Chapter 2
Gaussian Means

2.1 Integral Representation

In this chapter we are dealing with the standard GUE matrix integrals in the absence of any external source. The Gaussian random matrix theory has a probability distribution $P(M)$

$$P(M) = \frac{1}{Z_0}\exp[-\frac{\lambda}{2}\mathrm{tr}M^2] \tag{2.1}$$

where M is an $N \times N$ Hermitian matrix. The partition function Z_0 is thus given by

$$Z_0 = \int dM \exp[-\frac{\lambda}{2}\mathrm{tr}M^2] \tag{2.2}$$

The integration is performed with the standard $U(N)$ invariant measure over the N^2 matrix elements. The Gaussian average of a product of vertices $\prod \mathrm{tr}M^{k_i}$ is given by

$$\langle \prod_{i=1}^{n} \mathrm{tr}M^{k_i} \rangle = \int dMP(M) \prod_{i=1}^{n} \mathrm{tr}M^{k_i} \tag{2.3}$$

where k_i, $(i = 1, \ldots, n)$ is integer. The Gaussian means may be computed with the help of Wick's theorem, which counts the pairings of the vertices. In Wick's theorem for matrices averages, the size N of the matrices appears in the combinatorics and it is at the origin of the topological properties. The topological dependence on N was first pointed out by t' Hooft [125] for $U(N)$ gauge theories. In the large N limit, planar diagrams dominate. For the non-Gaussian case, planar diagrams are discussed in the approach to two dimensional quantum gravity in [13, 15].

Wick's theorem for the matrix M_{ij} provides the Gaussian means of products of vertices. For instance,

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$$\langle \frac{1}{N} \mathrm{tr} M^2 \rangle = \langle \frac{1}{N} \sum_{i,j=1}^{N} M_{ij} M_{ji} \rangle = N/\lambda \tag{2.4}$$

$$\langle \frac{1}{N} \mathrm{tr} M^4 \rangle = \sum_{i,j,k,l} \frac{1}{N} \langle M_{ij} M_{jk} M_{kl} M_{li} \rangle = \frac{N^2}{\lambda^2}(2 + \frac{1}{N^2}) \tag{2.5}$$

Wick's theorem for matrix elements is based upon

$$\langle M_{ij} M_{lk} \rangle = \frac{1}{\lambda} \delta_{i,k} \delta_{j,l} \tag{2.6}$$

where $\delta_{i,k}$ is the Kronecker delta function. In the rest of this section we simply take $\lambda = N$ in the probability measure (2.1). The n-point function, which is a generating function of the above Gaussian means, is given by

$$U(\sigma_1, \ldots, \sigma_n) = \langle \prod_{i=1}^{n} \frac{1}{N} \mathrm{tr} e^{\sigma_i M} \rangle = \frac{1}{N^n} \sum_{k_1, \ldots, k_n} (\prod_{i=1}^{n} \frac{1}{k_i!} \sigma_i^{k_i}) \langle \mathrm{tr} M^{k_1} \cdots \mathrm{tr} M^{k_n} \rangle \tag{2.7}$$

This generating function is also the evolution operator of the n-point resolvent G_n, defined as

$$G_n(z_1, \cdots, z_n) = \langle \prod_{a=1}^{n} \frac{1}{N} \mathrm{tr} \frac{1}{z_a - M} \rangle. \tag{2.8}$$

For instance, the average resolvent $G(z)$ is written in terms of the evolution operator as

$$G(z) = \frac{1}{N} \langle \mathrm{tr} \frac{1}{z - M} \rangle = i \int_0^{\infty} dt e^{-itz} U(\sigma). \tag{2.9}$$

From the definition of (2.7), n-point density correlation function R_n is written as

$$\begin{aligned} R_n(\lambda_1, \ldots, \lambda_n) &= \frac{1}{N^n} \langle \prod_{i=1}^{n} \mathrm{tr} \delta(\lambda_i - M) \rangle \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{dt_i}{2\pi} e^{-i \sum_i t_i \lambda_i} U(\sigma_1, \ldots, \sigma_n). \end{aligned} \tag{2.10}$$

We have obtained an exact representation for those generating functions $U(\sigma_1, \ldots, \sigma_n)$, which will be derived in the next section.

Note that $R_n(\lambda_1, \ldots, \lambda_n)$ is obtained also from the probability distribution function $P_N(x_1, \ldots, x_N)$, which becomes after the integration of unitary degree of freedom,

$$P_N(\lambda_1, \ldots, \lambda_N) = C \prod (\lambda_i - \lambda_j)^2 e^{-\frac{\lambda}{2} \sum \lambda_i^2} \tag{2.11}$$

$$R_n(\lambda_1,\ldots,\lambda_n) = \frac{N!}{(N-n)!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d\lambda_{n+1}\cdots d\lambda_N P_N(\lambda_1,\ldots,\lambda_N) \quad (2.12)$$

This n-point correlation function becomes the determinant of the kernel by the orthogonal polynomial method [97].

$$R_n(\lambda_1,\ldots,\lambda_n) = \det[K_N(\lambda_i,\lambda_j)] \quad (2.13)$$

We will discuss this determinant formula for the case of external source, which orthogonal polynomial method does not work, in Chap. 3 (Theorem 3.2.2).

