Chapter 2
Framework of Crystal Elasticity

As the knowledge of crystals is a benefit for understanding quasicrystals, it is worthwhile having a concise review of the crystal elasticity or classical elasticity before learning elasticity of quaicrystals. Here is a brief description to the theory. The detailed material for this theory can be found in many monographs and textbooks, e.g. Landau and Lifshitz [1]. Though the discussion here is limited within the framework of continuum medium mechanics, there are still connections to physical nature of the elasticity of crystals reflected by phonon concept (discussed in Sect. 1.5). The readers are advised to refer to the relevant chapters and sections of monographs of Born and Huang [2] and Anderson [3] which would help us understanding the phonon concept so the phason concept and elasticity of quasicrystals, which will be presented in the following chapters. The practice shows that it would be hard to understand phason concept and the phason elasticity if we limited our knowledge only within the classical continuum medium and complete intuition.

For simplicity, the tensor algebra will be used in the text.

2.1 Review on Some Basic Concepts

2.1.1 Vector

A quantity with both magnitude and direction is named vector, denoted by $\mathbf{a}$, and $a = |\mathbf{a}|$ represents its magnitude. The scalar product $\mathbf{a} \cdot \mathbf{b} = \mathrm{abcos}(\mathbf{a}, \mathbf{b})$ of two vectors $\mathbf{a}$ and $\mathbf{b}$. The vector product $\mathbf{a} \times \mathbf{b} = \mathbf{n}\mathrm{absin}(\mathbf{a}, \mathbf{b})$, in which $\mathbf{n}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so $|\mathbf{n}| = 1$. A more general definition on vector is given later.

© Science Press and Springer Science+Business Media Singapore 2016

T.-Y. Fan, *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*,
Springer Series in Materials Science 246, DOI 10.1007/978-981-10-1984-5_2

2.1.2 Coordinate Frame

To describe vector and tensor, it is convenient to introduce the coordinate frame. We will consider the orthogonal frame. Assume that $\mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$ are three unit vectors and mutually perpendicular, i.e. $\mathbf{e_1} \cdot \mathbf{e_2} = 0, \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \mathbf{e}_3 \cdot \mathbf{e}_1 = 0,$ and $\mathbf{e_3} = \mathbf{e_1} \times \mathbf{e_2}, \mathbf{e_2} = \mathbf{e_3} \times \mathbf{e_1}, \mathbf{e_1} = \mathbf{e_2} \times \mathbf{e_3}$, then they are base vectors of an orthogonal coordinate frame, and often called base vectors briefly.

In the orthogonal coordinate frame $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$, any vector $\mathbf{a}$ can be expressed by

$$\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e}_3 \tag{2.1.1}$$

or

$$\mathbf{a} = (a_\mathbf{1}, a_\mathbf{2}, a_\mathbf{3})$$

2.1.3 Coordinate Transformation

Consider another orthogonal frame $\mathbf{e}'_\mathbf{1}, \mathbf{e}'_\mathbf{2}, \mathbf{e}'_\mathbf{3}$ which can be expressed in terms of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. From (2.1.1), there are

$$\begin{aligned}
\mathbf{e}'_\mathbf{1} &= c_{11}\mathbf{e_1} + c_{12}\mathbf{e_2} + c_{13}\mathbf{e_3} \\
\mathbf{e}'_\mathbf{2} &= c_{21}\mathbf{e_1} + c_{22}\mathbf{e_2} + c_{23}\mathbf{e_3} \\
\mathbf{e}'_\mathbf{3} &= c_{31}\mathbf{e_1} + c_{32}\mathbf{e_2} + c_{33}\mathbf{e_3}
\end{aligned} \tag{2.1.2}$$

where $c_{11}, c_{12}, \ldots, c_{33}$ are some scalar constants. The relation (2.1.2) is named coordinate transformation too, which can also be denoted by matrix

$$\begin{bmatrix} \mathbf{e}'_\mathbf{1} \\ \mathbf{e}'_\mathbf{2} \\ \mathbf{e}'_\mathbf{3} \end{bmatrix} = [C] \begin{bmatrix} \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{e_3} \end{bmatrix} \tag{2.1.3}$$

where

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

which is an orthogonal matrix, consequently

$$[C]^T = [C]^{-1} \tag{2.1.4}$$

here notation "T" marks transpose operation and "− 1" the inversion operation. It is natural that

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = [C]^T \begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{bmatrix} = [C]^{-1} \begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{bmatrix} \tag{2.1.5}$$

