Chapter 2
Linear Algebra

We need to use both *vector* and *matrix* ideas in this course. This was covered already in the first text (Peterson 2015), so we will assume you can review that material before you start into this chapter. Here we will introduce some new ideas as well as tools in MatLab we can use to solve what are called linear algebra problems; i.e. systems of equations. Let's begin by looking at inner products more closely.

2.1 The Inner Product of Two Column Vectors

We can also define the *inner product* of two vectors. If $\boldsymbol{V}$ and $\boldsymbol{W}$ are two column vectors of size $n \times 1$, then the product $\boldsymbol{V}^{\boldsymbol{T}}\,\boldsymbol{W}$ is a matrix of size 1×1 which we identify with a real number. We see if

$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix} \text{ and } \boldsymbol{W} = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_n \end{bmatrix}$$

then we define the 1×1 matrix

$$\boldsymbol{V}^{\boldsymbol{T}}\,\boldsymbol{W} = \boldsymbol{W}^{\boldsymbol{T}}\,\boldsymbol{V} = \langle \boldsymbol{V}, \boldsymbol{W} \rangle\,[V_1 W_1 + V_2 W_2 + V_3 W_3 + \cdots + V_n W_n]$$

and we identify this one by one matrix with the real number

$$V_1 W_1 + V_2 W_2 + V_3 W_3 + \cdots + V_n W_n$$

This product is so important, it is given a special name: it is the **inner product** of the two vectors $\boldsymbol{V}$ and $\boldsymbol{W}$. Let's make this formal with Definition 2.1.1.

J.K. Peterson, *Calculus for Cognitive Scientists*,
Cognitive Science and Technology, DOI 10.1007/978-981-287-877-9_2

Definition 2.1.1 (*The Inner Product Of Two Vectors*)
If $\boldsymbol{V}$ and $\boldsymbol{W}$ are two column vectors of size $n \times 1$, the inner product of these vectors is denoted by $\boldsymbol{< V, W >}$ which is defined as the matrix product $\boldsymbol{V^T}\ \boldsymbol{W}$ which is equivalent to the $\boldsymbol{W^T}\ \boldsymbol{V}$ and we interpret this 1×1 matrix product as the real number

$$V_1 W_1 + V_2 W_2 + V_3 W_3 + \cdots + V_n W_n$$

where V_i are the components of $\boldsymbol{V}$ and W_i are the components of $\boldsymbol{W}$.

2.1.1 Homework

Exercise 2.1.1 *Find the dot product of the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \ \textit{and} \ \boldsymbol{W} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

Exercise 2.1.2 *Find the dot product of the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} -6 \\ -8 \end{bmatrix} \ \textit{and} \ \boldsymbol{W} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

Exercise 2.1.3 *Find the dot product of the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} 10 \\ -4 \end{bmatrix} \ \textit{and} \ \boldsymbol{W} = \begin{bmatrix} 2 \\ 80 \end{bmatrix}.$$

We add, subtract and scalar multiply vectors and matrices as usual. We also suggest you review how to do matrix–vector multiplications. Multiplication of matrices is more complex as we discussed in the volume (Peterson 2015). Let's go through it again a bit more abstractly. Recall the dot product of two vectors $\boldsymbol{V}$ and $\boldsymbol{V W}$ is defined to be

$$\langle \boldsymbol{V}, \boldsymbol{W} \rangle = \sum_{i=1}^{n} V_i W_i$$

where n is the number of components in the vectors. Using this we can define the multiplication of the matrix $\boldsymbol{A}$ of size $n \times p$ with the matrix $\boldsymbol{B}$ of size $p \times m$ as follows.

$$\begin{bmatrix} \text{Row 1 of A} \\ \text{Row 2 of A} \\ \vdots \\ \text{Row n of A} \end{bmatrix} \left[\text{Column 1 of B} \mid \cdots \mid \text{Column n of B}\right]$$

$$= \begin{bmatrix} < \text{Row 1 of A, Column 1 of B} > \ldots < \text{Row 1 of A, Column n of B} > \\ < \text{Row 2 of A, Column 1 of B} > \ldots < \text{Row 2 of A, Column n of B} > \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ < \text{Row n of A, Column 1 of B} > \ldots < \text{Row n of A, Column n of B} > \end{bmatrix}$$

We can write this more succinctly with we let $\boldsymbol{A}_i$ denote the rows of $\boldsymbol{A}$ and $\boldsymbol{B}^i$ be the columns of $\boldsymbol{B}$. Note the use of subscripts for the rows and superscripts for the columns. Then, we can rewrite the matrix multiplication algorithm more compactly as

$$\begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_n \end{bmatrix} \left[\boldsymbol{B}^1 \mid \cdots \mid \boldsymbol{B}^n\right] = \begin{bmatrix} \langle \boldsymbol{A}_1, \boldsymbol{B}^1 \rangle & \ldots & \langle \boldsymbol{A}_1, \boldsymbol{B}^n \rangle \\ \langle \boldsymbol{A}_2, \boldsymbol{B}^1 \rangle & \ldots & \langle \boldsymbol{A}_2, \boldsymbol{B}^n \rangle \\ \vdots & \vdots & \vdots \\ \langle \boldsymbol{A}_n, \boldsymbol{B}^1 \rangle & \ldots & \langle \boldsymbol{A}_n, \boldsymbol{B}^1 \rangle \end{bmatrix}$$

Thus, the entry in row i and column j of the matrix product $\boldsymbol{AB}$ is

$$\boldsymbol{AB}_{ij} = \langle \boldsymbol{A}_i, \boldsymbol{B}^j \rangle.$$

Comment 2.1.1 *If $\boldsymbol{A}$ is a matrix of any size and $\boldsymbol{0}$ is the appropriate zero matrix of the same size, then both $\boldsymbol{0} + \boldsymbol{A}$ and $\boldsymbol{A} + \boldsymbol{0}$ are nicely defined operations and the result is just $\boldsymbol{A}$.*

