

Chapter 2

Fundamental Equations

Conservation equations are the foundation for Fluid Mechanics and MHD, but others are needed to close the mathematical models. Although fluid pressure was at first assumed to be isotropic, when viscous stress was considered early in the eighteenth century it was evident that the assumption of incompressibility or the inclusion of a simple equation of state was no longer sufficient. The classical macroscopic equations (for mass, momentum and energy) follow from underlying microscopic theory, which also provides the relevant pressure tensor to incorporate viscosity. Except near magnetic null points, the pressure tensor for a plasma in a magnetic field is found to differ significantly from the classical shear viscosity form for a neutral fluid. There is also a brief introduction to the additional equation of magnetic induction required in MHD. The bibliography includes some references that provide further background to our presentation in this chapter, a worthy source on thermodynamics, and two books by Lamb and Prager particularly recommended for supplementary reading (more books on Fluid Mechanics are listed for Chap. 3).

2.1 Conservation Equations

The conservation of physical quantities such as mass, momentum and energy is an essential feature of any fluid mechanics or MHD model. A conservation equation is a differential equation of the form

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \Gamma = s, \quad (2.1)$$

where σ , Γ and s are fields of appropriate tensor rank.

To appreciate the important notion of “conservation” introduced here, consider the integral of (2.1) over any fixed volume V in space bounded by a closed surface S , and use a dot contraction of (1.59) to get

$$\frac{d}{dt} \int_V \sigma d\tau = - \int_S d\mathbf{S} \cdot \boldsymbol{\Gamma} + \int_V s d\tau. \quad (2.2)$$

Equation (2.2) states that the quantity $\int_V \sigma d\tau$ would be conserved, if it were not for the flux $\int_S d\mathbf{S} \cdot \boldsymbol{\Gamma}$ of the quantity $\boldsymbol{\Gamma}$ across the boundary S and the contribution from any term s (a source if $s > 0$, or a sink if $s < 0$). Thus if some closed surface S exists such that $\boldsymbol{\Gamma}$ vanishes on S , and there is no source or sink term in V , then $\int_V \sigma d\tau$ is *conserved* in the corresponding volume V and σ may be called the *density* of this conserved quantity. In passing, we note that the fixed volume in space is often called a “control volume” in the engineering literature.

We first consider the identification of σ with the field $\rho(\mathbf{r}, t)$ denoting the mass density of a fluid, with the mass flux vector $\rho\mathbf{v}$ where $\mathbf{v}(\mathbf{r}, t)$ is the fluid velocity, to produce the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0 \quad (2.3)$$

if there is no source or sink ($s = 0$).¹ This scalar conservation equation is often called the continuity equation, due to an alternative derivation where the evolution of any finite volume of the fluid is considered (cf. Sect. 2.5). Equations of energy conservation constitute another important class, where σ is seen to be identified with some scalar field later.

A vector conservation equation has the form (2.1) with σ and s vectors and $\boldsymbol{\Gamma}$ a dyadic. Of specific interest is the momentum conservation equation

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla \cdot \mathbf{T} = \mathbf{F}, \quad (2.4)$$

where the dyadic $\mathbf{T}$ defines the total momentum flux, with both advective and pressure components as discussed in the following section, and $\mathbf{F}$ denotes the external force density.² The corresponding form of (2.2) in this case is of course

$$\frac{d}{dt} \int_V \rho\mathbf{v} d\tau = - \int_S d\mathbf{S} \cdot \mathbf{T} + \int_V \mathbf{F} d\tau. \quad (2.5)$$

The volume integral on the left-hand side of (2.5) is the total momentum of the fluid in V , so the mass flux vector $\rho\mathbf{v}$ is alternatively called the fluid momentum density. The surface integral on the right-hand side represents the internal surface forces, and the volume integral represents the total external force acting on the fluid in the volume V . Note that the corresponding vector quantity $-\hat{\mathbf{n}} \cdot \mathbf{T}$ in the surface integral,

¹This outcome may be recognised by considering the motion of an infinitesimal cylinder of fluid, slanted in the direction of $\mathbf{v}$, crossing the surface S . The cylindrical volume $\Delta\tau = dS \hat{\mathbf{n}} \cdot \mathbf{v} dt$ carries the mass $\Delta m = \rho\Delta\tau = \rho\mathbf{v} \cdot d\mathbf{S}dt$, with (2.2) obtained from the Divergence Theorem (1.60) in this case.

²In solid mechanics the dyadic $\mathbf{T}$ in the equation for the displacement field corresponding to (2.5) is usually called the stress tensor—but in Fluid Mechanics and MHD we prefer to emphasise the pressure tensor $\mathbf{p}$, which is the pressure component of $\mathbf{T}$.

where the unit normal $\hat{\mathbf{n}}$ such that $d\mathbf{S} = \hat{\mathbf{n}}dS$ points outward from the volume V , usually has both tangential and normal components at any point on the surface S .

The fluid pressure tensor in the total momentum flux dyadic $\mathbf{T}$ is first discussed in the next section. Some basic concepts (gravity, the two classical alternative viewpoints, and material rates of change) are then considered in Sects. 2.3–2.5, before we proceed to derive the macroscopic equations of Fluid Mechanics and MHD in Sects. 2.6 and 2.7. We explore extended mathematical models in Sect. 2.8, and distinguish the pressure tensor for magnetised plasma from the fluid pressure tensor in Sects. 2.9 and 2.10. There is then a brief discussion on the heat flux in the starred Sect. 2.11, before the final Sect. 2.12 on the additional equation of magnetic induction introduced in MHD.

2.2 Fluid Pressure Tensor

Fluid is often transported across the boundary S of a fixed volume V in space (i.e. $\mathbf{n} \cdot \mathbf{v}$ is not zero), when one part of the surface term $-\hat{\mathbf{n}} \cdot \mathbf{T}$ above represents advection of momentum across the boundary. (We prefer to use the term “advection” rather than “convection”, which more strictly applies to transport due to buoyancy.) The microscopic fluid constituents (atoms or molecules) carry momentum, as they random walk back and forth between collisions with each other. As discussed later in this chapter, the microscopic particle motion is often described by kinetic theory, and the conservation equations of fluid mechanics or MHD may be derived in detail from a resultant mathematically exact basic equation of change. However, let us now first identify the advective and pressure components of the tensor $\mathbf{T}$ for a typical fluid in an heuristic fashion.

If the atoms or molecules of a fluid with velocities in a narrow range are grouped to constitute a beam with density ρ_i and average velocity $\mathbf{u}_i$, then summing over many such beams yields the aggregate quantities

$$\rho = \sum_i \rho_i \quad (\text{total density}) \quad (2.6)$$

$$\rho\mathbf{v} = \sum_i \rho_i \mathbf{u}_i \quad (\text{total mass flux}) \quad (2.7)$$

$$\mathbf{T} = \sum_i \rho_i \mathbf{u}_i \mathbf{u}_i \quad (\text{total momentum flux}). \quad (2.8)$$

Note that the velocity $\mathbf{v}$ is actually the mass-weighted mean of the various average velocities of the constituents. It is often convenient to think of a composite fluid element moving with this velocity $\mathbf{v}$ and the related average *peculiar* velocity of the i th beam

$$\mathbf{w}_i \equiv \mathbf{u}_i - \mathbf{v}, \quad (2.9)$$

which immediately splits out the advective and pressure components. Thus substituting (2.9) in (2.8) we have

$$\mathbf{T} = \begin{array}{ccc} \text{advective} & & \text{pressure} \\ \text{component} & & \text{component} \\ & \downarrow & \downarrow \\ & \rho \mathbf{v} \mathbf{v} & + \quad \mathbf{p} \end{array} \quad (2.10)$$

where

$$\mathbf{p} \equiv \sum_i \rho_i \mathbf{w}_i \mathbf{w}_i, \quad (2.11)$$

since from (2.6) and (2.7)

$$\sum_i \rho_i \mathbf{w}_i = 0. \quad (2.12)$$

The total momentum conservation equation (2.5) may therefore be rewritten as

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + \mathbf{p}) = \mathbf{F}, \quad (2.13)$$

where the term $\nabla \cdot \mathbf{p}$ represents the *surface force* density within the fluid. As discussed below, the diagonal components of the pressure tensor combine to produce a normal total *hydrostatic pressure*, and tangential viscous forces arise from the off-diagonal components in its representation—cf. the integrand of the surface integral in the corresponding integral form (2.5).

Since the *pressure tensor* $\mathbf{p}$ is symmetric, there is an orthonormal set $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ of principal axes such that $\mathbf{p}$ is diagonal—i.e.

$$\mathbf{p} = \lambda_1 \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \lambda_2 \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \lambda_3 \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3,$$

where λ_i are the eigenvalues of $\mathbf{p}$. Now let us suppose that $\lambda_{\text{mfp}} \ll L$, where λ_{mfp} is the mean free path length between the microscopic particle collisions and L is the characteristic length scale for gradients in the macroscopic fields, such as the density ρ and the fluid velocity $\mathbf{v}$ say. Thus for a typical fluid, to the zeroth order in λ_{mfp}/L the $\mathbf{w}_i$ distribution is isotropic—i.e. there is no preferred direction, which implies that *any* orthonormal set $\{\mathbf{e}_i\}$ defines principal axes such that $\lambda_1 = \lambda_2 = \lambda_3$ ($= p$ say). Let us therefore anticipate a pressure tensor of form $\mathbf{p} = p\mathbf{I} + \mathbf{t}$, in which the second term $\mathbf{t}$ of order λ_{mfp}/L is linear in the velocity gradient $\nabla \mathbf{v}$ ($\sim v/L$ by dimensional analysis) that defines the rate of deformation, because $d\mathbf{v} = d\mathbf{r} \cdot \nabla \mathbf{v}$ is the relative velocity of neighbouring fluid elements distance $d\mathbf{r}$ apart.

Now the only two symmetric dyadics that can be formed from $\nabla \mathbf{v}$ are $\nabla \cdot \mathbf{v} \mathbf{I}$ and $\nabla \mathbf{v} + (\nabla \mathbf{v})^T$, so it follows that $\mathbf{t}$ must be a linear combination of them. In solid mechanics, the two “constants” (scalar fields) in an analogous linear combination are

the Lamé parameters, although two other related coefficients (viz. Young's modulus and Poisson's ratio) are often preferred. We may write the corresponding relation for fluids as

$$\mathbf{t} = -2\mu\mathbf{s} - \mu_v \nabla \cdot \mathbf{v} \mathbf{I} \quad (2.14)$$

on defining the *rate of deformation tensor*

$$\mathbf{s} \equiv \{\nabla \mathbf{v}\} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T] - \frac{1}{3} \nabla \cdot \mathbf{v} \mathbf{I}, \quad (2.15)$$

where μ and μ_v denote the *shear viscosity coefficient* and the *volume viscosity coefficient* (or “coefficient of expansive friction”), respectively.³ Thus the fluid pressure tensor is

$$\mathbf{p} = p \mathbf{I} - \mu \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^T - \frac{2}{3} \nabla \cdot \mathbf{v} \mathbf{I} \right] - \mu_v \nabla \cdot \mathbf{v} \mathbf{I}, \quad (2.16)$$

where the combination $p - \mu_v \nabla \cdot \mathbf{v}$ is the total hydrostatic pressure.

The traceless nature of the rate of deformation tensor $\mathbf{s}$ ensures it makes no contribution to the pressure tensor if the fluid dilates (expands or contracts) isotropically, and the associated shear viscosity coefficient μ is often small enough for that viscosity contribution to be negligible except in regions of strong velocity shear (boundary layers—cf. Sect. 3.8). Although the volume viscosity coefficient μ_v in a fluid other than a monatomic gas is non-zero, many authors do not explicitly include the volume viscosity term $\mu_v \nabla \cdot \mathbf{v} \mathbf{I}$ that corresponds to dilatation (cf. Sect. 2.5), as we now do too—i.e. henceforth, we also adopt the Newtonian fluid relation $\mathbf{t} = -2\mu\mathbf{s}$, where the volume viscosity may be regarded as implicit in the total hydrostatic pressure field p . Finally, we note that any variation in the coefficient μ (and μ_v if the volume viscosity is retained) is usually assumed to be negligible over the characteristic timescale for the flow. There are fluids where this is not an appropriate approximation, such as a good house paint!

Exercise

Determine the shear and volume viscosity contributions to the pressure tensor if:

- (a) the fluid velocity $\mathbf{v} = ky\hat{\mathbf{i}}$, where y is a Cartesian coordinate in the direction perpendicular to the basis vector $\hat{\mathbf{i}}$, and k is a constant; and
- (b) the fluid velocity $\mathbf{v} = k\mathbf{r}/3$, where $\mathbf{r}$ is the position vector and k is a constant.

³The operator $\{\}$ introduced here and often invoked later is defined by

$$\{\mathbf{F}\} \equiv \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \frac{1}{3} \mathbf{F} : \mathbf{I} \mathbf{I}$$

for any dyadic $\mathbf{F}$, where $\mathbf{F}^T$ denotes its transpose. Since $\text{Tr } \mathbf{F} \equiv \mathbf{F} : \mathbf{I}$ and $\text{Tr } \mathbf{I} = 3$, the resulting symmetric dyadic is also traceless.