Based on (2.1.1), if in the frame $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ then

$$\mathbf{a} = a'_1\mathbf{e}'_1 + a'_2\mathbf{e}'_2 + a'_3\mathbf{e}'_3 \tag{2.1.6}$$

Substituting (2.1.5) into (2.1.1) yields

$$\mathbf{a} = (c_{11}a_1 + c_{12}a_2 + c_{13}a_3)\mathbf{e}'_1 + (c_{21}a_1 + c_{22}a_2 + c_{23}a_3)\mathbf{e}'_2 + (c_{31}a_1 + c_{32}a_2 + c_{33}a_3)\mathbf{e}'_3 \tag{2.1.7}$$

It follows that by the comparison between (2.1.6) and (2.1.7)

$$\begin{aligned} a'_1 &= c_{11}a_1 + c_{12}a_2 + c_{13}a_3 \\ a'_2 &= c_{21}a_1 + c_{22}a_2 + c_{23}a_3 \\ a'_3 &= c_{31}a_1 + c_{32}a_2 + c_{33}a_3 \end{aligned} \tag{2.1.8}$$

or by the matrix expression, i.e.

$$\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = [C] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2.1.8$'$}$$

Whatever (2.1.8) or (2.1.8′), there is

$$a'_i = \sum_{j=1}^{3} c_{ij}a_j = c_{ij}a_j \tag{2.1.9}$$

The summation sign in the right-hand side of (2.1.9) is omitted, when the repeated indexes in $c_{ij}a_j$ represent summing. Henceforth, the summation convention will be used throughout.

A set of numbers (a_1, a_2, a_3) satisfying the relation (2.1.9) under linear transformation (2.1.2) will be called vector regardless its physical meaning. This is an algebraic definition of vector; it is more general than saying that the vector has both magnitude and direction.

2.1.4 Tensor

Let us define 9 numbers in the orthogonal frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{2.1.10}$$

in which the components satisfy the relation

$$A'_{kl} = \sum_{i,j=1}^{3} c_{ki} c_{lj} A_{ij} = c_{ki} c_{lj} A_{ij} \tag{2.1.11}$$

under the linear transformation, then $\mathbf{A}$ is a tensor of rank 2, where c_{ij} are given by (2.1.3) and the summing sign is omitted in the right-hand side of (2.1.11). It is evident that the concept of tensor is an extension of that of vector. According to the definition A_{ij} represents a tensor where i = 1, 2, 3, j = 1, 2, 3. It is understood that it represents a component with the indexes i and j of the tensor.

2.1.5 Algebraic Operation of Tensor

(i) Unit tensor

$$I = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{2.1.12}$$

which is named the Kronecker sign conventionally.

(ii) Transpose of tensor

$$A^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \tag{2.1.13}$$

(iii) Algebraic sum of tensors

$$A \pm B = A_{ij} \pm B_{ij} \tag{2.1.14}$$

(iv) Product of scalar and tensor

$$mA = mA_{ij} \tag{2.1.15}$$

(v) Product of tensors

$$AB = A_{ij}B_{kl} \tag{2.1.16}$$

Other operations about tensors will be provided in the description of the subsequent text.

2.2 Basic Assumptions of Theory of Elasticity

The theory of elasticity is a branch of continuum mechanics, it follows the basic assumptions thereof whereas there are:

(i) Continuity
In the theory one assumes that the medium fills the full space that it occupies, and this means the medium is continuous. Connected with this, the field variables concerning the medium are continuous and differentiable functions of coordinates.

(ii) Homogeneity
Physical constants describing the medium are independent from coordinates, so the medium is homogeneous.

(iii) Small deformation
Assume that displacements u_i are small and $\partial u_i/\partial x_j$ are less than unity. Due to small deformation, the boundary conditions are written at the boundaries before deformation though those boundaries have taken some deformation. This makes the problems linearized and simplifies the solution procedure.

2.3 Displacement and Deformation

That elastic body exhibiting deformation is connected to the relative displacement between points in it. So we first look for the displacement field.