Comment 2.1.2 *Matrix multiplication is not communicative: i.e. for square matrices $\mathbf{A}$ and $\mathbf{B}$, the matrix product $\mathbf{A}\ \mathbf{B}$ is not necessarily the same as the product $\mathbf{B}\ \mathbf{A}$.*

2.2 Interpreting the Inner Product

What could this number $<\boldsymbol{V}, \boldsymbol{W}>$ possibly mean? To figure this out, we have to do some algebra. Let's specialize to nonzero column vectors with only 2 components. Let

$$\boldsymbol{V} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \boldsymbol{W} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Since these vectors are not zero, only one of the terms in (a, c) and in (b, d) can be zero because otherwise both components would be zero and we are assuming these vectors are not the zero vector. We will use this fact in a bit. Now here $<\boldsymbol{V}, \boldsymbol{W}> = ab + cd$. So

$$(ab + cd)^2 = a^2b^2 + 2abcd + c^2d^2$$

$$\begin{aligned} \| \boldsymbol{V} \|^2 \| \boldsymbol{W} \|^2 &= \left(a^2 + c^2\right) \left(b^2 + d^2\right) \\ &= a^2b^2 + a^2d^2 + c^2b^2 + c^2d^2 \end{aligned}$$

Thus,

$$\begin{aligned} \| \boldsymbol{V} \|^2 \| \boldsymbol{W} \|^2 - (<\boldsymbol{V}, \boldsymbol{W}>)^2 &= a^2b^2 + a^2d^2 + c^2b^2 + c^2d^2 - a^2b^2 - 2abcd - c^2d^2 \\ &= a^2d^2 - 2abcd + c^2b^2 \\ &= (ad - bc)^2 . \end{aligned}$$

Now, this does look complicated, doesn't it? But this last term is something squared and so it must be non-negative! Hence, taking square roots, we have shown that

$$|<\boldsymbol{V}, \boldsymbol{W}>| \leq \| \boldsymbol{V} \| \| \boldsymbol{W} \|$$

Note, since a real number is always less than or equal to it absolute value, we can also say

$$<\boldsymbol{V}, \boldsymbol{W}> \leq \| \boldsymbol{V} \| \| \boldsymbol{W} \|$$

And we can say more. If it turned out that the term $(ad - bc)^2$ was zero, then $ad - bc = 0$. There are then a few cases to look at.

1. If all the terms a, b, c and d are not zero, then we can write $ad = bc$ implies $a/c = b/d$. We know the vector $\boldsymbol{V}$ can be interpreted as the line segment starting at $(0, 0)$ on the line with equation $y = (a/c)x$. Similarly, the vector $\boldsymbol{W}$ can be interpreted as the line segment connecting $(0, 0)$ and (b, d) on the line $y = (b/d)x$. Since $a/c = b/d$, these lines are the same. So both points (a, c) and (b, d) line on the same line. Thus, we see these vectors lay on top of each other or point directly opposite each other in the $x - y$ plane; i.e. the angle between these vectors is 0 or π radians (that is $0°$ or $180°$).
2. If $a = 0$, then bc must be 0 also. Since we know the vector $\boldsymbol{V}$ is not the zero vector, we can't have $c = 0$ also. Thus, b must be zero. This tells us $\boldsymbol{V}$ has components $(0, c)$ for some non zero c and $\boldsymbol{V}$ has components $(0, d)$ for some non zero d. These components also determine two lines like in the case above which either point in the same direction or opposite one another. Hence, again, the angle between the lines determined by these vectors is either 0 or π radians.
3. We can argue just like the case above if $d = 0$. We would find the angle between the lines determined by the vectors is either 0 or π radians.

We can summarize our results as a Theorem which is called the Cauchy–Schwartz Theorem for two dimensional vectors.

Theorem 2.2.1 (Cauchy Schwartz Theorem For Two Dimensional Vectors)
If $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *are two dimensional column vectors with components* (a, c) *and* (b, d) *respectively, then it is always true that*

$$< \boldsymbol{V}, \boldsymbol{W} > \leq |< \boldsymbol{V}, \boldsymbol{W} >| \leq || \boldsymbol{V} || \, || \boldsymbol{W} ||$$

Moreover,

$$|< \boldsymbol{V}, \boldsymbol{W} >| = || \boldsymbol{V} || \, || \boldsymbol{W} ||$$

if and only the quantity $ad - bc = 0$*. Further, this quantity is equal to* 0 *if and only if the angle between the line segments determined by the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *is* $0°$ *or* $180°$.

Here is yet another way to look at this: assume there is a non zero value of t so that the equation below is true.

$$\boldsymbol{V} + t\, \boldsymbol{W} = \begin{bmatrix} a \\ c \end{bmatrix} + t \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This implies

$$\begin{bmatrix} a \\ c \end{bmatrix} = -t \begin{bmatrix} b \\ d \end{bmatrix}$$

Since these two vectors are equal, their components must match. Thus, we must have

$$\begin{aligned} a &= -t\, b \\ c &= -t\, d \end{aligned}$$

Thus,

$$a\, d = (-t\, b)\, \frac{c}{-t} = b\, c$$

and we are back to $ad - bc = 0$! Hence, another way of saying that the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ are either $0°$ or $180°$ apart is to say that as vectors they are multiples of one another! Such vectors are called **collinear** vectors to save writing. In general, we say two n dimensional vectors are collinear if there is a nonzero constant t so that $\boldsymbol{V} = t\, \boldsymbol{W}$ although, of course, we can't really figure out a way to visualize these vectors!

