$$

where

$$F(t) = fe^{\delta t} + 2\delta v_t - \delta^2 v,\ G(t) = ge^{\delta t} + \delta\phi + \delta\beta v_x, \tag{2.2.27}$$

$$\int_0^1 \phi_0(x)dx = \int_0^1 \theta_0(x)dx = 0. \tag{2.2.28}$$

We easily derive from (2.1.2), (2.1.10), and (2.2.28) that

$$\int_0^1 \phi(x,t)dx = \int_0^1 \theta(x,t)dx = 0. \tag{2.2.29}$$

To facilitate our analysis, let us introduce the linear problem

$$\begin{cases} V_{tt} - V_{xx} + \alpha\Phi_x = \mathcal{F}, & (2.2.30)\\ \Phi_t - \hat{k} * \Phi_{xx} + \beta V_{xt} = \mathcal{G}, & (2.2.31)\\ P_t + (1-\delta)P + k\Phi_x = 0, & (2.2.32)\\ V(0,t) = V(1,t) = P(0,t) = P(1,t) = 0, & (2.2.33)\\ V(x,0) = V_0,\ V_t(x,0) = V_1,\ \Phi(x,0) = \Phi_0,\ P(x,0) = P_0, & (2.2.34) \end{cases}$$

with

$$\int_0^1 \Phi_0(x)dx = 0. \tag{2.2.35}$$

It follows from (2.2.29) and (2.2.35) that when $(V,\Phi) = (v,\phi)$, $(\mathcal{F},\mathcal{G}) = (F,G)$ or $(V,\Phi) = (v_t,\phi_t)$, $(\mathcal{F},\mathcal{G}) = (F_t, G_t + \hat{k}(t)\phi_{0xx})$,

$$\int_0^1 \Phi(x,t)dx = 0, \quad for \ \ all\ t > 0. \tag{2.2.36}$$

In the sequel, we shall study the linearized system (2.2.30)–(2.2.32). To this end, we define the following energy functions

$$\begin{cases} E_1(t,V,\Phi) = \dfrac{1}{2}\displaystyle\int_0^1 (V_t^2 + V_x^2 + \alpha\beta^{-1}\Phi^2)dx, & (2.2.37)\\ E_2(t,V,\Phi) = \dfrac{1}{2}\displaystyle\int_0^1 (V_{tt}^2 + V_{tx}^2 + \alpha\beta^{-1}\Phi_t^2)dx, & (2.2.38)\\ E_3(t,V,\Phi) = \dfrac{1}{2}\displaystyle\int_0^1 (V_{tx}^2 + V_{xx}^2 + \alpha\beta^{-1}\Phi_x^2)dx. & (2.2.39) \end{cases}$$

Multiplying (2.2.25) and (2.2.26) by V_t and $\alpha\beta^{-1}\Phi$, respectively, and summing the results, we have

$$\frac{d}{dt}E_1(t,V,\Phi) = -\alpha\beta^{-1}\int_0^1 \Phi_x \hat{k} * \Phi_x dx + \int_0^1 (\mathcal{F}V_t + \alpha\beta^{-1}\mathcal{G}\Phi)dx. \tag{2.2.40}$$

Assuming regular initial data and noting that V_t and Φ_t satisfy the same boundary conditions, we get

$$\begin{aligned}\frac{d}{dt}E_2(t,V,\Phi) &= -\alpha\beta^{-1}\int_0^1 \Phi_{tx}\hat{k} * \Phi_{tx}dx - \alpha\beta^{-1}\hat{k}(t)\int_0^1 \Phi_{0x}\Phi_{tx}dx \\ &\quad + \int_0^1 (\mathcal{F}_t V_{tt} + \alpha\beta^{-1}\mathcal{G}_t\Phi_t)dx \\ &= -\alpha\beta^{-1}\int_0^1 \Phi_{tx}\hat{k} * \Phi_{tx}dx - \alpha\beta^{-1}\frac{d}{dt}\Big(\hat{k}(t)\int_0^1 \Phi_{0x}\Phi_{tx}dx\Big) \\ &\quad + \alpha\beta^{-1}\frac{d}{dt}\hat{k}'(t)\int_0^1 \Phi_{0x}\Phi_{tx}dx + \int_0^1 (\mathcal{F}_t V_{tt} + \alpha\beta^{-1}\mathcal{G}_t\Phi_t)dx.\end{aligned} \tag{2.2.41}$$

Similarly, multiplying (2.2.30) and (2.2.31) by V_{xxt} and $\alpha\beta^{-1}\Phi_{xx}$, respectively, and summing the results, we have

$$\begin{aligned}\frac{d}{dt}E_3(t,V,\Phi) &= -\alpha\beta^{-1}\int_0^1 \Phi_{xx}\hat{k} * \Phi_{xx}dx + \int_0^1 (\mathcal{F}V_{xxt} + \alpha\beta^{-1}\mathcal{G}\Phi_{xx})dx \\ &= -\alpha\beta^{-1}\int_0^1 \Phi_{xx}\hat{k} * \Phi_{xx}dx - \frac{d}{dt}\int_0^1 \mathcal{F}V_{xx}dx \\ &\quad + \int_0^1 (\mathcal{F}_t V_{xt} + \alpha\beta^{-1}\mathcal{G}\Phi_{xx})dx.\end{aligned} \tag{2.2.42}$$

Now we introduce the following functions:

$$\begin{cases} E_4(t,V,\Phi) = -\int_0^1\int_0^x \Phi_t dy V_{tt}dx, \\ E_5(t,V,\Phi) = \int_0^1 V_{tx}\Phi dx, \quad E_6(t,V,\Phi) = \int_0^1 V_{tx}V_x dx, \\ E_7(t,V,\Phi) = -\int_0^1 \Phi_t\hat{k}\Phi_t dx, \quad E_8(t,V,\Phi) = -\int_0^1 \Phi_x\hat{k}\Phi_x dx. \end{cases}$$

Thus, integrating (2.2.31) over $(0,x)$, using the boundary conditions and (2.2.32), we derive

$$\int_0^x \Phi_t dy - \hat{k} * \Phi_x + \beta V_t = \int_0^x \mathcal{G}dy. \tag{2.2.43}$$

By (2.2.30) and (2.2.43), we easily get

$$E_4(t, V, \Phi) \le -\frac{\beta}{2} \parallel V_{tt} \parallel^2 + \frac{\beta}{8} \parallel V_{tx} \parallel^2 + \frac{1}{\beta} \left(k^2(0) \parallel \Phi_x \parallel^2 + \parallel \hat{k}' * \Phi_x \parallel^2 \right)$$
$$+ (\alpha + \frac{2}{\beta}) \parallel \Phi_t \parallel^2 + \int_0^1 \left(\int_0^x \mathcal{G}_t dy V_{tt} + \int_0^x \Phi_t dy \mathcal{F}_t \right) dx. \tag{2.2.44}$$

Now we define

$$n(t, V, \Phi) = \int_0^1 \left(V_{tt}^2 + V_{tx}^2 + V_{xx}^2 + \Phi_t^2 + \Phi_x^2 \right)(t) dx$$

and

$$L(t, V, \Phi) = N \left(E_1(t, V, \Phi) + E_2(t, V, \Phi) + E_3(t, V, \Phi) + \alpha\beta^{-1} \hat{k} \int_0^1 \Phi_{0x} \Phi_x dx \right)$$
$$+ E_4(t, V, \Phi) + E_5(t, V, \Phi) + \frac{\beta}{4} E_6(t, V, \Phi) + a_1 E_7(t, V, \Phi) + a_2 E_8(t, V, \Phi),$$

where $N > 0$ is a parameter sufficiently large and

$$a_1 = \frac{4}{\tilde{k}_1(0)} \left(\alpha + \frac{2}{\beta} \right), \quad a_2 = \frac{4}{\tilde{k}_1(0)} \left(\alpha + \frac{4 + \tilde{k}_1^2(0)}{\beta} + \frac{\alpha^2 \beta}{8} + a_1 \right).$$

Under the above notations, we can derive the following lemma.

