Chapter 2
General Concept with Filtering Vectors

In this chapter, we explain the principle of variable span (VS) linear filters and show how it can be applied to the signal enhancement problem. Then, we derive a large class of optimal filters with a great deal of flexibility in the sense that they can naturally compromise between noise reduction and desired signal distortion. We end this part by also showing how to estimate the noise signal, from which we can also estimate the desired signal with another class of variable span linear filters.

2.1 Signal Model and Problem Formulation

We consider the very general signal model:

$$\mathbf{y} = \mathbf{x} + \mathbf{v}, \tag{2.1}$$

where $\mathbf{y}$ is the observation or noisy signal vector of length M, $\mathbf{x}$ is the desired signal vector, and $\mathbf{v}$ is the noise signal vector. We assume that the components of the two vectors $\mathbf{x}$ and $\mathbf{v}$ are zero mean, stationary, and circular. We further assume that these two vectors are uncorrelated, i.e., $E\left(\mathbf{x}\mathbf{v}^H\right) = E\left(\mathbf{v}\mathbf{x}^H\right) = \mathbf{0}_{M\times M}$, where $E(\cdot)$ denotes mathematical expectation, the superscript ^{H} is the conjugate-transpose operator, and $\mathbf{0}_{M\times M}$ is a matrix of size $M \times M$ with all its elements equal to 0. In this context, the correlation matrix (of size $M \times M$) of the observations is

$$\begin{aligned}\mathbf{\Phi}_{\mathbf{y}} &= E\left(\mathbf{y}\mathbf{y}^H\right) \\ &= \mathbf{\Phi}_{\mathbf{x}} + \mathbf{\Phi}_{\mathbf{v}},\end{aligned} \tag{2.2}$$

where $\mathbf{\Phi}_{\mathbf{x}} = E\left(\mathbf{x}\mathbf{x}^H\right)$ and $\mathbf{\Phi}_{\mathbf{v}} = E\left(\mathbf{v}\mathbf{v}^H\right)$ are the correlation matrices of $\mathbf{x}$ and $\mathbf{v}$, respectively. In the rest of this chapter, we assume that the rank of

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the desired signal correlation matrix, $\mathbf{\Phi_x}$, is equal to $P \leq M$ and the rank of the noise correlation matrix, $\mathbf{\Phi_v}$, is equal to M.

Let x_1 be the first element of $\mathbf{x}$. It is assumed that x_1 is the desired signal sample. Then, the objective of signal enhancement (or noise reduction) is to estimate x_1 from $\mathbf{y}$. This should be done in such a way that the noise is reduced as much as possible with no or little distortion of the desired signal sample [1], [2], [3], [4].

2.2 Joint Diagonalization

The use of the joint diagonalization in noise reduction was first proposed in [5] and then in [6]. In this work, we give a different perspective as it will be shown later.

The two Hermitian matrices $\mathbf{\Phi_x}$ and $\mathbf{\Phi_v}$ can be jointly diagonalized as follows [7]:

$$\mathbf{B}^H \mathbf{\Phi_x} \mathbf{B} = \mathbf{\Lambda}, \tag{2.3}$$

$$\mathbf{B}^H \mathbf{\Phi_v} \mathbf{B} = \mathbf{I}_M, \tag{2.4}$$

where $\mathbf{B}$ is a full-rank square matrix (of size $M \times M$), $\mathbf{\Lambda}$ is a diagonal matrix whose main elements are real and nonnegative, and $\mathbf{I}_M$ is the $M \times M$ identity matrix. Furthermore, $\mathbf{\Lambda}$ and $\mathbf{B}$ are the eigenvalue and eigenvector matrices, respectively, of $\mathbf{\Phi_v}^{-1}\mathbf{\Phi_x}$, i.e.,

$$\mathbf{\Phi_v}^{-1}\mathbf{\Phi_x}\mathbf{B} = \mathbf{B}\mathbf{\Lambda}. \tag{2.5}$$

Since the rank of the matrix $\mathbf{\Phi_x}$ is equal to P, the eigenvalues of $\mathbf{\Phi_v}^{-1}\mathbf{\Phi_x}$ can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_P > \lambda_{P+1} = \cdots = \lambda_M = 0$. In other words, the last $M - P$ eigenvalues of the matrix product $\mathbf{\Phi_v}^{-1}\mathbf{\Phi_x}$ are exactly zero, while its first P eigenvalues are positive, with λ_1 being the maximum eigenvalue. We also denote by $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_P, \mathbf{b}_{P+1}, \ldots, \mathbf{b}_M$, the corresponding eigenvectors. A consequence of this joint diagonalization is that the noisy signal correlation matrix can also be diagonalized as

$$\mathbf{B}^H \mathbf{\Phi_y} \mathbf{B} = \mathbf{\Lambda} + \mathbf{I}_M. \tag{2.6}$$

We can decompose the matrix $\mathbf{B}$ as

$$\mathbf{B} = \left[\mathbf{B}'_Q \; \mathbf{B}''_Q \right], \tag{2.7}$$

where

$$\mathbf{B}'_Q = \left[\mathbf{b}_1 \; \mathbf{b}_2 \cdots \mathbf{b}_Q \right] \tag{2.8}$$

is an $M \times Q$ matrix,

$$\mathbf{B}''_Q = \left[\mathbf{b}_{Q+1}\ \mathbf{b}_{Q+2} \cdots\ \mathbf{b}_M \right] \tag{2.9}$$

is an $M \times (M-Q)$ matrix, and $1 \leq Q \leq M$. For the particular case $Q = P$, the matrices $\mathbf{B}'_P$ and $\mathbf{B}''_P$ span the desired signal-plus-noise subspace and the noise subspace, respectively. It can be verified from (2.3) and (2.4) that

$$\mathbf{B}''^H_P \mathbf{x} = \mathbf{0}_{(M-P)\times 1} \tag{2.10}$$

and

$$\begin{aligned} \mathbf{\Phi}_{\mathbf{v}}^{-1} &= \mathbf{B}\mathbf{B}^H \\ &= \mathbf{B}'_P \mathbf{B}'^H_P + \mathbf{B}''_P \mathbf{B}''^H_P \\ &= \mathbf{B}'_Q \mathbf{B}'^H_Q + \mathbf{B}''_Q \mathbf{B}''^H_Q. \end{aligned} \tag{2.11}$$

Another convenient way to express (2.3) and (2.4) is

$$\mathbf{B}'^H_Q \mathbf{\Phi}_{\mathbf{x}} \mathbf{B}'_Q = \mathbf{\Lambda}'_Q, \tag{2.12}$$

$$\mathbf{B}'^H_Q \mathbf{\Phi}_{\mathbf{v}} \mathbf{B}'_Q = \mathbf{I}_Q, \tag{2.13}$$

where

$$\mathbf{\Lambda}'_Q = \mathrm{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_Q\right) \tag{2.14}$$

is a diagonal matrix containing the first Q eigenvalues of $\mathbf{\Phi}_{\mathbf{v}}^{-1}\mathbf{\Phi}_{\mathbf{x}}$ and $\mathbf{I}_Q$ is the $Q \times Q$ identity matrix. As a consequence of (2.12) and (2.13), we also have

$$\mathbf{B}'^H_Q \mathbf{\Phi}_{\mathbf{y}} \mathbf{B}'_Q = \mathbf{\Lambda}'_Q + \mathbf{I}_Q. \tag{2.15}$$

The joint diagonalization is a very natural tool to use if we want to fully exploit the desired signal-plus-noise and noise subspaces in noise reduction and fully optimize the linear filtering process.