Proposition 2.1

$$U(\sigma_1,\ldots,\sigma_n) = \frac{1}{N^n}\oint\prod_{i=1}^{n}\frac{du_i}{2\pi i}\prod_{i=1}^{N}(1+\frac{\sigma_i}{u_i})^N e^{\sum u_i\sigma_i/\lambda+\frac{1}{2\lambda}\sum\sigma_i^2}\det\frac{1}{u_i-u_j+\sigma_i} \quad (2.14)$$

where the contours enclose all poles at $u_i = 0$.

This Proposition 2.1 follows from Theorem 3.2.1 in the next section which deals with a Gaussian model with an external source, when the external source vanishes. Remarkably enough introducing a source provides explicit formulae even for a vanishing source, which would be very difficult to derive directly.

For the connected part of the above correlation functions, one simply keeps the longest cycles in the expansion of the determinant which appears in (2.14). The Proposition 2.1 determines explicitly the coefficients of the expansions in powers of σ_i.

In the following, the n-point functions ($n = 1$, 2 and 3) are considered with the help of this formula.

(i) one point function

$$U(\sigma) = \langle\frac{1}{N}\mathrm{tr}e^{\sigma M}\rangle = \frac{e^{\frac{\sigma^2}{2\lambda}}}{N\sigma}\oint\frac{du}{2i\pi}(1+\frac{\sigma}{u})^N e^{u\sigma/\lambda} \quad (2.15)$$

In terms of the density of eigenvalues

$$\rho(x) = \langle\frac{1}{N}\mathrm{tr}\delta(x-M)\rangle \quad (2.16)$$

one has by definition of (σ) in (2.15),

$$U(\sigma) = \int_{-\infty}^{\infty} dx\rho(x)e^{itx} \quad (2.17)$$

where $\sigma = it$. In the large N limit, by the shift of $u \to Nu$ and putting $\lambda = N$, $U(\sigma)$ becomes

$$\lim_{N\to\infty} U(\sigma) = \frac{1}{\sigma}\oint \frac{du}{2i\pi} e^{\sigma(u+\frac{1}{U})} = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}\sigma^{2k} = -\frac{1}{i\sigma} J_1(-2i\sigma), \quad (2.18)$$

where $J_1(x)$ is Bessel function of order one. Its Fourier transform provides thus the density of eigenvalues

$$\rho(x) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-itx} U(it) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\frac{1}{t} J_1(2t) e^{-ixt} = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}, \quad (|x| \le 2) \quad (2.19)$$

One recovers for the density of state $\rho(x)$, the well known Wigner's semi-circle.

The finite N expression for $U(\sigma)$ is for $\lambda = N$, through a binomial expansion,

$$\begin{aligned} U(\sigma) &= \frac{1}{\sigma}\oint \frac{du}{2i\pi} \sum_{k=0}^{\infty} \frac{N!}{k!(N-k)!u^k} \frac{1}{N^k}\sigma^k \sum_m \frac{1}{m!}\sigma^m u^m e^{\frac{\sigma^2}{2N}} \\ &= e^{\frac{\sigma^2}{2N}} \sum_{k=0}^{\infty} \frac{N!}{N^{k+1}(N-k-1)!k!(k+1)!}\sigma^{2k} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{\sigma^{2k}}{2^{k-l}N^k} \frac{(N-1)!}{(N-l-1)!} \frac{1}{l!(l+1)!(k-l)!} \end{aligned} \quad (2.20)$$

This expression, provides an explicit finite N result for the Gaussian average from (2.15) :

$$\begin{aligned} \frac{1}{N}\langle \mathrm{tr} M^{2k}\rangle &= (2k)! \sum_{l=0}^{k} \frac{1}{2^{k-l}N^k} \frac{(N-1)!}{(N-l-1)!} \frac{1}{l!(l+1)!(k-l)!} \\ &= (2k)! \sum_{l=0}^{k} \Big(\prod_{j=1}^{l}\big(1-\frac{j}{N}\big)\Big) \frac{1}{(2N)^{k-l}} \frac{1}{l!(l+1)!(k-l)!} \end{aligned} \quad (2.21)$$

(Alternatively one may compute the resolvent

$$G(x) = \frac{1}{i}\int_0^{\infty} dt e^{itx} U(it) = -\frac{1}{x}\int_0^{\infty} \frac{dt}{t} e^{it} \oint \frac{du}{2i\pi} \big(1+\frac{it}{Nu}\big)^N e^{itu/x - t^2/2Nx^2}$$

with $\lambda = N$ in $U(\sigma)$ and expand it in powers of $1/x$.)