Lemma 2.2.2 *There exist positive constants $\beta_1, \beta_2, \beta_3, C_3, C_4$ and sufficiently large constant N such that $L(t, V, \Phi)$ satisfies the following inequalities:*

$$\frac{d}{dt} L(t, V, \Phi) \le -C_3 n(t, V, \Phi) + C_4 \left(\parallel \hat{k} * \Phi_x \parallel^2 + \parallel \hat{k} * \Phi_{xt} \parallel^2 + \parallel \hat{k} * \Phi_{xx} \parallel^2 \right)$$
$$- \alpha N \beta^{-1} \int_0^1 (\Phi_x \hat{k} * \Phi_x + \Phi_{tx} \hat{k} * \Phi_{tx} + \Phi_{xx} \hat{k} * \Phi_{xx}) dx$$
$$+ R(t, V, \Phi), \tag{2.2.45}$$

$$L(t, V, \Phi) \le \beta_2 \left(n(t, V, \Phi) + \parallel \hat{k} * \Phi_x \parallel^2 + \parallel \hat{k} * \Phi_t \parallel^2 + \hat{k}^2(t) \parallel \Phi_{0x} \parallel^2 \right), \tag{2.2.46}$$

$$L(t, V, \Phi) \ge \beta_1 n(t, V, \Phi) - \beta_3 \left(\parallel \hat{k} * \Phi_x \parallel^2 + \parallel \hat{k} * \Phi_t \parallel^2 + \hat{k}^2(t) \parallel \Phi_{0x} \parallel^2 \right), \tag{2.2.47}$$

where

$$\begin{aligned}
R(t, V, \Phi) &= N \int_0^1 \Big(\mathcal{F}V_t + \alpha\beta^{-1}\mathcal{G}\Phi + \mathcal{F}_t V_{tt} + \alpha\beta^{-1}\mathcal{G}_t\Phi_t + \mathcal{F}_t V_{xx} - \alpha\beta^{-1}\mathcal{G}\Phi_{xx}\Big)\, dx \\
&\quad -N\frac{d}{dt}\int_0^1 \mathcal{F}V_{xx}dx + \alpha N\beta^{-1}\hat{k}'(t)\int_0^1 \Phi_{0x}\Phi_x dx - a_1\int_0^1 \mathcal{G}_t\hat{k} * \Phi_t dx \\
&\quad +\int_0^1 \Big(\int_0^x \mathcal{G}_t dy V_{tt} + \int_0^x \Phi_t dy \mathcal{F}_t\Big)dx + a_2\int_0^1 \mathcal{G}\hat{k} * \Phi_{xx}dx \\
&\quad +\int_0^1 \Big(V_{tx}\mathcal{G} - \mathcal{F}\Phi_x - \frac{\beta}{4}\mathcal{F}V_{xx}\Big)dx.
\end{aligned} \tag{2.2.48}$$

Proof By (2.2.30)–(2.2.31) and integration by parts, we get

$$\begin{aligned}
\frac{d}{dt}E_5(t, V, \Phi) &= -\beta \parallel V_{tx} \parallel^2 + \alpha \parallel \Phi_x \parallel^2 - \int_0^1 (V_{xx} + \mathcal{F})\Phi_x dx \\
&\quad + \int_0^1 V_{tx}(\hat{k} * \Phi_{xx} + \mathcal{G})dx \\
&\le -\frac{\beta}{2} \parallel V_{tx} \parallel^2 + \frac{\beta}{16} \parallel V_{xx} \parallel^2 + \Big(\alpha + \frac{4}{\beta}\Big) \parallel \Phi_x \parallel^2 \\
&\quad + \frac{1}{2\beta} \parallel \hat{k} * \Phi_{xx} \parallel^2 + \int_0^1 (V_{tx}\mathcal{G} - \mathcal{F}\Phi_x)dx
\end{aligned} \tag{2.2.49}$$

and

$$\begin{aligned}
\frac{d}{dt}E_6(t, V, \Phi) &= \parallel V_{tx} \parallel^2 - \parallel V_{xx} \parallel^2 + \alpha\int_0^1 \Phi_x V_{xx}dx - \int_0^1 \mathcal{F}V_{xx}dx \\
&\le -\frac{1}{2} \parallel V_{xx} \parallel^2 + \frac{\alpha^2}{2} \parallel \Phi_x \parallel^2 + \parallel V_{tx} \parallel^2 - \int_0^1 \mathcal{F}V_{xx}dx.
\end{aligned} \tag{2.2.50}$$

Thus it follows from (2.2.44) and (2.2.49)–(2.2.50) that

$$\begin{aligned}
&\frac{d}{dt}\Big(E_4(t, V, \Phi) + E_5(t, V, \Phi) + \frac{\beta}{4}E_6(t, V, \Phi)\Big) \\
&\le -\frac{\beta}{2} \parallel V_{tt} \parallel^2 - \frac{\beta}{16} \parallel V_{xx} \parallel^2 - \frac{\beta}{8} \parallel V_{tx} \parallel^2 + \Big(\alpha + \frac{2}{\beta}\Big) \parallel \Phi_t \parallel^2 \\
&+ \Big(\alpha + \frac{4}{\beta} + \frac{\alpha^2\beta}{8} + \frac{\tilde{k}_1^2(0)}{\beta}\Big) \parallel \Phi_x \parallel^2 + \frac{1}{\beta} \parallel \hat{k}' * \Phi_x \parallel^2 + \frac{1}{2\beta} \parallel \hat{k} * \Phi_{xx} \parallel^2 \\
&+ \int_0^1 \Big(\int_0^x \mathcal{G}_t dy V_{tt} + \int_0^x \Phi_t dy \mathcal{F}_t\Big)dx \\
&+ \int_0^1 \Big(V_{tx}\mathcal{G} - \mathcal{F}\Phi_x - \frac{\beta}{4}\mathcal{F}V_{xx}\Big)dx.
\end{aligned} \tag{2.2.51}$$