Now, the scaled vectors $\boldsymbol{E} = \frac{\boldsymbol{V}}{||\boldsymbol{V}||}$ and $\boldsymbol{F} = \frac{\boldsymbol{W}}{||\boldsymbol{W}||}$ have magnitudes of 1. Their components are $(a/ \, || \, \boldsymbol{V} \, ||, c/ \, || \, \boldsymbol{V} \, ||$ and $(b/ \, || \, \boldsymbol{W} \, ||, d/ \, || \, \boldsymbol{W} \, ||$. These points lie on a circle of radius 1 centered at the origin. Let θ_1 be the angle $\boldsymbol{E}$ makes with the positive x-axis. Then, since the hypotenuse distance that defines the $\cos(\theta_1)$ and $\sin(\theta_1)$ is 1, we must have

$$\cos(\theta_1) = \frac{a}{|| \boldsymbol{V} ||}$$
$$\sin(\theta_1) = \frac{c}{|| \boldsymbol{V} ||}$$

We can do the same thing for the angle θ_2 that $\boldsymbol{F}$ makes with the positive x axis to see

$$\cos(\theta_2) = \frac{b}{|| \boldsymbol{W} ||}$$
$$\sin(\theta_2) = \frac{d}{|| \boldsymbol{W} ||}$$

The angle between vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ is the same as between vectors $\boldsymbol{E}$ and $\boldsymbol{F}$. Call this angle θ. Then $\theta = \theta_1 - \theta_2$ and using the formula for the cos of the difference of angles

$$\begin{aligned}
\cos(\theta) &= \cos(\theta_1 - \theta_2) \\
&= \cos(\theta_1)\ \cos(\theta_2) + \sin(\theta_1)\ \sin(\theta_2) \\
&= \frac{a}{|| \boldsymbol{V} ||}\ \frac{b}{|| \boldsymbol{W} ||} + \frac{c}{|| \boldsymbol{V} ||}\ \frac{d}{|| \boldsymbol{W} ||} \\
&= \frac{ab + cd}{|| \boldsymbol{V} ||\, || \boldsymbol{W} ||} \\
&= \frac{< \boldsymbol{V}, \boldsymbol{W} >}{|| \boldsymbol{V} ||\, || \boldsymbol{W} ||}
\end{aligned}$$

Hence, the ratio $< \boldsymbol{V}, \boldsymbol{W} > /(|| \boldsymbol{V} ||\, || \boldsymbol{W} ||)$ is the same as $\cos(\theta)$! So we can use this simple calculation to find the angle between a pair two dimensional vectors.

The more general proof of the Cauchy Schwartz Theorem for n dimensional vectors is a journey you can take in another mathematics class! We will state it though so we can use it later if we need it.

Theorem 2.2.2 (Cauchy Schwartz Theorem For n Dimensional Vectors)
If $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *are n dimensional column vectors with components* $(V_1, \ldots, V_n)$ *and* $(W_1, \ldots, W_n)$ *respectively, then it is always true that*

$$< \boldsymbol{V}, \boldsymbol{W} > \ \leq \ | < \boldsymbol{V}, \boldsymbol{W} > | \leq || \boldsymbol{V} ||\, || \boldsymbol{W} ||$$

Moreover,

$$|< \boldsymbol{V}, \boldsymbol{W} >| = || \boldsymbol{V} ||\, || \boldsymbol{W} ||$$

if and only the vector $\boldsymbol{V}$ *is a non zero multiple of the vector* $\boldsymbol{W}$.

Theorem 2.2.2 then tells us that if the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ are not zero, then

$$-1 \leq \frac{<\boldsymbol{V}, \boldsymbol{W}>}{||\ \boldsymbol{V}\ ||\ ||\ \boldsymbol{W}\ ||} \leq 1$$

and by analogy to what works for two dimensional vectors, we can use this ratio to define the cos of the angle between two n dimensional vectors even though we can't see them at all! We do this in Definition 2.2.1.

Definition 2.2.1 (*The Angle Between n Dimensional Vectors*)
If $\boldsymbol{V}$ and $\boldsymbol{W}$ are two non zero n dimensional column vectors with components $(V_1, \ldots, V_n)$ and $(W_1, \ldots, W_n)$ respectively, the angle θ between these vectors is defined by

$$\cos(\theta) = \frac{<\boldsymbol{V}, \boldsymbol{W}>}{||\ \boldsymbol{V}\ ||\ ||\ \boldsymbol{W}\ ||}$$

Moreover, the angle between the vectors is $0°$ if $<\boldsymbol{V}, \boldsymbol{W}> = 1$ and is $180°$ if $<\boldsymbol{V}, \boldsymbol{W}> = -1$.

2.2.1 Examples

Example 2.2.1 Find the angle between the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ given by

$$\boldsymbol{V} = \begin{bmatrix} -6 \\ 13 \end{bmatrix} \text{ and } \boldsymbol{W} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}.$$

Solution *Compute the inner product* $<\boldsymbol{V}, \boldsymbol{W}> = (-6)(-8) + (13)(1) = 61$. *Next, find the magnitudes of these vectors:* $||\ \boldsymbol{V}\ || = \sqrt{(-6)^2 + (13)^2} = \sqrt{205}$ *and* $||\ \boldsymbol{W}\ || = \sqrt{(-8)^2 + (1)^2} = \sqrt{65}$. *Then, if* θ *is the angle between the vectors, we know*

$$\cos(\theta) = \frac{<\boldsymbol{V}, \boldsymbol{W}>}{||\ \boldsymbol{V}\ ||\ ||\ \boldsymbol{W}\ ||} = \frac{61}{\sqrt{205}\sqrt{65}}$$
$$= 0.5284$$

Hence, since $\boldsymbol{V}$ *is in quadrant 2 and* $\boldsymbol{W}$ *is in quadrant 2 as well, we expect the angle between them should be between 0° and 90°. Your calculator should return* $\cos^{-1}(0.5284) = 58.10°$ *or* 1.0141 *rad. You should graph these vectors and see this visually too.*

