

Chapter 2

Structure Formation in the Universe

2.1 The Standard Cosmological Model

2.1.1 Friedmann Equation

The universe has a rich variety of structure. We know that galaxies are made of stars, and galaxies show a tendency to cluster into groups. Clusters of galaxies are a building block of larger structure such as superclusters and filaments. Even though the universe has the hierarchical structure, the matter distribution in the universe on a *sufficient* large scale should be homogenous and isotropic. This assumption is called the cosmological principle. In four space-time dimensions, the Friedmann-Lemaître-Robertson-Walker (FLRW) metric fulfills the requirement of the cosmological principle, i.e. a homogeneity and isotropy of space. This metric is given by

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (2.1)$$

where $a(t)$ is the scale factor and K is the spatial curvature of the space. The spatially closed, flat and open universe correspond to the case of $K > 0$, $K = 0$ and $K < 0$, respectively. In the FLRW metric, the scale factor determines the time evolution of the space. In this thesis, we normalize as $a = 1$ at the present. In an expanding universe, it is useful to define the *comoving* distance χ as follows;

$$d\chi^2 \equiv \frac{dr^2}{1 - Kr^2}. \quad (2.2)$$

With the definition of the comoving distance, the proper distance r is described as a function of χ , which is given by

$$r(\chi) = \begin{cases} \sinh(\sqrt{-K}\chi)/\sqrt{-K} & (K < 0) \\ \chi & (K = 0) \\ \sin(\sqrt{K}\chi)/\sqrt{K} & (K > 0) \end{cases}. \quad (2.3)$$

The time evolution of $a(t)$ can be determined by the Einstein equations;

$$G_{\mu\nu} = R_{\mu\nu} - \left(\frac{1}{2}R - \Lambda\right)g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (2.4)$$

Let us consider the case of a perfect isotropic fluid under the FLRW metric. In this case, the energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (2.5)$$

with density ρ and pressure p . The time-time and the space-space components of the Einstein equations then leads to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{c^2 K}{a^2} + \frac{\Lambda c^2}{3}, \quad (2.6)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3p) + \frac{\Lambda c^2}{3}, \quad (2.7)$$

where $\dot{}$ denotes d/dt . Equations (2.6) and (2.7) would reduce to the single equation as

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}. \quad (2.8)$$

The property of the fluid is specified by its equation of state, that is $p = w\rho$. The parameter w is zero for non-relativistic pressure-less components such as dark matter, while w is set to be one third for relativistic components, e.g., radiation. Using Eq. (2.8) and w of each component, we can derive the time evolution of the density as: $\rho_m \propto a^{-3}$ for non-relativistic component and $\rho_\gamma \propto a^{-4}$ for relativistic one. In this thesis, we call non-relativistic components “matter” and relativistic components “radiation”. In general, the time evolution of energy density $\rho(t)$ is given by

$$\rho \propto \exp\left(-3 \int \frac{da'}{a'}(1 + w(a'))\right). \quad (2.9)$$

Suppose that a hypothetical fluid corresponding to cosmological constant Λ . We can obtain the following conditions of energy fluid from Eqs. (2.6) and (2.7):

$$\rho_\Lambda = \frac{\Lambda c^4}{8\pi G}, \quad (2.10)$$

$$w_{\text{DE}} \equiv \frac{p_\Lambda}{\rho_\Lambda} = -1. \quad (2.11)$$

Hence, we can regard cosmological constant effectively as the energy fluid specified by Eqs. (2.10) and (2.11). Hereafter, we introduce dark energy ρ_Λ instead of cosmological constant. The density of dark energy with the same properties as Λ does not evolve in time.

In modern cosmology, the expansion history of the universe can be described by the following parameters called cosmological parameters: Hubble parameter H , density parameter Ω_α , critical density ρ_c and the curvature parameter Ω_K . The definition of these parameters are summarized as follows:

$$H = \frac{\dot{a}}{a}, \quad (2.12)$$

$$\Omega_\alpha = \frac{\rho_\alpha}{\rho_c}, \quad (2.13)$$

$$\rho_c = \frac{3H^2}{8\pi G}, \quad (2.14)$$

$$\Omega_K = \frac{Kc^2}{a^2 H^2}. \quad (2.15)$$

With these parameters, Eq. (2.6) is given by

$$H^2(a) = H_0^2 \left[\frac{\Omega_{m0}}{a^3} + \frac{\Omega_{\gamma 0}}{a^4} - \frac{\Omega_{K0}}{a^2} + \Omega_{\Lambda 0} \exp \left\{ -3 \int \frac{da'}{a'} (1 + w_{\text{DE}}(a')) \right\} \right], \quad (2.16)$$

where the index 0 denotes the present value of each parameter. Once cosmological parameters are specified at present, the expansion history of the universe can be determined by Eq. (2.16). In the following, we summarize the expansion rate of the universe at the dominant epoch of radiation Ω_γ , matter Ω_m , curvature K and dark energy Ω_Λ .