2.3 Gravity

A gravitational field can exert a force on a fluid. An external force $(\Delta m)\mathbf{g}$ acting on each fluid element of mass $\Delta m = \rho\Delta\tau$ corresponds to the force density $\mathbf{F} = \rho\mathbf{g}$, when the momentum conservation equation (2.13) becomes

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla \cdot (\rho\mathbf{v}\mathbf{v} + \mathbf{p}) = \rho\mathbf{g}. \quad (2.17)$$

In a self-gravitating system, such as an interstellar gas cloud, the force density $\rho\mathbf{g}$ may be written as $-\nabla \cdot \mathbf{G}$, where

$$\mathbf{G} = \frac{\mathbf{g}\mathbf{g}}{4\pi G} - \frac{|\mathbf{g}|^2}{8\pi G}\mathbf{I} \quad (2.18)$$

denotes the gravitational field stress dyadic and G is the gravitational constant. Thus we have $\nabla \times \mathbf{g} = 0$ since $\mathbf{g}$ is a conservative field, and Newton's law of gravitation in differential form is $\nabla \cdot \mathbf{g} = -4\pi G\rho$, so that

$$\begin{aligned} \nabla \cdot \mathbf{G} &= \frac{(\nabla \cdot \mathbf{g})\mathbf{g}}{4\pi G} + \frac{\mathbf{g} \cdot \nabla \mathbf{g}}{4\pi G} - \frac{(\nabla \mathbf{g}) \cdot \mathbf{g}}{4\pi G} \\ &= -\rho\mathbf{g} - \frac{\mathbf{g} \times (\nabla \times \mathbf{g})}{4\pi G} = -\rho\mathbf{g}. \end{aligned}$$

Consequently, the momentum equation can be expressed in conservation form

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla \cdot (\rho\mathbf{v}\mathbf{v} + \mathbf{p} + \mathbf{G}) = 0. \quad (2.19)$$

Note that this is a conservation equation for the *total* system, consisting of both the fluid subsystem and the gravitational subsystem, which just happens to have a vanishing momentum density! The previous form (2.17), where the interaction with the gravitational subsystem is represented as an external force density, describes the fluid subsystem alone.

Exercise

(Archimedes' Principle.) Show that when a body is partly or completely immersed in a static incompressible fluid (a "liquid"), the resultant force due to the fluid ("upthrust") is equal and opposite to the weight of fluid displaced. Assume constant gravity, and ignore the weight of the air above the fluid.

2.4 Eulerian and Lagrangian Descriptions

The density ρ in the frequently occurring combination $\partial_t(\rho\mathbf{F}) + \nabla \cdot (\rho\mathbf{vF})$, where ∂_t denotes $\partial/\partial t$ and $\mathbf{F}$ is an arbitrary tensor field, can be commuted outside the derivatives upon invoking the continuity equation (2.3). Thus although the momentum equation (2.17) is in conservation form, it is often replaced by the equation of motion

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} + \nabla \cdot \mathbf{p} = \rho \mathbf{g}, \quad (2.20)$$

since from (2.3)

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) = \rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} \quad (2.21)$$

for the velocity field $\mathbf{v}(\mathbf{r}, t)$. The differential operator $(\partial_t + \mathbf{v} \cdot \nabla)$ is called the *material derivative* (or convective derivative, although strictly this term should be “advective derivative”), or referred to as “differentiation following the motion”. It is often denoted by D/Dt , but the notation

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad (2.22)$$

is adopted in this book. The various terminology and notation originates from two distinct views of fluid behaviour, as described below and illustrated in the case of two-dimensional steady flow in Fig. 2.1.

In the *Eulerian* view, an observer at any fixed point in space $\mathbf{r}$ watches the fluid flow by, so that $\mathbf{r}$ is an independent coordinate and the partial time derivative

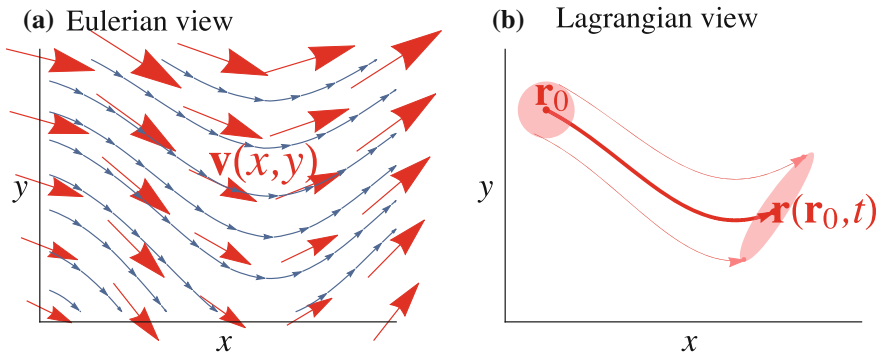


Fig. 2.1 **a** Eulerian picture of a fluid flow—a velocity field $\mathbf{v}$ indicated by the larger arrows. In this simple example $\mathbf{v}$ is tangential to streamlines, the continuous curves with small arrows. **b** The more dynamical Lagrangian picture is based on the map taking an ensemble of fluid elements from their positions $\mathbf{r}_0$ at the initial time, $t = 0$, along their trajectories to their positions $\mathbf{r}$ at time t

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t} \right)_{\mathbf{r}} \quad (2.23)$$

is used, where the subscript $\mathbf{r}$ means that $\mathbf{r}$ is held fixed. On the other hand, in the *Lagrangian* view the observer moves with a given fluid element and watches changes “as seen by the fluid”. Thus $\mathbf{r}$ is no longer an independent variable, since it changes with time as the fluid element moves [7]. The Lagrangian time derivative is defined by

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} \right)_{\mathbf{r}_0}, \quad (2.24)$$

where $\mathbf{r}_0$ is the initial position of the fluid element at some fixed initial time, say t_0 .

However, suppose $\mathbf{r}_0$ lies in a volume of fluid occupying the region V_0 at time t_0 , and advected to the region V at time t . Consider the trajectory of the fluid element as it moves from $\mathbf{r}_0$ at time t_0 to another position $\mathbf{r} = \mathbf{R}(t; \mathbf{r}_0, t_0)$ at time t . The trajectory function $\mathbf{R}$ is defined by

$$\dot{\mathbf{R}} \equiv \left(\frac{\partial \mathbf{R}}{\partial t} \right)_{\mathbf{r}_0} = \mathbf{v}(\mathbf{R}, t) \quad (2.25)$$

subject to the initial condition

$$\mathbf{R}(t_0; \mathbf{r}_0, t_0) = \mathbf{r}_0. \quad (2.26)$$

Provided the velocity field is smooth and finite, and fluid elements are neither created nor destroyed (with no two fluid elements occupying the same region in space at the same time), the mapping $\mathbf{r} = \mathbf{R}(t; \mathbf{r}_0, t_0)$ is a diffeomorphism between V_0 and V that is parameterised by t (and also by t_0 , regarded as a constant). Applying the chain rule of partial differentiation, (2.24) becomes

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} \right)_{\mathbf{r}} + \left(\frac{\partial \mathbf{R}}{\partial t} \right)_{\mathbf{r}_0} \cdot \frac{\partial}{\partial \mathbf{r}} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

from (2.23) and (2.25). Thus the two forms of derivative d/dt , the Eulerian (2.22) and the Lagrangian (2.24), are equivalent.

Consequently, the continuity equation (2.3) may be rewritten as

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0; \quad (2.27)$$

and since the material derivative $d\mathbf{v}/dt$ is the acceleration of the fluid element moving with velocity $\mathbf{v}$ at the position $\mathbf{r} = \mathbf{R}(t; \mathbf{r}_0, t)$ at time t , the equation of motion (2.20) may be rendered as

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla \cdot \mathbf{p}, \quad (2.28)$$

where the terms deliberately written on the right-hand side may be interpreted as the total force density acting on the fluid element, in analogy with the well-known Newton equation of motion for a particle. It is also often useful to envisage discrete fluid particles, each represented by the mass in a localised fluid element of infinitesimally small volume from the macroscopic viewpoint, such that the density field corresponds to their mass to volume ratios.

Sometimes, it is appropriate to consider motion relative to a moving system of coordinates—e.g. a Cartesian reference frame $Ox'y'z'$ rotating with (instantaneous) angular velocity ω_0 relative to another $Oxyz$ with common origin O . The velocity of a particle is

$$\mathbf{v} = \left. \frac{d\mathbf{r}}{dt} \right|_F = \left. \frac{d\mathbf{r}}{dt} \right|_M + \omega_0 \times \mathbf{r} = \left(\left. \frac{d}{dt} \right|_M + \omega_0 \times \right) \mathbf{r}$$

where $\mathbf{r}(t)$ is the position vector of a fluid particle at time t relative to O as seen in the moving frame, the subscript M referring to the moving reference frame $Ox'y'z'$ and F to the “fixed” (inertial) reference frame $Oxyz$. Moreover, the acceleration in the rotating reference frame is likewise

$$\begin{aligned} \left. \frac{d\mathbf{v}}{dt} \right|_F &= \left. \frac{d}{dt} \right|_F \left. \frac{d}{dt} \right|_F \mathbf{r} \\ &= \left(\left. \frac{d}{dt} \right|_M + \omega_0 \times \right)^2 \mathbf{r}, \end{aligned} \quad (2.29)$$

which may be expanded if desired. A moving reference frame with another origin O' , not coincident with the origin O of the inertial reference frame, is likewise readily considered (cf. Sect. 3.14).

Exercises

- (Q1) A cylindrical container rotates about its vertical axis with constant angular velocity ω_0 . If there is liquid in the container, and assuming the atmospheric pressure is uniform over the surface of the liquid, by integrating the equation of motion show that the surface of the liquid is a paraboloid of revolution about the vertical axis.
(Assume constant gravity and a centrifugal acceleration.)
- (Q2) A region of homogeneous self-gravitating fluid rotates about a fixed axis with constant angular velocity ω_0 . Assuming there is no external pressure, show that there is an upper limit to $|\omega_0|$.
(Introduce the gravitational potential V such that $\mathbf{g} = -\nabla V$, and consider $\nabla^2 p$ over the fluid volume.)
- (Q3) Since $\mathbf{p}$ is a symmetric dyadic, from the momentum conservation equation (2.17) derive the angular momentum conservation equation

$$\frac{\partial}{\partial t}(\mathbf{r} \times \rho \mathbf{v}) + \nabla \cdot (\mathbf{v} \mathbf{r} \times \rho \mathbf{v} - \mathbf{p} \times \mathbf{r}) = \mathbf{r} \times \rho \mathbf{g}, \quad (2.30)$$

where $\mathbf{r}$ is the position vector.

2.5 Rates of Change of Material Integrals

The material derivative d/dt may also be applied to any integral over a collection of fluid elements that remain identifiable and usually well connected during their motion [8]. Indeed, let us assume that the fluid is a continuum such that any closed fluid surface always consists of the same fluid elements, and any fluid element in the enclosed fluid volume always remains inside that fluid volume. (The explicit reference is to a material volume, in contrast to a fixed volume in space that is often called a control volume in the engineering literature.) Thus for any tensor field $\mathbf{F}(\mathbf{r}, t)$, the integral

$$I(\mathbf{F}) = \int_V \mathbf{F}(\mathbf{r}, t) d\tau \quad (2.31)$$

over a simply connected fluid volume V has the rate of change

$$\frac{dI}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\int_{V'} \mathbf{F}(\mathbf{r}', t + \Delta t) d\tau' - \int_V \mathbf{F}(\mathbf{r}, t) d\tau}{\Delta t}. \quad (2.32)$$

Here any element in the fluid volume V , with position vector $\mathbf{r}(t)$ and velocity $\mathbf{v}(\mathbf{r}, t)$ at time t , has the position vector $\mathbf{r}' = \mathbf{r} + \mathbf{v}\Delta t$ to linear order in Δt at time $t + \Delta t$; and the fluid volume V occupies the volume V in space (with closed surface S say) at time t , but then volume V' in space (with closed surface S' say) at time $t + \Delta t$.