On the other hand, differentiating (2.2.31) with respect to t, multiplying the resulting equation by $\hat{k} * \Phi_t$ and integrating it by parts, we deduce

$$\begin{aligned}\frac{d}{dt}E_7(t, V, \Phi) &= -\tilde{k}_1(0) \parallel \Phi_t \parallel^2 + \int_0^1 \left(\Phi_t \hat{k}' * \Phi_t + \tilde{k}_1(0)\Phi_x \hat{k} * \Phi_{tx} + \hat{k}' * \Phi_x \hat{k} * \Phi_{tx}\right) dx \\ &\quad - \int_0^1 (\beta V_{tt} \hat{k} * \Phi_{tx} - \mathcal{G}_t \hat{k} * \Phi_t) dx \\ &\le -\frac{\tilde{k}_1(0)}{2} \parallel \Phi_t \parallel^2 + \frac{\beta}{4a_1} \parallel V_{tt} \parallel^2 + \parallel \Phi_x \parallel^2 + \frac{1}{2\tilde{k}_1(0)} \parallel \hat{k}' * \Phi_t \parallel^2 \\ &\quad + \frac{\tilde{k}_1^2(0)}{4} \parallel \hat{k} * \Phi_{tx} \parallel^2 + \frac{1}{2}\left(\parallel \hat{k}' * \Phi_x \parallel^2 + \parallel \hat{k} * \Phi_{tx} \parallel^2\right) \\ &\quad + \beta a_1 \parallel \hat{k} * \Phi_{tx} \parallel^2 - \int_0^1 \mathcal{G}_t \hat{k} * \Phi_t dx. \end{aligned} \tag{2.2.52}$$

Similarly, differentiating (2.2.31) with respect to x, multiplying the resulting equation by $\hat{k} * \Phi_x$ and integrating it by parts, we infer

$$\begin{aligned}\frac{d}{dt}E_8(t, V, \Phi) &= -\tilde{k}_1(0) \parallel \Phi_x \parallel^2 + \parallel \hat{k} * \Phi_{xx} \parallel^2 + \int_0^1 \mathcal{G} \hat{k} * \Phi_{xx} dx \\ &\quad - \int_0^1 \left(\beta V_{tx} \hat{k} * \Phi_{xx} + \Phi_x \hat{k}' * \Phi_x\right) dx \\ &\le -\frac{\tilde{k}_1(0)}{2} \parallel \Phi_x \parallel^2 + \frac{\beta}{4a_2} \parallel V_{tx} \parallel^2 + \frac{1}{2\tilde{k}_1(0)} \parallel \hat{k}' * \Phi_x \parallel^2 \\ &\quad + (1 + \beta a_2) \parallel \hat{k} * \Phi_{xx} \parallel^2 + \int_0^1 \mathcal{G} \hat{k} * \Phi_{xx} dx. \end{aligned} \tag{2.2.53}$$

Combining (2.2.52) and (2.2.53) with (2.2.51) gives

$$\begin{aligned} &\frac{d}{dt}\left(E_4(t, V, \Phi) + E_5(t, V, \Phi) + \frac{\beta}{4}E_6(t, V, \Phi) + a_1 E_7(t, V, \Phi) + a_2 E_8(t, V, \Phi)\right) \\ &\le -C_3 n(t, V, \Phi) + \left(\frac{1}{\beta} + \frac{a_1}{2} + \frac{a_2}{2\tilde{k}_1(0)}\right) \parallel \hat{k}' * \Phi_x \parallel^2 \\ &\quad + \left[\frac{1}{\beta} + (1 + \beta a_2) a_2\right] \parallel \hat{k} * \Phi_{xx} \parallel^2 + \frac{a_1}{2\tilde{k}_1(0)} \parallel \hat{k}' * \Phi_t \parallel^2 \\ &\quad + \left(\tilde{k}_1^2(0)\frac{a_1}{4} + \frac{a_1}{2} + \beta a_1^2\right) \parallel \hat{k} * \Phi_{tx} \parallel^2 + R_1(t, V, \Phi), \end{aligned} \tag{2.2.54}$$

where $C_3 = \min\left(\beta/16, \tilde{k}_1(0)a_1/4, \tilde{k}_1(0)a_2/4\right)$. In view of (2.2.20), (2.2.36) and Poincare's inequality, we have

$$\parallel \hat{k}' * \Phi_t \parallel \le \parallel \hat{k}' * \Phi_{tx} \parallel \le C \parallel \hat{k} * \Phi_{tx} \parallel . \tag{2.2.55}$$

Thus it follows from (2.2.40)–(2.2.42) and (2.2.54)–(2.2.55) that (2.2.45) holds. From the definition of $L(t, V, \Phi)$, we easily know that there exist constants $\beta_1, \beta_2, \beta_3 > 0$ and a sufficiently large constant N such that (2.2.46) and (2.2.47) hold. The proof is complete. $\square$

Now we define

$$\mathcal{M}(t, V, \Phi) = n(t, v, \phi) + n(t, v_t, \phi_t) + \| \phi_{xx}(t) \|^2 .$$

Differentiating (2.2.23) with respect to t, we arrive at

$$\phi_{tt} - \tilde{k}_1(0)\phi_{xx} - \hat{k} * \phi_{xx} + \beta v_{ttx} = G_t,$$

which, combined with (2.2.20), (2.2.23) and (2.2.27), yields

$$\begin{aligned} \| \phi_{xx} \|^2 &\leq C\Big(\| \phi_{tt} \|^2 + \| v_{ttx} \|^2 + \| \phi_t \|^2 + \| v_{tx} \|^2 + \| G \|^2 + \| G_t \|^2 \Big) \\ &\leq C_4\Big(n(t, v, \phi) + n(t, v_t, \phi_t)\Big) + C_5\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2\Big) \| p_{0x} \|^2 . \end{aligned} \tag{2.2.56}$$

Thus,

$$\begin{aligned} n(t, v, \phi) + n(t, v_t, \phi_t) \leq \mathcal{M}(t, v, \phi) &\leq C_6\Big(n(t, v, \phi) + n(t, v_t, \phi_t)\Big) \\ &+ C_5\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2\Big) \| p_{0x} \|^2 . \end{aligned} \tag{2.2.57}$$

By the smallness condition (2.1.16) of initial data, there is a constant $\alpha_1 > 1$, independent of δ, such that

$$\mathcal{M}(0, u, \theta, q) < \alpha_1 \epsilon^2. \tag{2.2.58}$$

Using equations (2.2.18)–(2.2.21), there exists a constant $\alpha_2 > 1$, independent of δ, such that

$$n(0, v, \phi) + n(0, v_t, \phi_t) \leq \mathcal{M}(0, v, \phi) \leq \alpha_2 \mathcal{M}(0, u, \theta) < \alpha_1 \alpha_2 \epsilon^2. \tag{2.2.59}$$

We derive from (2.1.7), (2.2.16), (2.2.19)–(2.2.20) and (2.2.59) that there exists a constant $\eta_0 > 0$, independent of δ, such that

$$\begin{aligned} \int_0^{+\infty} \Big\{ &\lambda_3(\hat{k}'(t))^2 \Big(\| \phi_{0x} \|^2 + \| \phi_{1x} \|^2 \Big) + \Big[\lambda_1(\hat{k}'(t))^2 + \lambda_2(\hat{k}(t))^2\Big] \| \phi_{0xx} \|^2 \\ &+ \lambda_4\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2 + ((k_1 * k_2)'(t))^2\Big) \| p_{0x} \|^2 \Big\} dt < \eta_0 \epsilon^2, \end{aligned} \tag{2.2.60}$$

where

$$\lambda_1 = 1 + \frac{2C_6}{C_3} + \frac{4N^2\alpha^2 C_6}{C_3\beta^2}, \quad \lambda_2 = \frac{2C_6}{C_3} + \frac{2N^2\alpha^2 C_6}{C_3\beta^2},$$
$$\lambda_3 = \frac{2N^2\alpha^2 C_6}{C_3\beta^2}, \quad \lambda_4 = \lambda_2 + NC_5. \tag{2.2.61}$$