2.3 Variable Span (VS) Linear Filtering

One of the most convenient ways to estimate the desired signal, x_1, from the observation signal vector, $\mathbf{y}$, is through a filtering operation, i.e.,

$$z = \mathbf{h}^H \mathbf{y}, \tag{2.16}$$

where z is the estimate of x_1 and

$$\mathbf{h} = \begin{bmatrix} h_1 \; h_2 \cdots \; h_M \end{bmatrix}^T \tag{2.17}$$

is a complex-valued filter of length M. It is always possible to write $\mathbf{h}$ in a basis formed from the vectors $\mathbf{b}_m,\ m = 1, 2, \ldots, M$, i.e.,

$$\begin{aligned} \mathbf{h} &= \mathbf{B}\mathbf{a} \\ &= \mathbf{B}'_Q \mathbf{a}'_Q + \mathbf{B}''_Q \mathbf{a}''_Q, \end{aligned} \tag{2.18}$$

where the components of

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} a_1 \cdots \; a_Q \; a_{Q+1} \cdots \; a_M \end{bmatrix}^T \\ &= \begin{bmatrix} \mathbf{a}'^T_Q \; \mathbf{a}''^T_Q \end{bmatrix}^T \end{aligned} \tag{2.19}$$

are the coordinates of $\mathbf{h}$ in the new basis, and $\mathbf{a}'_Q$ and $\mathbf{a}''_Q$ are vectors of length Q and $M - Q$, respectively. Now, instead of estimating the coefficients of $\mathbf{h}$ as in conventional approaches, we can estimate, equivalently, the coordinates $a_m,\ m = 1, 2, \ldots, M$. When $\mathbf{a}$ is estimated, it is then easy to determine $\mathbf{h}$ from (2.18). Furthermore, for $Q = P$, several optimal noise reduction filters with at most P constraints will lead to $\mathbf{a}''_P = \mathbf{0}_{(M-P)\times 1}$ since there is no desired signal in the directions $\mathbf{B}''_P$. Therefore, we can sometimes simplify our problem and force $\mathbf{a}''_P = \mathbf{0}_{(M-P)\times 1}$; as a result, the filter and the estimate are, respectively, $\mathbf{h} = \mathbf{B}'_P \mathbf{a}'_P$ and $z = \mathbf{a}'^H_P \mathbf{B}'^H_P \mathbf{y}$.

From the previous discussion and from (2.18), we see that we can build a more flexible linear filter. We define our variable span (VS) linear filter of length M as

$$\mathbf{h}(Q) = \mathbf{B}'_Q \mathbf{a}'_Q. \tag{2.20}$$

Obviously, $\mathbf{h}(Q) \in \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_Q\}$. As a consequence, the estimate of x_1 is

$$\begin{aligned} z &= \mathbf{a}'^H_Q \mathbf{B}'^H_Q \mathbf{x} + \mathbf{a}'^H_Q \mathbf{B}'^H_Q \mathbf{v} \\ &= x_{\text{fd}} + v_{\text{rn}}, \end{aligned} \tag{2.21}$$

where

$$x_{\text{fd}} = \mathbf{a}'^H_Q \mathbf{B}'^H_Q \mathbf{x} \tag{2.22}$$

is the filtered desired signal and

$$v_{\text{rn}} = \mathbf{a}'^H_Q \mathbf{B}'^H_Q \mathbf{v} \tag{2.23}$$

is the residual noise. We deduce that the variance of z is

$$\begin{aligned}\phi_z &= E\left(|z|^2\right) \\ &= \mathbf{a}_Q'^H \mathbf{\Lambda}_Q' \mathbf{a}_Q' + \mathbf{a}_Q'^H \mathbf{a}_Q'.\end{aligned} \tag{2.24}$$

Notice that the proposed linear processing implies implicitly that we force the last $M - Q$ components of $\mathbf{a}$ to 0.

2.4 Performance Measures

In this section, we briefly define the most useful performance measures for noise reduction with VS linear filters. We can divide these measures into two categories. The first category evaluates the noise reduction performance while the second one evaluates the distortion of the desired signal. We also discuss the very convenient mean-squared error (MSE) criterion, which we tailor for variable span filters, and show how it is related to the performance measures.

2.4.1 Noise Reduction

One of the most fundamental measures in all aspects of signal enhancement is the signal-to-noise ratio (SNR). Since x_1 is the desired signal, we define the input SNR as

$$\text{iSNR} = \frac{\phi_{x_1}}{\phi_{v_1}}, \tag{2.25}$$

where $\phi_{x_1} = E\left(|x_1|^2\right)$ is the variance of x_1 and $\phi_{v_1} = E\left(|v_1|^2\right)$ is the variance of the first component of $\mathbf{v}$, i.e., v_1.

From (2.24), it is easy to find that the output SNR is

$$\begin{aligned}\text{oSNR}\left(\mathbf{a}_Q'\right) &= \frac{\mathbf{a}_Q'^H \mathbf{\Lambda}_Q' \mathbf{a}_Q'}{\mathbf{a}_Q'^H \mathbf{a}_Q'} \\ &= \frac{\sum_{q=1}^{Q} \lambda_q |a_q|^2}{\sum_{q=1}^{Q} |a_q|^2}\end{aligned} \tag{2.26}$$

and it can be shown that

$$\text{oSNR}\left(\mathbf{a}_Q'\right) \leq \lambda_1, \tag{2.27}$$

which means that the output SNR can never exceed the maximum eigenvalue, λ_1. The filters should be derived in such a way that $\text{oSNR}\left(\mathbf{a}'_Q\right) \geq \text{iSNR}$.

The noise reduction factor, which quantifies the amount of noise whose is rejected by the complex filter, is given by

$$\xi_{\text{nr}}\left(\mathbf{a}'_Q\right) = \frac{\phi_{v_1}}{\mathbf{a}'^H_Q \mathbf{a}'_Q}. \tag{2.28}$$

For optimal filters, we should have $\xi_{\text{nr}}\left(\mathbf{a}'_Q\right) \geq 1$.