1. Radiation domination

$$a = (2H_0)^{1/2} \Omega_{\gamma 0}^{1/4} t^{1/2}. \quad (2.17)$$

2. Matter domination

$$a = \left(\frac{9}{4} H_0^2 \Omega_{m0} \right)^{1/3} t^{2/3}. \quad (2.18)$$

3. Curvature domination

Curvature can dominate the universe when $\Omega_\Lambda = 0$, $K < 0$, $a > -\Omega_{m0}/\Omega_{K0}$.

$$a = H_0 \Omega_{K0}^{1/2} t \quad (2.19)$$

4. Dark energy domination

We consider the case of $w_{DE} = -1$ for simplicity.

$$a = \exp \left[\Omega_{\Lambda 0}^{1/2} H_0 (t - t_0) \right] \quad (2.20)$$

Current astrophysical observations yield the present value of dark energy density $\Omega_{\Lambda 0} \sim 0.7$. The simplest candidate of dark energy with $w_{DE} = -1$ is thought to be vacuum energy. However, if dark energy is vacuum energy, there appears to be a huge discrepancy between the observed amount of dark energy and the expected amount of vacuum energy at the present. This is one of the main motivation of other proposals for the candidate of dark energy. Among various models of dark energy, $w_{DE}(a)$ is one of the key parameters to identify dark energy. It is crucial to determine $w_{DE}(a)$ precisely by observation for understanding what dominates the present universe and why the current expansion of the universe is accelerating. In practice, the following parameterization of $w_{DE}(a)$ is often used:

$$w_{DE}(a) = w_0 + w_1(1 - a) + \dots \quad (2.21)$$

In this thesis, we pay particular attention to constraints on the parameter of w_0 .

2.1.2 Cosmological Redshift and Angular-Diameter Distance

Here, we consider cosmological redshift as the time coordinate and define the angular diameter distance. Cosmological redshift is caused by a stretch of the wavelength of photon due to the expansion of the universe. Consider that the photon emitted at $t = t_1$ from the point (r_1, θ_1, ϕ_1) . The photon path in a FLRW universe is determined by null geodesics, i.e. $ds = 0$ in Eq. (2.1). It is given by

$$\int_{t_1}^{t_0} \frac{cdt}{a(t)} = \int_0^{r_1} \frac{dr}{1 - Kr^2}, \quad (2.22)$$

where t_0 is the arrival time of photon and we set $d\theta = d\phi = 0$ because of homogeneity and isotropy of space. The right hand side of Eq. (2.22) is independent of the time. This leads to the following equation when considering the case of another emitted time $t_1 + \delta t_1$ and arrival time $t_0 + \delta t_0$:

$$\int_{t_1}^{t_0} \frac{cdt}{a(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{cdt}{a(t)}. \quad (2.23)$$

Suppose that the evolution of $a(t)$ is negligible during δt_1 and δt_0 . Then, we can obtain the following relation with the Taylor expansion of Eq. (2.23) around t_0 and t_1 :

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)}. \quad (2.24)$$

This result can be described in terms of redshift z as follows:

$$1 + z = \frac{\lambda_0}{\lambda_1} = \frac{1}{a(t_1)}, \quad (2.25)$$

where the wavelength of photon λ_i is defined by $c\delta t_i$ and $a(t_0)$ is set to be unity.

We next define the angular diameter distance d_A . The angular diameter distance to an object is defined by the object's size ℓ and the apparent angular size of the object $\Delta\theta$. In the FLRW metric, the relation of ℓ between $\Delta\theta$ is given by

$$\ell = ar\Delta\theta = \frac{r}{1+z}\Delta\theta. \quad (2.26)$$

Thus, d_A is obtained by

$$d_A = \frac{\ell}{\Delta\theta} = \frac{r}{1+z}. \quad (2.27)$$

In general, the angular diameter distance between two redshifts z_1 and z_2 ($z_1 < z_2$) can be calculated by

$$d_A(z_1, z_2) = \frac{r(z_1, z_2)}{1+z_2}, \quad (2.28)$$

where $r(z_1, z_2)$ is defined by $\int_{z_1}^{z_2} \frac{cdz}{H(z)}$.