Now if the spatial volume element $d\tau$ is not only an element of V but also of V' , there is a contribution $\partial_t \mathbf{F} d\tau$ in the integrand appearing on the right-hand side of (2.32), where the partial derivative is taken at the position of the element $d\tau$ at time t . However, if a spatial volume element $d\tau'$ in V' has no counterpart in V , then $d\tau' = d\mathbf{S} \cdot \mathbf{v} dt$ and this element contributes $\mathbf{F}\mathbf{v} \cdot d\mathbf{S}$ to the integrand arising from the volume integral over V' ; and a spatial element $d\tau$ in V with no counterpart in V' contributes $\mathbf{F}\mathbf{v} \cdot d\mathbf{S}$ to the integrand arising from the volume integral over V , where an opposite sign in $\hat{\mathbf{n}} \cdot \mathbf{v}$ (recall that $\hat{\mathbf{n}}$ is the unit outward normal to any closed surface) compensates for the negative sign on the volume integral over V . In summary, on taking the limit $\Delta t \rightarrow 0$ in (2.32) we obtain the Reynolds transport theorem as applied to material elements—i.e.

$$\frac{d}{dt} \int_V \mathbf{F}(\mathbf{r}, t) d\tau = \int_V \frac{\partial \mathbf{F}}{\partial t} d\tau + \int_S \mathbf{F} \mathbf{v} \cdot d\mathbf{S}, \quad (2.33)$$

or from the relevant dot contraction of (1.59)

$$\frac{d}{dt} \int_V \mathbf{F}(\mathbf{r}, t) d\tau = \int_V \left[\frac{\partial \mathbf{F}}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{F}) \right] d\tau, \quad (2.34)$$

or on invoking (2.22)

$$\frac{d}{dt} \int_V \mathbf{F}(\mathbf{r}, t) d\tau = \int_V \left(\frac{d\mathbf{F}}{dt} + \mathbf{F} \nabla \cdot \mathbf{v} \right) d\tau. \quad (2.35)$$

Identifying $\mathbf{F} \equiv 1$ in either (2.34) or (2.35) yields

$$\frac{d}{dt} \int_V d\tau = \int_V \nabla \cdot \mathbf{v} d\tau, \quad (2.36)$$

defining the rate of dilatation (expansion or contraction) of the fluid volume V , measured by the divergence $\nabla \cdot \mathbf{v}$ as mentioned in Sect. 2.2. In particular,

$$\frac{d}{dt} \int_V d\tau = 0$$

for any arbitrary fluid volume V implies the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ everywhere in the fluid, which from (2.35) is evidently necessary and sufficient for the material derivative and volume integral operations to commute. Identifying $\mathbf{F}$ in (2.34) with the fluid density $\rho(\mathbf{r}, t)$ yields

$$\frac{d}{dt} \int_V \rho(\mathbf{r}, t) d\tau = \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] d\tau = 0, \quad (2.37)$$

if the fluid body is conserved (there is no internal source or sink). Thus (2.37) implies the continuity equation (2.3) everywhere in the fluid, if (2.37) applies to any arbitrary fluid volume V . Identifying $\mathbf{F}$ in (2.35) with the fluid density $\rho(\mathbf{r}, t)$ likewise yields the alternative form (2.27).

The time rate of change of momentum of an arbitrary fluid volume corresponds to setting $\mathbf{F} = \rho \mathbf{v}$ in (2.34). Then by postulating that this rate of change of momentum equals the sum (over the volume elements) of the total force acting on each as in (2.28), on applying the relevant dot contraction of (1.59) we have

$$\frac{d}{dt} \int_V \rho \mathbf{v} d\tau = \int_V \left[\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) \right] d\tau = \int_V \rho \mathbf{g} d\tau - \int_S \mathbf{p} \cdot d\mathbf{S}, \quad (2.38)$$

which identifies the gravity $\rho \mathbf{g}$ as a “body force” density and the pressure tensor $\mathbf{p}$ with a “surface force” acting on the surface S of the fluid volume V . Conversely, since the fluid volume V is arbitrary, (2.38) may be regarded as the origin of (2.20) on invoking (2.21)—or of course (2.28), after also invoking (2.22).

The field $\mathbf{F}$ may be identified with other quantities such as the energy density, but there is often a simple relation between the fluid pressure p and density ρ that leads to a relatively simple complete dynamical description. This is discussed further in the following sections, which are devoted to the derivation of the macroscopic equations of Fluid Mechanics and MHD from a kinetic description of the motion of the microscopic constituents of fluids and plasmas.

For later reference, let us also note here that the rate of change of the material line integral of a vector field $\mathbf{f}$, taken over a collection of fluid elements forming a closed curve C that remains simple and connected, is

$$\begin{aligned} \frac{d}{dt} \oint_C \mathbf{f} \cdot d\mathbf{r} &= \oint_C \left(\frac{d\mathbf{f}}{dt} + (\nabla \mathbf{v}) \cdot \mathbf{f} \right) \cdot d\mathbf{r} \\ &= \oint_C \left(\frac{d\mathbf{f}}{dt} - (\nabla \mathbf{f}) \cdot \mathbf{v} \right) \cdot d\mathbf{r} \end{aligned} \quad (2.39)$$

since $\nabla(\mathbf{v} \cdot \mathbf{f}) = (\nabla \mathbf{v}) \cdot \mathbf{f} + (\nabla \mathbf{f}) \cdot \mathbf{v}$. In particular, identifying $\mathbf{f} \equiv \mathbf{v}$ in (2.39) yields the rate of change of the fluid circulation

$$\frac{d}{dt} \oint_C \mathbf{v} \cdot d\mathbf{r} = \oint_C \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} \quad (2.40)$$

since $(\nabla \mathbf{v}) \cdot \mathbf{v} = \nabla(\mathbf{v} \cdot \mathbf{v}/2) = \nabla(v^2/2)$ —i.e. the material derivative and line integral operations commute in this case. There is also the rate of change result for a material surface, discussed and invoked in Sect. 5.8.

Exercise

- (Q1) Sketch a proof for the result (2.39), noting that values of the velocity field at the end-points of any infinitesimal directed material line segment $d\mathbf{r}'$ are respectively $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$, where $d\mathbf{v} = d\mathbf{r} \cdot \nabla \mathbf{v}$.

2.6 Equations of Change

As mentioned earlier, on a microscopic scale the motion of gases and liquids (fluids) is described in terms of their constituent atoms or molecules—and in the case of ionised gases (plasmas), their constituent ions and electrons [3]. A kinetic equation that allows for encounters between the microscopic particles was first considered by Boltzmann in 1872, to describe the classical dynamics of gases. Thus for fluids it is normally assumed that the atoms or molecules are correlated due to binary collisions over distances much less than the macroscopic length scale and at frequencies much greater than any characteristic macroscopic frequency, as previously indicated in Sect. 2.2. The rapid mobility of electrons in plasma, due to their significantly smaller mass relative to any other species, ensures that they are almost instantaneously

positioned such that the electric field of any charge is screened out on a scale characterised by the Debye length—and consequently, any collision between the charged particles in a plasma is also predominantly binary over distances bounded below by the Coulomb length and above by the Debye length [10]. This notion of charged particle screening was introduced in the theory of electrolytes developed by Debye and Hückel.⁴ The term plasma normally implies that the Debye length scale is much less than the macroscopic length scale, and that electric charge oscillations due to small perturbations from the typical quasi-neutral configuration in plasmas occur at frequencies much greater than any characteristic macroscopic frequency.

For each constituent species s , a Boltzmann-type kinetic equation may be introduced to define the evolution in time t of a velocity distribution function $f_s(\mathbf{r}, \mathbf{c}, t)$ in phase space with coordinates the particle position $\mathbf{r}$ and velocity $\mathbf{c}$:

$$\frac{\partial f_s}{\partial t} + \mathbf{c} \cdot \nabla f_s + \mathbf{a}_s(\mathbf{r}, \mathbf{c}, t) \cdot \nabla_{\mathbf{c}} f_s = \frac{\delta f_s}{\delta t}, \quad (2.41)$$

where the acceleration of the particle of mass m_s and charge e_s is

$$\mathbf{a}_s(\mathbf{r}, \mathbf{c}, t) = \mathbf{g} + \frac{e_s}{m_s} [\mathbf{E}(\mathbf{r}, t) + \mathbf{c} \times \mathbf{B}(\mathbf{r}, t)],$$

if both gravitational and electromagnetic components are included ($\mathbf{g}$, $\mathbf{E}$ and $\mathbf{B}$ denote the respective gravitational, electric and magnetic fields) [3, 9]. The symbol $\nabla_{\mathbf{c}}$ denotes the gradient operator relative to the independent velocity vector $\mathbf{c}$, analogous to ∇ relative to the position vector $\mathbf{r}$ in phase space, and the right-hand side of (2.41) represents the time rate of change of the velocity distribution function due to the microscopic particle collisions.

Any macroscopic field equation of interest corresponds to taking some moment of Eq. (2.41). If $\mathbf{w} = \mathbf{c} - \mathbf{c}_0$ denotes the peculiar velocity for the species relative to some reference velocity $\mathbf{c}_0$, the moment corresponding to any related function $\Psi(\mathbf{w})$ is defined by

$$n_s \langle \Psi \rangle = \int f_s(\mathbf{r}, \mathbf{c}, t) \Psi(\mathbf{w}) d\mathbf{c}, \quad (2.42)$$

including the particle number density (number of particles in a unit volume)

$$n_s(\mathbf{r}, t) = \int f_s(\mathbf{r}, \mathbf{c}, t) d\mathbf{c}.$$

We prefer to identify $\mathbf{c}_0$ with the mass-weighted mean velocity $\mathbf{v}$, defined via the total density and total mass flux

$$\rho = \sum_s \rho_s, \quad \rho \mathbf{v} = \sum_s \rho_s \mathbf{v}_s,$$

⁴P. Debye and E. Hückel, *Physikalische Zeitschrift* **24**, 183 and 305 (1923).

analogous to (2.6) and (2.7) in Sect. 2.2 except that the summation is now over all species present (rather than beams)—i.e. where the summations involve the density ρ_s and the mean flow velocity $\mathbf{v}_s = \langle \mathbf{c} \rangle$ of the particular species (cf. below). The consequent general equation of change that follows from (2.41) may be written in the form

$$\frac{\partial(n_s \langle \Psi \rangle)}{\partial t} + \nabla \cdot (n_s \langle (\mathbf{v} + \mathbf{w}) \Psi \rangle) - n_s \left\langle \left(\mathbf{f}_s + \frac{e_s}{m_s} \mathbf{w} \times \mathbf{B} - \mathbf{w} \cdot \nabla \mathbf{v} \right) \cdot \nabla_{\mathbf{w}} \Psi \right\rangle = C_s(\Psi), \quad (2.43)$$

which involves the acceleration

$$\mathbf{f}_s = \mathbf{g} + \frac{e_s}{m_s} [\mathbf{E} + \mathbf{v} \times \mathbf{B}] - \frac{d\mathbf{v}}{dt}$$

and the collision integral

$$C_s(\Psi) = \int \frac{\delta f_s}{\delta t} \Psi d\mathbf{c}. \quad (2.44)$$

This form is a simplification of the Enskog equation, analogous to an equation of change due to Maxwell, because Ψ is only explicitly dependent on $\mathbf{w}$ (cf. Chapman and Cowling [3]). Equation (2.43) replaces (2.41) as our starting point. In passing, we note that the subscript s has been omitted from both $\mathbf{c}$ and $\mathbf{w} = \mathbf{c} - \mathbf{v}$, since $\mathbf{c}$ becomes a dummy integration variable.

Fundamental field quantities in Fluid Mechanics and MHD are related to the lower moments defined by (2.42) for each species s of particle mass m_s —viz.

$$\begin{aligned} \rho_s &\equiv n_s m_s && \text{(density)} \\ \mathbf{u}_s &\equiv \langle \mathbf{w} \rangle && \text{(mean relative velocity)} \\ p_s &\equiv n_s \left\langle \frac{1}{3} m_s w^2 \right\rangle && \text{(pressure)} \\ \mathbf{p}_s &\equiv n_s \langle m_s \mathbf{w} \mathbf{w} \rangle && \text{(pressure tensor)} \\ \mathbf{q}_s &\equiv n_s \left\langle \frac{1}{2} m_s w^2 \mathbf{w} \right\rangle && \text{(thermal flux).} \end{aligned}$$

In passing, we note that $\mathbf{u}_s = \mathbf{v}_s - \mathbf{v}$ is the mean species velocity relative to the reference velocity $\mathbf{v}$, the species pressure tensor $\mathbf{p}_s = p_s \mathbf{I} + \mathbf{t}_s$ has the symmetric traceless part $\mathbf{t}_s \equiv \{\mathbf{p}_s\} = n_s \langle m_s \{\mathbf{w} \mathbf{w}\} \rangle$, and the pressure p_s and thermal flux vector $\mathbf{q}_s$ respectively determine the species contribution to the pressure and heat transfer. (The fundamental macroscopic equations discussed in Sect. 2.7 involve the reference velocity $\mathbf{v}$ and the respective total fields ρ , p , $\mathbf{p}$ and $\mathbf{q}$ obtained by summation over the species present, together with the current density and electromagnetic fields in the case of MHD as considered there.)