Consider a region R in an elastic body, refer to Fig. 2.1, it becomes another region R' after deformation. The point O with radius vector **r**,

before deformation, which becomes point O' with radius vector $\mathbf{r}'$ after deformation, and **u** is the displacement vector of point O during the deformation process (see Fig. 2.1), i.e.

$$\mathbf{r}' = \mathbf{r} + \mathbf{u} \tag{2.3.1}$$

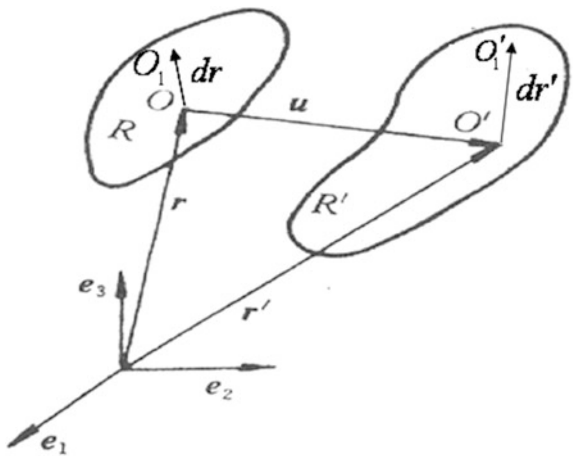

Fig. 2.1 Displacement of a point in an elastic body

or

$$\mathbf{u} = \mathbf{r}' - \mathbf{r} = x_i' - x_i \tag{2.3.1$'$}$$

In Fig. 2.1, frame $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ depicts any orthogonal coordinate system, especially we use the rectilinear coordinate system (x_1, x_2, x_3) or (x, y, z). Assume that O_1 in R is a point near the point O, the radius vector joining them is $\mathrm{d}\mathbf{r} = \mathrm{d}x_i$. The point O_1 becomes point O_1' in R' after deformation. The vector radius joining points O_1 and point O_1' is $\mathrm{d}\mathbf{r}' = \mathrm{d}x_i' = \mathrm{d}x_i + \mathrm{d}u_i$. The displacement of point O_1 is $\mathbf{u}'$, there is

$$\mathbf{u}' = \mathbf{u} + d\mathbf{u} \tag{2.3.2}$$

i.e.

$$\mathrm{d}u_i = u_i' - u_i \tag{2.3.3}$$

and

$$\mathrm{d}u_i = \frac{\partial u_i}{\partial x_j}\mathrm{d}x_j \tag{2.3.4}$$

Equation (2.3.4) expresses the Taylor expansion at point O and takes the first-order term only. Under the small deformation assumption, this reaches a very high accuracy. It denotes

$$\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij} \tag{2.3.5}$$

in which

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.3.6}$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.3.7}$$

here ε_{ij} is a symmetric tensor

$$\varepsilon_{ij} = \varepsilon_{ji} \tag{2.3.8}$$

and called the strain tensor, while ω_{ij} an asymmetric tensor, which has only three independent components

$$\begin{aligned} \Omega_x &= \omega_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) \\ \Omega_y &= \omega_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) \\ \Omega_z &= \omega_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) \end{aligned} \tag{2.3.9}$$

The physical meaning of ε_{ij} describes volume and shape changes of a cell, and that of ω_{ij} the rigid rotation, which is independent on deformation, so only ε_{ij} is considered afterwards.

The components $\varepsilon_{11}, \varepsilon_{22}$ and ε_{33} (if denote $x = x_1, y = x_2, z = x_3$, then we have $\varepsilon_{xx}, \varepsilon_{yy}$ and ε_{zz}) represent normal strains describing volume change of a cell, while $\varepsilon_{32} = \varepsilon_{23}, \varepsilon_{13} = \varepsilon_{31}$ and $\varepsilon_{12} = \varepsilon_{21}$ (or $\varepsilon_{yz} = \varepsilon_{zy}, \varepsilon_{zx} = \varepsilon_{xz}$ and $\varepsilon_{xy} = \varepsilon_{yx}$) represent shear strains describing shape change of a cell.

2.4 Stress Analysis

The internal forces per unit area due to deformation are called stresses, and denoted by σ_{ij}, which will be zero if there is no deformation for a body. When the body is in static equilibrium according to the law of momentum conservation, we have

$$\frac{\partial \sigma_{ij}}{\partial x_i} + f_j = 0 \tag{2.4.1}$$

in which the equation holds for any infinitesimal volume element of the body, and σ_{ij} represents the components of the stress tensor as mentioned above, and suffix j the acting direction of the component, i the direction of outward normal vector of the surface element that the component exerted and f_i the body force density vector. Amongst all, the components of σ_{ij}, σ_{xx}, σ_{yy} and σ_{zz} are normal to the surface elements which they exerted, and $\sigma_{yz}, \sigma_{zy}, \sigma_{zx}, \sigma_{xz}, \sigma_{xy}$ and σ_{yx} are along the tangent directions of the surface elements, the former are called normal stresses, and the latter shear stresses.

