Special Conformal Transformations and Contact Terms

Loriano Bonora

Abstract In this contribution I construct the Ward identity of special conformal transformations in momentum space and discuss some of its consequences on conformal field theory correlators. I show a few examples of covariant correlators in dimension 2 and 3 dimensions and in particular of those made of pure contact terms. I discuss in some detail the odd parity correlator in 3d and its connection with the gravitational Chern–Simons theory in 3d.

1 Introduction

Correlators in conformal field theories can be formulated both in configuration space and, via Fourier transform, in momentum space. In the first form they may happen to be singular at coincident insertion points and in need of regularization. In coordinate space they are therefore simply distributions. In the simplest cases such distributions have been studied and can be found in textbooks. But in general the correlators of CFT are very complicated expressions and their regularization has to be carried out from scratch. It is often convenient to do it in momentum space, [1] via Fourier transform, and regularize the Fourier transform of the relevant correlators. This procedure produces various types of terms, which we refer to as *non-local, partially local and local terms*. Local terms, a.k.a contact terms, are represented by polynomials of the external momenta in momentum space, or by delta functions and derivatives of delta functions in configuration space. The unregularized correlators will be referred to as *bare* correlators; they are ordinary regular functions at non-coincident points and are classified as non-local in the previous classification. While regularizing the latter one usually produces not only local terms, but also intermediate ones, which are product of bare functions and delta functions or derivatives thereof. These are referred to as partially local.

L. Bonora (✉)
International School for Advanced Studies (SISSA), Trieste and INFN, Sezione di Trieste,
Via Bonomea 265, 34136 Trieste, Italy
e-mail: bonora@sissa.it

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_2

Many general results are known nowadays about bare correlators in CFT, [2, 3]. But a complete analysis of the contact terms permitted by conformal symmetry in various dimensions is still lacking. In this contribution I would like to argue that such an analysis is possible and can be conveniently carried out in momentum space. The basic tool for this analysis is the special conformal transformation Ward identity in momentum space. The paper is intended to be an introduction to the subject and is mostly pedagogical. I start with some basic definitions about the conformal algebra in momentum space. Then I formulate the Ward Identities of special conformal transformations in momentum space and their consistency conditions, which lead to the corresponding cohomology, or K-cohomology. Finally I show a few examples of covariant correlators in 2 and 3 dimensions and in particular those made of pure contact terms. I discuss in some detail the odd parity correlator in 3d and its connection with the gravitational Chern–Simons theory in 3d.

2 The Conformal Algebra and SCT's

In this section we briefly introduce the conformal transformations in d dimensions, in particular the special conformal (SCT) ones, which are the main subject of this presentation. The conformal group is made of the usual Poincaré transformation plus dilatations $x^\mu \to \lambda x^\mu$, with generator D, and special conformal transformations with generator K_μ. A special conformal transformation (SCT)

$$x^{'\mu} = \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2 x^2} \approx x^\mu + b^\mu x^2 - 2b\cdot x\, x^\mu,$$

for b^μ small, can be seen as a diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ where $\xi^\mu = b^\mu x^2 - 2b\cdot x\, x^\mu$. Introducing a metric $\eta_{\mu\nu}$, this implies a transformation $\eta_{\mu\nu} \to \eta_{\mu\nu} + \delta_\xi \eta_{\mu\nu}$, where

$$\delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = -4b\cdot x\, \eta_{\mu\nu} \tag{1}$$

which is a Weyl rescaling. On the other hand the square line element

$$dx^{'2} \to x^2\,(1 - 4b\cdot x)$$

which confirms that SFT's are Weyl rescaling, because this can be viewed as a transformation $\eta_{\mu\nu} \to \eta_{\mu\nu}(1 - 4b\cdot x)$.

The conformal generators are

$$\begin{aligned}
P_\mu &= -i\partial\mu \\
D &= -ix^\mu\,\partial\mu \\
L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial\mu)
\end{aligned}$$

They form the Lie algebra

$$
\begin{aligned}
&[L_{\mu\nu}, L_{\lambda\rho}] = i\left(\eta_{\mu\lambda}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\lambda} - \eta_{\nu\lambda}L_{\mu\rho} + \eta_{\nu\rho}L_{\mu\lambda}\right)\\
&[P^{\mu}, P^{\nu}] = 0\\
&[L_{\mu\nu}, P_{\lambda}] = i\left(\eta_{\mu\lambda}P_{\nu} - \eta_{\nu\lambda}P_{\mu}\right)\\
&[P^{\mu}, D] = iP^{\mu}\\
&[K^{\mu}, D] = -iK^{\mu}\\
&[P^{\mu}, K^{\nu}] = 2i\eta^{\mu\nu}D + 2iL^{\mu\nu}\\
&[K^{\mu}, K^{\nu}] = 0\\
&[L^{\mu\nu}, D] = 0\\
&[L^{\mu\nu}, K^{\lambda}] = i\eta^{\lambda\mu}K^{\nu} - i\eta^{\lambda\nu}K^{\mu}
\end{aligned}
\tag{2}
$$

which is isomorphic to the Lie algebra of SO(d,2).