2.2 Growth of Matter Density

2.2.1 Evolution of Density Fluctuations

We can not explain the rich structure of the universe observed today only assuming FLRW metric because FLRW metric describes the *homogeneous* universe. According to the observation of cosmic microwave background (CMB), there exist tiny fluctuations in the CMB temperature map. These fluctuations are expected to amplify its amplitude mainly due to gravitational growth and develop the rich structure of the universe such as galaxies and clusters of galaxies. The gravitational growth of density fluctuations is governed by the fluid equation and the Poisson equation under the background expansion of the universe with FLRW metric. The matter density of fluid $\rho(\vec{x}, t)$ can be decomposed into the homogeneous and inhomogeneous part;

$$\rho(\vec{x}, t) = \bar{\rho}(t) + \delta\rho(\vec{x}, t), \quad (2.29)$$

$$\delta\rho(\vec{x}, t) \equiv \bar{\rho}(t)\delta(\vec{x}, t), \quad (2.30)$$

where $\bar{\rho}$ and $\delta\rho$ are the homogeneous and inhomogeneous part, respectively. The fluid equation and the Poisson equation under the background FLRW universe are given by

$$\dot{\delta} + \frac{1}{a} \nabla \cdot [(1 + \delta)\vec{u}] = 0, \quad (2.31)$$

$$\dot{\vec{u}} + \frac{\dot{a}}{a} \vec{u} + \frac{1}{a} (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla p}{a\bar{\rho}(1 + \delta)} - \nabla \frac{\Phi}{a}, \quad (2.32)$$

$$\Delta\Phi = 4\pi G\bar{\rho}a^2\delta, \quad (2.33)$$

where \vec{u} is the velocity field of matter fluid, $\Phi = \phi + \frac{a\ddot{a}}{2}x^2$, ϕ represents the gravitational potential and ∇ is the derivative operator by the comoving coordinate \vec{x} . The evolution of density fluctuations can be determined by a set of non-linear equations Eqs. (2.31)–(2.33).

2.2.2 Linear Perturbation

It is difficult to determine the evolution of matter density analytically in general. However, perturbations of matter density can be understood with the linearized equations when the amplitude of perturbations is sufficiently small, i.e. $\delta \ll 1$. For matter components with $p = 0$, we can obtain the following equation of δ by considering the first order of Eqs. (2.31)–(2.33)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\bar{\rho}\delta = 0. \quad (2.34)$$

Thus, the evolution of δ is determined as a function of time over various physical scales at the level of the first order in Eqs. (2.31)–(2.33). The linear growth of δ would be affected by the expansion history of the universe. Here, we summarize the linear growth of δ in various cosmological models.

1. Radiation domination and equality epoch

Let us consider the evolution of δ from radiation domination to the equality time. The equality time is defined by the cosmic epoch when the energy density of radiation in the universe is equal to that of matter. At this epoch, a mixture of radiation and matter dominates the universe. Hubble parameter H is then calculated by

$$\begin{aligned} H(a) &= H_0 \sqrt{\Omega_{m0}a^{-3} + \Omega_{\gamma0}a^{-4}} \\ &= \frac{H_0\Omega_{m0}^{1/2}}{a^2} \sqrt{a + a_{eq}}, \end{aligned} \quad (2.35)$$

where a_{eq} is the scale factor at the equality time, defined by $a_{eq} = \Omega_{\gamma0}/\Omega_{m0}$. We can rewrite Eq. (2.34) by the new time coordinate $y = a/a_{eq}$ instead of t as follows;

$$\frac{d^2\delta}{dy^2} + \frac{2+3y}{2y(1+y)} \frac{d\delta}{dy} - \frac{3\delta}{2y(1+y)} = 0. \quad (2.36)$$

There are two kinds of solutions of Eq. (2.36). One is given by

$$\delta \propto 1 + \frac{3y}{2}, \quad (2.37)$$

and another is expressed as

$$\delta \propto \left(1 + \frac{3y}{2}\right) \ln \left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y}. \quad (2.38)$$

This result indicates that the density fluctuation of matter density can grow *gradually* (i.e. by a factor of 2.5) in radiation domination.

2. Matter Domination

In matter domination, Hubble parameter H is equal to $2/(3t)$. Using the relation between $\bar{\rho}$ and the time at this epoch ($\bar{\rho} = 1/(6\pi Gt^2)$), we can rewrite Eq. (2.34) as follows;

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0. \quad (2.39)$$

We can solve Eq. (2.39) by considering the case of $\delta \propto t^n$. The solution can be expressed as

$$\delta = At^{2/3} + Bt^{-1}. \quad (2.40)$$

Thus, the evolution of δ is determined by $\delta \propto t^{2/3} \propto a$.