Other unnamed higher moments arising in the basic equations of change below are the third-degree $\mathbf{H}_s = n_s \langle m_s \mathbf{w} \mathbf{w} \mathbf{w} \rangle = (2/3) \mathbf{q}_s \mathbf{l} + \mathbf{h}_s$ where $\mathbf{h}_s = n_s \langle m_s \mathbf{w} \{ \mathbf{w} \mathbf{w} \} \rangle$, and the fourth-degree $\ell_s = n_s \langle \frac{1}{2} m_s w^2 \mathbf{w} \mathbf{w} \rangle$ or

$$\mathbf{L}_s = \ell_s - \frac{5}{2} \frac{p_s}{\rho_s} \mathbf{p}_s. \quad (2.45)$$

Thus if Ψ is identified with $m_s, m_s \mathbf{w}, \frac{1}{2} m_s w^2, m_s \{ \mathbf{w} \mathbf{w} \}, \frac{1}{2} m_s w^2 \mathbf{w}$, etc., successively, then for each species the general equation of change (2.43) produces the basic equations for the mass (continuity), momentum, thermal energy, $\mathbf{t}_s$ and $\mathbf{q}_s$, etc.:

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v} + \rho_s \mathbf{u}_s) = C_s(m_s) \quad (2.46)$$

$$\begin{aligned} \frac{\partial(\rho_s \mathbf{u}_s)}{\partial t} + \nabla \cdot (\rho_s \mathbf{v} \mathbf{u}_s + \mathbf{p}_s) \\ - \rho_s \mathbf{f}_s - \rho_s \frac{e_s}{m_s} \mathbf{u}_s \times \mathbf{B} + \rho_s \mathbf{u}_s \cdot \nabla \mathbf{v} = C_s(m_s \mathbf{w}) \end{aligned} \quad (2.47)$$

$$\frac{\partial(\frac{3}{2} p_s)}{\partial t} + \nabla \cdot (\frac{3}{2} p_s \mathbf{v} + \mathbf{q}_s) - \rho_s \mathbf{f}_s \cdot \mathbf{u}_s + \mathbf{p}_s : \nabla \mathbf{v} = C_s\left(\frac{1}{2} m_s w^2\right) \quad (2.48)$$

$$\begin{aligned} \frac{\partial \mathbf{t}_s}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{t}_s + \mathbf{h}_s) - 2 \rho_s \{ \mathbf{f}_s \mathbf{u}_s \} + 2 \{ \mathbf{p}_s \cdot \nabla \mathbf{v} \} \\ - 2 \frac{e_s}{m_s} \{ \mathbf{t}_s \times \mathbf{B} \} = C_s(m_s \{ \mathbf{w} \mathbf{w} \}) \end{aligned} \quad (2.49)$$

$$\begin{aligned} \frac{\partial \mathbf{q}_s}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{q}_s + \ell_s) - \mathbf{f}_s \cdot \mathbf{p}_s - \frac{3}{2} p_s \mathbf{f}_s - \frac{e_s}{m_s} \mathbf{q}_s \times \mathbf{B} \\ + \mathbf{q}_s \cdot \nabla \mathbf{v} + \mathbf{H}_s : \nabla \mathbf{v} = C_s\left(\frac{1}{2} m_s w^2 \mathbf{w}\right) \end{aligned} \quad (2.50)$$

etc. On writing $p_s = n_s k T_s$, where k denotes the Boltzmann constant and T_s the kinetic temperature of the species, Eq. (2.50) for $\mathbf{q}_s$ may be replaced by subtracting $\frac{5}{2} p_s / \rho_s$ times equation (2.48) from (2.50)—i.e.

$$\begin{aligned} \frac{\partial \mathbf{R}_s}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{R}_s + \mathbf{L}_s) - \mathbf{f}_s \cdot \mathbf{t}_s - \frac{e_s}{m_s} \mathbf{R}_s \times \mathbf{B} + \frac{5}{2} \frac{k}{m_s} \mathbf{p}_s \cdot \nabla T_s \\ + \frac{5}{2} n_s k \mathbf{u}_s \frac{dT_s}{dt} + \mathbf{R}_s \cdot \nabla \mathbf{v} + \mathbf{H}_s : \nabla \mathbf{v} = C_s\left(\frac{1}{2} m_s \left(w^2 - \frac{5kT_s}{m_s}\right) \mathbf{w}\right), \end{aligned} \quad (2.51)$$

which we invoke to analyse thermal effects (cf. Sect. 2.11).

Each equation in the hierarchy of basic equations of change introduces a higher moment. The necessary closure for the subsequent derivation of any related fluid mechanics or MHD model corresponds to either simply ignoring a higher moment at some level or approximating the velocity distribution function in some acceptable way, as discussed further in the following sections. Subsets of these basic equations of change, or an extension to even higher levels in the hierarchy, could be used as the macroscopic description for each species in terms of the associated variables one chooses to retain. In this book we continue to adopt the more traditional classical macroscopic equations obtained from (2.46)–(2.48) as discussed in the next section—but then we use (2.49) and (2.51) to derive appropriate explicit relations for $\mathbf{t}_s$ in Sect. 2.9 and $\mathbf{R}_s$ (and hence $\mathbf{q}_s$) in Sect. 2.11, in our subsequent discussion of the extended macroscopic models we do consider.

Finally, we remark that following Braginskii many plasma theorists have adopted the mean flow velocity $\mathbf{v}_s$ of the particular species as the reference velocity $\mathbf{c}_0$ in defining the moments [5], rather than the universal mass-weighted mean velocity $\mathbf{v}$ as in Chapman and Cowling [3] that we prefer. Thus recalling $\mathbf{w} = \mathbf{c} - \mathbf{v}$ and $\mathbf{v}_s = \langle \mathbf{c} \rangle$, from (2.42) we have

$$n_s \langle \mathbf{w} \rangle = \int f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}) d\mathbf{c} = \int f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{c} d\mathbf{c} - \mathbf{v} \int f_s(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} = n_s \langle \mathbf{c} \rangle - n_s \mathbf{v}$$

such that (as mentioned above) $\mathbf{u}_s = \mathbf{v}_s - \mathbf{v}$ or $\mathbf{v}_s = \mathbf{v} + \mathbf{u}_s$, so Eq. (2.46) may be rewritten as

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s) = \mathcal{C}(m_s). \quad (2.52)$$

The corresponding momentum equation for the species is

$$m_s n_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right) + \nabla \cdot \mathbf{P}_s - n_s e_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - m_s n_s \mathbf{g} = \mathbf{F}_s^{\text{coll}}, \quad (2.53)$$

involving the pressure tensor

$$\mathbf{P}_s(\mathbf{r}, t) = \int m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c}$$

defined using the mean flow velocity $\mathbf{v}_s$ of the particular species. Noting that $\mathbf{P}_s = \mathbf{p}_s - m_s n_s \mathbf{u}_s \mathbf{u}_s$ where $\mathbf{p}_s$ is our form for the pressure tensor, Eq. (2.53) is equivalent to (2.47) on writing $\mathbf{F}_s^{\text{coll}}$ for $\mathcal{C}_s(m_s \mathbf{w})$. Moreover, the heat balance equation for each species is written as

$$\frac{3}{2} n_s k \left(\frac{\partial T_s}{\partial t} + \mathbf{v}_s \cdot \nabla T_s \right) + \nabla \cdot \mathbf{Q}_s + \mathbf{P}_s : \nabla \mathbf{v}_s = G_s^{\text{coll}} \quad (2.54)$$

where T_s denotes the kinetic temperature defined such that $P_s = n_s k T_s$ is the associated hydrostatic pressure, the thermal flux

$$\mathbf{Q}_s = \int \frac{1}{2} m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c}$$

is likewise defined using the mean flow velocity $\mathbf{v}_s$ of the species, and G_s^{coll} is the relevant collision integral.⁵

Exercises

(Q1) Verify $\mathbf{P}_s = \mathbf{p}_s - m_s n_s \mathbf{u}_s \mathbf{u}_s$, and show that (2.53) is equivalent to (2.47).

(Q2) Show that

$$\frac{3}{2} n_s k \left(\frac{\partial T_s}{\partial t} + \mathbf{v}_s \cdot \nabla T_s \right) = \frac{\partial \left(\frac{3}{2} P_s \right)}{\partial t} + \nabla \cdot \left(\frac{3}{2} P_s \mathbf{v}_s \right),$$

and hence that the heat balance equation (2.54) may be rewritten as

$$\frac{\partial \left(\frac{3}{2} P_s \right)}{\partial t} + \nabla \cdot \left(\frac{3}{2} P_s \mathbf{v}_s + \mathbf{Q}_s \right) + \mathbf{P}_s : \nabla \mathbf{v}_s = G_s^{\text{coll}}.$$

Then show that this equation is consistent with (2.48), when the collision term G_s^{coll} is suitably identified.

Hint: Use (2.46), and an energy transport equation given by the dot product of (2.47) with $\mathbf{u}_s$.

(Q3) When the mean flow velocity of the species $\mathbf{v}_s$ is again used as the reference velocity, the next equation in the hierarchy involves the corresponding even higher moment $\mathbf{Q}_s = \int m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c}$ (note the heat flux $\mathbf{Q}_s$ defined in the text is an immediate contraction of $\mathbf{Q}_s$):

$$\frac{\partial \mathbf{P}_s}{\partial t} + \nabla \cdot (\mathbf{v}_s \mathbf{P}_s + \mathbf{Q}_s) + \mathbf{v}_s \overleftarrow{\nabla} \cdot \mathbf{P}_s + \mathbf{P}_s \cdot \nabla \mathbf{v}_s - 2 \frac{e_s}{m_s} \{ \mathbf{P}_s \times \mathbf{B} \} = \mathbf{G}_s^{\text{coll}},$$

where the leftward pointing arrow on the ∇ operator introduced here means it *applies to the left*, $\mathbf{G}_s^{\text{coll}}$ is the appropriate collisional term, and the other notation is the same as in the previous two Exercises. Derive this equation directly by expanding $\nabla \cdot \mathbf{Q}_s$.

Hint: Invoke the Boltzmann-type Eq. (2.41) in the tensor calculus. (We note the traceless part of this higher moment equation is analogous to (2.49), and the analogy to (2.50) is derived in the answer to this Exercise.)⁶

⁵In the case of a plasma consisting of electrons and only one ion species, the mean velocity is effectively the ion velocity ($\mathbf{v} \simeq \mathbf{v}_i$ corresponding to $m_e \ll m_i$ and $\mathbf{u}_i \simeq 0$) so the defined fields such as $\mathbf{P}_i$ and $\mathbf{Q}_i$ for example are essentially the same as $\mathbf{p}_i$ and $\mathbf{q}_i$, respectively. However, this is not so for the electrons, since the alternative reference velocities for the electron component are quite distinct corresponding to inter-species (electron) diffusion—i.e. $\mathbf{u}_e \neq 0$.

⁶The corresponding extension of the hierarchy of equations (where the moments are defined using the species mean flow velocity $\mathbf{v}_s$ as the reference velocity) was originally obtained by J.J. Ramos (*Physics of Plasmas* **14**, 052506, 2007).

2.7 Classical Macroscopic Equations

When the sum over all of the species present in the fluid or plasma is taken and it is assumed the combined contributions from the respective collision integrals $\mathcal{C}_s(m_s)$, $\mathcal{C}_s(m_s \mathbf{w})$ and $\mathcal{C}_s(\frac{1}{2}m_s w^2)$ vanish (i.e. the binary collisions summed over all species preserve mass, momentum and energy), noting that $\sum_s \rho_s \mathbf{u}_s = 0$ we obtain the classical macroscopic equations of mass conservation (continuity), motion and energy (cf. also [3]):

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \quad (2.55)$$

$$\rho \frac{d\mathbf{v}}{dt} + \nabla \cdot \mathbf{p} = \rho \mathbf{g} + \mathbf{j} \times \mathbf{B} \quad (2.56)$$

$$\frac{d}{dt} \left(\frac{3}{2} p \right) + \frac{3}{2} p \nabla \cdot \mathbf{v} + \mathbf{p} : \nabla \mathbf{v} + \nabla \cdot \mathbf{q} = \mathbf{j} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (2.57)$$

Here the material derivative $d/dt = \partial_t + \mathbf{v} \cdot \nabla$ appropriately refers to the mass-weighted mean flow velocity $\mathbf{v}$, the variable subset $\{\rho, \mathbf{p} = p \mathbf{I} + \mathbf{t}, \mathbf{q}\}$ and implicitly p are total quantities (the respective sums over all of the species present), and

$$\mathbf{j} \equiv \sum_s n_s e_s \langle \mathbf{c}_s \rangle = \sum_s \frac{e_s}{m_s} \rho_s \langle \mathbf{c}_s \rangle$$

denotes and defines the *total current density*. Equation (2.55) is identical to (2.27), and the equation of motion (2.56) is comparable with (2.28). An electric force term does not appear in (2.56) because the medium is assumed to be quasi-neutral on the macroscopic scale, due to the charged particle screening outlined in the previous section in the case of a plasma, but the gravitational force $\rho \mathbf{g}$ in the equation of motion is supplemented with the electromagnetic body force $\mathbf{j} \times \mathbf{B}$ that is of major interest in MHD—i.e. whenever the system is electrically conducting.