2.4.2 Desired Signal Distortion

In practice, the complex filter may distort the desired signal. In order to evaluate the level of this distortion, we define the desired signal reduction factor:

$$\xi_{\text{sr}}\left(\mathbf{a}'_Q\right) = \frac{\phi_{x_1}}{\mathbf{a}'^H_Q \mathbf{\Lambda}'_Q \mathbf{a}'_Q}. \tag{2.29}$$

For optimal filters, we should have $\xi_{\text{sr}}\left(\mathbf{a}'_Q\right) \geq 1$. The larger is the value of $\xi_{\text{sr}}\left(\mathbf{a}'_Q\right)$, the more the desired signal is distorted.

By making the appropriate substitutions, one can derive the relationship:

$$\frac{\text{oSNR}\left(\mathbf{a}'_Q\right)}{\text{iSNR}} = \frac{\xi_{\text{nr}}\left(\mathbf{a}'_Q\right)}{\xi_{\text{sr}}\left(\mathbf{a}'_Q\right)}. \tag{2.30}$$

This expression indicates the equivalence between gain/loss in SNR and distortion (for both desired signal and noise).

Another way to measure the distortion of the desired signal due to the complex filter is the desired signal distortion index, which is defined as the mean-squared error between the desired signal and the filtered desired signal, normalized by the variance of the desired signal, i.e.,

$$\upsilon_{\text{sd}}\left(\mathbf{a}'_Q\right) = \frac{E\left(\left|x_1 - \mathbf{a}'^H_Q \mathbf{B}'^H_Q \mathbf{x}\right|^2\right)}{\phi_{x_1}}. \tag{2.31}$$

The desired signal distortion index is usually upper bounded by 1 for optimal filters.

2.4.3 Mean-Squared Error (MSE) Criterion

The error signal between the estimated and desired signals is

$$\begin{aligned} e &= z - x_1 \\ &= \mathbf{a}_Q'^H \mathbf{B}_Q'^H \mathbf{y} - x_1, \end{aligned} \tag{2.32}$$

which can also be written as the sum of two uncorrelated error signals:

$$e = e_{\mathrm{ds}} + e_{\mathrm{rs}}, \tag{2.33}$$

where

$$e_{\mathrm{ds}} = \mathbf{a}_Q'^H \mathbf{B}_Q'^H \mathbf{x} - x_1 \tag{2.34}$$

is the distortion of the desired signal due to the filter and

$$e_{\mathrm{rs}} = \mathbf{a}_Q'^H \mathbf{B}_Q'^H \mathbf{v} \tag{2.35}$$

represents the residual noise. The mean-squared error (MSE) criterion is then

$$\begin{aligned} J\left(\mathbf{a}_Q'\right) &= E\left(|e|^2\right) \\ &= \phi_{x_1} - \mathbf{i}^T \mathbf{\Phi}_{\mathbf{x}} \mathbf{B}_Q' \mathbf{a}_Q' - \mathbf{a}_Q'^H \mathbf{B}_Q'^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{i} + \mathbf{a}_Q'^H \left(\mathbf{\Lambda}_Q' + \mathbf{I}_Q\right) \mathbf{a}_Q' \\ &= J_{\mathrm{ds}}\left(\mathbf{a}_Q'\right) + J_{\mathrm{rs}}\left(\mathbf{a}_Q'\right), \end{aligned} \tag{2.36}$$

where $\mathbf{i}$ is the first column of $\mathbf{I}_M$,

$$\begin{aligned} J_{\mathrm{ds}}\left(\mathbf{a}_Q'\right) &= E\left(|e_{\mathrm{ds}}|^2\right) \\ &= \phi_{x_1} - \mathbf{i}^T \mathbf{\Phi}_{\mathbf{x}} \mathbf{B}_Q' \mathbf{a}_Q' - \mathbf{a}_Q'^H \mathbf{B}_Q'^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{i} + \mathbf{a}_Q'^H \mathbf{\Lambda}_Q' \mathbf{a}_Q' \\ &= \upsilon_{\mathrm{sd}}\left(\mathbf{a}_Q'\right) \phi_{x_1}, \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} J_{\mathrm{rs}}\left(\mathbf{a}_Q'\right) &= E\left(|e_{\mathrm{rs}}|^2\right) \\ &= \mathbf{a}_Q'^H \mathbf{a}_Q' \\ &= \frac{\phi_{v_1}}{\xi_{\mathrm{nr}}\left(\mathbf{a}_Q'\right)}. \end{aligned} \tag{2.38}$$

We deduce that

$$\frac{J_{\mathrm{ds}}\left(\mathbf{a}'_Q\right)}{J_{\mathrm{rs}}\left(\mathbf{a}'_Q\right)} = \mathrm{iSNR} \times \xi_{\mathrm{nr}}\left(\mathbf{a}'_Q\right) \times \upsilon_{\mathrm{sd}}\left(\mathbf{a}'_Q\right) \tag{2.39}$$
$$= \mathrm{oSNR}\left(\mathbf{a}'_Q\right) \times \xi_{\mathrm{sr}}\left(\mathbf{a}'_Q\right) \times \upsilon_{\mathrm{sd}}\left(\mathbf{a}'_Q\right).$$

This shows how the different performances measures are related to the MSEs.

2.5 Optimal VS Linear Filters

In this section, we derive a large class of VS linear filters for noise reduction from the different MSEs developed in the previous section. We will see how all these filters, with different objectives, are strongly connected.

2.5.1 VS Minimum Distortion

The VS minimum distortion filter is obtained by minimizing the distortion-based MSE, $J_{\mathrm{ds}}\left(\mathbf{a}'_Q\right)$. We get

$$\mathbf{a}'_{Q,\mathrm{MD}} = \mathbf{\Lambda}'^{-1}_Q \mathbf{B}'^H_Q \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}, \tag{2.40}$$

where it is assumed that $Q \leq P$. Therefore, the VS minimum distortion filter is

$$\mathbf{h}_{\mathrm{MD}}(Q) = \mathbf{B}'_Q \mathbf{a}'_{Q,\mathrm{MD}} \tag{2.41}$$
$$= \sum_{q=1}^{Q} \frac{\mathbf{b}_q \mathbf{b}_q^H}{\lambda_q} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}, \ Q \leq P.$$

One important particular case of (2.41) is $Q = P$. In this situation, we obtain the celebrated minimum variance distortionless response (MVDR) filter:

$$\mathbf{h}_{\mathrm{MVDR}} = \mathbf{h}_{\mathrm{MD}}(P) \tag{2.42}$$
$$= \mathbf{B}'_P \mathbf{a}'_{P,\mathrm{MD}}$$
$$= \sum_{p=1}^{P} \frac{\mathbf{b}_p \mathbf{b}_p^H}{\lambda_p} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}$$
$$= \sum_{p=1}^{P} \mathbf{b}_p \mathbf{b}_p^H \mathbf{\Phi}_{\mathbf{v}} \mathbf{i}.$$