2.1 Momentum Space Algebra

If we Fourier transform the generators of the conformal algebra we get (a tilde represents the transformed generator and $\tilde{\partial} = \frac{\partial}{\partial k}$)

$$
\begin{aligned}
&\tilde{P}_{\mu} = -k_{\mu}\\
&\tilde{D} = i(d + k^{\mu}\tilde{\partial}_{\mu})\\
&\tilde{L}_{\mu\nu} = i(k_{\mu}\tilde{\partial}_{\nu} - k_{\nu}\tilde{\partial}_{\mu})\\
&\tilde{K}_{\mu} = 2d\,\tilde{\partial}_{\mu} + 2k_{\nu}\tilde{\partial}^{\nu}\tilde{\partial}_{\mu} - k_{\mu}\tilde{\Box}
\end{aligned}
$$

Notice that $\tilde{P}_{\mu}$ is a multiplication operator and $\tilde{K}_{\mu}$ is a quadratic differential operator. The Leibniz rule does not hold for $\tilde{K}_{\mu}$ and $\tilde{P}_{\mu}$ with respect to the ordinary product. However it does hold for the convolution product:

$$
\tilde{K}_{\mu}(\tilde{f} \star \tilde{g}) = (\tilde{K}_{\mu}\tilde{f}) \star \tilde{g} + \tilde{f} \star (\tilde{K}_{\mu}\,\tilde{g})
$$

where $(\tilde{f} \star \tilde{g})(k) = \int dp\, f(k-p)g(p)$.

Nevertheless these generators form a closed algebra under commutation

$$
\begin{aligned}
&[\tilde{D}, \tilde{P}_{\mu}] = i\tilde{P}_{\mu}\\
&[\tilde{D}, \tilde{K}_{\mu}] = i\tilde{K}_{\mu}\\
&[\tilde{K}_{\mu}, \tilde{K}_{\nu}] = 0\\
&[\tilde{K}_{\mu}, \tilde{P}_{\nu}] = i(\eta_{\mu\nu}\tilde{D} - \tilde{L}_{\mu\nu})\\
&[\tilde{K}_{\lambda}, \tilde{L}_{\mu\nu}] = i(\eta_{\lambda\mu}K_{\nu} - \eta_{\lambda\nu}K_{\mu})
\end{aligned}
$$

$$[\tilde{P}_\lambda, \tilde{L}_{\mu\nu}] = i(\eta_{\lambda\mu} P_\nu - \eta_{\lambda\nu} P_\mu)$$
$$[\tilde{L}_{\mu\nu}, \tilde{L}_{\lambda\rho}] = i(\eta_{\nu\lambda}\tilde{L}_{\mu\rho} + \eta_{\mu\rho}\tilde{L}_{\nu\lambda} - \eta_{\mu\lambda}\tilde{L}_{\nu\rho} - \eta_{\nu\rho}\tilde{L}_{\mu\lambda}$$

One should however remember that they do not generate infinitesimal transformation in momentum space.

Our purpose is to use this formulation in momentum space to study the cohomology of SCT's, referred to as K-cohomology, and in particular the polynomial K-cohomology. As explained in the introduction polynomials in momentum space represent contact terms in field theory and the latter are important in two respects, as action terms and as anomalies. To arrive at the cohomology corresponding to a given symmetry one needs the Ward identities of that symmetry. So the next step is to formulate the Ward identities of SCT's (the WI's of the scaling transformation is rather trivial and is understood to be always satisfied).

2.2 Ward Identities for SCT's

Since currents and energy-momentum tensor will play the main role in the sequel, we start with their transformation properties under SCT's

$$i[K_\lambda, J_\mu] = \big(2(d-1)x_\lambda + 2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\big)\, J_\mu + 2\left(x^\alpha J_\alpha \eta_{\lambda\mu} - x_\mu J_\lambda\right) \quad (3)$$

$$\begin{aligned} i[K_\lambda, T_{\mu\nu}] = &\big(2dx_\lambda + 2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\big)\, T_{\mu\nu} \\ &+ 2\left(x^\alpha T_{\alpha\nu}\eta_{\lambda\mu} + x^\alpha T_{\mu\alpha}\eta_{\lambda\nu} - x_\mu T_{\lambda\nu} - x_\nu T_{\mu\lambda}\right) \end{aligned} \quad (4)$$