Noting (2.55), the first two terms on the left-hand side of (2.57) are sometimes re-expressed as $(3/2)n k dT/dt$, where T denotes a systemic temperature. Alternatively, from (2.55) the energy equation (2.57) may be rewritten as

$$\frac{3}{2} \rho^{\frac{5}{3}} \frac{d}{dt} (p \rho^{-\frac{5}{3}}) = -\mathbf{t} : \nabla \mathbf{v} - \nabla \cdot \mathbf{q} + \mathbf{j} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (2.58)$$

where the first term on the right-hand side represents viscous dissipation, the second term the heat transfer (thermal conduction), and the third term the electromagnetic heating. The well-known *adiabatic equation of state*

$$\frac{d}{dt} (p \rho^{-\gamma}) = 0 \quad \text{or} \quad \frac{dp}{dt} + \gamma p \nabla \cdot \mathbf{v} = 0 \quad \left(\text{identifying } \gamma = \frac{5}{3} \right) \quad (2.59)$$

corresponds to neglecting all the terms on the right-hand side of (2.58)—i.e. we assume all three dissipation contributions are negligible. The term “adiabatic” (no heat transfer) used above is more common, but a more precise term is *isentropic*—i.e. (2.59) corresponds to no change in entropy (any *reversible* adiabatic process is isentropic) [2]. Another useful term is *barotropic*, which means that there is some relation between the fluid pressure p and fluid density ρ , but no third thermodynamic variable—e.g. $p\rho^{-\gamma} = \text{constant}$ (following the motion), corresponding to (2.59).

In ideal fluid mechanics and various inviscid MHD models, the traceless component $\mathbf{t}$ of the pressure tensor is simply ignored such that $\nabla \cdot \mathbf{p} = \nabla p$, when the system is described in terms of the field variables $\{\rho, \mathbf{v}, p\}$ by (2.55) and (2.56) together with a barotropic equation of state. There can be similar acceptable dynamical descriptions in viscous Fluid Mechanics and MHD too, if an appropriate expression for $\mathbf{p}$ is adopted (cf. Sects. 2.2 and 2.9). An expression for the thermal flux vector $\mathbf{q}$ may also be included if there is significant heat conduction (cf. Sect. 2.11). They are often called constitutive relations in the literature, and sometimes viewed as phenomenological inputs, but we emphasise that the explicit expressions for $\mathbf{t}_s$ and $\mathbf{R}_s$ (and hence $\mathbf{q}_s$) presented below are obtained from their respective higher basic equations of change. In the case of MHD, where the additional body force $\mathbf{j} \times \mathbf{B}$ is significant, supplementary electromagnetic equations must be included (cf. Sect. 2.12).

2.8 Extended Macroscopic Models

As indicated, ideal fluid mechanics and various inviscid MHD models correspond to truncation of the hierarchy of basic equations of change where only (2.46) and (2.47) are retained, to consider just five scalar field variables $\{\rho_s, \mathbf{v}_s, p_s\}$. This is sometimes referred to as the “five-moment” description, albeit with additional electromagnetic variables in MHD. However, the extension to non-ideal Fluid Mechanics and MHD corresponds to including $\mathbf{t}_s$ in a “ten-moment” description, or both $\mathbf{t}_s$ and $\mathbf{q}_s$ in a “thirteen-moment” description, and so on. Thus if the term $\nabla \cdot \mathbf{t}$ is to be retained in expressing $\nabla \cdot \mathbf{p}$ in (2.56), one may proceed to derive an extended macroscopic model by augmenting the lowest level Eqs. (2.46) and (2.47) with the additional basic equation of change (2.49) for $\mathbf{t}_s$ in the underlying hierarchy; and if thermal effects are to be considered, then the extension also involves including (2.50) for $\mathbf{q}_s$, or equivalently (2.51) for $\mathbf{R}_s$.

In collisionless theory, the terms on the right-hand sides of the fundamental equations of change are omitted entirely, but various representations have also been considered. Here we retain the collisional terms for each species s in the simplest way—viz. linear expressions for the integrals in (2.49) and (2.51). On tensor rank alone, these expressions must involve their respective moments and are thus

$$C_s(m_s\{\mathbf{w}\mathbf{w}\}) = - \sum_j \frac{\vartheta_{sj}}{\tau_{sj}} \mathbf{t}_j, \quad (2.60)$$

$$C_s \left(\frac{1}{2} m_s \left(w^2 - \frac{5kT_s}{m_s} \right) \mathbf{w} \right) = - \sum_j \left(\frac{1}{\tau_{sj}} \mathbf{R}_j + \zeta_{sj} \rho_j \mathbf{u}_j \right), \quad (2.61)$$

with the coefficients $\{\tau_{sj}, \vartheta_{sj}, \zeta_{sj}\}$ functions of number density and temperature. As previously indicated, the “ten moment” description is closed if we may ignore terms in the two higher moments $\mathbf{q}_s$ and $\mathbf{h}_s$ from (2.49), in truncating at one level beyond the “five moment” description to include the term $\nabla \cdot \mathbf{t}_s$ from $\nabla \cdot \mathbf{p}_s$ in (2.47). The “thirteen moment” description, corresponding to also including Eq. (2.50) for $\mathbf{q}_s$ or equivalently Eq. (2.51) for $\mathbf{R}_s$ together with Eqs. (2.46)–(2.49) as previously mentioned, could likewise be achieved at that level in the hierarchy of basic equations by omitting terms in $\mathbf{h}_s$ and ℓ_s —and yet higher “ n moment” ($n > 13$) closure could be considered by including even more of the hierarchy and neglecting terms in moments beyond the corresponding set.

Although the ultimate test of the accuracy of the closure at any level is of course experimental or observational agreement with the consequent mathematical model, there have been two major theoretical approaches to this issue. In deriving macroscopic field equations from the kinetic equation (2.41), various asymptotic schemes based upon some small parameter in a particular physical context have been proposed in order to estimate the approximation involved [5]. There is also an elegant procedure where the velocity distribution function is represented as a series expansion, appropriately truncated for an assumed level of closure of the hierarchy of exact equations of change. A well-known example consistent with the expressions (2.60) and (2.61) is Grad’s expansion in multidimensional Hermite polynomials, where the truncated expression

$$f_s \simeq f_s^0 \left[a_s^0 + \mathbf{a}_s^1 \cdot \mathbf{w} + \mathbf{a}_s^2 : \mathbf{w}\mathbf{w} + \mathbf{a}_s^3 \cdot \mathbf{w} \left(\alpha_s w^2 - \frac{5}{2} \right) \right] \quad (2.62)$$

with the Maxwellian zeroth-order form

$$f_s^0 = n_s \left(\frac{\alpha_s}{\pi} \right)^{\frac{3}{2}} \exp(-\alpha_s w^2) \quad (2.63)$$

provides his “thirteen moment approximation”.⁷ Particle collisions tend to drive the distribution function towards this Maxwellian form, which is associated with “local thermodynamic equilibrium”. Consequently, on adopting (2.62) and (2.63) and substituting into (2.42), from the respective lower moments $\langle 1 \rangle$, $\langle \mathbf{w} \rangle$, $\langle \frac{1}{2} m_s w^2 \rangle$, $\langle \{m_s \mathbf{w}\mathbf{w}\} \rangle$ and $\langle \{ \frac{1}{2} m_s \mathbf{w} w^2 \} \rangle$ we obtain

$$\begin{aligned} a_s^0 &= 1, & \mathbf{a}_s^1 &= 2\alpha_s \mathbf{u}_s, \\ \alpha_s &= m_s/2kT, & \mathbf{a}_s^2 &= (\alpha_s/p_s) \mathbf{t}_s, & \mathbf{a}_s^3 &= \frac{4}{5} (\alpha_s/p_s) \mathbf{R}_s, \end{aligned} \quad (2.64)$$

⁷H. Grad (*Communications in Pure and Applied Mathematics* **2**, 231–407, 1949—cf. also *Handbuch der Physik*, S. Flügge (Editor), Volume 12, Chapter X, Springer, Berlin, 1958).

together with the relevant results for the higher moments

$$\nabla \cdot \mathbf{h}_s = \frac{4}{5} \{ \nabla \mathbf{q}_s \} , \quad \mathbf{H}_s : \nabla \mathbf{v} = \frac{2}{5} [\mathbf{q}_s \cdot \nabla \mathbf{v} + \mathbf{q}_s \nabla \cdot \mathbf{v} + (\nabla \mathbf{v}) \cdot \mathbf{q}_s] , \quad \mathbf{L}_s = \frac{p_s}{\rho_s} \mathbf{t}_s \quad (2.65)$$

and the coefficients in the expressions (2.60) and (2.61) for the collision integrals. Closure therefore occurs at a level involving no more than the thirteen scalar fields corresponding to the variables $\rho_s = m_s n_s$, $\mathbf{u}_s$, $p_s = n_s k T_s$, $\mathbf{t}_s$, and $\mathbf{q}_s$. Retaining only the first three expansion coefficients listed in (2.64) produces the related “ten moment” description in $\{\rho_s, \mathbf{u}_s, p_s, \mathbf{t}_s\}$, and the heat flux vector $\mathbf{R}_s$ (and hence $\mathbf{q}_s$) included in the “thirteen moment” description is completely determined by the fourth coefficient. Balescu [1] followed Grad by expanding f_s in terms of Hermite polynomials, and other authors have considered truncated expansions involving alternative basis functions (e.g. Sonine-Laguerre polynomials in a “fifteen moment” approximation). The expansion could also be taken about some other zeroth-order distribution function. Indeed, an appropriate number of leading terms in an integrable series expansion of orthogonal functions about the most suitable zeroth-order form (not necessarily the Maxwellian) probably represents almost everywhere the best possible expression for the relevant velocity distribution function, given the corresponding number of basic equations of change for the species one chooses to retain. Thus the fluid or MHD models eventually obtained, upon truncation of the underlying hierarchies of exact equations of change, may be more widely applicable than is traditionally thought.

2.9 Plasma Pressure Tensor

2.9.1 Invariant Form for the Traceless Component

Let us now proceed to consider the traceless pressure tensor component $\mathbf{t}_s$, implicit in (2.49) for any species, with the ultimate objective to render an appropriate form for the pressure tensor in plasma. In a magnetic field, it emerges that the consequent explicit relation for $\mathbf{t}_s$ is usually quite different for charged species than for neutrals. Thus the total pressure tensor $\mathbf{p}$ in (2.56) for a magnetised plasma differs from the form for a fluid, which was previously obtained in Sect. 2.2 using an heuristic argument. We adopt the linear representation (2.60), *but for notational simplicity now suppress the subscript s denoting the chosen species where there is no chance of confusion* (including writing $\vartheta/\tau = \vartheta_{ss}/\tau_{ss}$), so (2.49) may be re-expressed as

$$\begin{aligned} \frac{d\mathbf{t}}{dt} + \frac{\vartheta}{\tau} \mathbf{t} + \mathbf{t} \nabla \cdot \mathbf{v} - 2\rho\{\mathbf{f} \mathbf{u}\} + 2\{\mathbf{p} \cdot \nabla \mathbf{v}\} \\ - 2\frac{e}{m}\{\mathbf{t} \times \mathbf{B}\} + \nabla \cdot \mathbf{h} = - \sum_{j \neq s} \frac{\vartheta_{sj}}{\tau_{sj}} \mathbf{t}_j. \end{aligned} \quad (2.66)$$

It is convenient to introduce a characteristic frequency ω to represent the time derivative term $d\mathbf{t}/dt$ as $\omega\mathbf{t}$ in Eq. (2.66), which may then be rewritten in the notationally convenient form as

$$\mathbf{t} - 2\{\mathbf{t} \times \mathbf{a}\} = -2\mu\mathbf{s} \quad (2.67)$$

where the symmetric and traceless generalised rate of deformation (or strain) tensor $\mathbf{s}$ is given by

$$2p\mathbf{s} = -2\rho\{\mathbf{f} \mathbf{u}\} + 2p\{\nabla \mathbf{v}\} + \mathbf{t} \nabla \cdot \mathbf{v} + 2\{\mathbf{t} \cdot \nabla \mathbf{v}\} + \nabla \cdot \mathbf{h} + \sum_{j \neq s} \frac{\vartheta_{sj}}{\tau_{sj}} \mathbf{t}_j. \quad (2.68)$$

The coefficient $\mu = \frac{p\tau}{\omega\tau + \vartheta}$ and the vector $\mathbf{a} = \frac{e\mathbf{B}}{m} \frac{\tau}{\omega\tau + \vartheta}$ (proportional to the gyrofrequency $\omega_c = eB/m$) introduced in (2.67) reflect the first time derivative and the second collisional terms in (2.66), with one or the other predominant if the magnitude of the parameter $\omega\tau$ is, respectively, sufficiently large or small. Note that we refer here to collisions between particles of the particular species under consideration (with its appropriate factor ϑ typically of order one), and any collisional coupling between the different species is represented by the summation for $j \neq s$ incorporated in $\mathbf{s}$. The form (2.67) isolates the term $\{\mathbf{t} \times \mathbf{a}\}$ proportional to the magnetic field $\mathbf{B}$, which enables us to obtain $\mathbf{t}$ as an explicit function of $\mathbf{s}$ below—an important outcome even though there are two residual terms involving $\mathbf{t}$ in the generalised deformation tensor defined by (2.68), a point we examine further in the supplementary discussion for a simple ion-electron plasma in Sect. 2.10.