The above equations provide also the $1/N$ expansions for $U(\sigma)$ and for the average vertices. The leading large N limit is given by (2.18) and beyond it one finds easily [28]:

$$\prod_{j=1}^{l}(1-\frac{j}{N}) = \exp[\sum_{j=1}^{l}\log(1-\frac{j}{N})]$$
$$= 1 - \frac{l(l+1)}{2N} + \frac{l(l+1)(l-1)(3l+2)}{24N^2} + \cdots \tag{2.22}$$

$$\frac{1}{N}\langle \mathrm{tr} M^{2k}\rangle = \frac{(2k)!}{k!(k+1)!}\left(1 + \frac{k(k-1)(k+1)}{12N^2} + \frac{k(k+1)(k-1)(k-2)(k-3)(5k-2)}{1440N^4}\right) + O(\frac{1}{N^6}) \tag{2.23}$$

Okounkov [106, 107] have shown that the Kontsevich intersection numbers [89], may be obtained by taking a simultaneous large N and large k limit of Gaussian averages. From (2.23), the limit for large N, large k, and finite k/N, is

$$\frac{1}{N}\langle \mathrm{tr} M^{2k}\rangle = \frac{(2k)!}{k!(k+1)!}\left(1 + \frac{1}{12}\frac{k^3}{N^2} + \frac{5}{1440}\frac{k^6}{N^4} + \cdots\right) \tag{2.24}$$

In the above equation, the coefficients of $\frac{k^{3g}}{N^{2g}}$, namely $\frac{1}{(12)^g g!}$, are the intersection numbers of the moduli space of curves with one marked point.

$$\langle \tau_{3g-2}\rangle_g = \frac{1}{(12)^g g! 2^g} \tag{2.25}$$

In our previous work, we have used the exact integral representation valid for finite N of those vertex correlation functions, and obtained explicitly the scaling region for large k_i and large N by a simple saddle-point [28]. This led to a practical way to compute intersection numbers from a pure Gaussian model, much simpler than with the Kontsevich's Airy matrix model. The intersection numbers derived by several different methods will be discussed in Chap. 6.

(ii) two point function

The connected part is given by the shifts $u_i \to Nu_i$ and $\lambda \to N$ in (2.14),

$$U_c(\sigma_1,\sigma_2) = -\frac{1}{N^2}\oint \frac{du_1 du_2}{(2i\pi)^2}\frac{(1+\frac{\sigma_1}{Nu_1})^N(1+\frac{\sigma_2}{Nu_2})^N}{(u_1-u_2+\frac{\sigma_1}{N})(u_2-u_1+\frac{\sigma_2}{N})}e^{\sigma_1 u_1+\sigma_2 u_2+\frac{1}{2N}(\sigma_1^2+\sigma_2^2)} \tag{2.26}$$

Expanding the denominator as

$$\frac{1}{u_2}\sum_{l=0}^{\infty}(\frac{u_1}{u_2})^l(1+\frac{\sigma_1}{Nu_1})^l\sum_{m=0}^{\infty}\frac{u_1^m}{(1+\frac{\sigma_2}{Nu_2})^{m+1}u_2^{m+1}} \tag{2.27}$$

the two point function follows from a residue calculation,

$$
\begin{aligned}
U_c(\sigma_1,\sigma_2) &= \frac{1}{N^2} e^{\frac{1}{2N}(\sigma_1^2+\sigma_2^2)} \sum_{k,t,l,m=0}^{\infty} \frac{\sigma_1^{2k+l+m+1}\sigma_2^{2t+l+m+1}}{k!(k+l+m+1)!t!(t+m+l+1)!} \\
&\times \frac{(N+l)!\,(N-m-1)!}{(N-m-k-1)!(N-m-t-1)!N^{k+t+l+m+1}} \\
&= \frac{1}{N^2} e^{\frac{1}{2N}(\sigma_1^2+\sigma_2^2)} (\sigma_1\sigma_2 + \frac{1}{2}\sigma_1^2\sigma_2^2 + \frac{1}{2}(1-\frac{1}{N})(\sigma_1^3\sigma_2+\sigma_1\sigma_2^3) + \sigma_1^3\sigma_2^3(\frac{1}{3}-\frac{1}{2N}+\frac{1}{3N^2}) \\
&\quad + \cdots) \\
&= \frac{1}{N^2}\Big(\sigma_1\sigma_2 + \frac{1}{2}\sigma_1^2\sigma_2^2 + \frac{1}{2}(\sigma_1^3\sigma_2+\sigma_1\sigma_2^3) + (\frac{1}{3}+\frac{1}{12N^2})\sigma_1^3\sigma_2^3 + \cdots\Big)
\end{aligned}
\tag{2.28}
$$