Using the continuity of the solutions, it follows that there exist constants $\alpha_0 > 0$ and $t_0 \in [0, T)$ such that

$$\mathcal{M}(t, v, \phi) \leq \alpha_0 \epsilon^2, \quad \textit{for all } t \in [0, t_0). \tag{2.2.62}$$

Now we define

$$t_1 = \sup \Big\{ \tau_1 > 0;\ \mathcal{M}(t, v, \phi) \leq \alpha_0 \epsilon^2 \ \text{ in } \ [0, \tau_1) \Big\}. \tag{2.2.63}$$

By Sobolev's embedding theorem and (2.2.63), we obtain that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$| v_x(x, t) | + | \phi(x, t) | + | \phi_x(x, t) | + | \phi_t(x, t) | \leq C_7 \epsilon, \tag{2.2.64}$$

which implies that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$| u_x(x, t) | + | \theta(x, t) | + | \theta_x(x, t) | + | \theta_t(x, t) | \leq C_8 \epsilon e^{-\delta t}. \tag{2.2.65}$$

Thus, if ϵ is small enough, we have that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$| u_x(x, t) | < \rho_0. \tag{2.2.66}$$

Define

$$\nu = \sup_{|x|+|y| \leq \rho_0} \{ |\partial^\rho \eta_i|;\ i = 1, 2;\ 0 \leq |\rho| \leq 3 \},$$

where ∂^ρ denotes the partial derivatives of order $|\rho|$. Recalling the definitions of $\eta_i (i = 1, 2)$ and using the above inequalities, we deduce

$$| \eta_i | \leq C_9 \epsilon, \quad i = 1, 2, \tag{2.2.67}$$

with $C_9 = C_9(\nu) > 0$ being a constant. By (2.2.63)–(2.2.67), we easily derive that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$| v_t(x, t) | + | v_{tx}(x, t) | + | v_{tt}(x, t) | \leq C_{10} \epsilon,$$

which, together with (2.2.19), (2.2.21), implies that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$| u_t(x, t) | + | u_{tx}(x, t) | + | u_{tt}(x, t) | \le C_{11} e^{-\delta t} \epsilon. \tag{2.2.68}$$

By (2.2.22), (2.2.64), and (2.2.67)–(2.2.68), we get

$$| v_{xx}(x, t) | \le C\epsilon + C\epsilon \, | v_{xx}(x, t) |,$$

which gives

$$| v_{xx}(x, t) | \le C_{12}\epsilon, \quad | u_{xx}(x, t) | \le C_{12}\epsilon e^{-\delta t}, \ \textit{for all} \ (x, t) \in [0, 1] \times [0, t_1). \tag{2.2.69}$$

Similarly, differentiating (2.2.22) with respect to x, we conclude

$$\| v_{xxx}(t) \|^2 \le C\Big[\mathcal{M}(t, v, \phi) + \| F_x \|^2 \Big] \le C\mathcal{M}(t, v, \phi) + C\epsilon^2 \, \| v_{xxx}(t) \|^2,$$

which gives that for any $t \in [0, t_1)$,

$$\| v_{xxx}(t) \|^2 \le C\mathcal{M}(t, v, \phi) \le C_{13}\epsilon^2, \quad \| u_{xxx}(t) \|^2 \le C\mathcal{M}(t, v, \phi) \le C_{13}\epsilon^2 e^{-\delta t} \tag{2.2.70}$$

provided that ϵ is small enough.

Lemma 2.2.3 *Under the same assumptions as in Theorem 2.1.1, the following inequalities hold for any $t \in [0, t_1)$:*

$$\int_0^1 \mathcal{F}_t V_{tt} dx \le C(\delta + \epsilon)\mathcal{M}(t, v, \phi) - \frac{1}{2}\frac{d}{dt}\int_0^1 \eta_1 V_{tx}^2 dx, \tag{2.2.71}$$

$$\int_0^1 \mathcal{F}_t V_{xx} dx \le C(\delta + \epsilon)\mathcal{M}(t, v, \phi) + \frac{1}{2}\frac{d}{dt}\int_0^1 \eta_1 V_{xx}^2 dx, \tag{2.2.72}$$

$$\int_0^1 \mathcal{F} V_t dx \le C(\delta + \epsilon)\mathcal{M}(t, v, \phi), \tag{2.2.73}$$

$$\int_0^1 \mathcal{F} V_{xx} dx \le C(\delta + \epsilon)\mathcal{M}(t, v, \phi), \tag{2.2.74}$$

$$\int_0^1 \mathcal{F}^2 dx \le C(\delta + \epsilon)\mathcal{M}(t, v, \phi). \tag{2.2.75}$$

Proof We only consider the case of $(V, \Phi) = (v_t, \phi_t)$ and $\mathcal{F} = F_t$ to prove (2.2.71). The case of $(V, \Phi) = (v, \phi)$ and $\mathcal{F} = F$ is simple. By (2.2.15) and noting that

$$\begin{cases} F_{tt} = f_{tt}e^{\delta t} + 2\delta f_t e^{\delta t} + \delta^2 f e^{\delta t} + 2\delta v_{ttt} - \delta^2 v_t, \\ f_{tt} = \eta_{1tt}u_{xx} + 2\eta_{1t}u_{xxt} + \eta_1 u_{xxtt} + \eta_{2tt}\theta_x + 2\eta_{2t}\theta_{xt} + \eta_2\theta_{xtt}, \\ \eta_{1tt} = (u_{xt}, \theta_t)\mathcal{H}_{\eta_1}(u_{xt}, \theta_t)^\tau + \nabla\eta_1 \cdot (u_{xtt}, \theta_{tt}), \ \ \eta_{1t} = \nabla\eta_1 \cdot (u_{xt}, \theta_t), \end{cases}$$

we have

$$e^{\delta t} \parallel f_t \parallel \leq C(\epsilon+\delta)\Big(\parallel v_{xx} \parallel + \parallel v_{txx} \parallel + \parallel \phi_x \parallel + \parallel \phi_{tx} \parallel \Big).$$

Here we only estimate the typical term in $\int_0^1 f_{tt} v_{ttt} e^{\delta t} dx$, that is, $\int_0^1 \eta_1 u_{xxtt} v_{ttt} e^{\delta t} dx$. Using (2.2.65) and (2.2.67)–(2.2.70), the other terms in $\int_0^1 f_{tt} v_{ttt} e^{\delta t} dx$ can be controlled by $C(\epsilon+\delta)\mathcal{M}(t, v, \phi)$ in the same way. Noting that

$$\begin{aligned}
&u_{xxtt} e^{\delta t} = v_{xxtt} - 2\delta v_{xxt} + \delta^2 v_{xx},\\
&\parallel v_{ttt}(t) \parallel^2 \leq C\left(\parallel v_{txx}(t) \parallel^2 + \parallel v_{xx}(t) \parallel^2 + \parallel \phi_x(t) \parallel^2 + \parallel \phi_{tx}(t) \parallel^2\right)
\end{aligned}$$

and using the integration by parts, we arrive at

$$\int_0^1 \eta_1 u_{xxtt} v_{ttt} e^{\delta t} dx \leq C(\epsilon+\delta)\mathcal{M}(t, v, \phi) - \frac{1}{2}\frac{d}{dt}\int_0^1 \eta_1 v_{ttx}^2 dx.$$

Thus estimate (2.2.71) is valid. Similarly, we can prove estimates (2.2.72)–(2.2.75). The proof is complete. □

By Lemma 2.2.3, we can obtain the next two lemmas.