3. Λ CDM model

Here, we consider the case of Λ CDM model that is consistent with current multiple astrophysical observations. In this model, the radiation density $\Omega_{\gamma 0}$ and the curvature K is negligible and the equation state of dark energy w_{DE} is set to be -1 . Therefore, Hubble parameter H is given by

$$H(a) = H_0 \sqrt{\Omega_{\text{m}0} a^{-3} + \Omega_{\Lambda 0}}. \quad (2.41)$$

One can find that Eq. (2.41) is actually the specific solution of Eq. (2.34) in Λ CDM model with the relation of $4\pi G \bar{\rho} = 3/2 H^2 \Omega_{\text{m}}$. Hence, one can obtain another solution of Eq. (2.34) by assuming $D(a) = H(a)f(a)$. The solution is given by

$$D(a) \propto H(a) \int^a \frac{da'}{(a' H(a'))^3}. \quad (2.42)$$

Note that $H(a)$ represents the decline of the linear growth and $D(a)$ describes the linear growth of δ in Λ CDM model. We also extend Λ CDM model by considering $w_{\text{DE}} = \text{const.} \neq -1$. In this model, the linear growth of matter density perturbation can be expressed as

$$D(a) \propto a F\left(-\frac{1}{3w_{\text{DE}}}, \frac{w_{\text{DE}} - 1}{2w_{\text{DE}}}, 1 - \frac{5}{6w_{\text{DE}}}, x\right), \quad (2.43)$$

$$x = -\frac{\Omega_{\Lambda 0}}{\Omega_{\text{m}0}} a^{-3w_{\text{DE}}}, \quad (2.44)$$

where $F(\alpha, \beta, \gamma, x)$ is known as the hypergeometric function. An integral giving the hypergeometric function is

$$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tx)^{-\alpha} dt. \quad (2.45)$$

We here emphasize that all the results above are correct only when matter overdensity δ is significantly small, i.e. $\delta \ll 1$. We can not predict the evolution of δ in the way as shown above once the amplitude of δ becomes larger and the mode coupling of δ (the coupling term such as $\delta \cdot \vec{u}$ etc.) becomes important.

2.2.3 Non-linear Perturbation

The spherical collapse model is one of the simplest models to describe non-linear growth of matter density in the universe. Suppose that matter density around a given point distributes in a spherical manner. The gravitational force at a shell with an offset distance of r from the center of system can be determined by the inner mass M within this shell. The equation of motion for the shell is given by

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2}, \quad (2.46)$$

and the solution of the above equation under the condition of $dr/dt > 0$ and $r = 0$ at $t = 0$ is

$$r = A^2(1 - \cos \theta), \quad (2.47)$$

$$t = \frac{A^3}{\sqrt{GM}}(\theta - \sin \theta). \quad (2.48)$$

When considering the matter domination for simplicity, one can find that the overdensity within the shell is given by

$$\delta = \frac{9GMt^2}{2r^3} - 1 = \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^3} - 1, \quad (2.49)$$

where we use the relation of mean matter density and cosmic time i.e. $\bar{\rho} = 1/(6\pi Gt^2)$. As you can see from Eq. (2.47), the shell will expand from $\theta = 0$ to $\theta = \pi$ and then contract from $\theta = \pi$. Finally, the overdensity within this shell would diverge when $\theta = 2\pi$. Let us assume this system would be virialized through the contraction of each shell and the formation of object with a finite size of r_{vir} occurs. In this scenario, the following relations should be realized according to the energy conservation of system and the virial theorem;

$$2K_{\text{vir}} + U_{\text{vir}} = 0, \quad (2.50)$$

$$K_{\text{vir}} + U_{\text{vir}} = U_{\text{turn}}, \quad (2.51)$$

where U_{turn} is the potential energy of the shell at $\theta = \pi$ and K_{vir} and U_{vir} represent the kinematic energy and the potential energy of shell, respectively. These equations provide $r_{\text{vir}} = A^2$. Thus, broadly speaking, the overdensity when the system is virialized can be evaluated as

$$\Delta_{\text{vir}} = \frac{3M}{4\pi r_{\text{vir}}^3} \frac{1}{\bar{\rho}(t_{\text{coll}})} - 1 = 18\pi^2 - 1 \simeq 177, \quad (2.52)$$

where t_{coll} is the free fall time given by $t(\theta = 2\pi) = 2A^3/\sqrt{GM}$. Therefore, we can estimate the overdensity within an virialized region as ~ 177 .

It is useful to consider the *linear* overdensity in a virialized system. The linear density in the spherical model is defined by the lowest order of Eq. (2.49) in terms of θ ,

$$\delta_L(t) = \frac{3}{20} \left(\frac{6\sqrt{GM}}{A^3} t \right)^{2/3}. \quad (2.53)$$

The linear overdensity at $t = t_{\text{coll}}$ is ~ 1.69 .