Example 2.2.2 Find the angle between the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ given by

$$\boldsymbol{V} = \begin{bmatrix} -6 \\ -13 \end{bmatrix} \text{ and } \boldsymbol{W} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

Solution *Compute the inner product* $< \boldsymbol{V}, \boldsymbol{W} >= (-6)(8) + (-13)(1) = -61$. *Next, find the magnitudes of these vectors:* $|| \boldsymbol{V} ||= \sqrt{(-6)^2 + (-13)^2} = \sqrt{205}$ *and* $|| \boldsymbol{W} ||= \sqrt{(8)^2 + (1)^2} = \sqrt{65}$. *Then, if* θ *is the angle between the vectors, we know*

$$\begin{aligned}\cos(\theta) &= \frac{< \boldsymbol{V}, \boldsymbol{W} >}{|| \boldsymbol{V} || \, || \boldsymbol{W} ||} = \frac{-61}{\sqrt{205}\sqrt{65}} \\ &= -0.5284\end{aligned}$$

Hence, since **V** *is in quadrant 3 and* **W** *is in quadrant 1, we expect the angle between them should be larger than* $90°$. *Your calculator should return* $\cos^{-1}(-0.5284) = 121.90°$ *or* 2.1275 *rad. You should graph these vectors and see this visually too.*

Example 2.2.3 Find the angle between the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ given by

$$\boldsymbol{V} = \begin{bmatrix} 6 \\ -13 \end{bmatrix} \text{ and } \boldsymbol{W} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

Solution *Compute the inner product* $< \boldsymbol{V}, \boldsymbol{W} >= (6)(8) + (-13)(1) = 35$. *Next, find the magnitudes of these vectors:* $|| \boldsymbol{V} ||= \sqrt{(6)^2 + (-13)^2} = \sqrt{205}$ *and* $|| \boldsymbol{W} ||= \sqrt{(8)^2 + (1)^2} = \sqrt{65}$. *Then, if* θ *is the angle between the vectors, we know*

$$\begin{aligned}\cos(\theta) &= \frac{< \boldsymbol{V}, \boldsymbol{W} >}{|| \boldsymbol{V} || \, || \boldsymbol{W} ||} = \frac{35}{\sqrt{205}\sqrt{65}} \\ &= 0.3032\end{aligned}$$

Hence, since **V** *is in quadrant 4 and* **W** *is in quadrant 1, we expect the angle between them should be between* $0°$ *and* $180°$. *Your calculator should return* $\cos^{-1}(0.3032) = 72.35°$ *or* 1.2627 *rad. You should graph these vectors and see this visually too.*

2.2.2 Homework

Exercise 2.2.1 *Find the angle between the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \textit{ and } \boldsymbol{W} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

Exercise 2.2.2 *Find the angle between the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} -6 \\ -8 \end{bmatrix} \textit{ and } \boldsymbol{W} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}.$$

Exercise 2.2.3 *Find the angle between the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} 10 \\ -4 \end{bmatrix} \textit{ and } \boldsymbol{W} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

Exercise 2.2.4 *Find the angle between the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \textit{ and } \boldsymbol{W} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

Exercise 2.2.5 *Find the angle between the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \textit{ and } \boldsymbol{W} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Exercise 2.2.6 *Find the angle between the vectors* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *given by*

$$\boldsymbol{V} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \textit{ and } \boldsymbol{W} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

2.3 Determinants of 2 × 2 Matrices

Since the number $ad - bc$ is so important is all of our discussions about the relationship between the two dimensional vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ with components (a, c) and (b, d) respectively, we will define this number to be the **determinant** of the matrix $\boldsymbol{A}$ formed by using $\boldsymbol{V}$ for column 1 and $\boldsymbol{W}$ for column 2 of $\boldsymbol{A}$. That is

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix} = \left[\boxed{\begin{bmatrix} a \\ c \end{bmatrix}} \boxed{\begin{bmatrix} b \\ d \end{bmatrix}} \right]$$

We then formally define the **determinant** of the 2×2 matrix $\boldsymbol{A}$ by Definition 2.3.1.

Definition 2.3.1 (*The Determinant Of A* 2×2 *Matrix*)
Given the 2×2 matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the determinant of $\boldsymbol{A}$ is the number $ad - bc$. We denote the determinant by $det(\boldsymbol{A})$ or $\mid \boldsymbol{A} \mid$.

Comment 2.3.1 *It is also common to denote the determinant by*

$$det \boldsymbol{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Also, note that if we looked at the transpose of $\boldsymbol{A}$, we would find

$$\boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{Z} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{bmatrix} a \\ b \end{bmatrix}} & \boxed{\begin{bmatrix} c \\ d \end{bmatrix}} \end{bmatrix}.$$

Notice that $det\left(\boldsymbol{A}^{\boldsymbol{T}}\right)$ is $(a)(d) - (b)(c)$ also. Hence, if $det\left(\boldsymbol{A}^{\boldsymbol{T}}\right)$ is zero, it means that $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are collinear. Hence, it the $det\ (\boldsymbol{A})$ is zero, both the vectors determined by the rows of $\boldsymbol{A}$ and the columns of $\boldsymbol{A}$ are collinear. Let's summarize what we know about this new thing called the **determinant** of $\boldsymbol{A}$.

1. If $\mid \boldsymbol{A} \mid$ is 0, then the vectors determined by the columns of $\boldsymbol{A}$ are collinear. This also means that the vectors determined by the columns are multiples of one another. Also, the vectors determined by the columns of $\boldsymbol{A}^{\boldsymbol{T}}$ are also collinear.
2. If $\mid \boldsymbol{A} \mid$ is not 0, then the vectors determined by the columns of $\boldsymbol{A}$ are not collinear which means these vectors point in different directions. Another way of saying this is that these vectors are **not** multiples of one another. The same is true for the columns of the transpose of $\boldsymbol{A}$.