Lemma 2.2.4 *Under the same assumptions as in Theorem 2.1.1, the following inequalities hold for any* $t \in [0, t_1)$:

$$\begin{aligned}
R(t, v, \phi) \leq\ & C(\epsilon+\delta)\mathcal{M}(t, v, \phi) + C\delta(\parallel \hat{k} * \phi_{tx} \parallel^2 + \parallel \hat{k} * \phi_{xx} \parallel^2)\\
&+ \frac{C_3}{8C_6} \parallel \phi_x \parallel^2 + \frac{2N^2\alpha^2 C_6}{C_3\beta^2}(\hat{k}'(t))^2 \parallel \phi_{0x} \parallel^2\\
&+ \frac{2N^2\alpha^2 C_6}{C_3\beta^2}\left((k_2(t))^2 + ((k_1 * k_2)(t))^2\right) \parallel p_{0x} \parallel^2 - N\frac{d}{dt}\int_0^1 F v_{xx} dx,\\
N\int_0^1 F v_{xx} dx \leq\ & C(\epsilon+\delta)\mathcal{M}(t, v, \phi).
\end{aligned}$$

Lemma 2.2.5 *Under the same assumptions as in Theorem 2.1.1, the following inequalities hold for any* $t \in [0, t_1)$:

$$\begin{aligned}
R(t, v_t, \phi_t)\ \leq\ & C(\epsilon+\delta)\mathcal{M}(t, v, \phi) + (C\delta + \frac{a_1}{4}) \parallel \hat{k} * \phi_{tx} \parallel^2 + C\delta \parallel \hat{k} * \phi_{txx} \parallel^2\\
&+ \frac{C_3}{8C_6}\Big(\parallel v_{ttt} \parallel^2 + \parallel v_{ttx} \parallel^2 + \parallel \phi_t \parallel^2 + \parallel \phi_{tt} \parallel^2 + \parallel \phi_{tx} \parallel^2 + \parallel \phi_{xx} \parallel^2\Big)\\
&+ \Big(\lambda_1(\hat{k}'(t))^2 + \lambda_2(\hat{k}(t))^2\Big) \parallel \phi_{0xx} \parallel^2 + \lambda_3(\hat{k}'(t))^2 \parallel \phi_{1x} \parallel^2\\
&+ \lambda_4\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2 + ((k_1 * k_2)'(t))^2\Big) \parallel p_{0x} \parallel^2\\
&+ \frac{d}{dt}\int_0^1 \Big[\frac{N}{2}(\eta_1 v_{txx}^2 - v_{ttx}^2) - N\alpha\beta^{-1}\hat{k}(t)\phi_{0xx}\phi_{xx} - N F_t v_{txx}\Big] dx, \quad (2.2.76)
\end{aligned}$$

$$
\begin{aligned}
N\int_0^1 F_t dx &\le C(\epsilon+\delta)\mathcal{M}(t,v,\phi) - N\alpha\beta^{-1}\hat{k}(t)\int_0^1 \phi_{0xx}\phi_{xx}dx \\
&\le \frac{\beta_1}{2}\Big(n(t,v,\phi)+n(t,v_t,\phi_t)\Big) + \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}\hat{k}^2(t)\parallel \phi_{0xx}\parallel^2 . \qquad (2.2.77)
\end{aligned}
$$

Let us introduce the following function

$$
\begin{aligned}
\Upsilon_1(t) &= L(t,v,\phi)+L(t,v_t,\phi_t)+N\int_0^1 (Fv_{xx}+F_t v_{txx})dx \\
&\quad + \frac{N}{2}\int_0^1 \eta_1(v_{ttx}^2 - v_{txx}^2)dx + N\alpha\beta^{-1}\hat{k}(t)\int_0^1 \phi_{0xx}\phi_{xx}dx.
\end{aligned}
$$

Then it follows from (2.2.57), (2.2.65)–(2.2.70) and Lemmas 2.2.2–2.2.5. that if $\epsilon+\delta$ is small enough,

$$
\begin{aligned}
\Upsilon_1(t) &\le \Big(\beta_2+\frac{\beta_1}{2}\Big)\Big(n(t,v,\phi)+n(t,v_t,\phi_t)\Big) + C(\epsilon+\delta)\mathcal{M}(t,v,\phi) \\
&\quad + \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}\hat{k}^2(t)\parallel \phi_{0xx}\parallel^2 + \beta_2\Big(\parallel \hat{k}*\phi_t\parallel^2 + \parallel \hat{k}*\phi_{tt}\parallel^2 + \parallel \hat{k}*\phi_x\parallel^2 \\
&\quad + \parallel \hat{k}*\phi_{tx}\parallel^2 + \hat{k}^2(t)\parallel \phi_{0x}\parallel^2 + \hat{k}^2(t)\parallel \phi_{1x}\parallel^2\Big) \\
&\le \frac{(\beta_2+\beta_1)}{2}[n(t,v,\phi)+n(t,v_t,\phi_t)] \qquad (2.2.78) \\
&\quad + C_5 C(\epsilon+\delta)\Big((k_2(t))^2 + ((k_1*k_2)(t))^2\Big)\parallel p_{0x}\parallel^2 \\
&\quad + \Big(\beta_2\parallel \phi_{0x}\parallel^2 + \beta_2\parallel \phi_{1x}\parallel^2 + \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}\parallel \phi_{0xx}\parallel^2\Big)\hat{k}^2(t) \\
&\quad + \beta_2(\parallel \hat{k}*\phi_t\parallel^2 + \parallel \hat{k}*\phi_{tt}\parallel^2 + \parallel \hat{k}*\phi_x\parallel^2 + \parallel \hat{k}*\phi_{tx}\parallel^2) \qquad (2.2.79)
\end{aligned}
$$

and

$$
\begin{aligned}
\Upsilon_1(t) &\ge \frac{\beta_1}{2}[n(t,v,\phi)+n(t,v_t,\phi_t)] - C(\epsilon+\delta)\mathcal{M}(t,v,\phi) \\
&\quad - \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}\hat{k}^2(t)\parallel \phi_{0xx}\parallel^2 - \beta_3\Big(\parallel \hat{k}*\phi_t\parallel^2 + \parallel \hat{k}*\phi_{tt}\parallel^2 \\
&\quad + \parallel \hat{k}*\phi_x\parallel^2 + \parallel \hat{k}*\phi_{tx}\parallel^2 + \hat{k}^2(t)\parallel \phi_{0x}\parallel^2 + \hat{k}^2(t)\parallel \phi_{1x}\parallel^2\Big) \\
&\ge \frac{\beta_1}{4}[n(t,v,\phi)+n(t,v_t,\phi_t)] \\
&\quad - C_5 C(\epsilon+\delta)\Big((k_2(t))^2 + ((k_1*k_2)(t))^2\Big)\parallel p_{0x}\parallel^2 \\
&\quad - \Big(\beta_3\parallel \phi_{0x}\parallel^2 + \beta_2\parallel \phi_{1x}\parallel^2 + \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}\parallel \phi_{0xx}\parallel^2\Big)\hat{k}^2(t) \\
&\quad - \beta_3(\parallel \hat{k}*\phi_t\parallel^2 + \parallel \hat{k}*\phi_{tt}\parallel^2 + \parallel \hat{k}*\phi_x\parallel^2 + \parallel \hat{k}*\phi_{tx}\parallel^2). \qquad (2.2.80)
\end{aligned}
$$