Nothing that

$$
U_c(\sigma_1,\sigma_2) = \frac{1}{N^2}\sum_{k_1,k_2} \langle \mathrm{tr}M^{k_1}\mathrm{tr}M^{k_2}\rangle_c \frac{1}{k_1!k_2!}\sigma_1^{k_1}\sigma_2^{k_2} \tag{2.29}
$$

the Gaussian means are obtained,

$$
\begin{aligned}
&\frac{1}{N^2}\langle \mathrm{tr}M\mathrm{tr}M\rangle_c = \frac{1}{N^2}, \quad \frac{1}{N^2}\langle \mathrm{tr}M^2\mathrm{trM}^2\rangle_c = \frac{2}{N^2}, \\
&\frac{1}{N^2}\langle \mathrm{tr}M\mathrm{tr}M^3\rangle_c = \frac{3}{N^2}, \quad \frac{1}{N^2}\langle \mathrm{tr}M^3\mathrm{tr}M^3\rangle_c = \frac{12}{N^2} + \frac{3}{N^4}, \ldots
\end{aligned}
\tag{2.30}
$$

(iii) three point function

There are two longest cycles in the determinant (2.14), which contribute to the connected part $U_c(\sigma_1,\sigma_2,\sigma_3)$, which are after the shifts $u_i \to Nu_i$ and $\lambda \to N$,

$$
\begin{aligned}
I &= \frac{1}{(u_1-u_2+\frac{\sigma_1}{N})(u_2-u_3+\frac{\sigma_2}{N})(u_3-u_1+\frac{\sigma_3}{N})} \\
&+ \frac{1}{(u_1-u_3+\frac{\sigma_1}{N})(u_3-u_2+\frac{\sigma_3}{N})(u_2-u_1+\frac{\sigma_2}{N})}
\end{aligned}
\tag{2.31}
$$

Computing the residues in the contour integrals, the three point function reads

$$
\begin{aligned}
U_c(\sigma_1,\sigma_2,\sigma_3) &= -e^{(\sigma_1^2+\sigma_2^2+\sigma_3^2)/2N} \sum_{k,l,m,n_1,n_2,n_3=0}^{\infty} \frac{(N+k)!(N+l)!(N+m)!}{n_1!n_2!n_3!N^{n_1+n_2+n_3+3}} \\
&\times\Bigg(\frac{\sigma_1^{2n_1+k-l}\sigma_2^{2n_2+l-m}\sigma_3^{2n_3+m-k}}{(n_1+k-l)!(n_2+l-m)!(n_3+m-k)!(N+l-n_1)!(N+m-n_2)!(N+k-n_3)!} \\
&+ \frac{\sigma_1^{2n_1+m-l}\sigma_2^{2n_2+k-m}\sigma_3^{2n_3+l-k}}{(n_1+m-l)!(n_2+k-m)!(n_3+l-k)!(N+l-n_1)!(N+m-n_2)!(N+k-n_3)!}\Bigg)
\end{aligned}
\tag{2.32}
$$

2.2 Generating Functions of Gaussian Means

The evaluation of Gaussian means has attracted considerable interests in various fields. A different generating function for the Gaussian means appears for instance in the work of Harer and Zagier [76] and Morozov and Shakirov [102, 103]. Harer and Zagier have found a generating function for $C_{2k} = \frac{1}{N}\langle \mathrm{tr} M^{2k}\rangle$ which reads

$$\sum_{k=0}^{\infty} C_{2k} \frac{N^k x^{2k}}{(2k-1)!!} = \frac{1}{2Nx^2}((\frac{1+x^2}{1-x^2})^N - 1) \tag{2.33}$$

This has the very simple large N limit

$$\lim_{N\to\infty} \sum_{k=0}^{\infty} C_{2k} \frac{x^{2k}}{(2k-1)!!} = \frac{e^{2x^2}-1}{2x^2} \tag{2.34}$$

From (2.33) one finds also

$$\sum_{N,k=0}^{\infty} C_{2k} \frac{x^{2k}\mu^N N^{k+1}}{(2k-1)!!} = \frac{\mu}{1-\mu} \frac{1}{(1-\mu)-(1+\mu)x^2} \tag{2.35}$$

where N is the size of the Hermitian matrix M. These formulae can be derived directly from the above result (2.21). For instance

$$\begin{aligned}
\sum_{k=0}^{\infty} C_{2k} \frac{N^k x^{2k}}{(2k-1)!!} &= \int_0^\infty dte^{-t} U(x\sqrt{2Nt}) \\
&= \frac{1}{x}\int_0^\infty dt \oint \frac{du}{2i\pi} e^{-t(1-x^2-2Nux)}[(1+\frac{x}{Nu})^N - 1] \\
&= \frac{1}{2Nx^2}((\frac{1+x^2}{1-x^2})^N - 1)
\end{aligned} \tag{2.36}$$