Let us show why (2.42) corresponds to the MVDR filter. With $\mathbf{h}_{\mathrm{MVDR}}$, the filtered desired signal is

$$\begin{aligned} x_{\text{fd}} &= \left(\mathbf{B}'_P \mathbf{B}'^H_P \mathbf{\Phi}_{\mathbf{v}} \mathbf{i}\right)^H \mathbf{x} \\ &= \left(\mathbf{i} - \mathbf{B}''_P \mathbf{B}''^H_P \mathbf{\Phi}_{\mathbf{v}} \mathbf{i}\right)^H \mathbf{x} \\ &= x_1 - \mathbf{i}^T \mathbf{\Phi}_{\mathbf{v}} \mathbf{B}''_P \mathbf{B}''^H_P \mathbf{x} \\ &= x_1, \end{aligned} \tag{2.43}$$

where we have used (2.10) and (2.11) in the previous expression. Then, it is clear that

$$\upsilon_{\text{sd}}\left(\mathbf{a}'_{P,\text{MD}}\right) = 0, \tag{2.44}$$

proving that, indeed, $\mathbf{h}_{\text{MVDR}}$ is the MVDR filter.

Another interesting case of (2.41) is $Q = 1$. In this scenario, we obtain the maximum SNR filter:

$$\begin{aligned} \mathbf{h}_{\text{max},0} &= \mathbf{h}_{\text{MD}}(1) \\ &= \mathbf{b}_1 a_{1,\text{MD}} \\ &= \frac{\mathbf{b}_1 \mathbf{b}_1^H}{\lambda_1} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}. \end{aligned} \tag{2.45}$$

Indeed, it can be verified that

$$\text{oSNR}\left(a_{1,\text{MD}}\right) = \lambda_1. \tag{2.46}$$

We should always have

$$\text{oSNR}\left(\mathbf{a}'_{P,\text{MD}}\right) \leq \text{oSNR}\left(\mathbf{a}'_{P-1,\text{MD}}\right) \leq \cdots \leq \text{oSNR}\left(a_{1,\text{MD}}\right) = \lambda_1 \tag{2.47}$$

and

$$\upsilon_{\text{sd}}\left(\mathbf{a}'_{P,\text{MD}}\right) \leq \upsilon_{\text{sd}}\left(\mathbf{a}'_{P-1,\text{MD}}\right) \leq \cdots \leq \upsilon_{\text{sd}}\left(a_{1,\text{MD}}\right) \leq 1. \tag{2.48}$$

If $\mathbf{\Phi}_{\mathbf{x}}$ is a full-rank matrix, i.e., $P = M$, then

$$\mathbf{h}_{\text{MD}}(M) = \mathbf{i}, \tag{2.49}$$

which is the identity filter. Assume that the rank of $\mathbf{\Phi}_{\mathbf{x}}$ is $P = 1$. In this case, $Q = 1$, the filter is $\mathbf{h}_{\text{MD}}(1) = \mathbf{h}_{\text{MVDR}}$, and the output SNR is maximized, i.e., equal to λ_1. Also, we can write the desired signal correlation matrix as

$$\mathbf{\Phi}_{\mathbf{x}} = \phi_{x_1} \mathbf{d}\mathbf{d}^H, \tag{2.50}$$

where $\mathbf{d}$ is a vector of length M, whose first element is equal to 1. As a consequence,

$$\begin{aligned}\lambda_1 &= \phi_{x_1}\mathbf{d}^H\boldsymbol{\Phi}_{\mathbf{v}}^{-1}\mathbf{d} \\ &= \phi_{x_1}\left|\mathbf{d}^H\mathbf{b}_1\right|^2\end{aligned} \tag{2.51}$$

and

$$\mathbf{h}_{\mathrm{MVDR}} = \frac{\boldsymbol{\Phi}_{\mathbf{v}}^{-1}\mathbf{d}}{\mathbf{d}^H\boldsymbol{\Phi}_{\mathbf{v}}^{-1}\mathbf{d}}. \tag{2.52}$$

2.5.2 VS Wiener

The VS Wiener filter is obtained from the optimization of the MSE criterion, $J\left(\mathbf{a}'_Q\right)$. The minimization of $J\left(\mathbf{a}'_Q\right)$ leads to

$$\mathbf{a}'_{Q,\mathrm{W}} = \left(\boldsymbol{\Lambda}'_Q + \mathbf{I}_Q\right)^{-1}\mathbf{B}'^H_Q\boldsymbol{\Phi}_{\mathbf{x}}\mathbf{i}, \tag{2.53}$$

where $Q \leq M$. We deduce that the VS Wiener filter is

$$\begin{aligned}\mathbf{h}_{\mathrm{W}}(Q) &= \mathbf{B}'_Q\mathbf{a}'_{Q,\mathrm{W}} \\ &= \sum_{q=1}^{Q}\frac{\mathbf{b}_q\mathbf{b}_q^H}{1+\lambda_q}\boldsymbol{\Phi}_{\mathbf{x}}\mathbf{i}.\end{aligned} \tag{2.54}$$

It is interesting to compare $\mathbf{h}_{\mathrm{W}}(Q)$ to $\mathbf{h}_{\mathrm{MD}}(Q)$. The two VS filters are very close to each other; they differ by the weighting function, which strongly depends on the eigenvalues of the joint diagonalization. For the VS Wiener filter, this function is equal to $(1+\lambda_q)^{-1}$ while it is equal to λ_q^{-1} for the VS minimum distortion filter. Also, in the latter filter, Q must be smaller than or equal to P, while Q can be greater than P in the former one.

One important particular case of (2.54) is $Q = M$. In this situation, we obtain the classical Wiener filter:

$$\begin{aligned}\mathbf{h}_{\mathrm{W}} &= \mathbf{h}_{\mathrm{W}}(M) \\ &= \mathbf{B}'_M\mathbf{a}'_{M,\mathrm{W}} \\ &= \sum_{m=1}^{M}\frac{\mathbf{b}_m\mathbf{b}_m^H}{1+\lambda_m}\boldsymbol{\Phi}_{\mathbf{x}}\mathbf{i} \\ &= \boldsymbol{\Phi}_{\mathbf{y}}^{-1}\boldsymbol{\Phi}_{\mathbf{x}}\mathbf{i}.\end{aligned} \tag{2.55}$$