2.3.1 Worked Out Problems

Example 2.3.1 Compute the determinant of

$$\boldsymbol{A} = \begin{bmatrix} 16.0 & 8.0 \\ -6.0 & -5.0 \end{bmatrix}$$

Solution $\mid \boldsymbol{A} \mid = (16)(-5) - (8)(-6) = -32.$

Example 2.3.2 Compute the determinant of

$$\boldsymbol{A} = \begin{bmatrix} -2.0 & 3.0 \\ 6.0 & -9.0 \end{bmatrix}$$

Solution $\mid \boldsymbol{A} \mid = (-2)(-9) - (3)(6) = 0.$

Example 2.3.3 Determine if the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ given by

$$\boldsymbol{V} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \text{ and } \boldsymbol{W} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

are collinear.

Solution *Form the matrix* **A** *using these vectors as the columns. This gives*

$$\boldsymbol{A} = \begin{bmatrix} 4 & -2 \\ 5 & 3 \end{bmatrix}$$

The calculate $\mid \boldsymbol{A} \mid = (4)(3) - (-2)(5)$. *Since this value is* 22 *which is not zero, these vectors are not collinear.*

Example 2.3.4 Determine if the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ given by

$$\boldsymbol{V} = \begin{bmatrix} -6 \\ 4 \end{bmatrix} \text{ and } \boldsymbol{W} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

are collinear.

Solution *Form the matrix* $\boldsymbol{A}$ *using these vectors as the columns. This gives*

$$\boldsymbol{A} = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$$

The calculate $\mid \boldsymbol{A} \mid = (-6)(-2) - (3)(4)$. *Since this value is* 0, *these vectors are collinear. You should graph them in the* $x-y$ *plane to see this visually.*

2.3.2 Homework

Exercise 2.3.1 *Compute the determinant of*

$$\begin{bmatrix} 2.0 & -3.0 \\ 6.0 & 5.0 \end{bmatrix}$$

Exercise 2.3.2 *Compute the determinant of*

$$\begin{bmatrix} 12.0 & -1.0 \\ 4.0 & 2.0 \end{bmatrix}$$

2.4 Systems of Two Linear Equations

We can use all of this material to understand simple two linear equations in two unknowns x and y. Consider the problem

$$2\,x + 4\,y = 7 \tag{2.1}$$

$$3\,x + 4\,y = -8 \tag{2.2}$$

Now consider the equation below written in terms of vectors:

$$x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$$

Using the standard ways of multiplying vectors by scalars and adding vectors, we see the above can be rewritten as

$$\begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} 4y \\ 4y \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$$

or

$$\begin{bmatrix} 2x + 4y \\ 3x + 4y \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$$

This last vector equation is clearly the same as the original Eqs. 2.1 and 2.2:

$$\begin{aligned} 2x + 4y &= 7 \\ 3x + 4y &= -8 \end{aligned}$$

Further, in this example, letting

$$\boldsymbol{V} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{D} = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$$

we see Eqs. 2.1 and 2.2 are equivalent to the vector equation

$$x\,\boldsymbol{V} + y\,\boldsymbol{W} = \boldsymbol{D}.$$

We can also write the system Eqs. 2.1 and 2.2 in an equivalent matrix–vector form. Recall the original system which is written below:

$$\begin{aligned} 2\,x + 4\,y &= 7 \\ 3\,x + 4\,y &= -8 \end{aligned}$$

We have already identified this system is equivalent to the vector equation

$$x\,\boldsymbol{V} + y\,\boldsymbol{W} = \boldsymbol{D}$$

where

$$\boldsymbol{V} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \text{ and } \boldsymbol{D} = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$$

Now use $\boldsymbol{V}$ and $\boldsymbol{W}$ as column one and column two of the matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$$

Then, the original system can be written in the matrix–vector form

$$\begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$$

We call the matrix $\boldsymbol{A}$ the **coefficient** matrix of the system given by Eqs. 2.1 and 2.2.

Now we can introduce a new type of notation. Think of the column vector

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

as being a **vector** variable. We will use a bold font and a capital letter for this and set

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Then, the original system can be written as

$$\boldsymbol{A}\,\boldsymbol{X} = \boldsymbol{D}.$$

We typically refer to the vector $\boldsymbol{D}$ as the **data** vector associated with the system given by Eqs. 2.1 and 2.2.

2.4.1 Worked Out Examples

Example 2.4.1 Consider the system of equations

$$\begin{aligned} 1\,x \;+\; 2\,y &= 9 \\ -5\,x \;+\; 12\,y &= -1 \end{aligned}$$

Find the matrix vector equation form of this system.

Solution *Define **V**, **W** and **D** as follows:*

$$\boldsymbol{V} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} 2 \\ 12 \end{bmatrix} \text{ and } \boldsymbol{D} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$$

*Then define the matrix **A** using **V** and **W** as its columns:*

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ -5 & 12 \end{bmatrix}$$

and the system is equivalent to

$$\boldsymbol{A}\,\boldsymbol{X} = \boldsymbol{D}.$$

Example 2.4.2 Consider the system of equations

$$\begin{aligned} 7\,x \;+\; 5\,y &= 2 \\ -3\,x \;+\; -4\,y &= 1 \end{aligned}$$

Find the matrix vector equation form of this system.

Solution *Define **V**, **W** and **D** as follows:*

$$\boldsymbol{V} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \text{ and } \boldsymbol{D} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

*Then define the matrix **A** using **V** and **W** as its columns:*

$$\boldsymbol{A} = \begin{bmatrix} 7 & 5 \\ -3 & -4 \end{bmatrix}$$

and the system is equivalent to

$$\boldsymbol{A}\,\boldsymbol{X} = \boldsymbol{D}.$$

2.4.2 Homework

Exercise 2.4.1 *Consider the system of equations*

$$\begin{aligned} 1\,\alpha \;+\; 2\,\beta &= 3 \\ 4\,\alpha \;+\; 5\,\beta &= 6 \end{aligned}$$

Find the matrix vector equation form of this system.