Using tensor identities, we may re-express (2.67) as (cf. Exercises)

$$\begin{aligned} \mathbf{t} = - \frac{2\mu}{(1 + |\mathbf{a}|^2)(1 + 4|\mathbf{a}|^2)} [(\mathbf{s} + 2\{\mathbf{s} \times \mathbf{a}\})(1 + |\mathbf{a}|^2) \\ + 6\{(\mathbf{s} \cdot \mathbf{a}\mathbf{a}) + 2\{(\mathbf{s} \cdot \mathbf{a}\mathbf{a}) \times \mathbf{a}\} + 6\mathbf{s} : \mathbf{a}\mathbf{a}\{\mathbf{a}\mathbf{a}\}]. \end{aligned} \quad (2.69)$$

The Cartesian representation of this exact form (2.69) appears in Chapman and Cowling [3], who assumed a uniform magnetic field $\mathbf{B}$ —but we now recognise that (2.69) is an invariant result (i.e. valid in any system of coordinates) without restriction on the magnetic field. Equation (2.67) and likewise the result (2.69) obviously always reduce to $\mathbf{t} = -2\mu\mathbf{s}$ for a neutral species and for every species if there is no magnetic field (i.e. for $\mathbf{a} = 0$), when we recover the shear viscosity form familiar from Sect. 2.2

on identifying $\mathbf{s} = \{\nabla \mathbf{v}\}$. However, the terms involving the vector $\mathbf{a}$ usually dominate for charged particles in the presence of a magnetic field, producing characteristically anisotropic plasma viscosity contributions as follows.

2.9.2 Parallel, Cross and Perpendicular Components

For any species of charged particles in magnetised plasma (i.e. with a permeating magnetic field), typically $|\mathbf{a}| \gg 1$ except in the near neighbourhood of magnetic null points. Thus on noting

$$-\frac{2\mu}{(1 + |\mathbf{a}|^2)(1 + 4|\mathbf{a}|^2)} = \frac{\mu}{2} \left[-|\mathbf{a}|^{-4} + \frac{5}{4}|\mathbf{a}|^{-6} + \dots \right],$$

the general explicit form (2.69) may consequently be expanded to obtain

$$\mathbf{t} = \mathbf{t}_{\parallel} + \mathbf{t}_{\text{g}} + \mathbf{t}_{\perp} + \dots, \quad (2.70)$$

where the *parallel*, *cross* (or *gyroviscous*) and *perpendicular* components

$$\mathbf{t}_{\parallel} = -3\mu \mathbf{s} : \hat{\mathbf{b}}\hat{\mathbf{b}}\{\hat{\mathbf{b}}\hat{\mathbf{b}}\}, \quad (2.71)$$

$$\mathbf{t}_{\text{g}} = -\frac{\mu}{|\mathbf{a}|} \left\{ \mathbf{s} \times \hat{\mathbf{b}} + 6\{\mathbf{s} \cdot \hat{\mathbf{b}}\hat{\mathbf{b}}\} \times \hat{\mathbf{b}} \right\} \quad \text{and} \quad (2.72)$$

$$\mathbf{t}_{\perp} = -\frac{\mu}{2|\mathbf{a}|^2} \left[\mathbf{s} + 6\{\mathbf{s} \cdot \hat{\mathbf{b}}\hat{\mathbf{b}}\} - \frac{15}{2}\mathbf{s} : \hat{\mathbf{b}}\hat{\mathbf{b}}\{\hat{\mathbf{b}}\hat{\mathbf{b}}\} \right] \quad (2.73)$$

are named with reference to the magnetic field direction $\hat{\mathbf{b}} \equiv \mathbf{B}/|\mathbf{B}|$. The coefficient $\mu/|\mathbf{a}|$ in (2.72) is entirely independent of the collision time τ , and this second-order component is often called the gyroviscous or alternatively the “finite Larmor radius” (FLR) contribution due to an association with the charged particle gyration in the magnetic field. It has been derived elsewhere from the Vlasov (“collisionless Boltzmann”) equation, on assuming that the microscopic particles are correlated through their interaction with the magnetic field. However, the leading parallel component (2.71) remains in the low collisional limit (cf. μ as defined above and also Sect. 2.10), and the magnetic field need not be strong to satisfy $|\mathbf{a}| \gg 1$ nor otherwise restricted in any way.

The parallel component (2.71) produces an anisotropic plasma pressure on its own, for the resulting pressure tensor may be written as $\mathbf{p} = p_{\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} + p_{\perp}\mathbf{I}_{\perp}$ with $p_{\parallel} = p + t_{\parallel,\parallel}$ and $p_{\perp} = p - (1/2)t_{\parallel,\parallel}$ where the projector into the plane locally perpendicular to $\mathbf{B}$ is defined by

$$\mathbf{I}_{\perp} = \mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}, \quad (2.74)$$

since we have $t_{\parallel,\parallel} \equiv \mathbf{t} : \hat{\mathbf{b}}\hat{\mathbf{b}} = \mathbf{t}_{\parallel} : \hat{\mathbf{b}}\hat{\mathbf{b}} = -2\mu\mathbf{s} : \hat{\mathbf{b}}\hat{\mathbf{b}}$ and

$$\{\hat{\mathbf{b}}\hat{\mathbf{b}}\} = \hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathbf{I} = -\frac{1}{3}(\mathbf{I}_{\perp} - 2\hat{\mathbf{b}}\hat{\mathbf{b}}). \quad (2.75)$$

Further, on noting $\mathbf{s} : \hat{\mathbf{b}}\hat{\mathbf{b}} = \hat{\mathbf{b}} \cdot \mathbf{s} \cdot \hat{\mathbf{b}}$ we may re-express (2.71) as

$$\mathbf{t}_{\parallel} = \mu \hat{\mathbf{b}} \cdot \mathbf{s} \cdot \hat{\mathbf{b}} (\mathbf{I}_{\perp} - 2\hat{\mathbf{b}}\hat{\mathbf{b}}). \quad (2.76)$$

The right-hand sides of (2.72) and (2.73) may also be expanded to yield alternative expressions for the gyroviscous (cross) and perpendicular components (Exercise Q3):

$$\mathbf{t}_g = \frac{\mu}{2|\mathbf{a}|} \left[\hat{\mathbf{b}} \times \mathbf{s} \cdot \mathbf{I}_{\perp} - \mathbf{I}_{\perp} \cdot \mathbf{s} \times \hat{\mathbf{b}} + 4 \left(\hat{\mathbf{b}} \times \mathbf{s} \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} - \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \mathbf{s} \times \hat{\mathbf{b}} \right) \right], \quad (2.77)$$

$$\mathbf{t}_{\perp} = -\frac{\mu}{2|\mathbf{a}|^2} \left[\mathbf{I}_{\perp} \cdot \mathbf{s} \cdot \mathbf{I}_{\perp} + \frac{1}{2} \hat{\mathbf{b}} \cdot \mathbf{s} \cdot \hat{\mathbf{b}} \mathbf{I}_{\perp} + 4 \left(\mathbf{I}_{\perp} \cdot \mathbf{s} \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} + \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \mathbf{s} \cdot \mathbf{I}_{\perp} \right) \right]. \quad (2.78)$$

In summary, except in the near neighbourhood of a magnetic null, in a magnetic field the components (2.71)–(2.73)—or their equivalents (2.76)–(2.78)—provide the appropriate representation for the traceless pressure tensor component $\mathbf{t}_s$ for either ions or electrons, in any coordinate system.⁸

Exercises

(Q1) If $\mathbf{t}$ is a symmetric traceless dyadic and $\mathbf{a}$ is a vector, show that

$$2\{\{\mathbf{t} \times \mathbf{a}\} \times \mathbf{a}\} = 3\{\mathbf{t} \cdot \mathbf{a}\mathbf{a}\} - 2|\mathbf{a}|^2\mathbf{t},$$

$$6\{\mathbf{t} \cdot \mathbf{a}\mathbf{a}\} \cdot \mathbf{a} = 3\mathbf{t} \cdot \mathbf{a} |\mathbf{a}|^2 + \mathbf{a} \mathbf{t} : \mathbf{a}\mathbf{a}$$

and

$$\{\{\mathbf{t} \times \mathbf{a}\} \cdot \mathbf{a}\mathbf{a}\} = \{\mathbf{t} \cdot \mathbf{a}\mathbf{a}\} \times \mathbf{a}.$$

(Q2) Take a post-cross product of (2.67) with $\mathbf{a}$, followed by the bracket operator defined in the footnote in Sect. 2.2, to obtain

⁸The result (2.69) was originally obtained by B.S. Liley, and the successive terms (2.71)–(2.73) for a sufficiently large magnetic field (when $|\mathbf{a}| \gg 1$) were identified by R.J. Hosking and G.M. Marinoff (*Plasma Physics* **15**, 327–341, 1971). The alternative forms (2.76)–(2.78) were presented by J.D. Callen, W.X. Qu, K.D. Siebert, B.A. Carreras, K.C. Shaing and D.A. Spong (*Plasma Physics and Controlled Nuclear Fusion Research*, Volume II, IAEA, Vienna, 1987), and essentially earlier by S.I. Braginskii (*Reviews of Plasma Physics* **1**, 205–311, 1965) in the collisional limit—cf. the traceless component of his pressure tensor. In the absence of collisions, an anisotropic plasma pressure $\mathbf{p} = p_{\parallel} \hat{\mathbf{b}}\hat{\mathbf{b}} + p_{\perp} \mathbf{I}_{\perp}$ was first discussed by G.E. Chew, M.L. Goldberger and F.E. Low (*Proceedings of the Royal Society of London A* **236**, 112–118, 1956).

$$-4\mu\{\mathbf{s} \times \mathbf{a}\} + 4\{\{\mathbf{t} \times \mathbf{a}\} \times \mathbf{a}\} = 2\{\mathbf{t} \times \mathbf{a}\} = 2\mu\mathbf{s} + \mathbf{t} \quad (*).$$

Then use the first identity above, to obtain

$$-4\mu\{\mathbf{s} \times \mathbf{a}\} + 6\{\mathbf{t} \cdot \mathbf{aa}\} - 2\mu\mathbf{s} = (1 + 4|\mathbf{a}|^2)\mathbf{t};$$

and remove the dot product $\mathbf{t} \cdot \mathbf{a}$ using a post-dot product of (*) with $\mathbf{a}$ and the second identity, to obtain

$$(1 + |\mathbf{a}|^2) \mathbf{t} \cdot \mathbf{a} = -4\mu\{\mathbf{s} \times \mathbf{a}\} \cdot \mathbf{a} - 2\mu(\mathbf{s} : \mathbf{aaa} + \mathbf{s} \cdot \mathbf{a})$$

since $\mathbf{t} : \mathbf{aa} = -2\mu\mathbf{s} : \mathbf{aa}$ also follows from (2.67). Finally, eliminate $\mathbf{t} \cdot \mathbf{a}$ between these last two equations with the help of the third identity above, to obtain the result (2.69).

- (Q3) Expand the original expressions (2.72) and (2.73) for the components $\mathbf{t}_g$ and $\mathbf{t}_\perp$, to obtain the respective alternative forms (2.77) and (2.78).
- (Q4) Deduce the representations for the successive components $\mathbf{t}_\parallel$, $\mathbf{t}_g$ and $\mathbf{t}_\perp$ in a local orthonormal coordinate system where $\hat{\mathbf{b}} = \hat{\mathbf{e}}_3$ in terms of the entries in the tensor $\mathbf{s}$, recalling that $\mathbf{s}$ is also symmetric and traceless.

(Analogous representations were obtained by Braginskii for the traceless component of his pressure tensor, where the peculiar velocity is identified with the mean species velocity—cf. the Exercises in Sect. 2.6, and for example Ref. [4]. The distinction between fluid and plasma viscosity is also emphasised in the celebrated L.D. Landau & E.M. Lifshitz *Course of Theoretical Physics* (Elsevier)—cf. Volume 10 by L.P. Pitaevskii and E.M. Lifshitz in particular.)

2.10 Parallel Viscosity in an Ion-Electron Plasma

Let us now specifically consider the leading parallel component $\mathbf{t}_\parallel$ in the case of a magnetised fully ionised plasma consisting of electrons and only one ion species (usually protons). There are then useful simplifications due to the much smaller electron mass relative to the ion mass ($m_e \ll m_i$). Collisional coupling between the ion and electron species is negligible, so the collision integral for the ions has the simple form $C_i(m_i\{\mathbf{ww}\}) \simeq -(\vartheta_{ii}/\tau_{ii})\mathbf{t}_i$. Further, the viscosity coefficient μ is much greater for the ions than for the electrons when the electron and ion temperatures are comparable, so the dominant $\nabla \cdot \mathbf{t}$ contribution to render $\nabla \cdot \mathbf{p}$ in the macroscopic equation of motion (2.56) is due to the ion species. The ion velocity $\mathbf{v}_i$ is also approximately the mass-weighted mean velocity $\mathbf{v}$ (i.e. the ion diffusion velocity

$\mathbf{u}_i \simeq 0$),⁹ so we may conveniently continue to entirely omit subscripts when discussing the ion species but later include the subscript e to specify the electrons.