We can easily extend the above calculation to the case of various cosmological models. Reference [1] provided the useful fitting formula in Λ CDM model as follows;

$$\Delta_{\text{vir}} \simeq 18\pi^2(1 + 0.4093w_f^{0.9052}), \quad (2.54)$$

$$\delta_L \simeq \frac{3(12\pi)^{2/3}}{20}(1 - 0.00123 \log_{10} \Omega_f), \quad (2.55)$$

$$w_f = \frac{1}{\Omega_f} - 1, \quad (2.56)$$

$$\Omega_f = \frac{\Omega_{m0}(1+z)^3}{\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}}. \quad (2.57)$$

How does matter distribute in virialized system such as galaxy and cluster of galaxies? Reference [2] has performed cosmological N-body simulation with various cosmological model and the authors found that matter density profile of virialized dark matter halo can be described by the *universal* function as follows;

$$\rho_h(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}, \quad (2.58)$$

where ρ_s and r_s are the scale density and the scale radius, respectively. These parameters can be condensed into one parameter, the concentration $c_{\text{vir}}(M, z)$, by the use of two halo mass relations; namely, $M = 4\pi r_{\text{vir}}^3 \Delta_{\text{vir}}(z) \rho_{\text{crit}}(z)/3$, where r_{vir} is the virial radius corresponding to the overdensity criterion $\Delta_{\text{vir}}(z)$ as shown, e.g., in Eq. (2.54), and $M = \int dV \rho_h(\rho_s, r_s)$ with the integral performed out to r_{vir} . At present, the density profile shown in Eq. (2.58) is called NFW profile. NFW profile have been conformed for wide range of mass scales (from earth-size halos to clusters of galaxies) at different epochs in current (dark matter only) N-body simulations [3–6]. Once NFW profile is assumed, one can easily calculate the various observables such as rotation curve of galaxies (e.g., [7, 8]), gravitational lensing effect of clusters of galaxies (e.g., [9]), hot gas distribution in galaxy clusters (e.g., [10]), and two-point statistics of density perturbations (e.g., [11]).

2.3 Statistics of Matter Density Perturbation

2.3.1 Two Point Statistics

One needs the statistical method to investigate gravitational growth of density perturbations in the universe with a large data set obtained from photometric and/or spectroscopic astronomical surveys. One of the simplest statistics is two point correlation function. Two point correlation function represents the clustering of astrophysical sources such as galaxies, which is defined by

$$\xi(|\vec{x}_1 - \vec{x}_2|) = \langle \delta_g(\vec{x}_1) \delta_g(\vec{x}_2) \rangle, \quad (2.59)$$

$$\delta_g(\vec{x}) = (n(\vec{x}) - \bar{n})/\bar{n}, \quad (2.60)$$

where $n(\vec{x})$ is the number density of objects and \bar{n} represents the mean number density. In general, number density of astrophysical objects $n(\vec{x})$ can be biased from underlying matter density $\rho(\vec{x})$. We here consider the simplest case that $n(\vec{x})$ can be proportional to $\rho(\vec{x})$, i.e. matter overdensity δ is equal to δ_g .

It is useful to consider two point correlation function in fourier space instead of real space. In fourier space, density perturbation $\tilde{\delta}(\vec{k})$ is related to $\delta(\vec{x})$ as follows;

$$\delta(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3k \tilde{\delta}(\vec{k}) \exp(i\vec{k} \cdot \vec{x}), \quad (2.61)$$

$$\tilde{\delta}(\vec{k}) = \int d^3x \delta(\vec{x}) \exp(-i\vec{k} \cdot \vec{x}). \quad (2.62)$$

Thus, two point correlation between two different wave numbers \vec{k} and \vec{k}' is given by

$$\langle \tilde{\delta}(\vec{k}) \tilde{\delta}(\vec{k}') \rangle = (2\pi)^3 \delta_D^{(3)}(\vec{k} + \vec{k}') \int d^3r \xi(r) \exp(-i\vec{k} \cdot \vec{r}), \quad (2.63)$$

where $\delta_D^{(3)}(\vec{r})$ represents three-dimensional dirac function. The integral in Eq. (2.63) is called power spectrum and often is denoted by $P(k)$. Power spectrum depends on only the amplitude of \vec{k} if the universe is isotropic.

The initial condition of power spectrum of matter density is usually assumed to be a power law function, i.e. $P_{\text{init}}(k) \propto k^{n_s}$. This originates from early works by Harrison [12] and Zeldovich [13] in 1970s (The similar approximation is also found in Ref. [14]). In their prescription, all perturbations that come within the horizon have the same amplitude. In this case, n_s is found to be unity and the case of $n_s = 1$ is called Harrison–Zeldovich spectrum. Most of inflation models also predict the power law type of primordial power spectrum.