Exercise 2.4.2 *Consider the system of equations*

$$\begin{aligned} -1\,w \;+\; 3\,z &= 21 \\ 6\,w \;+\; 7\,z &= 12 \end{aligned}$$

Find the matrix vector equation form of this system.

Exercise 2.4.3 *Consider the system of equations*

$$\begin{aligned} -7\,u \;+\; 14\,v &= 8 \\ 25\,u \;+\; -2\,v &= 8 \end{aligned}$$

Find the matrix vector equation form of this system.

2.5 Solving Two Linear Equations in Two Unknowns

We now know how to write the system of two linear equations in two unknowns given by Eqs. 2.3 and 2.4

$$a\,x + b\,y \;=\; D_1 \tag{2.3}$$
$$c\,x + d\,y \;=\; D_2 \tag{2.4}$$

in an equivalent matrix–vector form. This system is equivalent to the vector equation

$$x\,\boldsymbol{V} + y\,\boldsymbol{W} = \boldsymbol{D}$$

where

$$\boldsymbol{V} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} b \\ d \end{bmatrix} \text{ and } \boldsymbol{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

Finally, using $\boldsymbol{V}$ and $\boldsymbol{W}$ as column one and column two of the matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, the original system was written in vector and matrix–vector form as

$$x\,\boldsymbol{V} + y\,\boldsymbol{W} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

Now, we can solve this system very easily as follows. We have already discussed the inner product of two vectors. So we could compute the inner product of both sides of $x\,\boldsymbol{V} + y\,\boldsymbol{W} = \boldsymbol{D}$ with any vector $\boldsymbol{U}$ we want and get

$$< \boldsymbol{U}, x\,\boldsymbol{V} + y\,\boldsymbol{W} > = < \boldsymbol{U}, \boldsymbol{D} >$$

We can simplify the left hand side to get

$$x < \boldsymbol{U}, \boldsymbol{V} > + y\ < \boldsymbol{U}, \boldsymbol{W} > = < \boldsymbol{U}, \boldsymbol{D} >$$

Since this is true for any vector $\boldsymbol{U}$, let's try to find useful ones! Any vector $\boldsymbol{U}$ that satisfies $< \boldsymbol{U}, \boldsymbol{W} > = 0$ would be great as then the y would drop out and we could solve for x. The angle between such vector $\boldsymbol{U}$ and $\boldsymbol{W}$ would then be $90°$ or $270°$. We will call such vectors **orthogonal** as the lines associated with the vectors are perpendicular.

We can easily find such a vector. Since $\boldsymbol{W}$ defines a line through the origin with slope d/b, from our usual algebra and pre-calculus courses, we know the line through the origin which is perpendicular to it has negative reciprocal slope: i.e. $-b/d$. A line with the slope $-b/d$ corresponds with a vector with components $(d, -b)$. The usual symbol for *perpendicularity* is $\perp$ so we will label our vector orthogonal to $\boldsymbol{W}$ as $\boldsymbol{W}^{\perp}$. We see that

$$\boldsymbol{W}^{\perp} = \begin{bmatrix} d \\ -b \end{bmatrix}$$

and as expected

$$< \boldsymbol{W}^{\perp}, \boldsymbol{W} > = (d)(b) + (-b)(d) = 0$$

Thus, we have

$$\begin{aligned} < \boldsymbol{W}^{\perp}, \boldsymbol{D} > &= x < \boldsymbol{W}^{\perp}, \boldsymbol{V} > \ + \ y \ < \boldsymbol{W}^{\perp}, \boldsymbol{W} > \\ &= x < \boldsymbol{W}^{\perp}, \boldsymbol{V} > \end{aligned}$$

This looks complicated, but it can be written in terms of things we understand. Let's actually calculate the inner products. We find

$$< \boldsymbol{W}^{\perp}, \boldsymbol{V} > = (d)(a) + (-b)(c) = det\ (\boldsymbol{A})$$

and

$$< \boldsymbol{W}^{\perp}, \boldsymbol{D} > = (d)(D_1) + (-b)(D_2) = det\ \begin{bmatrix} D_1 & b \\ D_2 & d \end{bmatrix}.$$

Hence, by taking the inner product of both sides with $\boldsymbol{W}^{\perp}$, we find the y term drops out and we have

$$x\ det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = det \begin{bmatrix} D_1 & b \\ D_2 & d \end{bmatrix}$$

Thus, if $det\ (\boldsymbol{A})$ is not zero, we can solve for x to get

$$x = \frac{det \begin{bmatrix} D_1 & b \\ D_2 & d \end{bmatrix}}{det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \frac{det \begin{bmatrix} \boldsymbol{D} & \boldsymbol{W} \end{bmatrix}}{det \begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}}$$

We can do a similar thing to find out what the variable y is by taking the inner product of both sides of of $x\ \boldsymbol{V} + y\ \boldsymbol{W} = \boldsymbol{D}$ with the vector $\boldsymbol{V}^{\perp}$ and get

$$x\ < \boldsymbol{V}^{\perp}, \boldsymbol{V} > + y\ < \boldsymbol{V}^{\perp}, \boldsymbol{W} > = < \boldsymbol{V}^{\perp}, \boldsymbol{D} >$$

where

$$\boldsymbol{V}^{\perp} = \begin{bmatrix} c \\ -a \end{bmatrix}$$

and as expected

$$< \boldsymbol{V}^{\perp}, \boldsymbol{V} > = (c)(a) + (-a)(c) = 0$$

Going through the same steps as before, we would find that if $det\ (\boldsymbol{A})$ is non zero, we could solve for y to get

$$y = \frac{det \begin{bmatrix} a & D_1 \\ c & D_2 \end{bmatrix}}{det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \frac{det \begin{bmatrix} \boldsymbol{V} & \boldsymbol{D} \end{bmatrix}}{det \begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}}$$