Morozov and Shakirov [102] have considered the two point function, for odd powers $C_{2k_1+1,2k_2+1}$, as a generating function of μ,

$$\begin{aligned}
&\frac{\mu}{(1-\mu)^{\frac{3}{2}}} \frac{1}{\sqrt{1-\mu+(1+\mu)(x_1^2+x_2^2)}} \arctan(\frac{x_1x_2\sqrt{1-\mu}}{\sqrt{1-\mu+(1+\mu)(x_1^2+x_2^2)}}) \\
&= \sum_{N,k_1,k_2=0}^{\infty} C_{2k_1+1,2k_2+1} \frac{x_1^{2k_1+1} x_2^{2k_2+1} \mu^N}{(2k_1+1)!!(2k_2+1)!!}
\end{aligned} \tag{2.37}$$

with $C_{2k_1+1,2k_2+1} = \langle \frac{1}{N}\mathrm{tr}M^{2k_1+1}\frac{1}{N}\mathrm{tr}M^{2k_2+1}\rangle$. These results can also be derived from the generating function (2.15) and (2.26) [103]. The Gaussian means have applications for various problems and further investigated, for instance in [8].

The generating function of Gaussian means is given by the introduction of the parameters t_k coefficients of $\mathrm{tr}M^{k_i}$.

$$Z = \frac{1}{Z_0}\int dM \exp[-\frac{N}{2}\mathrm{tr}M^2 + \sum_{k=1}^{\infty}\frac{t_k}{N}\mathrm{tr}M^k] \tag{2.38}$$

with $Z_0 = \int dM e^{-N/2\ \mathrm{tr}M^2}$. Those Gaussian averages include non-connected parts. Therefore, it is useful to expand the free energy $F = \log Z$. The expansion of F reads (the suffix c indicates the connected parts) :

$$\begin{aligned} F &= \frac{t_1^2}{2N^2}\langle(\mathrm{tr}M)^2\rangle_c + \frac{t_2}{N}\langle\mathrm{tr}M^2\rangle_c + \frac{1}{N^2}t_1t_3\langle\mathrm{tr}M\mathrm{tr}M^3\rangle_c + \frac{t_4}{N}\langle\mathrm{tr}M^4\rangle_c \\ &+ \frac{1}{2N^3}t_1^2t_2\langle(\mathrm{tr}M)^2(\mathrm{tr}M^2)\rangle_c + \frac{t_2^2}{2N^2}\langle(\mathrm{tr}M^2)^2\rangle_c + \cdots \\ &= \frac{1}{2N^2}t_1^2 + t_2 + \frac{3}{N^2}t_1t_3 + t_4(2+\frac{1}{N^2}) + \frac{1}{N^4}t_1^2t_2 + \frac{1}{N^2}t_2^2 + \cdots . \end{aligned} \tag{2.39}$$

This expansion may be generated with the help of the Virasoro differential operators. They are defined through the identites

$$\int dM \frac{\partial}{\partial M_{ab}}\left((M^n)_{cd}\exp[-\frac{N}{2}\mathrm{tr}M^2 + \sum_{i=1}^{\infty}\frac{t_i}{N}\mathrm{tr}M^i]\right) = 0 \tag{2.40}$$

Let us start with the simple $n = 0$ case, the so-called string equation, i.e.

$$\int dM \frac{\partial}{\partial M_{ab}}\left(\exp[-\frac{N}{2}\mathrm{tr}M^2 + \sum_{k=1}^{\infty}\frac{t_k}{N}\mathrm{tr}M^k]\right) = 0 \tag{2.41}$$

which gives

$$\langle -NM_{ba} + \frac{1}{N}t_1\delta_{ab} + \frac{1}{N}\sum_{k=2} kt_k(M^{k-1})_{ba}\rangle = 0 \tag{2.42}$$

and after summing on $a = b = 1, \cdots, N$

$$L_{-1}Z = 0 \tag{2.43}$$

with

$$L_{-1} = -N^2\partial_1 + t_1 + \sum(k+1)t_{k+1}\partial_k \tag{2.44}$$

in which ∂_k stands for $\frac{\partial}{\partial t_k}$. Similarly for $n = 1$ the identity(2.40) gives

$$\langle \delta_{ac}\delta_{bd} - NM_{cd}M_{ba} + \frac{1}{N}M_{cd}\sum kt_k(M^{k-1})_{ba}\rangle = 0 \tag{2.45}$$

and, after summing over $b = d$ and $a = c$, one obtains the 'dilaton' equation

$$L_0 Z = 0 \tag{2.46}$$

with

$$L_0 = -N^2\partial_2 + N^2 + \sum_{k=1} kt_k\partial_k \tag{2.47}$$

The same identity for general n provides the full algebra $L_n Z = 0$ for $n \geq -1$ with, for $n \geq 1$,

$$L_n = -N^2\partial_{n+2} + 2N^2\partial_n + N^2\sum_1^{n-1}\partial_p\partial_{n-p} + \sum_1 kt_k\partial_{n+k} \tag{2.48}$$

in which L_1 and L_2 are given by

$$\begin{aligned} L_1 &= -N^2\partial_3 + 2N^2\partial_1 + \sum kt_k\partial_{k+1} \\ L_2 &= -N^2\partial_4 + 2N^2\partial_2 + N^2\partial_1^2 + \sum kt_k\partial_{k+2} \end{aligned} \tag{2.49}$$