In momentum representation they are given by

$$\tilde{K}_\mu \tilde{J}_\lambda(k) = (-2\tilde{\partial}_\mu - 2k\cdot\tilde{\partial}\,\tilde{\partial}_\mu + k_\mu\tilde{\Box})\tilde{J}_\lambda + 2(\tilde{\partial}^\alpha \tilde{J}_\alpha \eta_{\mu\lambda} - \tilde{\partial}_\lambda \tilde{J}_\mu) \quad (5)$$

$$\begin{aligned} \tilde{K}_\mu \tilde{T}_{\lambda\rho}(k) = &(-2k\cdot\tilde{\partial}\,\tilde{\partial}_\mu + k_\mu\tilde{\Box})\tilde{T}_{\lambda\rho} \\ &+ 2(\tilde{\partial}^\alpha \tilde{T}_{\alpha\rho}\eta_{\mu\lambda} - \tilde{\partial}_\lambda \tilde{T}_{\mu\rho} + \tilde{\partial}^\alpha \tilde{T}_{\lambda\alpha}\eta_{\mu\rho} - \tilde{\partial}_\rho \tilde{T}_{\lambda\mu}) \end{aligned} \quad (6)$$

where $\tilde{T}_{\mu\nu}(k)$, $\tilde{J}_\mu(k)$ denote the Fourier transforms of $T_{\mu\nu}(x)$, $J_\mu(x)$, respectively.

In order to formulate Ward Identities (WI) on correlators let us couple $T_{\mu\nu}$ to an external source $h_{\mu\nu}$ (this will eventually be identified with the background metric fluctuation: $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$), [6]. The generating function of connected Green functions is

$$W[h_{\mu\nu}] = \sum_{n=1}^{\infty} \frac{i^{n+1}}{2^n n!} \int \prod_{i=1}^{n} dx_i\;\, h^{\mu_i\nu_i}(x_i)\, \langle 0|\mathcal{T}\{T_{\mu_1\nu_1}(x_1)\ldots T_{\mu_n\nu_n}(x_n)\}|0\rangle_c,$$

In order for W to be invariant under SCT's the external source $h_{\mu\nu}$ must transform as $\delta_b h_{\mu\nu} = [b^\lambda K_\lambda(x), h_{\mu\nu}(x)] \equiv [b\cdot K(x), h_{\mu\nu}(x)]$, where

$$\begin{aligned} & i[K_\lambda(x), h_{\mu\nu}(x)] \\ & = \left(2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\right) h_{\mu\nu} + 2\left(x^\alpha h_{\alpha\nu}\eta_{\lambda\mu} + x^\alpha h_{\mu\alpha}\eta_{\lambda\nu} - x_\mu h_{\lambda\nu} - x_\nu h_{\mu\lambda}\right) \end{aligned} \tag{7}$$

Invariance of $W[h]$ leads to

$$\begin{aligned} 0 = \delta_b W &= \int d^d x \frac{\delta W}{\delta h^{\mu\nu}} \delta h^{\mu\nu} = \int d^d x\, [b\cdot K, h^{\mu\nu}(x)] \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle \\ &= -\int d^d x\, h^{\mu\nu}(x) [b\cdot K, \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle] = 0 \end{aligned} \tag{8}$$

where

$$\begin{aligned} \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle = 2\frac{\delta W[h]}{\delta h^{\mu\nu}(x)} &= \frac{1}{n!}\sum_{n=1}^{\infty} \int dx_1 \ldots \int dx_n h^{\mu_1\nu_1}(x_1)\ldots h^{\mu_n\nu_n}(x_n) \\ &\times \langle 0|\mathcal{T}\{T_{\mu_1\nu_1}(x_1)\ldots T_{\mu_n\nu_n}(x_n)\}|0\rangle_c \end{aligned} \tag{9}$$

Differentiating twice (8) with respect $h_{\mu\nu}$ and integrating by parts we get

$$(b\cdot K(x) + b\cdot K(y))\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y)\rangle|0\rangle = 0 \tag{10}$$

Differentiating three times (8)

$$(b\cdot K(x) + b\cdot K(y) + b\cdot K(z))\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(z)\rangle|0\rangle = 0 \tag{11}$$

In both equations it is understood that the Lorentz part of $b\cdot K(x)$ acts on the indices $\mu\nu$ only, $b\cdot K(y)$ on the indices $\lambda\rho$ and $b\cdot K(z)$ on $\alpha\beta$ alone.