Let's summarize:

1. Given any system of two linear equations in two unknowns, there is a coefficient matrix $\boldsymbol{A}$ with first column $\boldsymbol{V}$ and second column $\boldsymbol{W}$ that is associated with it. Further, the right hand side of the system defines a data vector $\boldsymbol{D}$.
2. If $det\ (\boldsymbol{A})$ is not zero, we can solve for the unknowns x and y as follows:

$$x = \frac{det\ \left(\left[\boldsymbol{D}\ \boldsymbol{W}\right]\right)}{det\ \left(\left[\boldsymbol{V}\ \boldsymbol{W}\right]\right)}$$
$$y = \frac{det\ \left(\left[\boldsymbol{V}\ \boldsymbol{D}\right]\right)}{det\ \left(\left[\boldsymbol{V}\ \boldsymbol{W}\right]\right)}$$

This method of solution is known as **Cramer's Rule**.

3. This system of two linear equations in two unknowns is associated with two column vectors $\boldsymbol{V}$ and $\boldsymbol{W}$. You can see there is a **unique** solution if and only if $|\ \boldsymbol{A}\ |$ is not zero. This is the same as saying there is a unique solution if and only if the vectors are not *collinear*.

We can state this as Theorem 2.5.1.

Theorem 2.5.1 (Cramer's Rule)
Consider the system of equations

$$\begin{aligned} a\,x + b\,y &= D_1 \\ c\,x + d\,y &= D_2. \end{aligned}$$

Define the vectors

$$\boldsymbol{V} = \begin{bmatrix} a \\ c \end{bmatrix},\quad \boldsymbol{W} = \begin{bmatrix} b \\ d \end{bmatrix},\ \textit{and}\ \boldsymbol{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

Also, define the matrix $\boldsymbol{A}$ *by*

$$\boldsymbol{A} = \left[\boldsymbol{V}\ \boldsymbol{W}\right]$$

Then, if $det(\boldsymbol{A}) \neq 0$*, the unique solution to this system of equations is given by*

$$x = \frac{det\ \left(\left[\boldsymbol{D}\ \boldsymbol{W}\right]\right)}{det\ (\boldsymbol{A})}$$
$$y = \frac{det\ \left(\left[\boldsymbol{V}\ \boldsymbol{D}\right]\right)}{det\ (\boldsymbol{A})}$$

2.5.1 Worked Out Examples

Example 2.5.1 Solve the system

$$\begin{aligned} -2\,x\ +\ 4\,y &= 6 \\ 8\,x\ +\ -1\,y &= 2 \end{aligned}$$

using Cramer's Rule.

Solution *Solve*

$$
\begin{aligned}
-2\,x \;+\; 4\,y &= 6\\
8\,x \;+\; -1\,y &= 2
\end{aligned}
$$

using Cramer's Rule. We have

$$
\boldsymbol{V} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \;\; and \;\; \boldsymbol{D} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}
$$

$$
\begin{aligned}
x &= \frac{det\ \left(\left[\boldsymbol{D}\ \boldsymbol{W}\right]\right)}{det\ \left(\left[\boldsymbol{V}\ \boldsymbol{W}\right]\right)}\\
&= \frac{det\ \left(\begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix}\right)}{det\ \left(\begin{bmatrix} -2 & 4 \\ 8 & -1 \end{bmatrix}\right)}\\
&= \frac{-30}{-30} = 1,\\
y &= \frac{det\ \left(\left[\boldsymbol{V}\ \boldsymbol{D}\right]\right)}{det\ \left(\left[\boldsymbol{V}\ \boldsymbol{W}\right]\right)}\\
&= \frac{det\ \left(\begin{bmatrix} -2 & 6 \\ 8 & 2 \end{bmatrix}\right)}{det\ \left(\begin{bmatrix} -2 & 4 \\ 8 & -1 \end{bmatrix}\right)}\\
&= \frac{-52}{-30} = -\frac{26}{15}
\end{aligned}
$$

Example 2.5.2 Solve the system

$$
\begin{aligned}
-5\,x \;+\; 1\,y &= 8\\
9\,x \;+\; -10\,y &= 2
\end{aligned}
$$

using Cramer's Rule.

Solution *We have*

$$
\boldsymbol{V} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} 1 \\ -10 \end{bmatrix} \;\; and \;\; \boldsymbol{D} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}
$$

$$
\begin{aligned}
x &= \frac{det\ \left(\begin{bmatrix} \boldsymbol{D} & \boldsymbol{W} \end{bmatrix}\right)}{det\ \left(\begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}\right)} \\
&= \frac{det\ \left(\begin{bmatrix} 8 & 1 \\ 2 & -10 \end{bmatrix}\right)}{det\ \left(\begin{bmatrix} -5 & 1 \\ 9 & -10 \end{bmatrix}\right)} \\
&= \frac{-82}{41} = -\frac{-82}{41}. \\
y &= \frac{det\ \left(\begin{bmatrix} \boldsymbol{V} & \boldsymbol{D} \end{bmatrix}\right)}{det\ \left(\begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}\right)} \\
&= \frac{det\ \left(\begin{bmatrix} -5 & 8 \\ 9 & 2 \end{bmatrix}\right)}{det\ \left(\begin{bmatrix} -5 & 1 \\ 9 & -10 \end{bmatrix}\right)} \\
&= \frac{-82}{41}
\end{aligned}
$$

2.5.2 Homework

Exercise 2.5.1 *Solve the system*

$$
\begin{aligned}
-3\,x \ + \ 4\,y &= 6 \\
8\,x \ + \ 9\,y &= -1
\end{aligned}
$$

using Cramer's Rule.