Define

$$
\begin{aligned}
\Upsilon(t) = {}& \Upsilon_1(t) + C_5C(\epsilon+\delta)\Big((k_2(t))^2 + ((k_1*k_2)(t))^2\Big) \parallel p_{0x} \parallel^2 \\
&+ \Big(\beta_3 \parallel \phi_{0x} \parallel^2 + \beta_2 \parallel \phi_{1x} \parallel^2 + \frac{N^2\alpha^2C_4}{2\beta_1\beta_2} \parallel \phi_{0xx} \parallel^2 \Big)\hat{k}^2(t) \\
&+ \beta_3\Big(\parallel \hat{k}*\phi_t \parallel^2 + \parallel \hat{k}*\phi_{tt} \parallel^2 + \parallel \hat{k}*\phi_x \parallel^2 + \parallel \hat{k}*\phi_{tx} \parallel^2 \Big). \qquad (2.2.81)
\end{aligned}
$$

Then it follows from (2.2.80), (2.2.20) and (2.2.57) that if $\epsilon+\delta$ is small enough, then we have

$$
\Upsilon(t) \geq \frac{\beta_1(n(t,v,\phi)+n(t,v_t,\phi_t))}{4} \geq \frac{\beta_1}{4C_6}\mathcal{M}(t,v,\phi), \qquad (2.2.82)
$$

$$
\begin{aligned}
\frac{d}{dt}\Upsilon(t) \leq {}& \frac{d}{dt}\Upsilon_1(t) + \frac{C_3}{8}(n(t,v,\phi)+n(t,v_t,\phi_t)) \\
&+ C_{14}\beta_3\Big(\parallel \hat{k}*\phi_{tx} \parallel^2 + \parallel \hat{k}*\phi_{ttx} \parallel^2 + \parallel \hat{k}*\phi_x \parallel^2 \Big). \qquad (2.2.83)
\end{aligned}
$$

Proof of Theorem 2.1.1 We shall assume that the initial data belong to $H^4(0,1)$. Our result will follow the standard density argument. By virtue of Lemmas 2.2.4 and 2.2.5, we easily obtain

$$
\begin{aligned}
\frac{d}{dt}\Upsilon_1(t) \leq {}& -N\alpha\beta^{-1}\int_0^1 \Big(\phi_x\hat{k}*\phi_x + 2\phi_{tx}\hat{k}*\phi_{tx} + \phi_{ttx}\hat{k}*\phi_{ttx} \\
&+ \phi_{xx}\hat{k}*\phi_{xx} + \phi_{txx}\hat{k}*\phi_{txx}\Big)dx \\
&- C_3(n(t,v,\phi)+n(t,v_t,\phi_t)) + \frac{C_3}{8C_6}\mathcal{M}(t,v,\phi) \\
&+ C_{15}(\epsilon+\delta)\mathcal{M}(t,v,\phi) + C_4\Big(\parallel \hat{k}*\phi_x \parallel^2 + 2 \parallel \hat{k}*\phi_{tx} \parallel^2 \\
&+ \parallel \hat{k}*\phi_{ttx} \parallel^2 + \parallel \hat{k}*\phi_{xx} \parallel^2 + \parallel \hat{k}*\phi_{txx} \parallel^2 \Big) + \frac{a_1^2}{4} \parallel \hat{k}*\phi_{tx} \parallel^2 \\
&+ C_{16}\delta\Big(\parallel \hat{k}*\phi_{tx} \parallel^2 + \parallel \hat{k}*\phi_{xx} \parallel^2 + \parallel \hat{k}*\phi_{txx} \parallel^2 \Big) \\
&+ \Big[\lambda_1(\hat{k}'(t))^2 + \lambda_2(\hat{k}(t))^2\Big] \parallel \phi_{0xx} \parallel^2 + \lambda_3(\hat{k}'(t))^2(\parallel \phi_{1x} \parallel^2 + \parallel \phi_{0x} \parallel^2) \\
&+ \lambda_4\Big[(k_2(t))^2 + ((k_1*k_2)(t))^2 + ((k_1*k_2)'(t))^2\Big] \parallel p_{0x} \parallel^2, \qquad (2.2.84)
\end{aligned}
$$

which, together with (2.2.82)–(2.2.83), yields that if $\epsilon + \delta$ is small enough,

$$\begin{aligned}\frac{d}{dt}\Upsilon(t) \le & -N\alpha\beta^{-1}\int_0^1 \Big(\phi_x\hat{k}*\phi_x + 2\phi_{tx}\hat{k}*\phi_{tx} + \phi_{ttx}\hat{k}*\phi_{ttx} \\ & +\phi_{xx}\hat{k}*\phi_{xx} + \phi_{txx}\hat{k}*\phi_{txx}\Big)dx \\ & -\frac{C_3}{2}\Big(n(t,v,\phi)+n(t,v_t,\phi_t)\Big) \\ & +\Big(2C_4+\frac{a_1^2}{4}+C_{16}\delta+C_{14}\beta_3\Big)\Big(\parallel \hat{k}*\phi_x\parallel^2 + \parallel \hat{k}*\phi_{tx}\parallel^2 \\ & + \parallel \hat{k}*\phi_{ttx}\parallel^2 + \parallel \hat{k}*\phi_{xx}\parallel^2 + \parallel \hat{k}*\phi_{txx}\parallel^2\Big) \\ & +\Big[\lambda_1(\hat{k}'(t))^2+\lambda_2(\hat{k}(t))^2\Big]\parallel \phi_{0xx}\parallel^2 + \lambda_3(\hat{k}'(t))^2(\parallel\phi_{1x}\parallel^2+\parallel\phi_{0x}\parallel^2) \\ & +\lambda_4\Big[(k_2(t))^2+((k_1*k_2)(t))^2+((k_1*k_2)'(t))^2\Big]\parallel p_{0x}\parallel^2 . \end{aligned} \tag{2.2.85}$$

Integrating (2.2.85) with respect to t, using Lemma 2.2.2, (2.2.79)–(2.2.82) and (2.2.59), taking δ and ϵ small enough, we deduce

$$\begin{aligned}&\Upsilon(t)+\frac{C_3}{2}\int_0^t\Big(n(\tau,v,\phi)+n(\tau,v_t,\phi_t)\Big)d\tau \\ &\quad +C_{17}\int_0^t\Big[\sum_{i=0}^{2}\parallel \hat{k}*\partial_t^i\phi_x\parallel^2 + \sum_{i=0}^{1}\parallel \hat{k}*\partial_t^i\phi_x\parallel^2\Big]d\tau \\ &\le (\beta_1+\beta_2)\frac{\alpha_1\alpha_2\epsilon^2}{2}+\tilde{k}_1(0)\Big[2(\beta_1+\beta_2)+\frac{N^2\alpha^2C_4}{\beta_1\beta_2}\Big]\alpha_1\alpha_2\epsilon^2+\eta_0\epsilon^2 \\ &= \alpha_3\epsilon^2. \end{aligned} \tag{2.2.86}$$