The $\hat{\mathbf{b}}\hat{\mathbf{b}}$ projection of the basic equation (2.66) for $\mathbf{t}_s$ produces the equation defining the scalar part $(3/2)t_{\parallel,\parallel} = p_{\parallel} - p_{\perp}$ of the parallel viscosity tensor $\mathbf{t}_{\parallel}$ defined in (2.71) for either the ions or the electrons, since $t_{\parallel,\parallel} \equiv \mathbf{t} : \hat{\mathbf{b}}\hat{\mathbf{b}} = \mathbf{t}_{\parallel} : \hat{\mathbf{b}}\hat{\mathbf{b}} = -2\mu \mathbf{s} : \hat{\mathbf{b}}\hat{\mathbf{b}}$ as we noted in the previous subsection. Let us first consider the usually dominant ion contribution, where since $\mathbf{u} \simeq 0$ we have

$$\frac{dt_{\parallel,\parallel}}{dt} + \frac{\vartheta}{\tau} t_{\parallel,\parallel} + \frac{4}{3} t_{\parallel,\parallel} \nabla \cdot \mathbf{v} + t_{\parallel,\parallel} \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} + 2p \{ \nabla \mathbf{v} \} : \hat{\mathbf{b}}\hat{\mathbf{b}} = 0 \quad (2.79)$$

on ignoring the $\nabla \cdot \mathbf{h}$ term, which corresponds to neglecting thermal effects under the Grad approximation—cf. (2.65). Equation (2.79) could be invoked directly in simulations involving parallel ion viscosity, but let us proceed to extract some approximations for $t_{\parallel,\parallel}$ by considering the various terms in this equation.

In the collisional Braginskii limit (i.e. for sufficiently small τ), balancing the second term with the last term on the left-hand side of (2.79) yields $t_{\parallel,\parallel} = -2\mu \{ \nabla \mathbf{v} \} : \hat{\mathbf{b}}\hat{\mathbf{b}}$ where $\mu = p \tau / \vartheta$. When ion–ion collisions are rarer (i.e. at larger τ), the other $t_{\parallel,\parallel}$ terms in (2.79) may also be significant. In fast evolving system phases for example—i.e. at large Strouhal numbers $\omega L / U$, where ω denotes the characteristic macroscopic frequency and L and U a characteristic length and velocity—the time derivative term may be important. Indeed, when the first two terms in $t_{\parallel,\parallel}$ jointly balance the last term on the left-hand side of (2.79), the appropriate coefficient is $\mu = p / (\omega + \vartheta / \tau) = p \tau / (\omega \tau + \vartheta)$ in $t_{\parallel,\parallel} = -2\mu \{ \nabla \mathbf{v} \} : \hat{\mathbf{b}}\hat{\mathbf{b}}$ as defined in Sect. 2.9. Consequently, in either case

$$t_{\parallel,\parallel} = -2\mu \{ \nabla \mathbf{v} \} : \hat{\mathbf{b}}\hat{\mathbf{b}} = -2\mu \left[\hat{\mathbf{b}} \cdot \nabla (\mathbf{v} \cdot \hat{\mathbf{b}}) - \mathbf{v} \cdot (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) - \frac{1}{3} \nabla \cdot \mathbf{v} \right], \quad (2.80)$$

where it is notable that the second term on the right-hand side of (2.80) proportional to the magnetic field curvature

$$\kappa = \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \quad (2.81)$$

supplements the two velocity gradient terms. Moreover, further curvature terms result on then taking the divergence, to express $\nabla \cdot \mathbf{t}_{\parallel}$ with $\mathbf{t}_{\parallel} = (3/2)t_{\parallel,\parallel} \{ \hat{\mathbf{b}}\hat{\mathbf{b}} \}$ in the macroscopic equation of motion (2.56).

In brief, so far we have proceeded beyond the collisional Braginskii limit for the ions to allow for the contribution $dt_{\parallel,\parallel}/dt$ in (2.79) from the time derivative term in the basic equation (2.66). Reference back to (2.68) shows that representing $\mathbf{t}_{\parallel}$ with $t_{\parallel,\parallel}$ rendered by (2.80) corresponds to approximating the generalised rate of

⁹The definition of all *ion* field quantities following Braginskii can therefore be considered to coincide with ours in an ion–electron plasma. On the other hand, if the electron temperature is so much higher than the ion temperature such that the electron viscosity contribution remains of interest, recall that the Braginskii pressure tensor for the electrons is $\mathbf{p}_e - m_e n_e \mathbf{u}_e \mathbf{u}_e$ in our notation—cf. Exercise (Q1) of Sect. 2.6.

deformation tensor $\mathbf{s}$ with $\{\nabla \mathbf{v}\}$. This velocity gradient approximation $\mathbf{s} \simeq \{\nabla \mathbf{v}\}$, previously familiar in the fluid mechanics context (cf. Sect. 2.2) but that we now find arises more generally from the second term in (2.68), has also been widely adopted for the ion species in plasmas. However, there is a further generalisation due to the residual terms in (2.68) involving $\mathbf{t}$ that we previously noted in Sect. 2.9, which produce additional contributions from the third and fourth terms involving $t_{\parallel, \parallel}$ on the left-hand side of Eq. (2.79) for the ions—i.e. we find—cf. Exercise (Q2):

$$t_{\parallel, \parallel} = -2p\tau\{\nabla \mathbf{v}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} \left[\vartheta + \tau \left(\omega + \frac{4}{3} \nabla \cdot \mathbf{v} + \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} \right) \right]^{-1}, \quad (2.82)$$

involving additional contributions from the $\nabla \mathbf{v}$ dyadic that could be significant when ion–ion collisions are rare (when the ion–ion collision time τ is relatively large).

In the occasional application where the electron temperature is so very much higher than the ion temperature such that the electron viscosity contribution cannot be ignored, we observe that additional terms from $\{\mathbf{f}_e \mathbf{u}_e\}$ due to relative inter-species diffusion arise in the basic equation (2.66) for the electrons. For example, in an ion-electron plasma where the ions are singly charged such that $n_e = n_i = n$ for quasi-neutrality, on retaining only the electron–electron collision term (and ignoring the $\nabla \cdot \mathbf{h}_e$ term) the corresponding $\hat{\mathbf{b}}\hat{\mathbf{b}}$ projection is

$$\begin{aligned} \frac{dt_{\parallel, \parallel e}}{dt} + \frac{\vartheta_{ee}}{\tau_{ee}} t_{\parallel, \parallel e} + \frac{4}{3} t_{\parallel, \parallel e} \nabla \cdot \mathbf{v}_e + t_{\parallel, \parallel e} \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v}_e \\ + 2p_e\{\nabla \mathbf{v}_e\} : \hat{\mathbf{b}}\hat{\mathbf{b}} - 2\mathbf{E} \cdot \hat{\mathbf{b}} \mathbf{j} \cdot \hat{\mathbf{b}} + \frac{2}{3} \mathbf{j} \cdot (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) = 0 \end{aligned} \quad (2.83)$$

where the electron velocity $\mathbf{v}_e \simeq \mathbf{v} - \mathbf{j}/(ne)$ and $(e/m_e)(\mathbf{E} + \mathbf{v}_e \times \mathbf{B})$ is a notable contribution in $\mathbf{f}_e$. Thus there are further terms to incorporate in the generalised rate of deformation tensor for the electrons, in addition to the term $2p_e\{\nabla \mathbf{v}_e\} : \hat{\mathbf{b}}\hat{\mathbf{b}}$ with form similar to that arising for the ions—viz. those arising from the inter-species contribution, the term involving the product of the parallel electric field $E_{\parallel} = \mathbf{E} \cdot \hat{\mathbf{b}}$ and the parallel current density $j_{\parallel} = \mathbf{j} \cdot \hat{\mathbf{b}}$, and perhaps also the last electromagnetic heating term. Incidentally, the possible inclusion of electron viscosity therefore provides one example where the electric field would need to be retained, whereas usually both $\mathbf{E}$ and $\mathbf{j}$ are readily eliminated in MHD by invoking two electromagnetic equations (cf. Chap. 5).

Finally, if there are significant gradients in the thermal flux for either species such that the basic equation (2.51) for the heat flux vector $\mathbf{R}_s = \mathbf{q}_s - (5/2)p_s \mathbf{u}_s$ should be considered together with the lower level basic equations (2.46)–(2.49), we observe that the $\nabla \cdot \mathbf{h}_s$ term then appropriately re-enters Eqs. (2.79) and (2.83) to produce associated thermal contributions to $\mathbf{t}_{\parallel}$. Thus for the Grad approximation in particular, from (2.65) we have $(\nabla \cdot \mathbf{h}) : \hat{\mathbf{b}}\hat{\mathbf{b}} = 4/5\{\nabla \mathbf{q}\} : \hat{\mathbf{b}}\hat{\mathbf{b}}$ and hence

$$t_{\parallel,\parallel} = -2p\tau\{\nabla\mathbf{v} + \frac{2}{5p}\nabla\mathbf{q}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} \left[\vartheta + \tau \left(\omega + \frac{4}{3}\nabla\cdot\mathbf{v} + \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{v} \right) \right]^{-1} \quad (2.84)$$

as the relevant modification of (2.80) for the ions, with the subscript omitted once again—and there would be a similar modification for the electrons.

Exercises

- (Q1) Deduce Eq. (2.79) in $t_{\parallel,\parallel}$ for the ions from (2.66), on neglecting the $\nabla\cdot\mathbf{h}$ term and assuming negligible collisional coupling between the ions and electrons—i.e. from the relevant moment equation (2.49) with the collisional integral $\mathcal{C}(m\{\mathbf{w}\mathbf{w}\}) = (\vartheta/\tau)\mathbf{t}$, on omitting the subscript.
- (Q2) Retain the two terms involving $\mathbf{t}$ in (2.68) to derive

$$t_{\parallel,\parallel} = -2p\tau\{\nabla\mathbf{v}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} \left[\vartheta + \tau \left(\omega + \frac{4}{3}\nabla\cdot\mathbf{v} + \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{v} \right) \right]^{-1}$$

for the scalar part in the parallel ion viscosity component $\mathbf{t}_{\parallel} = (3/2)t_{\parallel\parallel}\{\hat{\mathbf{b}}\hat{\mathbf{b}}\}$ in a simple ion-electron plasma, consistent with retaining all of the terms in (2.79).

- (Q3) Noting $\rho_e = n_e m_e$ and now choosing to omit subscript e , show that

$$-2\rho \left[\frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \frac{\mathbf{j}}{ne} \right] : \hat{\mathbf{b}}\hat{\mathbf{b}} = -2\mathbf{E} \cdot \hat{\mathbf{b}}\mathbf{j} \cdot \hat{\mathbf{b}} + \frac{2}{3}\mathbf{j} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

representing some of the additional electromagnetic terms arising in equation (2.83) for the electrons.

- (Q4) It is easy to expand $\nabla\cdot\mathbf{t}_{\parallel} = \nabla\cdot[(3/2)t_{\parallel\parallel}\{\hat{\mathbf{b}}\hat{\mathbf{b}}\}]$ to express the predominant ion viscosity contribution to the macroscopic equation of motion (2.56). It takes much more effort to derive the lower order gyroviscous contribution $\nabla\cdot\mathbf{t}_g$, which leads to a consequence known as “gyroviscous cancellation” in the ion momentum equation and thus in (2.56) for an ion-electron plasma. Although this cancellation in the advective derivative is now mainly of historical interest, show that the gyroviscous component for the ions may be rewritten as

$$\mathbf{t}_g = \frac{\mu}{2|\mathbf{a}|} \left[\hat{\mathbf{b}} \times \mathbf{s} \cdot (\mathbf{I} + 3\hat{\mathbf{b}}\hat{\mathbf{b}}) - (\mathbf{I} + 3\hat{\mathbf{b}}\hat{\mathbf{b}}) \cdot \mathbf{s} \times \hat{\mathbf{b}} \right],$$

and then assume the velocity gradient approximation $\mathbf{s} = \{\nabla\mathbf{v}\}$ to obtain

$$\begin{aligned} \nabla\cdot\mathbf{t}_g &= -\rho(\mathbf{v}_* \cdot \nabla)\mathbf{v} - \nabla\chi + 2\mathbf{B} \cdot \nabla(B^{-1}\mathbf{A}) \\ &\quad - \nabla \times \left[\frac{\mu}{|\mathbf{a}|} \left(\hat{\mathbf{b}} \cdot \nabla\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{v} - 3\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{v})\hat{\mathbf{b}} \right) \right] \end{aligned}$$

where three entities introduced here (with $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ the vorticity vector) are

$$\mathbf{v}_* = -\frac{1}{\rho} \nabla \times \left(\frac{\mu}{|\mathbf{a}|} \hat{\mathbf{b}} \right), \quad \chi = \frac{\mu}{|\mathbf{a}|} \hat{\mathbf{b}} \cdot \boldsymbol{\omega}, \quad \mathbf{A} = \frac{\mu}{|\mathbf{a}|} \hat{\mathbf{b}} \times (3 \hat{\mathbf{b}} \cdot \nabla \mathbf{v} + \hat{\mathbf{b}} \times \boldsymbol{\omega}) + \chi \hat{\mathbf{b}}.$$

[The first term on the right-hand side of this expression for $\nabla \cdot \mathbf{t}_g$, involving the amended “magnetisation velocity” $\mathbf{v}_*$, produces the legendary cancellation.]