For $Q = 1$, we obtain another form of the maximum SNR filter:

$$\begin{aligned}\mathbf{h}_{\max,1} &= \mathbf{h}_{\mathrm{W}}(1) \\ &= \mathbf{b}_1 a_{1,\mathrm{W}} \\ &= \frac{\mathbf{b}_1 \mathbf{b}_1^H}{1+\lambda_1}\boldsymbol{\Phi}_{\mathbf{x}}\mathbf{i},\end{aligned} \tag{2.56}$$

since

$$\mathrm{oSNR}\left(a_{1,\mathrm{W}}\right) = \lambda_1. \tag{2.57}$$

We should always have

$$\mathrm{oSNR}\left(\mathbf{a}'_{M,\mathrm{W}}\right) \le \mathrm{oSNR}\left(\mathbf{a}'_{M-1,\mathrm{W}}\right) \le \cdots \le \mathrm{oSNR}\left(a_{1,\mathrm{W}}\right) = \lambda_1 \tag{2.58}$$

and

$$\upsilon_{\mathrm{sd}}\left(\mathbf{a}'_{M,\mathrm{W}}\right) \le \upsilon_{\mathrm{sd}}\left(\mathbf{a}'_{M-1,\mathrm{W}}\right) \le \cdots \le \upsilon_{\mathrm{sd}}\left(a_{1,\mathrm{W}}\right) \le 1. \tag{2.59}$$

2.5.3 VS Tradeoff

Another interesting approach that can compromise between noise reduction and desired signal distortion is the VS tradeoff filter obtained by

$$\min_{\mathbf{a}'_Q} J_{\mathrm{ds}}\left(\mathbf{a}'_Q\right) \quad \text{subject to} \quad J_{\mathrm{rs}}\left(\mathbf{a}'_Q\right) = \beta\phi_{v_1}, \tag{2.60}$$

where $0 \le \beta \le 1$, to ensure that filtering achieves some degree of noise reduction. We easily find that the optimal filter is

$$\mathbf{h}_{\mathrm{T},\mu}(Q) = \sum_{q=1}^{Q} \frac{\mathbf{b}_q \mathbf{b}_q^H}{\mu+\lambda_q}\boldsymbol{\Phi}_{\mathbf{x}}\mathbf{i}, \tag{2.61}$$

where $\mu \ge 0$ is a Lagrange multiplier[1]. Clearly, for $\mu = 0$ and $\mu = 1$, we get the VS minimum distortion and VS Wiener filters, respectively.

For $Q = M$, we obtain the classical tradeoff filter:

$$\begin{aligned}\mathbf{h}_{\mathrm{T},\mu} &= \mathbf{h}_{\mathrm{T},\mu}(M) \\ &= \sum_{m=1}^{M} \frac{\mathbf{b}_m \mathbf{b}_m^H}{\mu+\lambda_m}\boldsymbol{\Phi}_{\mathbf{x}}\mathbf{i} \\ &= \left(\boldsymbol{\Phi}_{\mathbf{x}} + \mu\boldsymbol{\Phi}_{\mathbf{v}}\right)^{-1}\boldsymbol{\Phi}_{\mathbf{x}}\mathbf{i}\end{aligned} \tag{2.62}$$

and for $Q = 1$, we obtain the maximum SNR filter:

[1] For $\mu = 0$, Q must be smaller than or equal to P.

Table 2.1 Optimal VS linear filters for signal enhancement.

VS MD:	$\mathbf{h}_{\mathrm{MD}}(Q) = \sum_{q=1}^{Q} \frac{\mathbf{b}_q \mathbf{b}_q^H}{\lambda_q} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i},\ Q \leq P$
MVDR:	$\mathbf{h}_{\mathrm{MVDR}} = \sum_{p=1}^{P} \frac{\mathbf{b}_p \mathbf{b}_p^H}{\lambda_p} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}$
VS Wiener:	$\mathbf{h}_{\mathrm{W}}(Q) = \sum_{q=1}^{Q} \frac{\mathbf{b}_q \mathbf{b}_q^H}{1+\lambda_q} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i},\ Q \leq M$
Wiener:	$\mathbf{h}_{\mathrm{W}} = \sum_{m=1}^{M} \frac{\mathbf{b}_m \mathbf{b}_m^H}{1+\lambda_m} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i} = \mathbf{\Phi}_{\mathbf{y}}^{-1} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}$
VS Tradeoff:	$\mathbf{h}_{\mathrm{T},\mu}(Q) = \sum_{q=1}^{Q} \frac{\mathbf{b}_q \mathbf{b}_q^H}{\mu+\lambda_q} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i},\ \mu \geq 0$
Tradeoff:	$\mathbf{h}_{\mathrm{T},\mu} = \sum_{m=1}^{M} \frac{\mathbf{b}_m \mathbf{b}_m^H}{\mu+\lambda_m} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i} = \left(\mathbf{\Phi}_{\mathbf{x}} + \mu \mathbf{\Phi}_{\mathbf{v}}\right)^{-1} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}$
Maximum SNR:	$\mathbf{h}_{\max,\mu} = \frac{\mathbf{b}_1 \mathbf{b}_1^H}{\mu+\lambda_1} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}$

$$\begin{aligned} \mathbf{h}_{\max,\mu} &= \mathbf{h}_{\mathrm{T},\mu}(1) \\ &= \frac{\mathbf{b}_1 \mathbf{b}_1^H}{\mu+\lambda_1} \mathbf{\Phi}_{\mathbf{x}} \mathbf{i}. \end{aligned} \tag{2.63}$$

In Table 2.1, we summarize all optimal VS filters developed in this section, showing how they are strongly related.

2.6 Indirect Optimal VS Linear Filters

The indirect approach is based on two successive stages. In the first stage, we find an estimate of the noise signal. This estimate is then used in the second stage by subtracting it from the observation signal. This will lead to an estimate of the desired signal.

2.6.1 Indirect Approach

Let

$$\mathbf{h}' = \left[h'_1 \ h'_2 \ \cdots \ h'_M \right]^T \tag{2.64}$$

be a complex-valued filter of length M. By applying this filter to the observation signal vector, we obtain

$$\widehat{v} = \mathbf{h}'^H \mathbf{x} + \mathbf{h}'^H \mathbf{v} \tag{2.65}$$

and the corresponding output SNR is

$$\text{oSNR}_{\widehat{v}}(\mathbf{h}') = \frac{\mathbf{h}'^H \mathbf{\Phi}_{\mathbf{x}} \mathbf{h}'}{\mathbf{h}'^H \mathbf{\Phi}_{\mathbf{v}} \mathbf{h}'}. \tag{2.66}$$

Then, we find $\mathbf{h}'$ that minimizes $\text{oSNR}_{\widehat{v}}(\mathbf{h}')$. It is easy to check that the solution is

$$\mathbf{h}' = \mathbf{B}''_P \mathbf{a}''_P, \tag{2.67}$$

where $\mathbf{B}''_P$ and $\mathbf{a}''_P$ are defined in the previous sections. With (2.67), $\text{oSNR}_{\widehat{v}}(\mathbf{h}') = 0$; therefore, $\widehat{v}$ can be seen as the estimate of the noise.