According to the angular momentum conservation, one finds that

$$\sigma_{ij} = \sigma_{ji} \tag{2.4.2}$$

this means the stress tensor is a symmetric tensor and (2.4.2) is named as the shear stress mutual equal law.

External surface forces density (tractions) T_i subjected to the surface of a body should be balanced with the internal stresses, this leads to

$$\sigma_{ij} n_j = T_i \tag{2.4.3}$$

where n_j is the unit vector along the outward normal to the surface element. People also call T_i area force density.

Equation (2.4.3) describes the stress boundary conditions which play a very important role for elasticity.

2.5 Generalized Hooke's Law

Between stresses σ_{ij} and strains ε_{ij}, there exists a certain relationship depending upon the material behaviour of the body. Hereafter, we consider only the linear elastic behaviour of materials and the state without initial stresses. In the case, the classical experimental law–Hooke's law can be extended as

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = C_{ijkl} \varepsilon_{kl} \tag{2.5.1}$$

in which U denotes the free energy density, or the strain energy density, i.e.

$$U = F = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \tag{2.5.2}$$

and C_{ijkl} is the elastic constant tensor, consisting of 81 components. Due to the symmetry of σ_{ij} and ε_{ij}, each of them has independent 6 components only, such that the independent components of C_{ijkl} reduce to 36. Formula (2.5.2) shows that U is a homogenous quantity of ε_{ij} of rank two, considering the symmetry of ε_{ij}, then we have

$$C_{ijkl} = C_{klij} \tag{2.5.3}$$

so the independent components amongst 36 reduce to 21.

The relation (2.5.1) with 21 independent elastic constants is named as generalized Hooke's law.

The generalized Hooke's law describes anisotropic elastic bodies including crystals. Stress and strain tensors can also be expressed by corresponding vectors with 6 independent elements, then can be denoted by corresponding elastic constants matrix $[B_{ijkl}]$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} B_{1111} & B_{1122} & B_{1133} & B_{1123} & B_{1131} & B_{1112} \\ & B_{2222} & B_{2233} & B_{2223} & B_{2231} & B_{2212} \\ & & B_{3333} & B_{3323} & B_{3331} & B_{3312} \\ & & & B_{2323} & B_{2331} & B_{2312} \\ & (symmetry) & & & B_{3131} & B_{3112} \\ & & & & & B_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix} \tag{2.5.4}$$

Applying formula (2.5.4) to crystals, between the elements C_{ijkl} (or B_{ijkl}), there are some relations by considering certain symmetry of the crystals, so that the resulting number of the elastic constants for certain individual crystal systems may be less than 21. In the following, we give a brief discussion on the argument.

1. Triclinic system (classes 1 or C_1 and C_i)
   The triclinic symmetry does not add any restrictions to the components of tensor C_{ijkl} (or B_{ijkl} in (2.5.4)); however, appropriate choice of the coordinate system enables us to reduce the number of nonzero independent elastic constants. Because the orientation of the coordinate system is determined by three rotation angles, this provides three conditions to restrict some components in C_{ijkl} (or B_{ijkl} in (2.5.4)); for example, one can take three of them to be zero, such that, the triclinic crystal system has 18 components of elastic moduli.
2. Monoclinic system (classes C_s, C_2 and C_{2h})
   In the class C_s, there is a plane of symmetry, we take it as $x_3 = 0$ $(z = 0)$ in coordinate frame $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$. Making a coordinate transformation with this plane of symmetry one can obtain a new coordinate frame $\mathbf{e}'_\mathbf{1}, \mathbf{e}'_\mathbf{2}, \mathbf{e}'_\mathbf{3}$. Between these two coordinate frames, there are relations

$$\mathbf{e}'{}_1 = \mathbf{e}_1, \mathbf{e}'_\mathbf{2} = \mathbf{e}_2, \mathbf{e}'_\mathbf{3} = -\mathbf{e}_3 \tag{2.5.5}$$

   This operation is the reflection or mapping. In addition, we know that between σ'_{ij} in $\mathbf{e}'_\mathbf{1}, \mathbf{e}'_\mathbf{2}, \mathbf{e}'_\mathbf{3}$ and σ_{ij} in $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$, there are (refer to §2.1)

$$\sigma'_{kl} = \alpha_{kj}\alpha_{li}\sigma_{ji} \tag{2.5.6}$$

   in which α_{ij} are the coefficients of linear transformation, i.e.