The shape of power spectrum would be affected by growth of primordial density perturbation through various physical processes. In the linear regime (i.e. $\delta \ll 1$), power spectrum would be modified in an independent way of wave number mode k

because the evolution of δ can be determined by a function of time not k as shown in Eq. (2.34). Thus, overdensity $\tilde{\delta}(k, a)$ should be decomposed into a function of k and a as follows;

$$\tilde{\delta}(k, a) = \frac{T(k)D(a)}{D(a_{\text{init}})} \tilde{\delta}(k, a_{\text{init}}), \quad (2.64)$$

where $D(a)$ is the growth factor given by the solution of Eq. (2.34) and $T(k)$ represents the evolution of density perturbation at different scales, which is called the transfer function. Therefore, power spectrum at a given k and a can be written as

$$P(k, a) = \frac{T^2(k)D^2(a)}{D^2(a_{\text{init}})} P_{\text{init}}(k). \quad (2.65)$$

Note that the above formula should be valid in the linear regime (i.e. $\delta \ll 1$). In order to obtain the specific shape of $T(k)$, we need to solve the Boltzmann equation coupled with General Relativity. Although it is difficult to derive $T(k)$ analytically, one can obtain $T(k)$ numerically with the Boltzmann equation solver [15] or the fitting formula shown in Ref. [16].

The normalization of power spectrum is determined by observations. One possible way is based on the variance of smoothed overdensity σ_R with comoving scale of R Mpc/ h at present. σ_R is given by

$$\sigma_R^2 = \int \frac{d^3k}{(2\pi)^3} P(k, a=1) |W_R(k)|^2 \quad (2.66)$$

where $W_R(k)$ is the window function, which is set to be the top-hat function in practice. $R = 8$ Mpc/ h is the conventional scale for the normalization of power spectrum. We can also use another observational result in the early universe, e.g., the power spectrum of primordial curvature perturbation generated by inflation at some pivot wave number of k_0 . In matter domination, power spectrum of curvature perturbation is related to one of matter density through Poisson equation as follows;

$$\frac{4\pi k^3 P(k, a)}{(2\pi)^3} = \Delta_{\mathcal{R}}^2(k_0) \left(\frac{2c^2 k^2}{5H_0^2 \Omega_{m0}} \right)^2 D^2(a) T^2(k) \left(\frac{k}{k_0} \right)^{-1+n_s} \quad (2.67)$$

$$\Delta_{\mathcal{R}}^2(k) = \frac{4\pi k^3 P_{\mathcal{R}}(k)}{(2\pi)^3} \quad (2.68)$$

where $P_{\mathcal{R}}(k_0)$ represents power spectrum of curvature perturbation which is determined by observations.

2.3.2 Mass Function and Halo Bias

The abundance of massive objects such as clusters of galaxies is one of the powerful tools to probe cosmology at lower redshift. Let us consider the number of virialized objects with mass range of $M \sim M + dM$ which is called mass function. One of the simplest way to calculate the mass function is Press-Schechter formalism [17]. In Press-Schechter formalism, virialized objects with mass of M are assumed to form where the density perturbation in an sphere with radii of $R = (3M/4\pi\bar{\rho})^{1/3}$ is larger than the critical value δ_c . δ_c is often considered to be ~ 1.69 as shown in Eq. (2.53).

Smoothed density perturbation with smoothing scale of R is given by

$$\delta_R(\vec{x}) = \int d^3x' W_R(|\vec{x} - \vec{x}'|) \delta(\vec{x}'), \quad (2.69)$$

where $W_R(|\vec{x} - \vec{x}'|)$ represents window function for smoothing. Top-hat window function is often used in literature. Suppose that smoothed density perturbation follows Gaussian, the probability of formation of virialized objects with mass of $M = 4\pi/3\bar{\rho}R^3$ can be written as

$$F(M) = 2 \int_{\delta_c}^{\infty} d\delta_R \frac{1}{\sqrt{2\pi\sigma_R^2}} \exp\left(-\frac{1}{2} \frac{\delta_R^2}{\sigma_R^2}\right) = 2 \times \frac{1}{2} \text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R}\right), \quad (2.70)$$

where σ_R is the variance of smoothed density perturbation given by Eq. (2.66). The factor of 2 in Eq. (2.70) is the multiplicative correction so that $F(0)$ would be equal to unity when $R \rightarrow 0$. (Note that $\sigma_R \rightarrow \infty$ with limit of $R \rightarrow 0$.) In this thesis, $n(M)dM$ denotes the number density of virialized objects with mass range of $M - M + dM$. Thus, the mass fraction of virialized objects with mass of M can be written as $n(M)M dM/\bar{\rho}$. This fraction should be equal to be $F(M + dM) - F(M) = |\partial F/\partial M|_M dM$. Therefore, we can evaluate $n(M)$ by equating $n(M)M dM/\bar{\rho}$ with $|\partial F/\partial M|_M dM$;