We consider the more general scenario:

$$\mathbf{h}'(\mathcal{Q}) = \mathbf{B}''_{\mathcal{Q}} \mathbf{a}''_{\mathcal{Q}}, \tag{2.68}$$

where $0 \leq \mathcal{Q} < M$,

$$\mathbf{B}''_{\mathcal{Q}} = \left[\mathbf{b}_{\mathcal{Q}+1} \; \mathbf{b}_{\mathcal{Q}+2} \cdots \; \mathbf{b}_M \right] \tag{2.69}$$

is a matrix of size $M \times (M - \mathcal{Q})$,

$$\mathbf{a}''_{\mathcal{Q}} = \left[a_{\mathcal{Q}+1} \; a_{\mathcal{Q}+2} \cdots \; a_M \right]^T \tag{2.70}$$

is a vector of length $M - \mathcal{Q}$, and $\mathbf{h}'(\mathcal{Q}) \in \text{Span}\{\mathbf{b}_{\mathcal{Q}+1}, \mathbf{b}_{\mathcal{Q}+2}, \ldots, \mathbf{b}_M\}$. As a consequence,

$$\widehat{v} = \mathbf{h}'^H(\mathcal{Q})\mathbf{x} + \mathbf{h}'^H(\mathcal{Q})\mathbf{v}. \tag{2.71}$$

Now, in general, $\text{oSNR}_{\widehat{v}}[\mathbf{h}'(\mathcal{Q})] \neq 0$, and this implies distortion as it will become clearer soon. This concludes the first stage.

In the second stage, we estimate the desired signal, x_1, as the difference between the observation, y_1, and the estimate of the noise obtained from the first stage, i.e.,

$$\begin{aligned} z' &= y_1 - \mathbf{h}'^H(\mathcal{Q})\mathbf{x} - \mathbf{h}'^H(\mathcal{Q})\mathbf{v} \\ &= x_1 - \mathbf{a}''^H_{\mathcal{Q}} \mathbf{B}''^H_{\mathcal{Q}} \mathbf{x} + v_1 - \mathbf{a}''^H_{\mathcal{Q}} \mathbf{B}''^H_{\mathcal{Q}} \mathbf{v} \\ &= \overline{\mathbf{h}}'^H(\mathcal{Q})\mathbf{y}, \end{aligned} \tag{2.72}$$

where

$$\overline{\mathbf{h}}'(\mathcal{Q}) = \mathbf{i} - \mathbf{B}''_{\mathcal{Q}} \mathbf{a}''_{\mathcal{Q}} \tag{2.73}$$

is the equivalent filter applied to the observation signal vector. We deduce that the variance of z' is

$$\begin{aligned}\phi_{z'} &= E\left(|z'|^2\right) \\ &= \phi_{x_1} - \mathbf{i}^T\mathbf{\Phi_x}\mathbf{B}_{\mathcal{Q}}''\mathbf{a}_{\mathcal{Q}}'' - \mathbf{a}_{\mathcal{Q}}''^H\mathbf{B}_{\mathcal{Q}}''^H\mathbf{\Phi_x}\mathbf{i} + \mathbf{a}_{\mathcal{Q}}''^H\mathbf{\Lambda}_{\mathcal{Q}}''\mathbf{a}_{\mathcal{Q}}'' \\ &\quad + \phi_{v_1} - \mathbf{i}^T\mathbf{\Phi_v}\mathbf{B}_{\mathcal{Q}}''\mathbf{a}_{\mathcal{Q}}'' - \mathbf{a}_{\mathcal{Q}}''^H\mathbf{B}_{\mathcal{Q}}''^H\mathbf{\Phi_v}\mathbf{i} + \mathbf{a}_{\mathcal{Q}}''^H\mathbf{a}_{\mathcal{Q}}''.\end{aligned} \tag{2.74}$$

2.6.2 MSE Criterion and Performance Measures

We define the error signal between the estimated and desired signals as

$$\begin{aligned}e' &= z' - x_1 \\ &= -\mathbf{a}_{\mathcal{Q}}''^H\mathbf{B}_{\mathcal{Q}}''^H\mathbf{x} + v_1 - \mathbf{a}_{\mathcal{Q}}''^H\mathbf{B}_{\mathcal{Q}}''^H\mathbf{v}.\end{aligned} \tag{2.75}$$

This error can be written as the sum of two uncorrelated error signals, i.e.,

$$e' = e'_{\mathrm{ds}} + e'_{\mathrm{rs}}, \tag{2.76}$$

where

$$e'_{\mathrm{ds}} = -\mathbf{a}_{\mathcal{Q}}''^H\mathbf{B}_{\mathcal{Q}}''^H\mathbf{x} \tag{2.77}$$

is the distortion of the desired signal due to the filtering operation and

$$e'_{\mathrm{rs}} = v_1 - \mathbf{a}_{\mathcal{Q}}''^H\mathbf{B}_{\mathcal{Q}}''^H\mathbf{v} \tag{2.78}$$

is the residual noise. Then, the MSE criterion is

$$\begin{aligned}J\left(\mathbf{a}_{\mathcal{Q}}''\right) &= E\left(|e'|^2\right) \\ &= \phi_{v_1} - \mathbf{i}^T\mathbf{\Phi_v}\mathbf{B}_{\mathcal{Q}}''\mathbf{a}_{\mathcal{Q}}'' - \mathbf{a}_{\mathcal{Q}}''^H\mathbf{B}_{\mathcal{Q}}''^H\mathbf{\Phi_v}\mathbf{i} + \mathbf{a}_{\mathcal{Q}}''^H\left(\mathbf{\Lambda}_{\mathcal{Q}}'' + \mathbf{I}_{M-\mathcal{Q}}\right)\mathbf{a}_{\mathcal{Q}}'' \\ &= J_{\mathrm{ds}}\left(\mathbf{a}_{\mathcal{Q}}''\right) + J_{\mathrm{rs}}\left(\mathbf{a}_{\mathcal{Q}}''\right),\end{aligned} \tag{2.79}$$

where $\mathbf{I}_{M-\mathcal{Q}}$ is the $(M-\mathcal{Q})\times(M-\mathcal{Q})$ identity matrix,

$$\mathbf{\Lambda}_{\mathcal{Q}}'' = \mathrm{diag}\left(\lambda_{\mathcal{Q}+1}, \lambda_{\mathcal{Q}+2}, \ldots, \lambda_M\right) \tag{2.80}$$

is a diagonal matrix containing the last $M-\mathcal{Q}$ eigenvalues of $\mathbf{\Phi_v}^{-1}\mathbf{\Phi_x}$,

$$\begin{aligned}J_{\mathrm{ds}}\left(\mathbf{a}_{\mathcal{Q}}''\right) &= E\left(|e'_{\mathrm{ds}}|^2\right) \\ &= \mathbf{a}_{\mathcal{Q}}''^H\mathbf{\Lambda}_{\mathcal{Q}}''\mathbf{a}_{\mathcal{Q}}'' \\ &= \upsilon_{\mathrm{sd}}\left(\mathbf{a}_{\mathcal{Q}}''\right)\phi_{x_1}\end{aligned} \tag{2.81}$$