Due to translational invariance we can set $y = 0$ in (10) and $z = 0$ in (11). These equations become

$$b\cdot K(x)\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(0)\rangle|0\rangle = 0 \tag{12}$$

and

$$(b\cdot K(x) + b\cdot K(y))\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(0)\rangle|0\rangle = 0 \tag{13}$$

In these equations $K_\mu(\cdot)$ is understood as the differential operator at the RHS's of (3), (4). So far the results are classical. But we know that a SCT produces a conformal factor $\sim b\cdot x$. Therefore the RHS of (10) and (11) may no vanish if we take the trace of the e.m. tensor:

$$\begin{aligned} &(b\cdot K(x) + b\cdot K(y))\langle 0|\mathcal{T} T^\mu_\mu(x) T_{\lambda\rho}(y)\rangle|0\rangle = \mathcal{A}_{\lambda\rho}(x, y) \\ &(b\cdot K(x) + b\cdot K(y) + b\cdot K(z))\langle 0|\mathcal{T} T^\mu_\mu(x) T_{\lambda\rho}(y) T_{\alpha\beta}(z)\rangle|0\rangle = \mathcal{A}_{\lambda\rho\alpha\beta}(x, y, z) \end{aligned} \tag{14}$$

The RHS's are linear in b. They are unintegrated anomalies. Using translational invariance we can set

$$b\cdot K(x)\langle 0|\mathcal{T}T^{\mu}_{\mu}(x)T_{\lambda\rho}(0)\rangle|0\rangle = \mathcal{A}_{\lambda\rho}(x) \tag{15}$$

$$(b\cdot K(x)+b\cdot K(y))\langle 0|\mathcal{T}T^{\mu}_{\mu}(x)T_{\lambda\rho}(y)T_{\alpha\beta}(0)\rangle|0\rangle = \mathcal{A}_{\lambda\rho\alpha\beta}(x,y) \tag{16}$$

As is well known the above anomalies have to satisfy consistency conditions, which we are going to derive next.

Coupling the current $J_{\mu}(x)$ with a background gauge field $A^{\mu}(x)$, it is easy to derive similar WI's also for current correlators.

2.3 Consistency Conditions

Let us start again from $W[h]$ and perform two SCT's on a row. We get

$$\begin{aligned}
\delta_{b_2}\delta_{b_1}W &= \delta_{b_2}\int d^dx\,\frac{\delta W}{\delta h^{\mu\nu}(x)}\delta_{b_1}h^{\mu\nu}(x)\\
&= \int d^dy\int d^dx\,\left\{\frac{\delta^2 W}{\delta h^{\mu\nu}(x)\delta h^{\lambda\rho}(y)}\delta_{b_1}h^{\mu\nu}(x)\delta_{b_2}h^{\lambda\rho}(y)\right.\\
&\quad\left.+\frac{\delta W}{\delta h^{\mu\nu}(x)}\frac{\delta\delta_{b_1}h^{\mu\nu}(x)}{\delta h^{\lambda\rho}(y)}\delta_{b_2}h^{\lambda\rho}(y)\right\}\\
&= \int d^dy\int d^dx\,\left\{[b_1\cdot K(x),h^{\mu\nu}(x)][b_2\cdot K(y),h^{\lambda\rho}(y)]\frac{\delta^2 W}{\delta h^{\mu\nu}(x)\delta h^{\lambda\rho}(y)}\right.\\
&\quad\left.+\frac{\delta W}{\delta h^{\mu\nu}(x)}[b_1\cdot K(x),\delta(x-y)][b_2\cdot K(y),h^{\mu\nu}(y)]\right\}\\
&= \int d^dy\int d^dx\,\left\{[b_1\cdot K(x),h^{\mu\nu}(x)][b_2\cdot K(y),h^{\mu\nu}(y)]\frac{\delta^2 W}{\delta h^{\mu\nu}(x)\delta h^{\lambda\rho}(y)}\right.\\
&\quad\left.-\left[b_1\cdot K(x),\frac{\delta W}{\delta h^{\mu\nu}(x)}\right]\delta(x-y)\left[b_2\cdot K(y),h^{\mu\nu}(y)\right]\right\}
\end{aligned}$$

after integration by parts in x. Integrating over y and integrating again by parts one finally gets

$$\begin{aligned}
\delta_{b_2}\delta_{b_1}W = \int d^dy\int d^dx\,&\left\{[b_1\cdot K(x),h^{\mu\nu}(x)][b_2\cdot K(y),h^{\mu\nu}(y)]\ \times\right.\\
&\left.\times\frac{\delta^2 W}{\delta h^{\mu\nu}(x)\delta h^{\lambda\rho}(y)}\ +\ \left[b_1\cdot K(x),\left[b_2\cdot K(x),h^{\mu\nu}(x)\right]\right]\frac{\delta W}{\delta h^{\mu\nu}(x)}\right\}
\end{aligned} \tag{17}$$

Making the transformations in reverse order and taking the difference one gets

$$0 = \delta_{b_2}\delta_{b_1} W - \delta_{b_1}\delta_{b_2} W = \int d^d x\, h^{\mu\nu}(x) \left\{ \left[b_1\cdot K(x), \left[b_2\cdot K(x), \frac{\delta W}{\delta h^{\mu\nu}(x)} \right]\right] - \left[b_2\cdot K(x), \left[b_1\cdot K(x), \frac{\delta W}{\delta h^{\mu\nu}(x)} \right]\right] \right\} \tag{18}$$

This is equivalent to promoting b to an anticommuting parameter and writing

$$\int d^d x\, h^{\mu\nu}(x) \left[b\cdot K(x), \left[b\cdot K(x), \frac{\delta W}{\delta h^{\mu\nu}(x)} \right]\right] = 0 \tag{19}$$

In fact differentiating (18) with respect to b_1^μ and b_2^ν and (19) first with respect to b^μ and then wrt to b^ν one gets the same result. From now on we will use the second formulation, i.e. b anticommuting.