Thus it follows from (2.2.57), (2.2.60), (2.2.82) and (2.2.86) that for any $t \in [0, t_1)$,

$$\begin{aligned}&\mathcal{M}(t,v,\phi)+\frac{2C_3}{\beta_1}\int_0^t\mathcal{M}(\tau,v,\phi)d\tau \\ &\quad +\frac{4C_6C_{17}}{\beta_1}\int_0^t\Big(\sum_{i=0}^{2}\parallel \hat{k}*\partial_t^i\phi_x\parallel^2+\sum_{i=0}^{1}\parallel \hat{k}*\partial_t^i\phi_x\parallel^2\Big)d\tau \\ &\quad \le \frac{4C_6\alpha_3\epsilon^2}{\beta_1} = (\alpha_0-\alpha_1\alpha_2)\epsilon^2. \end{aligned} \tag{2.2.87}$$

Letting $t \to t_1$ in (2.2.87), we have

$$\mathcal{M}(t_1,v,\phi) \le (\alpha_0-\alpha_1\alpha_2)\epsilon^2 < \alpha_0\epsilon^2,$$

which contradicts the definition of t_1, By repeating the same procedure, taking ϵ even smaller if necessary, and using the continuity of $\mathcal{M}(t, v, \phi)$, (2.2.87) is established for all $t > 0$. On the other hand, it is easy to verify that for all $t > 0$,

$$C_{18}^{-1}\mathcal{M}(t,\theta,v)e^{2\delta t} \le \mathcal{M}(t,v,\phi) \le C_{18}\mathcal{M}(t,\theta,v)e^{2\delta t}. \tag{2.2.88}$$

By (2.2.18), we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=0}^{1} \| (\hat{k} * \partial_t^i \phi_x)(t) \|^2 + \| (\hat{k} * \partial_t^i \phi_{xx})(t) \|^2\right)\\
&\quad \le C\Big(\sum_{i=0}^{1}(\| (\hat{k} * \partial_t^i \phi_x)(t) \|^2 + \| (\partial_t^i \phi_x)(t) \|^2)\\
&\qquad + \| (\hat{k} * \partial_t^i \phi_{xx})(t) \|^2 + \| \phi_{xx}(t) \|^2 \Big)\\
&\quad \le C\Big(\sum_{i=0}^{1} \| (\hat{k} * \partial_t^i \phi_x)(t) \|^2 + \| (\hat{k} * \partial_t^i \phi_{xx})(t) \|^2 + \mathcal{M}(t,v,\phi)\Big).
\end{aligned} \tag{2.2.89}$$

Integrating (2.2.89) with respect to t, and exploiting (2.2.87), we finally obtain

$$\sum_{i=0}^{1}\Big(\| (\hat{k} * \partial_t^i \phi_x)(t) \|^2 + \| (\hat{k} * \partial_t^i \phi_{xx})(t) \|^2\Big) \le C. \tag{2.2.90}$$

Then by (2.2.11)–(2.2.14), (2.2.21), (2.2.87), (2.2.88) and (2.2.90), the proof of Theorem 2.1.1 is now complete. □

2.3 Bibliographic Comments

Thermoelastic equations describe the elastic and the thermal behavior of elastic, heat conductive media, in particular the reciprocal actions between elastic stresses and temperature differences [27, 53, 122].

The classical model of thermoelasticity, constructed on the basis of the Fourier's law, provides good approximations for the description in a wide range of engineering applications. In the simplest case of a homogeneous isotropic medium, we have the following equations in two or three space dimensions

$$\begin{aligned}
&U_{tt} - \Big((2\mu+\lambda)\nabla\nabla' - \mu\nabla\times\nabla\times\Big)U + \gamma\nabla\theta = b,\\
&\delta\theta_t - \kappa\Delta\theta + \gamma\nabla' U_t = r,
\end{aligned}$$

where $\mu, \lambda, \gamma, \delta, b, r$ and κ are constants (μ, λ are the Lamé moduli) satisfying

$$\mu > 0, \quad 2\mu + \lambda > 0,$$

and

$$\delta, \kappa > 0, \quad \gamma \neq 0.$$

Notice that in two space dimensions, the rotation of a scalar field f in $\mathbb{R}^2$ is defined to be the vector field

$$\nabla \times f := (\partial_2 f, -\partial_1 f)'$$

and the rotation of a vector field $F = (F_1, F_2)'$ in $\mathbb{R}^2$ is defined to the scalar

$$\nabla \times F := \partial_1 F_2 - \partial_2 F_1.$$

In particular, the formula

$$\Delta = \nabla\nabla' - \nabla \times \nabla$$

holds in $\mathbb{R}^2$ and $\mathbb{R}^3$.

However, this model leads to the paradox of the infinite propagation speed of heat pulse and in some practical situations may lead to an inadequate description of heat conduction. In order to eliminate these shortcomings of classical thermoelasticity, many hyperbolic thermoelastic models have been developed from the middle of the last century. Recently, Green and Naghdi [89, 90] re-examined the classical Fourier's law in thermoelasticity, instead of the classical entropy inequality, used a general entropy balance and, introducing a new thermal variable, proposed three models, based on the different material responses, labeled as types I, II and III. The linearized version of the first model leads to the Fourier law, the linearized version of both types II and III models whose constitutive assumptions on the heat flux vector are different from the Fourier's law allows heat transmission at a finite speed.

Let us now recall the classical model for linear thermoelastic systems of types I, II and III which take the following forms, respectively.
Classical model of type I:

$$\begin{cases} U_{tt} - \Big((2\mu + \lambda)\nabla\nabla' - \mu\nabla \times \nabla \times\Big)U + \gamma\nabla\theta = 0, \\ \delta\theta_t - \kappa\Delta\theta + \gamma\nabla \cdot U_t = 0. \end{cases} \tag{2.3.1}$$

Model of type II:

$$\begin{cases} U_{tt} - \Big((2\mu + \lambda)\nabla\nabla' - \mu\nabla \times \nabla \times\Big)U + \gamma\nabla\theta = 0, \\ \delta\theta_{tt} - \kappa\Delta\theta + \gamma\nabla \cdot U_{tt} = 0. \end{cases} \tag{2.3.2}$$

Model of type III:

$$\begin{cases} U_{tt} - \Big((2\mu + \lambda)\nabla\nabla' - \mu\nabla \times \nabla \times\Big)U + \gamma\nabla\theta = 0, \\ \delta\theta_{tt} - \kappa\Delta\theta - \kappa\Delta\theta_t + \gamma\nabla \cdot U_{tt} = 0. \end{cases} \tag{2.3.3}$$

In the above, $U = U(x, t)$ is the displacement vector, $\theta = \theta(x, t)$ denotes the temperature.

Moreover, the three-dimensional thermoelastic equations with second sound obeying Cattaneo's law takes the form

$$\begin{cases} U_{tt} - \mu\Delta U - (\mu + \lambda)\nabla divU + \beta\nabla\theta = 0, \\ \theta_t + \gamma divq + \delta divU_t = 0, \\ \tau_0 q_t + q + \kappa\nabla\theta = 0, \end{cases} \tag{2.3.4}$$

here, $\mu, \beta, \gamma, \delta$ and κ are positive constants, U and q are two unknown vector functions, while θ is a unknown scalar function. Constant $\tau_0 > 0$ is the so-called relaxation parameter.