is the distortion-based MSE,

$$\upsilon_{\mathrm{sd}}\left(\mathbf{a}_{\mathrm{Q}}''\right)=\frac{\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{\Lambda}_{\mathrm{Q}}''\mathbf{a}_{\mathrm{Q}}''}{\phi_{x_1}} \tag{2.82}$$

is the desired signal distortion index,

$$\begin{aligned} J_{\mathrm{rs}}\left(\mathbf{a}_{\mathrm{Q}}''\right)&=E\left(\left|e_{\mathrm{rs}}'\right|^2\right) \\ &=\phi_{v_1}-\mathbf{i}^T\mathbf{\Phi}_{\mathbf{v}}\mathbf{B}_{\mathrm{Q}}''\mathbf{a}_{\mathrm{Q}}''-\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{B}_{\mathrm{Q}}''^{H}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}+\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{a}_{\mathrm{Q}}'' \\ &=\frac{\phi_{v_1}}{\xi_{\mathrm{nr}}\left(\mathbf{a}_{\mathrm{Q}}''\right)} \end{aligned} \tag{2.83}$$

is the MSE corresponding to the residual noise, and

$$\xi_{\mathrm{nr}}\left(\mathbf{a}_{\mathrm{Q}}''\right)=\frac{\phi_{v_1}}{\phi_{v_1}-\mathbf{i}^T\mathbf{\Phi}_{\mathbf{v}}\mathbf{B}_{\mathrm{Q}}''\mathbf{a}_{\mathrm{Q}}''-\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{B}_{\mathrm{Q}}''^{H}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}+\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{a}_{\mathrm{Q}}''} \tag{2.84}$$

is the noise reduction factor. We deduce that

$$\begin{aligned} \frac{J_{\mathrm{ds}}\left(\mathbf{a}_{\mathrm{Q}}''\right)}{J_{\mathrm{rs}}\left(\mathbf{a}_{\mathrm{Q}}''\right)}&=\mathrm{iSNR}\times\xi_{\mathrm{nr}}\left(\mathbf{a}_{\mathrm{Q}}''\right)\times\upsilon_{\mathrm{sd}}\left(\mathbf{a}_{\mathrm{Q}}''\right) \\ &=\mathrm{oSNR}\left(\mathbf{a}_{\mathrm{Q}}''\right)\times\xi_{\mathrm{sr}}\left(\mathbf{a}_{\mathrm{Q}}''\right)\times\upsilon_{\mathrm{sd}}\left(\mathbf{a}_{\mathrm{Q}}''\right), \end{aligned} \tag{2.85}$$

where

$$\mathrm{oSNR}\left(\mathbf{a}_{\mathrm{Q}}''\right)=\frac{\phi_{x_1}-\mathbf{i}^T\mathbf{\Phi}_{\mathbf{x}}\mathbf{B}_{\mathrm{Q}}''\mathbf{a}_{\mathrm{Q}}''-\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{B}_{\mathrm{Q}}''^{H}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}+\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{\Lambda}_{\mathrm{Q}}''\mathbf{a}_{\mathrm{Q}}''}{\phi_{v_1}-\mathbf{i}^T\mathbf{\Phi}_{\mathbf{v}}\mathbf{B}_{\mathrm{Q}}''\mathbf{a}_{\mathrm{Q}}''-\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{B}_{\mathrm{Q}}''^{H}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}+\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{a}_{\mathrm{Q}}''} \tag{2.86}$$

is the output SNR and

$$\xi_{\mathrm{sr}}\left(\mathbf{a}_{\mathrm{Q}}''\right)=\frac{\phi_{x_1}}{\phi_{x_1}-\mathbf{i}^T\mathbf{\Phi}_{\mathbf{x}}\mathbf{B}_{\mathrm{Q}}''\mathbf{a}_{\mathrm{Q}}''-\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{B}_{\mathrm{Q}}''^{H}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}+\mathbf{a}_{\mathrm{Q}}''^{H}\mathbf{\Lambda}_{\mathrm{Q}}''\mathbf{a}_{\mathrm{Q}}''} \tag{2.87}$$

is the desired signal reduction factor.

2.6.3 Optimal Filters

2.6.3.1 Indirect VS Minimum Residual Noise

The indirect VS minimum residual noise filter is derived from $J_{\mathrm{rs}}\left(\mathbf{a}_{\mathrm{Q}}''\right)$. Indeed, by minimizing $J_{\mathrm{rs}}\left(\mathbf{a}_{\mathrm{Q}}''\right)$, we easily get

$$\mathbf{a}_{\mathrm{Q,MR}}''=\mathbf{B}_{\mathrm{Q}}''^{H}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}. \tag{2.88}$$

Therefore, the indirect VS minimum residual noise filter is

$$\begin{aligned}\overline{\mathbf{h}}'_{\mathrm{MR}}(\mathcal{Q}) &= \mathbf{i} - \mathbf{B}''_{\mathcal{Q}}\mathbf{B}''^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \\ &= \mathbf{B}'_{\mathcal{Q}}\mathbf{B}'^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}\end{aligned} \tag{2.89}$$

for $\mathcal{Q} \geq 1$ and $\overline{\mathbf{h}}'_{\mathrm{MR}}(0) = \mathbf{0}_{M\times 1}$. We can express the previous filter as

$$\begin{aligned}\overline{\mathbf{h}}'_{\mathrm{MR}}(\mathcal{Q}) &= \sum_{\mathrm{q}=1}^{\mathcal{Q}} \mathbf{b}_{\mathrm{q}}\mathbf{b}_{\mathrm{q}}^{H}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \\ &= \sum_{q=1}^{Q} \frac{\mathbf{b}_q\mathbf{b}_q^H}{\lambda_q}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i} + \sum_{\mathrm{i}=Q+1}^{\mathcal{Q}} \mathbf{b}_{\mathrm{i}}\mathbf{b}_{\mathrm{i}}^{H}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \\ &= \mathbf{h}_{\mathrm{MD}}(Q) + \sum_{\mathrm{i}=Q+1}^{\mathcal{Q}} \mathbf{b}_{\mathrm{i}}\mathbf{b}_{\mathrm{i}}^{H}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \\ &= \overline{\mathbf{h}}'_{\mathrm{MR}}(Q) + \sum_{\mathrm{i}=Q+1}^{\mathcal{Q}} \mathbf{b}_{\mathrm{i}}\mathbf{b}_{\mathrm{i}}^{H}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}\end{aligned} \tag{2.90}$$

for $\mathcal{Q} < M$ [and $\overline{\mathbf{h}}'_{\mathrm{MR}}(M) = \mathbf{i}$]. We observe that for $\mathcal{Q} \leq Q \leq P$, $\overline{\mathbf{h}}'_{\mathrm{MR}}(\mathcal{Q}) = \mathbf{h}_{\mathrm{MD}}(\mathcal{Q})$. But for $\mathcal{Q} > P$, the two filters are different since $\mathbf{h}_{\mathrm{MD}}(\mathcal{Q})$ is not defined in this context.