Differentiating (19) wrt to h several times one gets the consistency conditions for (10) and (11). For instance

$$b\cdot K(x)\, b\cdot K(x) \langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y)\rangle|0\rangle \; + \\ + \, b\cdot K(y)\, b\cdot K(y) \langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y)\rangle|0\rangle = 0$$

The RHS is strictly 0 even in the quantum theory. Due to translational invariance we can rewrite this equation as

$$b\cdot K(x)\, b\cdot K(x) \langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(0)\rangle|0\rangle = 0 \tag{20}$$

and (14) becomes the consistency condition

$$b\cdot K(x)\, \mathcal{A}_{\lambda\rho}(x) = b\cdot K(x)\, \mathcal{A}_{\lambda\rho}(x) = 0 \tag{21}$$

We can Fourier transform this equation and obtain

$$b\cdot \tilde{K}(k)\, \tilde{\mathcal{A}}_{\lambda\rho}(k) = 0 \tag{22}$$

where $\tilde{K}(k)$ is given by Eq. (6).

3 Examples

We consider now a few simple examples of the approach outlined above. Here we limit ourselves 0-cocycles (i.e. invariants) of the K-cohomology. The analysis of 1-cocycles, i.e. anomalies, requires additional tools and will not be considered here.

In momentum representation the CFT correlators must be annihilated by $b\cdot\tilde{K}$. For instance the 2-pt function of a scalar field of weight Δ is $\sim (k^2)^{\Delta-\frac{d}{2}}$ and

$$\tilde{K}_\mu (k^2)^{\Delta-\frac{d}{2}} = (2\Delta - d)\cdot 0 \cdot (k^2)^{\Delta-\frac{d}{2}-1} = 0 \tag{23}$$

in any dimension. A less trivial, but still simple, example is the 2-pt function of two currents in 3d

$$\langle \tilde{J}_i(k)\tilde{J}_j(-k)\rangle = \frac{\delta_{ij}k^2 - k_i k_j}{|k|} \tag{24}$$

Working out the expression

$$\begin{aligned}&\Big(2(b\cdot\tilde{\partial}) - (b\cdot k\tilde{\Box} - 2k\cdot\tilde{\partial}\, b\cdot\tilde{\partial})\Big)\langle \tilde{J}_i(k)\tilde{J}_j(-k)\rangle\\ &+2(b^l\partial_i - b_i\tilde{\partial}^l)\langle \tilde{J}_l(k)\tilde{J}_j(-k)\rangle + 2(b^l\partial_j - b_j\tilde{\partial}^l)\langle \tilde{J}_l(k)\tilde{J}_l(-k)\rangle\end{aligned} \tag{25}$$

one can check that it is 0.

The 2-pt function of the energy momentum tensor in 3d has three possible (conserved) tensorial structures, which are given by the expression

$$\big\langle T_{\mu\nu}(k)\, T_{\rho\sigma}(-k)\big\rangle = -\frac{i\,\tau}{|k|}\left(k_\mu k_\nu - \eta_{\mu\nu}k^2\right)\left(k_\rho k_\sigma - \eta_{\rho\sigma}k^2\right)$$

$$-\frac{i\,\tau'}{|k|}\left[\left(k_\mu k_\rho - \eta_{\mu\rho}k^2\right)\left(k_\nu k_\sigma - \eta_{\nu\sigma}k^2\right) + \mu\leftrightarrow\nu\right] \tag{26}$$

$$+\frac{\kappa}{192\pi}\left[\epsilon_{\mu\rho\tau}k^\tau\left(k_\nu k_\sigma - \eta_{\nu\sigma}k^2\right) + \epsilon_{\mu\sigma\tau}k^\tau\left(k_\nu k_\rho - \eta_{\nu\rho}k^2\right) + \mu\leftrightarrow\nu\right] \tag{27}$$

where τ, τ', κ are (model-dependent) constants, [4, 5].