For the thermoelasticity of type I, there are many works (see, e.g., [47, 110, 120, 154, 183, 184, 185, 186, 274, 106, 122, 155, 240]) on the existence, uniqueness and asymptotic behavior of solutions of the linear system. Slemrod [257] proved the global existence, uniqueness and asymptotic stability of classical smooth solutions; Shibata [255] considered the initial boundary value problem with the boundary conditions $u_x - \gamma\theta = 0, \theta_x = 0 \quad (x = 0, l)$; Racke, Shibata and Zheng [240] obtained the global existence and uniqueness of solutions for the nonlinear thermoelastic system of type I with small initial data; Jiang [120] proved an exponential decay result for solutions of the equations of linear, homogeneous, isotropic thermoelasticity in bounded regions in two or three space dimensions; Racke [234] considered the Cauchy problem in three-dimensional nonlinear thermoelasticity for a medium which is homogeneous and initially isotropic; Lebeau and Zuazua [143], by a decoupling method, reduced the problem to an observability inequality for the Lamé system in linear elasticity and more precisely to whether the total energy of solutions can be estimated in terms of the energy concentrated on its longitudinal component, and showed that when the domain is convex, the decay rate is never uniform, and, in three space dimensions, the lack of uniform decay may be due to a critical polarization of the energy on the transversal component of the displacement; Muñoz Rivera and Qin [186] proved the global existence, uniqueness, and asymptotic behavior of solutions for $1D$ nonlinear thermoelasticity with thermal memory subject to Dirichlet-Dirichlet boundary conditions; Muñoz Rivera and Qin [189] proved the global existence, uniqueness and asymptotic behavior of solutions for the one-dimensional nonlinear thermoelasticity with thermal memory subject to Dirichlet-Dirichlet boundary conditions. When Cattaneo's law substitutes Fourier's law, results concerning existence, blow-up, and asymptotic behavior of smooth (weak) solutions have been established by several authors (see, e.g., [166, 167, 169, 235, 236, 266, 275, 281]).

Now we recall some results on the thermoelastic systems with second sound. Tarabek [266] treated problems related to system (2.1.1)–(2.1.3) in both bounded and unbounded domains and established global existence results for small initial data, and showed that these classical solutions tend to their equilibria as t tends to infinity. Concerning the asymptotic behavior, Racke [235] discussed the global existence and decay exponentially to the equilibrium state for one-dimensional case. For the multi-dimensional case ($n = 2, 3$), Racke [236] established an existence result for homogeneous linear problem; Messaoudi [164] investigated the situation where a nonlinear source term is competing with a damping caused by the heat conduction and established a local existence result, and further prove that solutions with negative energy blow up in a finite time. Later on, Messaoudi and Said-Houari [169] proved that the exponential stability in one-dimensional nonlinear thermoelasticity with second sound. Their work generalized earlier ones in [165, 166, 167] to thermoelasticity with second sound. Racke and Wang [239] proved asymptotic behavior of discontinuous solutions to thermoelastic systems with second sound. The Cauchy problem of the linear thermoelastic system with second sound was also studied by Wang and Wang [275], and Yang and Wang [281]. In particular, if $k_1 = 0,\ q = -k\theta_x$, then the system (2.1.1)–(2.1.3) reduces to the system of a classical thermoelasticity, in which the heat flux is given by Fourier's law instead of Cattaneo's law and results concerning existence, blow-up, and asymptotic behavior of smooth (weak) solutions have been established by several authors over the past two decades (see, e.g., [47, 89, 90, 120, 169, 183, 210, 226, 257, 274]); if $k_1 \neq 0$, $q = -k_1 * \theta_x$, Muñoz Rivera and Qin [189] studied the global existence, uniqueness, and asymptotic behavior of solutions to the equations of thermoelastic system subject to Dirichlet-Dirichlet boundary conditions. When the heat flux obeys the theory of Gurtin and Pipkin, that is, $q = -k_1 * \theta_x$, Fatori and Muñoz Rivera [74] established the energy decay for a linear hyperbolic thermoelastic system provided the relaxation kernel $k_1(t)$ is a strongly positive definite and decays exponentially. For the models of linear and nonlinear thermoelastic plates, Bucci and Chueshov [24] proved the existence of a compact, finite dimensional, global attractor for a coupled PDE system comprising a nonlinearly damped semilinear wave equation and a nonlinear system of thermoelastic plate equations, without any mechanical (viscous or structural) dissipation in the plate component, a major part in the proof is played by an estimate-known as stabilizability estimate-which shows that the difference of any two trajectories can be exponentially stabilized to zero, modulo a compact perturbation. Chueshov and Lasiecke [40] studied asymptotic behavior of solutions corresponding to von Karman thermoelastic plates, a distinct feature of this work is that the model considered has no added dissipation-particularly mechanical dissipation typically added to plate equation when long time-behavior is considered, thus, the model consists of undamped oscillatory plate equation strongly coupled with heat equation, nevertheless the authors were able to showed that the ultimate (asymptotic) behavior of the von Karman evolution is described by finite dimensional global attractor. In addition, the obtained estimate for the dimension and the size of the attractor are independent of the rotational inertia parameter γ and heat/thermal capacity k, where the former was known to change the character of dynamics from hyperbolic ($\gamma > 0$) to parabolic like ($\gamma = 0$). Meyvaci [180] studied the problem of continuous dependence on parameters of strong solutions of the initial boundary value problem for the linear thermoelastic plate equation.

For the 2D Timoshenko systems with second sound, we notice the works of Fastovska [72] and Grobbelaar-Van Dalsen. In [94], the authors are concerned with the strong stabilization of models for the Reissner-Mindlin plate equations with second sound, that is, models that include thermal effects described according to Cattaneo's law of heat conduction instead of Fourier's law in classical thermoelasticity. Two models will be considered which are distinct with respect to the property of compactness or non-compactness of the resolvent of the generator of the underlying semigroup. In accordance with the compactness or non-compactness of the resolvent operator, a different criterion for strong stability is implemented to achieve the strong stabilization of each model. Recently, Clark et al. [43] gave a mathematical treatment of a model for small vertical vibrations of an elastic membrane coupled with a heat equation and subject to nonlinear mixed boundary conditions. They established the existence, uniqueness, and a uniform decay rate for global solutions to this nonlinear nonlocal thermoelastic coupled system under nonlinear boundary conditions. Later on, Clark and Guardia [42] dealt with the global existence and uniqueness of solutions, and uniform stabilization of the energy of an initial-boundary value problem for a thermoelastic system with nonlinear terms of nonlocal kind. The asymptotic stabilization of the energy of system is obtained without any dissipation mechanism acting in the displacement variable u of the membrane equation. Fatori et al. [73] considered the long-time behavior of a class of thermoelastic plates with nonlinear strain, they established the existence of global and exponential attractors for the strongly damped problem through a stabilizability inequality. In addition, for the weakly damped problem, they established the exponential stability of its Galerkin semiflows.

We would like to mention other works in [2, 26, 41, 44, 139, 145, 138, 188, 272] for related models.

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