$$\begin{aligned} n(M) &= -\frac{\bar{\rho}}{M} \frac{\partial F}{\partial M} \\ &= -\frac{\bar{\rho}}{M} \sqrt{\frac{2}{\pi}} \frac{\delta_c}{\sigma_R^2} \exp\left(-\frac{1}{2} \frac{\delta_c^2}{\sigma_R^2}\right) \frac{\partial \sigma_R}{\partial M} \\ &= f_{\text{PS}}\left(\frac{\delta_c}{\sigma_R}\right) \frac{\bar{\rho}}{M^2} \left| \frac{d \ln \sigma_R^{-1}}{d \ln M} \right|, \end{aligned} \quad (2.71)$$

$$f_{\text{PS}}(v) = \sqrt{\frac{2}{\pi}} v e^{-v^2/2}. \quad (2.72)$$

Although Press-Schechter formalism relies on various approximations, it can explain overall feature of mass function found in cosmological N-body simulation. The detailed shape of mass function have been calibrated with numerical simulations (e.g., [18–20]). There are some previous works based on the analytic approach with ellipsoidal collapse model [21, 22]. In the case of ellipsoidal collapse model, f_{PS} can be replaced with the following function;

$$f_{\text{ST}}(v) = A \sqrt{\frac{2\alpha}{\pi}} \left[1 + (\alpha v^2)^{-p} \right] v e^{-\alpha v^2/2}, \quad (2.73)$$

where α and p represent the parameter of ellipsoidal collapse model and A is the normalization factor. Numerical simulation have been utilized for calibration of these parameters, which are given by $A = 0.322$, $\alpha = 0.707$, and $p = 0.3$.

Virialized objects such as galaxies and galaxy clusters are *biased* tracer of underlying matter distribution. Thus, the clustering of virialized objects would be different from one of matter density perturbation. The peak-background split formalism [23] give a simple framework to calculate the clustering of virialized objects at large scale. One can split underlying density perturbation into long-wavelength mode δ_ℓ and short-wavelength mode δ_s ;

$$\rho(\vec{q}) = \bar{\rho}(1 + \delta_\ell + \delta_s), \quad (2.74)$$

where \vec{q} represents the coordinate in the Lagrangian space. The number density of virialized object with mass of M at the position of \vec{q} would be modulated by presence of the long-wavelength mode of density perturbation. Hence, the simple model of the number density field of virialized objects is given by a local shift in the density threshold, i.e. replacing δ_c with $\delta_c - \delta_\ell(\vec{q})$ in Eq. (2.71). In this context, the number density contrast of virialized objects in the Lagrangian space is given by

$$\delta_h(\vec{q}|M) = \frac{n_h(\vec{q}|M)}{n(M)} - 1, \quad (2.75)$$

where $n_h(\vec{q}|M)$ is the number density field of objects with mass of M at \vec{q} and $n(M)$ represents the mean number density which is given by e.g., Eq. (2.71). By expanding this equation into Taylor series of δ_ℓ in Eq. (2.75), one can relate δ_h with δ_ℓ as follows;

$$\delta_h(\vec{q}|M) = b_L(M) \delta_\ell(\vec{q}), \quad (2.76)$$

$$b_L = \frac{1}{n(M)} \left(\frac{\partial n_h}{\partial \delta_\ell} \right)_{\delta_\ell=0}. \quad (2.77)$$

b_L is the large-scale bias of virialized objects in Lagrangian space. In Eulerian space, one needs to add the correction of the Eulerian space clustering. Therefore, the total or Eulerian bias is given by $b = b_L + 1$. b_L can be calculated once the function

form of mass function is specified. For example, in the case of the functional form in Eq. (2.72), b_L is given by

$$b_L(M) = \frac{v^2 - 1}{\delta_c}, \quad (2.78)$$

where v is given by $\delta_c/\sigma_R(M)$.