Let us show that these structures satisfy the SCT Ward identities. We have

$$\begin{aligned}b\cdot\tilde{K}\,\frac{k_\mu k_\nu k_\lambda k_\rho}{|k|} &= (d-3)\frac{b_\mu k_\nu k_\lambda k_\rho + k_\mu b_\nu k_\lambda k_\rho + k_\mu k_\nu b_\lambda k_\rho + k_\mu k_\nu k_\lambda b_\rho}{|k|}\\ &\quad -(d-3)b\cdot k\,\frac{k_\mu k_\nu k_\lambda k_\rho}{|k|^3}\end{aligned} \tag{28}$$

$$b\cdot\tilde{K}\,\frac{k_\mu k_\nu k^2}{|k|} = (d-3)(b_\mu k_\nu + k_\mu b_\nu)|k| + (d-3)\frac{b\cdot k}{|k|}k_\mu k_\nu \tag{29}$$

and

$$b\cdot\tilde{K}\,|k|^3 = 3(d-3)b\cdot k\,|k| \tag{30}$$

Therefore the even (nonlocal) tensorial structures (26) satisfy the SCT WI.

The third tensorial structure in 3d is parity-odd, traceless and local

$$\langle \tilde{T}_{\mu\nu}(k)\tilde{T}_{\lambda\rho}(-k)\rangle \sim \epsilon_{\mu\lambda\sigma}k^{\sigma}\left(k_{\nu}k_{\rho}-\eta_{\nu\rho}k^{2}\right)+\begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix}\equiv\tilde{F}_{\mu\nu\lambda\rho}(k) \tag{31}$$

Acting on it with $b\cdot K$ we find

$$\begin{aligned} b\cdot K\ \tilde{F}_{\mu\nu\lambda\rho} &= \left(-2k\cdot\tilde{\partial}\, b\cdot\tilde{\partial}+b\cdot k\tilde{\Box}\right)\tilde{F}_{\mu\nu\lambda\rho}+2(b_{\mu}\tilde{\partial}^{\tau}-b^{\tau}\tilde{\partial}_{\mu})\tilde{F}_{\tau\nu\lambda\rho}\\ &\quad +2(b_{\nu}\tilde{\partial}^{\tau}-b^{\tau}\tilde{\partial}_{\nu})\tilde{F}_{\mu\tau\lambda\rho}\\ &= -2(d-2)\, b\cdot k\ \epsilon_{\mu\lambda\sigma}k^{\sigma}\eta_{\nu\rho}-2b^{\sigma}\epsilon_{\sigma\mu\lambda}(k_{\nu}k_{\rho}-\eta_{\nu\rho}k^{2})\\ &\quad +2(d-2)k^{\sigma}\epsilon_{\sigma\mu\lambda}b_{\nu}k_{\rho}+2\,b^{\tau}k^{\sigma}\epsilon_{\lambda\tau\sigma}(k_{\rho}\eta_{\mu\nu}+k_{\nu}\eta_{\mu\rho})\\ &\quad +4b^{\tau}k^{\sigma}\epsilon_{\tau\lambda\sigma}k_{\mu}\eta_{\nu\rho}+\begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix} \end{aligned} \tag{32}$$

This vanishes thanks to the identities

$$b^{\sigma}\epsilon_{\sigma\mu\lambda}k_{\nu}-b_{\nu}\epsilon_{\tau\mu\lambda}k^{\tau}+b^{\tau}\epsilon_{\tau\lambda\sigma}k^{\sigma}\eta_{\mu\nu}-b^{\tau}\epsilon_{\tau\mu\sigma}k^{\sigma}\eta_{\nu\lambda}=0 \tag{33}$$

$$b^{\sigma}\epsilon_{\sigma\mu\lambda}k^{2}+b^{\sigma}b^{\sigma}\epsilon_{\sigma\lambda\tau}k_{\mu}k^{\tau}-b^{\sigma}\epsilon_{\sigma\mu\tau}k_{\tau}k_{\lambda}-b\cdot k\,k^{\tau}\epsilon_{\tau\mu\lambda}=0 \tag{34}$$

which are consequences of

$$\eta_{\mu\nu}\epsilon_{\lambda\rho\sigma}-\eta_{\mu\lambda}\epsilon_{\nu\rho\sigma}+\eta_{\mu\rho}\epsilon_{\nu\lambda\sigma}-\eta_{\mu\sigma}\epsilon_{\nu\lambda\rho}=0$$

Therefore also the parity-odd structure satisfies the SCT Ward identity. Actually the two terms in the RHS of (31) are separately invariant under a SCT. What determines the relative $-$ sign is the em tensor conservation.