Hint: From the identity $\partial v_j / \partial x_i = \partial v_i / \partial x_j + \epsilon_{ijk} \omega_k$, first verify that

$$\begin{aligned} & \hat{\mathbf{b}} \times [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \cdot (\mathbf{I} + 3\hat{\mathbf{b}}\hat{\mathbf{b}}) \\ &= 2 \hat{\mathbf{b}} \times \nabla \mathbf{v} - \hat{\mathbf{b}} \cdot \boldsymbol{\omega} \mathbf{I} + [6\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \mathbf{v}) - 2\boldsymbol{\omega} + 3\hat{\mathbf{b}} \cdot \boldsymbol{\omega} \hat{\mathbf{b}}] \hat{\mathbf{b}}, \end{aligned}$$

and then also consider the identity

$$\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \times \nabla \mathbf{v} = -\nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \mathbf{v} \right) + \left[\nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \right] \mathbf{v}.$$

2.11 Heat Flux★

The derivation of an explicit form for the heat flux vector $\mathbf{R}_s$ is more straightforward than that for $\mathbf{t}_s$, although its basic equation may appear more cumbersome. Thus adopting the collision integral representation (2.61) and again suppressing the subscript s where there is no chance of confusion, the relevant basic equation of change (2.51) is

$$\begin{aligned} \frac{d\mathbf{R}}{dt} + \frac{1}{\tau} \mathbf{R} + \mathbf{R} \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{L} - \mathbf{f} \cdot \mathbf{t} - \frac{e}{m} \mathbf{R} \times \mathbf{B} + \frac{5}{2} \frac{k}{m} \mathbf{p} \cdot \nabla T \\ + \frac{5}{2} n k \mathbf{u} \frac{dT}{dt} + \mathbf{R} \cdot \nabla \mathbf{v} + \mathbf{H} : \nabla \mathbf{v} = - \sum_{s \neq j} \frac{1}{\tau_{js}} \mathbf{R}_s - \sum_s \zeta_{js} \rho_s \mathbf{u}_s. \end{aligned} \quad (2.85)$$

Equation (2.85) may be rewritten in a notationally convenient vector form, analogous to the tensor form (2.67) for the traceless component of the pressure tensor $\mathbf{t}$ —viz. as

$$\mathbf{R} - \mathbf{R} \times \mathbf{a}_0 = -\lambda \mathbf{d}, \quad (2.86)$$

where the first two terms and the sixth term are represented on the left-hand side, and all the other terms are incorporated in $\mathbf{d}$ on the right-hand side. Thus on writing $d\mathbf{R}/dt = \omega \mathbf{R}$, the thermal conductivity coefficient and the vector

$$\lambda = \frac{5}{2} \frac{k}{m} \frac{p\tau}{\omega\tau + 1} \quad \text{and} \quad \mathbf{a}_0 = \frac{e\mathbf{B}}{m} \frac{\tau}{\omega\tau + 1}$$

introduced here reflect the time derivative and collisional terms. Well-known vector identities then immediately yield the corresponding general explicit form

$$\mathbf{R} = -\frac{\lambda}{1 + |\mathbf{a}_0|^2}(\mathbf{d} + \mathbf{d} \times \mathbf{a}_0 + \mathbf{d} \cdot \mathbf{a}_0 \mathbf{a}_0), \quad (2.87)$$

which reduces to the classical Fourier law of heat conduction in the limit $\mathbf{a}_0 = 0$.

Once again, the explicit form can be expanded in typical magnetised plasma (where $|\mathbf{a}_0| \gg 1$), except in the neighbourhood of null points in the magnetic field. Thus from (2.87) we obtain

$$\mathbf{R} = \mathbf{R}_{\parallel} + \mathbf{R}_g + \mathbf{R}_{\perp} + \dots \quad (2.88)$$

in notation analogous to that used for the pressure tensor, where

$$\mathbf{R}_{\parallel} = -\lambda \mathbf{d} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}}, \quad \mathbf{R}_g = -\frac{\lambda}{a_0} \mathbf{d} \times \hat{\mathbf{b}} \quad \text{and} \quad \mathbf{R}_{\perp} = -\frac{\lambda}{a_0^2} \mathbf{d} \cdot \mathbf{l}_{\perp}. \quad (2.89)$$

Further, in an ion-electron plasma the negligible collisional coupling between the two species again simplifies the relevant collision integral. Thus in this context we have $C_s(\frac{1}{2}m_s(w^2 - 5kT_s/m_s)\mathbf{w}) = -(\mathbf{R}_s/\tau_{ss} + \zeta_s \rho_s \mathbf{u}_s)$, and we may proceed to consider the leading component $\mathbf{R}_{\parallel}$ in conjunction with $\mathbf{t}_{\parallel}$. For the ions $\mathbf{R} \simeq \mathbf{q}$, so on again omitting the subscript and noting that $\hat{\mathbf{b}} \cdot \mathbf{p} = p_{\parallel} \hat{\mathbf{b}}$ where $p_{\parallel} = p + t_{\parallel, \parallel}$, the $\hat{\mathbf{b}}$ projection of the basic equation of change (2.85) for $q_{\parallel} = R_{\parallel} = \mathbf{R} \cdot \hat{\mathbf{b}}$ is

$$\begin{aligned} \frac{dq_{\parallel}}{dt} + \frac{1}{\tau} q_{\parallel} + q_{\parallel} \nabla \cdot \mathbf{v} + q_{\parallel} \hat{\mathbf{b}} \hat{\mathbf{b}} : \nabla \mathbf{v} - t_{\parallel, \parallel} \mathbf{f} \cdot \hat{\mathbf{b}} \\ + \frac{5}{2} p_{\parallel} \frac{k}{m} \hat{\mathbf{b}} \cdot \nabla T + \hat{\mathbf{b}} \cdot (\nabla \cdot \mathbf{L}) + \hat{\mathbf{b}} \cdot \mathbf{H} : \nabla \mathbf{v} = 0. \end{aligned} \quad (2.90)$$

It is conceivable that the last two terms on the left-hand side are negligible, such that the temperature gradient term familiar from the classical Fourier law predominates and therefore determines the heat flux in the direction of the magnetic field. On the other hand, we may invoke Grad's approximation such that

$$\begin{aligned} \hat{\mathbf{b}} \cdot \nabla \cdot \mathbf{L} &= \frac{kT}{m} \nabla \cdot (t_{\parallel, \parallel} \hat{\mathbf{b}}) + \frac{kT}{m} t_{\parallel, \parallel} \hat{\mathbf{b}} \hat{\mathbf{b}} : \nabla \hat{\mathbf{b}} + t_{\parallel, \parallel} \frac{k}{m} \hat{\mathbf{b}} \cdot \nabla T \\ \text{and} \quad \hat{\mathbf{b}} \cdot \mathbf{H} : \nabla \mathbf{v} &= \frac{2}{5} q_{\parallel} (2\hat{\mathbf{b}} \hat{\mathbf{b}} : \nabla \mathbf{v} + \nabla \cdot \mathbf{v}), \end{aligned}$$

from (2.65), associated with $t_{\parallel, \parallel}$ given by (2.84). For the electrons, there are again additional terms to include due to inter-species diffusion ($\mathbf{u}_e \neq 0$).

2.12 Magnetic Induction

As indicated in Sect. 2.7, any dynamical macroscopic model includes equations of continuity and motion, supplemented by a barotropic equation of state (i.e. a relation between the pressure p and density ρ) and a constitutive relation for either the fluid or the plasma pressure tensor $\mathbf{p}$ rendered in Sects. 2.2 and 2.9, respectively. Let us also recall that this is a complete (closed) system of equations unless the medium carries an electric current, when the electromagnetic body force term $\mathbf{j} \times \mathbf{B}$ in the equation of motion becomes significant. As discussed further in Sect. 5.3, the pre-Maxwell electromagnetic equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad (\mu_0 \text{ a constant}) \quad (2.91)$$

may then be adopted to eliminate $\mathbf{j}$, but another equation involving the magnetic field variable $\mathbf{B}$ is obviously required to complete the system of equations. This equation is obtained from another constitutive relation known as *Ohm's law* for a conducting fluid, or a *generalised Ohm's law* in the case of plasma.

The usual form of Ohm's law for an electrically conducting fluid is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{\sigma} \mathbf{j}, \quad (2.92)$$

where σ here denotes the *conductivity coefficient*, inversely related to the *resistivity coefficient* $\eta \equiv 1/(\mu_0 \sigma)$. The terms on the left-hand side of (2.92) are often referred to as *ideal*, as they remain in the limit of zero resistivity, when σ is infinite. Both the current density $\mathbf{j}$ and the electric field $\mathbf{E}$ may be eliminated using (2.91) and the additional electromagnetic equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.93)$$

respectively (cf. Chap. 5). Equation (2.92) yields the equation of *magnetic induction* as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}), \quad (2.94)$$

although another term due to the Hall effect may also arise if a generalised Ohm's law is used [cf. (2.100)].

For a plasma, a generalised Ohm's law is often identified as the second level of another hierarchy of moment equations, obtained by multiplying the Boltzmann equation (2.41) by e_s/m_s beforehand. Thus there is the charge conservation equation

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad (2.95)$$

at the first level, where q denotes the charge density (cf. also Chap. 5); and at the second level,

$$\mathbf{E} + \mathbf{v}_e \times \mathbf{B} = \frac{1}{\sigma} \mathbf{j} - \frac{1}{n_e e} \nabla p_e + \dots \quad (2.96)$$

where the subscript e again denotes electron field quantities.¹⁰ In passing, we note that on introducing the unit vector $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$ in the direction of the magnetic field, Eq. (2.96) implies that the component of the electron velocity perpendicular to the magnetic field is

$$\mathbf{v}_{e\perp} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{1}{n_e e B} \hat{\mathbf{b}} \times \nabla p_e + \dots \quad (2.97)$$

The terms on the right-hand side of (2.97) may be identified from quite elementary physical considerations as “polarisation”, “diamagnetic”, etc., contributions—although plasma physicists may also use such terminology to identify analogous contributions in the perpendicular velocity of the much more massive positive ions in a plasma, derived from the corresponding ion momentum equation.

In a neutral plasma containing an equal number of single-charge ions and electrons ($n_i = n_e = n$), on noting the electron velocity is $\mathbf{v}_e \simeq \mathbf{v} - \mathbf{j}/(ne)$ we may rewrite Eq. (2.96) as

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{\sigma} \mathbf{j} + \frac{1}{ne} \mathbf{j} \times \mathbf{B} - \frac{1}{ne} \nabla p_e + \dots, \quad (2.98)$$

where the second term $(1/ne) \mathbf{j} \times \mathbf{B}$ on the right-hand side is due to the *Hall effect* corresponding to the relative inter-species. The finite conductivity or *resistive* term arising when the conductivity coefficient σ is finite, and also the Hall term or any other term retained on the right-hand side of (2.98), may be referred to as *non-ideal*.

The current density $\mathbf{j}$ and electric field $\mathbf{E}$ appearing in (2.98) can be eliminated using the electromagnetic Eqs. (2.91) and (2.93). The electron pressure term in (2.96) or (2.98) has often been neglected—indeed, it has been argued that when the electrons are largely isothermal ($T_e \simeq \text{constant}$) we have

$$\nabla \times \left(\frac{1}{ne} \nabla p_e \right) \simeq -\frac{k T_e}{n^2 e} \nabla n \times \nabla n = 0, \quad (2.99)$$

on writing $p_e = nk T_e$ (where the subscript is again suppressed on the electron number density and k is the Boltzmann constant). Retaining the other four terms in (2.98) then produces the magnetic induction equation

¹⁰Most authors have associated the generalised Ohm’s law with the electron momentum equation in some way. Another viewpoint that produces the terms of interest as in (2.96), but with an alternative identification of the conductivity coefficient σ , was proposed by H.S. Green (*Physics of Fluids* **2**, 341–349, 1959).

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) - \nabla \times \left(\frac{1}{\mu_0 n e} (\nabla \times \mathbf{B}) \times \mathbf{B} \right), \quad (2.100)$$

with the resistivity coefficient $\eta \equiv 1/(\mu_0 \sigma)$ and $1/(\mu_0 n e)$ the Hall coefficient. However, the electron pressure term would necessarily be retained if we were to account for electron temperature gradients.

On incorporating $\mathbf{p} = p\mathbf{I}$ and invoking (2.91), Eqs. (2.55), (2.56), (2.59) and the appropriate equation of magnetic induction constitute a complete system in the variables $\{\rho, \mathbf{v}, p, \mathbf{B}\}$. The ideal MHD model corresponds to omitting all non-ideal terms from the equation of magnetic induction, as discussed further in Sect. 5.7 in the plasma context. Resistive MHD corresponds to the retention of the term $\nabla \times (\eta \nabla \times \mathbf{B})$, and when the Hall term is retained we have a Hall MHD model (ideal or resistive)—cf. Sect. 5.5 on conducting liquids, and Sect. 5.14 and Sect. 5.15 in the plasma context. All of these models may be modified further, such as when the plasma viscosity is important—using the pressure tensor discussed in Sect. 2.2 for a conducting fluid, or as in Sect. 2.9 for a plasma.

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