References

1. T. Kitayama, Y. Suto, Semianalytical predictions for statistical properties of x-ray clusters of galaxies in cold dark matter universes. *Astrophys. J.* **469**, 480 (1996). [astro-ph/9604141](#)
2. J.F. Navarro, C.S. Frenk, S.D. White, The structure of cold dark matter halos. *Astrophys. J.* **462**, 563–575 (1996). [astro-ph/9508025](#)
3. J.S. Bullock, T.S. Kolatt, Y. Sigad, R.S. Somerville, A.V. Kravtsov, A.A. Klypin, J.R. Primack, A. Dekel, Profiles of dark haloes. Evolution, scatter, and environment. *Mon. Not. Roy. Astron. Soc.* **321**, 559–575 (2001). [astro-ph/9908159](#)
4. F. Prada, A.A. Klypin, A.J. Cuesta, J.E. Betancort-Rijo, J. Primack, Halo concentrations in the standard LCDM cosmology. *Mon. Not. Roy. Astron. Soc.* **428**, 3018–3030 (2012). [arXiv:1104.5130](#)
5. T. Ishiyama, S. Rieder, J. Makino, S.P. Zwart, D. Groen, K. Nitadori, C. de Laat, S. McMillan, K. Hiraki, S. Harfst, The cosmogrid simulation: statistical properties of small dark matter halos. *Astrophys. J.* **767**, 146 (2013). [arXiv:1101.2020](#)
6. T. Ishiyama, Hierarchical formation of dark matter halos and the free streaming scale. *Astrophys. J.* **788**, 27 (2014). [arXiv:1404.1650](#)
7. M. Persic, P. Salucci, F. Stel, The Universal rotation curve of spiral galaxies: 1. The dark matter connection. *Mon. Not. Roy. Astron. Soc.* **281**, 27 (1996). [astro-ph/9506004](#)
8. P. Salucci, M. Persic, Dark Halos around Galaxies, in *Dark and Visible Matter in Galaxies and Cosmological Implications*, eds. by M. Persic, P. Salucci vol. 117 of Astronomical Society of the Pacific Conference Series (1997), p. 1. [astro-ph/9703027](#)
9. C.O. Wright, T.G. Brainerd, Gravitational lensing by nfw halos. [astro-ph/9908213](#)
10. E. Komatsu, U. Seljak, Universal gas density and temperature profile. *Mon. Not. Roy. Astron. Soc.* **327**, 1353–1366 (2001). [astro-ph/0106151](#)
11. A. Cooray, R.K. Sheth, Halo models of large scale structure. *Phys. Rept.* **372**, 1–129 (2002). [astro-ph/0206508](#)
12. E.R. Harrison, Fluctuations at the threshold of classical cosmology. *Phys. Rev. D* **1**, 2726–2730 (1970)
13. Y. Zeldovich, A. Hypothesis, Unifying the structure and the entropy of the universe. *Mon. Not. Roy. Astron. Soc.* **160**, 1P–3P (1972)
14. P. Peebles, J. Yu, Primeval adiabatic perturbation in an expanding universe. *Astrophys. J.* **162**, 815–836 (1970)
15. A. Lewis, A. Challinor, A. Lasenby, Efficient computation of CMB anisotropies in closed FRW models. *Astrophys. J.* **538**, 473–476 (2000). [astro-ph/9911177](#)
16. D.J. Eisenstein, W. Hu, Baryonic features in the matter transfer function. *Astrophys. J.* **496**, 605 (1998). [astro-ph/9709112](#)
17. W.H. Press, P. Schechter, Formation of galaxies and clusters of galaxies by selfsimilar gravitational condensation. *Astrophys. J.* **187**, 425–438 (1974)
18. A. Jenkins, C. Frenk, S.D. White, J. Colberg, S. Cole, A.E. Evrard, H.M.P. Couchman, N. Yoshida, The mass function of dark matter halos. *Mon. Not. Roy. Astron. Soc.* **321**, 372 (2001). [astro-ph/0005260](#)

19. M.S. Warren, K. Abazajian, D.E. Holz, L. Teodoro, Precision determination of the mass function of dark matter halos. *Astrophys. J.* **646**, 881–885 (2006). [astro-ph/0506395](#)
20. J. Tinker, A.V. Kravtsov, A. Klypin, K. Abazajian, M. Warren, G. Yepes, S. Gottlöber, D.E. Holz, Toward a halo mass function for precision cosmology: The limits of universality. *Astrophys. J.* **688**, 709–728 (2008). [arXiv:0803.2706](#)
21. R.K. Sheth, G. Tormen, Large scale bias and the peak background split. *Mon. Not. Roy. Astron. Soc.* **308**, 119 (1999). [astro-ph/9901122](#)
22. R.K. Sheth, G. Tormen, An Excursion set model of hierarchical clustering: Ellipsoidal collapse and the moving barrier. *Mon. Not. Roy. Astron. Soc.* **329**, 61 (2002). [astro-ph/0105113](#)
23. S. Cole, N. Kaiser, Biased clustering in the cold dark matter cosmogony. *Mon. Not. Roy. Astron. Soc.* **237**, 1127–1146 (1989)

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