4 Massive Fermions and Chern–Simons Theory in 3d

The examples of CFT correlators we have met before (31) were polynomials of the coordinates divided by powers of the relative distances between the insertion points (or their Fourier transforms). Equation (31) represents a new kind of correlator, which corresponds in momentum space to a polynomial of the momenta. By Fourier anti-transforming it we get,

$$F_{\mu\nu\lambda\rho}(x,y)\sim\epsilon_{\mu\lambda\sigma}\partial^{\sigma}\left(\partial_{\nu}\partial_{\rho}-\eta_{\nu\rho}\Box\right)\delta^{(3)}(x-y)+\begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix} \tag{35}$$

This expression is completely localized in coordinate space, that is made solely of delta functions and derivative of delta functions. Such expressions are called *contact* *terms*. The previous ones, like the even parity structures in $3d$, are *nonlocal terms*. It is interesting to dwell on (31) and (35) for several reasons. These formulas are a 2-point correlator of the e.m. tensor, which has been derived only on the basis of conformal symmetry properties. One question we may ask is whether, like in other cases, this correlator can be obtained from the regularization of a bare one. Another question is whether this may come from some free matter field theory, as it often happens in other cases. The answer is negative for both questions. So it is legitimate to ask: what is the conformal theory that supports such correlator? Well, in a sense (31) can indeed be obtained from a free field theory, but not in the usual way, and in another sense there is a theory that supports such correlators, but it is not free. Let us see how.

Consider the theory of a massive fermion in $3d$, minimally coupled to a metric $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$:

$$S[g] = \int d^3x\, e\left[i\bar{\psi} E_a^\mu \gamma^a \nabla_\mu \psi - m\bar{\psi}\psi\right], \tag{36}$$
$$\nabla_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu bc}\Sigma^{bc}, \qquad \Sigma^{bc} = \frac{1}{4}\left[\gamma^b, \gamma^c\right].$$

The corresponding energy momentum tensor

$$T_{\mu\nu} = \frac{i}{4}\bar{\psi}\left(\gamma_\mu \overleftrightarrow{\nabla}_\nu + \gamma_\nu \overleftrightarrow{\nabla}_\mu\right)\psi. \tag{37}$$

is covariantly conserved on shell as a consequence of the diffeomorphism invariance of the action.

$$\nabla^\mu T_{\mu\nu}(x) = 0 \tag{38}$$

The presence of a mass term breaks parity. From (7), the lowest term of the effective action in an expansion in $h_{\mu\nu}$ comes from the two-point function of the e.m. tensor. So let us compute the two-point function of the e.m. tensor in this theory with the Feynman diagram technique. The corresponding contribution comes from the bubble diagram (one graviton entering and one graviton exiting with momentum k, one fermionic loop):

$$\tilde{T}_{\mu\nu\lambda\rho}(k) = \tag{39}$$
$$= \frac{1}{64}\int \frac{d^3p}{(2\pi)^3}\left[\mathrm{Tr}\left(\frac{1}{\not{p}-m}(2p-k)_\mu\gamma_\nu \frac{1}{\not{p}-\not{k}-m}(2p-k)_\lambda\gamma_\rho\right) + \begin{pmatrix}\mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho\end{pmatrix}\right]$$

Working out the calculations involved (which requires also subtracting a divergent term) yields

$$\langle T_{\mu\nu}(k)\, T_{\lambda\rho}(-k)\rangle_{\text{P-odd}} = \frac{\kappa_g(k^2/m^2)}{192\pi}\, \epsilon_{\sigma\nu\rho}\, k^\sigma \left(k_\mu k_\lambda - k^2\eta_{\mu\lambda}\right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho \end{pmatrix} \tag{40}$$

with

$$\kappa_g(k^2/m^2) = \frac{3m}{k^2}\left(2m + \frac{k^2-4m^2}{|k|}\arctan\frac{|k|}{2m}\right) \quad , \qquad |k| \equiv \sqrt{-k^2} \tag{41}$$

It is worth recalling that (40) is conserved and traceless.

Now let us take the IR limit of κ_g, i.e. the limit in which the energy $|k| = \sqrt{k^2}$ becomes much smaller than the mass $|m|$. We get

$$\kappa_{IR} = \lim_{\frac{|k|}{m}\to 0} \kappa_g = \kappa = \frac{m}{|m|} \tag{42}$$

Therefore we recover the form of (31) with a precise coefficient in front, which is the same as in (27) with $\kappa = \pm 1$. It is remarkable that also in the UV there exists a similar limit, [8].