These differential operators satisfy the zero central charge Virasoro algebra

$$[L_k, L_m] = L_k L_m - L_m L_k = (k-m)L_{k+m} \tag{2.50}$$

Note that the commutation relations may also be used to determine successively all the L_n for $n \geq 2$. Finally one notes that L_{-1}, L_0, L_1 are linear in the derivatives with respect to the parameters t_k and thus act simply also on the free energy. However the non-linearity of the L_n for $n\rangle 1$ gives non-linear constraints on F. Therefore if the Virasoro constraints can be used easily to determine the lower moments of the Gaussian distribution, the integral form of $U(\sigma_1, \ldots, \sigma_s)$ turns out to be a much more efficient method for computing all the n-points moments.

Instead of Virasoro constraints on the partition functions $Z(t_1, \cdots, t_n, \cdots)$ one can also use recursion relations directly on pure Gaussian means, i.e. with the weight $\exp[-N/2\ \mathrm{trM}^2]$. Consider an operator $\mathcal{O}(M) = \mathrm{trM}^{\mathrm{k}_1}\cdots\mathrm{trM}^{\mathrm{k}_\mathrm{n}}$; one can use systematically the identities

$$\int dM \frac{\partial}{\partial M_{ab}}\Big((M^q)_{cd}\mathcal{O}(M)\exp[-N/2\ \mathrm{trM}^2]\Big) = 0 \tag{2.51}$$

For instance for $q = 0$ one obtains

$$N\langle \mathrm{trM\, trM^{k_1}} \cdots \mathrm{trM^{k_n}} \rangle = \sum_{\mathrm{i}} \mathrm{k_i} \langle \mathrm{trM^{k_1}} \cdots \mathrm{trM^{k_i-1}} \cdots \mathrm{trM^{k_n}} \rangle \tag{2.52}$$

(which follows also immediately from Wick's theorem; $\langle \cdots \rangle$ refers to the expectation value with the normalized weight $\frac{1}{Z_0} \exp[-\frac{N}{2} \mathrm{tr} M^2]$ as same as (2.3)). For q=1 similarly for the same $\mathscr{O}(M)$

$$N\langle \mathrm{trM^2}\ \mathscr{O}(\mathrm{M}) \rangle = (\mathrm{N^2} + \sum_{\mathrm{i}} \mathrm{k_i}) \langle \mathscr{O}(\mathrm{M}) \rangle \tag{2.53}$$

and so on.

2.3 Gaussian Means in the Large N Limit

The Gaussian means in the large N limit are easily derived from the expression of $U(\sigma_1, \ldots, \sigma_n)$, by the replacement

$$\lim_{N\to\infty} \prod_{i=1}^{n} (1 + \frac{\sigma_i}{N u_i})^N = \exp\left(\sum_{i=1}^{n} \frac{\sigma_i}{u_i} \right), \tag{2.54}$$

or from the explicit finite N expressions of Gaussian means in the $N \to \infty$ limit. In this limit, the n-point correlations $U(\sigma_1, \ldots, \sigma_n)$ becomes from (2.14) by the shift $u_i \to N u_i$ and $\lambda \to N$,

$$U(\sigma_1, \ldots, \sigma_n) = \oint \prod_{i=1}^{n} \frac{du_i}{2\pi i} e^{\sum (u_i + \frac{1}{u_i})\sigma_i} \det \frac{1}{N u_i - N u_j + \sigma_i} \tag{2.55}$$

From this formula, we obtain the following Gaussian means in the large N limit.

Proposition 2.3
In the large N limit, the Gaussian connected averages become

$$\begin{aligned}
&\langle \frac{1}{N} \mathrm{tr} M^{2k} \rangle = \frac{(2k)!}{k!(k+1)!} \\
&\langle \frac{1}{N} \mathrm{tr} M^{2k_1} \frac{1}{N} \mathrm{tr} M^{2k_2} \rangle_c = \frac{1}{N^2} \frac{(2k_1)!(2k_2)!}{(k_1!)^2 (k_2!)^2} \frac{k_1 k_2}{k_1 + k_2} \\
&\langle \frac{1}{N} \mathrm{tr} M^{2k_1+1} \frac{1}{N} \mathrm{tr} M^{2k_2+1} \rangle_c = \frac{1}{N^2} \frac{(2k_1+1)!(2k_2+1)!}{(k_1!)^2 (k_2!)^2} \frac{1}{k_1 + k_2 + 1} \\
&\langle \frac{1}{N} \mathrm{tr} M^{2k_1} \frac{1}{N} \mathrm{tr} M^{2k_2} \frac{1}{N} \mathrm{tr} M^{2k_3} \rangle_c = \frac{1}{N^4} \frac{(2k_1)!(2k_2)!(2k_3)!}{(k_1!)^2 (k_2!)^2 (k_3!)^2} k_1 k_2 k_3 \\
&\langle \frac{1}{N} \mathrm{tr} M^{2k_1} \frac{1}{N} \mathrm{tr} M^{2k_2+1} \frac{1}{N} \mathrm{tr} M^{2k_3+1} \rangle_c = \frac{1}{N^4} \frac{(2k_1)!(2k_2+1)!(2k_3+1)!}{(k_1!)^2 (k_2!)^2 (k_3!)^2} k_1
\end{aligned}$$