$$\mathbf{e}'_i = \alpha_{ij}\mathbf{e}_j \tag{2.5.7}$$

Under the transformation (2.5.5), there are

$$\alpha_{11} = 1, \alpha_{22} = 1, \alpha_{33} = -1, \quad others = 0 \tag{2.5.8}$$

Therefore, under the transformation, for C_{ijkl} in (2.5.1) (or B_{ijkl} in (2.5.4)) whose suffixes containing 3 with an odd number of times (1 or 3) will change sign, while the others will remain invariant. Considering the symmetry of the crystal, however, the physical properties including C_{ijkl} (or B_{ijkl} in (2.5.4)) should remain unchanged under symmetric operation (including the reflection). So it is obvious that all components with an odd number of suffixes 3 must vanish, i.e.

$$B_{1123} = B_{1131} = B_{2223} = B_{2231} = B_{3323} = B_{3331} = B_{2312} = B_{3112} = 0 \tag{2.5.9}$$

Consequently, there are only 13 independent elastic constants.
A similar discussion can be done for the classes C_2 and C_{2h}.

3. Orthorhombic system (classes C_{2v}, D_2 and D_{2h})
This crystal system has a macroscopic corresponding, i.e. the orthotropic materials, in which there exist two planes of symmetry perpendicular to each other. Let us take $x_3 = 0$ and $x_1 = 0$ as the planes. If on reflection in plane $x_3 = 0$, it is just the case for monoclinic system mentioned above. Subsequently, considering the mapping in plane $x_1 = 0$, between the new and old coordinate systems, there is the relation such as

$$\begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

By a similar description to that for monoclinic system, one finds that

$$B_{1112} = B_{2212} = B_{3312} = B_{2331} = 0 \tag{2.5.10}$$

Collecting to (2.5.9), the system contains 9 independent elastic constants.

4. Tetragonal system (classes C_{4v}, D_{2d}, D_4 and D_{4h})
This crystal system has 4 axes of symmetry. Similar to previous discussion, those independent elastic moduli are

$$B_{1111}, B_{3333}, B_{1122}, B_{1212}, B_{1133}, B_{1313}$$

The total number of these is six.

5. Rhombohedral system (classes C_{3v}, 3 or C_3, D_3, D_{3d} and S_6)
In this system, there is a third-order axis of symmetry (or threefold symmetric axis). We can take axis of symmetry as the axis $\mathbf{e_3}$, after a lengthy description that six independent elastic constants are as follows:

$$B_{3333}, B_{\xi\eta\xi\eta}, B_{\xi\xi\eta\eta}, B_{\xi\eta33}, B_{\xi3\eta3}, B_{\xi\xi\xi3}$$

with

$$\xi = x_1 + ix_2, \eta = x_1 - ix_2$$

The moduli can also be written in conventional version as

$$B_{3333}, B_{1212}, B_{1122}, B_{1233}, B_{1323}, B_{1113}$$

6. Hexagonal system (class C_6)
   The crystal system has a macroscopic correspondence—the transverse isotropic material, whose elasticity presents fundamental importance to elasticity of one- and two-dimensional quasicrystals.
   There is a sixth-order axis of symmetry (or say sixfold symmetric axis) in the system. By taking this axis as x_3-axis and using the coordinate substitution $\xi = x_1 + ix_2, \eta = x_1 - ix_2$. In a rotation with angle $2\pi/6$ about the x_3-axis, the coordinates ξ and η are experienced a transformation $\xi \rightarrow \xi e^{i2\pi/6}$, $\eta \rightarrow \eta e^{-i2\pi/6}$. Then one can see that only those components C_{ijkl} do not vanish which have the same number of suffixes ξ and η. These are

$$B_{3333}, B_{\xi\eta\xi\eta}, B_{\xi\xi\eta\eta}, B_{\xi\eta33}, B_{\xi3\eta\xi}$$