Now let us Fourier anti-transform (31)

$$\begin{aligned}\langle T_{\mu\nu}(x) T_{\lambda\rho}(y)\rangle_{\text{P-odd}} = &\frac{\kappa}{192\pi}\epsilon_{\mu\lambda\sigma}\partial^\sigma \left(\partial_\nu\partial_\rho - \eta_{\nu\rho}\Box\right)\delta^{(3)}(x-y)\ + \\ &+ \begin{pmatrix} \mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho \end{pmatrix}\end{aligned} \tag{43}$$

Saturating it with $h^{\mu\nu}(x)$ and $h^{\lambda\rho}(y)$ and integrating over x and y (according to the formula (2.2), one gets

$$\frac{\kappa}{192\pi}\int \epsilon_{\mu\lambda\sigma}\left(\partial^\sigma h^{\mu\nu}\partial_\nu\partial_\rho h^{\lambda\rho} - \partial^\sigma h^{\mu\nu}\Box h^\lambda_\nu\right) \tag{44}$$

This represents, to lowest order of approximation, the 3d CS action. It can in fact be obtained from

$$CS = -\frac{\kappa}{96\pi}\int d^3x\, \epsilon^{\mu\nu\lambda}\left(\partial_\mu \omega_\nu^{ab}\omega_{\lambda ba} + \frac{2}{3}\omega_{\mu a}{}^b\omega_{\nu b}{}^c\omega_{\lambda c}{}^a\right) \tag{45}$$

by expanding the spin connection ω in terms of $h_{\mu\nu}$, [7].

5 Comments

In this paper I have defined K-cohomology, and discussed some of its 0-cocycles, i.e. correlators that satisfy the WI of special conformal transformations. It is interesting to find out that there are correlators made out only of contact terms, that is corresponding to local action terms. I have shown the well-known example of 3d, where there exists a two-point function of the e.m. tensor, which is of this type, and corresponds to the lowest order expansion of the gravitational CS action. What is not so well-known, perhaps, is that the higher order terms of the CS action correspond to three, four, ... -point functions of the e.m. tensor. However these correlators are not included in the usual classification of the conformal correlators, because the latter are only required to be naively conserved, i.e. in momentum representation they are required to be transverse to the total momentum, or in configuration space to divergenceless. Such a requirement is totally adequate for the bare correlators, but not for correlators containing contact terms, such as (31). For the latter the usual requirement of transversality is only adequate for two-point functions, not for higher order ones. For instance for a three-point e.m. tensor correlator, its divergence does not vanish but satisfies an equation that involves also the two-point correlators, and so on, [6]. To be more concrete we show the example of 2- and 3-point function for a current J^a_μ. Their conservation laws takes the form

$$k^\mu \tilde{J}^{ab}_{\mu\nu}(k) = 0 \tag{46}$$

$$-iq^\mu \tilde{J}^{abc}_{\mu\nu\lambda}(k_1, k_2) + f^{abd}\tilde{J}^{dc}_{\nu\lambda}(k_2) + f^{acd}\tilde{J}^{db}_{\lambda\nu}(k_1) = 0 \tag{47}$$

where $q = k_1 + k_2$ and $\tilde{J}^{ab}_{\mu\nu}(k)$ and $\tilde{J}^{abc}_{\mu\nu\lambda}(k_1, k_2)$ are Fourier transform of the 2- and 3-point functions, respectively. A similar relation holds for the e.m. tensor. This part of the research program on conformal correlators is still largely unexplored, [8].

Acknowledgements I would like to thank Bruno Lima de Souza for allowing me to use common research material. I would like to thank Vladimir Dobrev, Ivan Todorov and Nikolai Nikolov for hospitality, discussions and suggestions.

References

1. A. Bzowski, P. McFadden and K. Skenderis, *Implications of conformal invariance in momentum space,* JHEP **1403** (2014) 111.
2. Y. S. Stanev, *Correlation Functions of Conserved Currents in Four Dimensional Conformal Field Theory*, Nucl. Phys. B **865** (2012) 200 [arXiv:1206.5639 [hep-th]].
3. A. Zhiboedov, *A note on three-point functions of conserved currents*, arXiv:1206.6370 [hep-th].
4. Cyril Closset, Thomas T. Dumitrescu, Guido Festuccia, Zohar Komargodski, and Nathan Seiberg, *Comments on Chern-Simons Contact Terms in Three Dimensions*, JHEP **1209** (2012), 091.
5. Simone Giombi, Shiroman Prakash and Xi Yin, *A note on CFT correlators in Three Dimensions*, arXiv:1104.4317[hep-th].
6. L. Bonora, A. D. Pereira and B. L. de Souza, *Regularization of energy-momentum tensor correlators and parity-odd terms,* JHEP **1506**, 024 (2015) [arXiv:1503.03326 [hep-th]].
7. I. Vuorio, *Parity Violation and the Effective Gravitational Action in Three-dimensions,* Phys. Lett. B **175** (1986) 176.
8. L. Bonora, M. Cvitan, P. Dominis Prester, B. Lima de Souza and I. Smolić, *Massive fermion model in 3d and higher spin currents*. JHEP **1605**, 072 (2016) doi:10.1007/JHEP05(2016)072.

http://www.springer.com/978-981-10-2635-5

Lie Theory and Its Applications in Physics
Varna, Bulgaria, June 2015
Dobrev, V. (Ed.)
2016, XV, 614 p. 29 illus., 17 illus. in color., Hardcover
ISBN: 978-981-10-2635-5