$$\langle \frac{1}{N}\mathrm{tr}M^{2k_1}\frac{1}{N}\mathrm{tr}M^{2k_2}\frac{1}{N}\mathrm{tr}M^{2k_3}\frac{1}{N}\mathrm{tr}M^{2k_4}\rangle_c = \frac{1}{N^6}\prod_{i=1}^{4}\frac{(2k_i)!}{(k_i!)^2}k_1k_2k_3k_4 \times(k_1+k_2+k_3+k_4-1)$$

$$\langle \frac{1}{N}\mathrm{tr}M^{2k_1}\frac{1}{N}\mathrm{tr}M^{2k_2}\frac{1}{N}\mathrm{tr}M^{2k_3+1}\frac{1}{N}\mathrm{tr}M^{2k_4+1}\rangle_c = \frac{1}{N^6}\prod_{i=1}^{4}\frac{1}{(k_i!)^2}(2k_1)!(2k_2)! \times(2k_3+1)!(2k_4+1)!k_1k_2(k_1+k_2+k_3+k_4)$$

$$\langle \frac{1}{N}\mathrm{tr}M^{2k_1+1}\frac{1}{N}\mathrm{tr}M^{2k_2+1}\frac{1}{N}\mathrm{tr}M^{2k_3+1}\frac{1}{N}\mathrm{tr}M^{2k_4+1}\rangle_c = \frac{1}{N^6}\prod_{i=1}^{4}\frac{(2k_i+1)!}{(k_i!)^2} \times(k_1+k_2+k_3+k_4) \tag{2.56}$$

It is easy to check that the Virasoro equation of the previous section holds for the above correlation functions in the large N limit. The Gaussian means appear in various applications. Fat graphs, used in topology (and in biology), use the planar character of the large N limit of Gaussian means [8]. The universal character of Chern-Simon invariants also uses Gaussian means [96]. In Chap. 10, the Gromov-Witten invariants of $\mathbf{P}^1$ will be compared with the expressions of Gaussian means of Proposition 2.3.

2.4 Gaussian Means in the Replica Limit $N \to 0$

We now return to the probability distribution (2.1) with a parameter $\lambda \neq N$. In the study of the intersection numbers of the moduli space of curves, we have used a replica limit ($N \to 0$) in [30] and the reason for this replica limit will be explained below. The integral representation (2.14) for $\lambda \neq N$ reads

$$U(\sigma_1,\ldots,\sigma_n) = \frac{1}{N^n}\oint\prod_{i=1}^{n}\frac{du_i}{2\pi i}(1+\frac{\sigma_i}{u_i})^N e^{\sum u_i\sigma_i/\lambda+\frac{1}{2}\sigma_i^2/\lambda}\det\frac{1}{u_i-u_j+\sigma_i}. \tag{2.57}$$

Let us consider first the $N \to 0$ limit of the one-point function

$$U(\sigma) = \frac{1}{N\sigma}e^{\frac{\sigma^2}{2\lambda}}\oint\frac{du}{2i\pi}e^{\sigma u/\lambda}(1+\frac{\sigma}{U})^N \tag{2.58}$$

from which one gets

$$\lim_{N\to 0}U(\sigma) = \frac{1}{\sigma}e^{\frac{\sigma^2}{2\lambda}}\oint\frac{du}{2i\pi}e^{\sigma u/\lambda}\log(1+\frac{\sigma}{U}) \tag{2.59}$$

The contour integral reduces to the integral of the discontinuity of the logarithm leading to

$$\lim_{N\to 0} U(\sigma) = \frac{\sinh(\frac{\sigma^2}{2\lambda})}{(\frac{\sigma^2}{2\lambda})} \tag{2.60}$$

From above formula, in the replica limit $N \to 0$ of $U(\sigma) = \frac{1}{N}\langle \mathrm{tr} e^{\sigma M}\rangle$, one obtain

$$\lim_{N\to 0} \frac{1}{N}\langle \mathrm{tr} M^k\rangle = \frac{4k!}{\lambda^{2k}4^k(2k+1)!}.$$

Proposition 2.4 ([30])

For the n-point functions the replica limit is given by

$$\lim_{N\to 0} U(\sigma_1,\ldots,\sigma_n) = \frac{\lambda}{\chi^2}\prod_{i=1}^{n} 2\sinh\frac{\chi\sigma_i}{2\lambda} \tag{2.61}$$

with $\chi = \sum_{i=1}^{n}\sigma_i$.