   or in conventional expressions

$$C_{1111} = C_{2222}, C_{3333}, C_{2323} = C_{3131}, C_{1122},$$
$$C_{1133} = C_{2233}, C_{1212}$$

   in which $2C_{1212} = C_{1111} - C_{1122}$, so the number of independent elastic constants is five.
7. Cubic system
   For this system there are 3 fourfold symmetric axes, in which there is tetragonal symmetry. If taking the fourfold symmetric axis of the tetragonal symmetry in the x_3-direction, the number of independent components of C_{ijkl} (or B_{ijkl} in (2.5.4)) are $B_{1111}, B_{1122}, B_{1212}$
8. Isotropic body
   In this case there are two elastic moduli, e.g. the Young's modulus and Poisson's ratio

$$E, v$$

   respectively, or the Lamé constants

$$\lambda = \frac{vE}{(1+v)(1-2v)}, \mu = \frac{E}{2(1+v)} \tag{2.5.11}$$

or the bulk modulus of compression and shear modulus $K = \frac{E}{3(1-2v)}, \mu = \frac{E}{2(1+v)} = G$

In this case the generalized Hooke's law presents very simple form, i.e.

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} \tag{2.5.12}$$

where $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}, \delta_{ij}$ is the unit tensor. An equivalent form of (2.5.12) is

$$\varepsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij} \tag{2.5.13}$$

in which $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$

2.6 Elastodynamics, Wave Motion

When the inertia effect is considered in (2.4.1), then it becomes

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{2.6.1}$$

where ρ is the mass density of the material.

Considering isotropic medium and omitting body forces from (2.6.1), (2.3.6) and (2.5.12), the equations of wave motion are obtained as

$$(c_1^2 - c_2^2)\frac{\partial^2 u_i}{\partial x_i \partial x_j} + c_2^2 \frac{\partial^2 u_j}{\partial x_i^2} = \frac{\partial^2 u_j}{\partial t^2} \tag{2.6.2}$$

where c_1 and c_2 defined by

$$c_1 = \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}}, c_2 = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \tag{2.6.3}$$

which are speeds of elastic longitudinal and transverse waves, respectively. If put

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi} \tag{2.6.4}$$

then (2.6.2) can be reduced to

$$\nabla^2\phi = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2}, \nabla^2\boldsymbol{\psi} = \frac{1}{c_2^2}\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} \tag{2.6.5}$$

where ϕ is scalar potential, $\boldsymbol{\psi}$ the vector potential and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, (2.6.5) are typical wave equations of mathematical physics. To solve the problem apart from the boundary conditions one needs initial conditions, i.e.

$$\begin{aligned} &u_i(x_i, 0) = u_{i0}(x_i) \\ &\qquad\qquad\qquad\qquad x_i \in \Omega \\ &\dot{u}_i(x_i, 0) = \dot{u}_{i0}(x_i). \end{aligned}$$

2.7 Summary

The classical theory of elasticity is concluded to solve the following initial-boundary value problem

$$\begin{aligned} &\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ &\frac{\partial \sigma_{ij}}{\partial x_i} = \rho\frac{\partial^2 u_j}{\partial t^2} - f_j, (t > 0, x_i \in \Omega) \\ &\sigma_{ij} = C_{ijkl}\varepsilon_{ijkl} \\ &u_i(x_i, 0) = u_{i0}(x_i) \\ &\dot{u}_i(x_i, 0) = \dot{u}_{i0}(x_i), (x_i \in \Omega) \\ &\sigma_{ij}n_j = T_i, t > 0, x_i \in S_t \\ &u_i = \bar{u}_i, t > 0, x_i \in S_u \end{aligned}$$

where $u_{i0}(x_i), \dot{u}_{i0}(x_i), T_i$ and $\bar{u}_i$ are known functions, Ω denotes the region of materials we studied, S_t and S_u are parts of boundary S on which the tractions and displacements are prescribed, respectively, and $S = S_t + S_u$. If $\frac{\partial^2 u_j}{\partial t^2} = 0$, the problem reduces to a static problem as pure-boundary value problem, there are no initial conditions at all.

References

1. Landau L D and Lifshitz E M, 1986, *Theoretical Physics V: Theory of Elasticity*, Pergamon Press, Oxford.
2. Born M and Huang K, 1954, *Dynamic Theory of Crystal Lattices*, Clarendon Press, Oxford.
3. Anderson P W, 1984, *Basic Notations of Condensed Matter Physics*, Benjamin-Cummings, Menlo Park.

http://www.springer.com/978-981-10-1982-1

Mathematical Theory of Elasticity of Quasicrystals and
Its Applications
Fan, T.-Y.
2016, XVI, 452 p. 103 illus., 40 illus. in color., Hardcover
ISBN: 978-981-10-1982-1