We have at least two interesting particular cases:

- $\overline{\mathbf{h}}'_{\mathrm{MR}}(1) = \mathbf{h}_{\mathrm{max},0}$, which corresponds to the maximum SNR filter; and
- $\overline{\mathbf{h}}'_{\mathrm{MR}}(P) = \mathbf{h}_{\mathrm{MVDR}}$, which corresponds to the MVDR filter.

2.6.3.2 Indirect VS Wiener

The indirect VS Wiener filter is obtained from the optimization of the MSE criterion, $J\left(\mathbf{a}''_{\mathcal{Q}}\right)$. The minimization of $J\left(\mathbf{a}''_{\mathcal{Q}}\right)$ leads to

$$\mathbf{a}''_{\mathcal{Q},\mathrm{W}} = \left(\mathbf{\Lambda}''_{\mathcal{Q}} + \mathbf{I}_{M-\mathcal{Q}}\right)^{-1}\mathbf{B}''^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}. \tag{2.91}$$

We deduce that the indirect VS Wiener filter is

$$\overline{\mathbf{h}}'_{\mathrm{W}}(\mathcal{Q}) = \mathbf{i} - \mathbf{B}''_{\mathcal{Q}}\left(\mathbf{\Lambda}''_{\mathcal{Q}} + \mathbf{I}_{M-\mathcal{Q}}\right)^{-1}\mathbf{B}''^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \tag{2.92}$$

for $\mathcal{Q} \geq 1$ and $\overline{\mathbf{h}}'_{\mathrm{W}}(0) = \mathbf{h}_{\mathrm{W}} = \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}$, which is the classical Wiener filter. Expression (2.92) can be rewritten as

$$\begin{aligned}\overline{\mathbf{h}}'_{\mathrm{W}}(\mathcal{Q}) &= \mathbf{i} - \mathbf{B}''_{\mathcal{Q}}\left(\mathbf{\Lambda}''_{\mathcal{Q}} + \mathbf{I}_{M-\mathcal{Q}}\right)^{-1}\mathbf{B}''^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{y}}\mathbf{i} \\ &\quad + \mathbf{B}''_{\mathcal{Q}}\left(\mathbf{\Lambda}''_{\mathcal{Q}} + \mathbf{I}_{M-\mathcal{Q}}\right)^{-1}\mathbf{B}''^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i} \qquad (2.93)\\ &= \mathbf{B}'_{\mathcal{Q}}\left(\mathbf{\Lambda}'_{\mathcal{Q}} + \mathbf{I}_{\mathcal{Q}}\right)^{-1}\mathbf{B}'^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{y}}\mathbf{i} + \mathbf{B}''_{\mathcal{Q}}\left(\mathbf{\Lambda}''_{\mathcal{Q}} + \mathbf{I}_{M-\mathcal{Q}}\right)^{-1}\mathbf{B}''^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i} \\ &= \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i} + \mathbf{B}'_{\mathcal{Q}}\left(\mathbf{\Lambda}'_{\mathcal{Q}} + \mathbf{I}_{\mathcal{Q}}\right)^{-1}\mathbf{B}'^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \\ &= \mathbf{h}_{\mathrm{W}} + \sum_{\mathrm{q}=1}^{\mathcal{Q}}\frac{\mathbf{b}_{\mathrm{q}}\mathbf{b}_{\mathrm{q}}^{H}}{1+\lambda_{\mathrm{q}}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \\ &= \overline{\mathbf{h}}'_{\mathrm{W}}(0) + \sum_{\mathrm{q}=1}^{\mathcal{Q}}\frac{\mathbf{b}_{\mathrm{q}}\mathbf{b}_{\mathrm{q}}^{H}}{1+\lambda_{\mathrm{q}}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}\end{aligned}$$

for $\mathcal{Q} < M$ [and $\overline{\mathbf{h}}'_{\mathrm{W}}(M) = \mathbf{i}$]. It is of interest to observe that $\overline{\mathbf{h}}'_{\mathrm{W}}(P) = \mathbf{h}_{\mathrm{MVDR}}$.

2.6.3.3 Indirect VS Tradeoff

The indirect VS tradeoff filter is obtained from the optimization problem:

$$\min_{\mathbf{a}''_{\mathcal{Q}}} J_{\mathrm{rs}}\left(\mathbf{a}''_{\mathcal{Q}}\right) \quad \text{subject to} \quad J_{\mathrm{ds}}\left(\mathbf{a}''_{\mathcal{Q}}\right) = \beta'\phi_{x_1}, \qquad (2.94)$$

where $0 \leq \beta' \leq 1$. We find that

$$\mathbf{a}''_{\mathcal{Q},\mathrm{T},\mu'} = \left(\mu'\mathbf{\Lambda}''_{\mathcal{Q}} + \mathbf{I}_{M-\mathcal{Q}}\right)^{-1}\mathbf{B}''^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i}, \qquad (2.95)$$

where $\mu' \geq 0$ is a Lagrange multiplier. As a result, the indirect VS tradeoff filter is

$$\overline{\mathbf{h}}'_{\mathrm{T},\mu'}(\mathcal{Q}) = \mathbf{i} - \mathbf{B}''_{\mathcal{Q}}\left(\mu'\mathbf{\Lambda}''_{\mathcal{Q}} + \mathbf{I}_{M-\mathcal{Q}}\right)^{-1}\mathbf{B}''^{H}_{\mathcal{Q}}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \qquad (2.96)$$

for $\mathcal{Q} \geq 1$ and

$$\begin{aligned}\overline{\mathbf{h}}'_{\mathrm{T},\mu'}(0) &= \mathbf{i} - \left(\mu'\mathbf{\Phi}_{\mathbf{x}} + \mathbf{\Phi}_{\mathbf{v}}\right)^{-1}\mathbf{\Phi}_{\mathbf{v}}\mathbf{i} \qquad (2.97)\\ &= \left(\mathbf{\Phi}_{\mathbf{x}} + \mu'^{-1}\mathbf{\Phi}_{\mathbf{v}}\right)^{-1}\mathbf{\Phi}_{\mathbf{x}}\mathbf{i} \\ &= \mathbf{h}_{\mathrm{T},1/\mu'}.\end{aligned}$$

Obviously, $\overline{\mathbf{h}}'_{\mathrm{T},0}(\mathcal{Q}) = \overline{\mathbf{h}}'_{\mathrm{MR}}(\mathcal{Q})$ and $\overline{\mathbf{h}}'_{\mathrm{T},1}(\mathcal{Q}) = \overline{\mathbf{h}}'_{\mathrm{W}}(\mathcal{Q})$. Also, we have $\overline{\mathbf{h}}'_{\mathrm{T},\mu'}(P) = \mathbf{h}_{\mathrm{MVDR}}$.

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