Using this replica result, the intersection numbers of p-spin curves of the moduli space of curves with one marked point have been derived in [30]. In Kontsevich's model for the intersection numbers of curves, a trivalent vertex $\mathrm{tr}M^3$ has been used. With one marked point, a one stroke line around a marked point characterizes the moduli space of a Riemann surface. This one stroke Feynman diagram is obtained by a $N \to 0$ limit (replica limit).

(2.61) is a generating function for the $N = 0$ limit of $\frac{1}{N}\langle \mathrm{tr}M^{p_1}\cdots \mathrm{tr}M^{p_k}\rangle$ by expanding in the σ_i's. Selecting the coefficients of equal powers for every σ_i, for instance of $(\sigma_1\cdots\sigma_k)^3$, we find

$$\lim_{N\to 0}\frac{1}{N}\langle(\mathrm{tr}M^3)^{4g-2}\rangle = \frac{3^{3g-2}2^{-2g}(6g-4)!}{\lambda^{6g-3}}\cdot\frac{(4g-2)!}{g!(3g-2)!} \tag{2.62}$$

all other powers of $\lim_{N\to 0}\frac{1}{N}\langle(\mathrm{tr}M^3)^k\rangle$ vanishes unless $k = 2$ (mod4). This leads to Kontsevich's intersection numbers $\langle\tau_l\rangle$ for one marked point. Indeed, these intersection numbers $\langle\tau_l\rangle$ are expressed for one marked point, from the partition function Z of following Kontsevich Airy matrix model with $\mathrm{tr}M^3$ term in exponent, as will be shown in Chap. 6,

$$\lim_{N\to 0}\frac{1}{N}\log Z = \sum_{m=1}^{\infty}\langle\tau_m\rangle t_m,\quad Z = \frac{1}{Z'}\int dM e^{\frac{i}{3}\mathrm{tr}M^3 - \mathrm{tr}\Lambda M^2} \tag{2.63}$$

with $Z' = \int dM e^{-\mathrm{tr}M^2}$, and $t_m = (-2)^{-(2m+1)/3}\prod_{l=0}^{m-1}(2l+1)(\frac{2}{\lambda})^{2m+1}$, where all eigenvalues of Λ in Kontsevich model are put equally to λ.

This provides

$$\langle \tau_{3g-2} \rangle_g = \frac{1}{(24)^g g!} \tag{2.64}$$

which agrees with (2.25).

From (2.61), the intersection numbers $\langle \tau_n \rangle$ of one marked point for p-spin curve (the algebraic geomerical definition will be given in Chap. 7) are obtained. The partition function for this higher Airy matrix model with p-spin is

$$Z = \frac{1}{Z_0} \int dM \exp[\frac{1}{p+1} \mathrm{tr}(B^{p+1} - \Lambda^{p+1}) - \mathrm{tr})(B - \Lambda)\Lambda^p] \tag{2.65}$$

with normalization constant Z_0 [30] and $F = \sum_{k_{m,j}} \langle \prod_{m,j} \tau_{m,j}^{k_{m,j}} \rangle \prod_{m,j} t_{m,j}^{k_{m,j}} / k_{m,j}!$, where

$$t_{m,j} = (-p)^{\frac{j-p-m(p+2)}{2(p+1)}} \prod_{l=0}^{m-1} (lp + j + 1) \mathrm{tr} \frac{1}{\Lambda^{mp+j+1}}. \tag{2.66}$$

Expanding the partition function of (2.65) and using (2.61), we obtain the intersection numbers of one marked point by this replica method, for instance we obtain $p = 3$ case,

$$\lim_{N \to 0} \frac{1}{N} \langle (\mathrm{tr} M^4)^{2m-1} \rangle = \frac{2^{2m-2}}{\lambda^{4m-2}} (4m - 3)! \frac{(2m-1)!}{m!(m-1)!}, \tag{2.67}$$

$$\langle \tau_{(8g-5-j)/3,j} \rangle_g = \frac{1}{(12)^g g!} \frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})} \tag{2.68}$$

A later section is devoted to the intersection numbers of the moduli spaces of curves by the use of the function of $U(\sigma)$, instead of above formula of $N \to 0$ limit (2.61). In Sect. 5, we will show that, by tuning an external source, the n-point functions $U(\sigma_1, \ldots, \sigma_n)$ are indeed generating functions of the intersection numbers of moduli space of curves. Note that the n-point function of (2.14) is valid only for a Gaussian ensemble without external source. For external source, the formula will be given by (3.6) in Theorem 3.2.1.

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