Exercise 2.5.2 *Solve the system*

$$
\begin{aligned}
2\,x \ + \ 3\,y &= 6 \\
-4\,x \ + \ 0\,y &= 8
\end{aligned}
$$

using Cramer's Rule.

Exercise 2.5.3 *Solve the system*

$$
\begin{aligned}
18\,x \ + \ 1\,y &= 1 \\
-9\,x \ + \ 3\,y &= 17
\end{aligned}
$$

using Cramer's Rule.

Exercise 2.5.4 *Solve the system*

$$\begin{aligned} -7\,x \;+\; 6\,y &= -4 \\ 8\,x \;+\; 1\,y &= 1 \end{aligned}$$

using Cramer's Rule.

Exercise 2.5.5 *Solve the system*

$$\begin{aligned} -90\,x \;+\; 1\,y &= 1 \\ 80\,x \;+\; -1\,y &= 1 \end{aligned}$$

using Cramer's Rule.

2.6 Consistent and Inconsistent Systems

So what happens if $det\,(\boldsymbol{A}) = 0$? By the remark above, we know that the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ are collinear. We also know from our discussions in Sect. 2.3 that the columns of $\boldsymbol{A}^{\boldsymbol{T}}$ are collinear. Hence, there is a non zero constant r so that

$$\begin{bmatrix} a \\ b \end{bmatrix} = r\,\begin{bmatrix} c \\ d \end{bmatrix}$$

Thus, $a = r\,c$ and $b = r\,d$ and the original system can be written as

$$\begin{aligned} r\,c\,x + r\,d\,y &= D_1 \\ c\,x + d\,y &= D_2 \end{aligned}$$

or

$$\begin{aligned} r\,(c\,x + d\,y) &= D_1 \\ c\,x + d\,y &= D_2 \end{aligned}$$

You can see we do not really have two equations in two unknowns since the top equation on the left hand side is just a multiple of the left hand side of the bottom equation. This can only make sense if $D_1/r = D_2$ or $D_1 = r\, D_2$. We can conclude that the relationship between the components of $\boldsymbol{D}$ must be just right! Hence, we have the system

$$\begin{aligned} c\,x + d\,y &= D_1/r \\ c\,x + d\,y &= D_2 \end{aligned}$$

Now subtract the top equation from the bottom equation. You find

$$0\,x + 0\,y = 0 = D_2 - D_1/r$$

This equation only makes sense if when you subtract the top from the bottom equation, the new right hand side is 0! We call such systems **consistent** if the right hand side becomes 0 and *inconsistent* if not zero. So we have a great test for *inconsistency*. We scale the top or bottom equation just right to make them identical and subtract the two equations. If we get $0 = \alpha$ for a nonzero α, the system is *inconsistent*.

Here is an example. Consider the system

$$\begin{aligned} 2\,x + 3\,y &= 8 \\ 4\,x + 6\,y &= 9 \end{aligned}$$

Here, the column vectors of $\boldsymbol{A}^T$ are

$$\boldsymbol{Y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \boldsymbol{Z} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

We see $\boldsymbol{Z} = 2\,\boldsymbol{Y}$ and we have the system

$$\begin{aligned} 2\,x + 3\,y &= 8 = D_1 \\ 2\ (2\,x + 3\,y) &= 9 = D_2 \end{aligned}$$

This system would be *consistent* if the bottom equation was exactly two times the top equation. For this to happen, we need $D_2 = 2\,D_1$; i.e., we need $9 = 2 \times 8$ which is impossible. So these equations are *inconsistent*. As mentioned earlier, an even better way to see these equations are inconsistent is to subtract the top equation from the bottom equation to get

$$0x + 0\,y = 0 = 1$$

which again is not possible. Remember, *consistent* equations when $det\,(\boldsymbol{A}) = 0$ would have the some multiple of **top** $-$ **bottom** equation = zero.

Another way to look at this situation is to note that the column vectors, $\boldsymbol{V}$ and $\boldsymbol{W}$, of $\boldsymbol{A}$ are collinear. Hence, there is another non zero scalar s so that $\boldsymbol{V} = s\,\boldsymbol{W}$. We can then rewrite the usual vector form of our system as

$$\begin{aligned} \boldsymbol{D} &= x\,\boldsymbol{V} + y\,\boldsymbol{W} \\ &= x\,\boldsymbol{s}\boldsymbol{W} + y\,\boldsymbol{W} \end{aligned}$$

This says that the data vector $\boldsymbol{D} = (xs + y)\,\boldsymbol{W}$. Hence, if there is a solution x and y, it will only happen in $\boldsymbol{D}$ is a multiple of $\boldsymbol{W}$. This says $\boldsymbol{D}$ is collinear with $\boldsymbol{W}$ which in turn is collinear with $\boldsymbol{V}$. Going back to our sample

$$\begin{aligned} 2\,x + 3\,y &= 8 \\ 4\,x + 6\,y &= 9. \end{aligned}$$

We see $\boldsymbol{D}$ with components (8, 9) is not a multiple of $\boldsymbol{V}$ with components (2, 4) or $\boldsymbol{W}$ with components (3, 6). Thus, the system must be inconsistent.

2.6.1 Worked Out Examples

Example 2.6.1 Consider the system

$$\begin{aligned} 4\,x + 5\,y &= 11 \\ -8\,x - 10\,y &= -22 \end{aligned}$$

Determine if this system is consistent or inconsistent.

Solution *We see immediately that the determinant of the coefficient matrix* $\boldsymbol{A}$ *is zero. So the question of consistency is reasonable to ask. Here, the column vectors of* $\boldsymbol{A}^{\boldsymbol{T}}$ *are*

$$\boldsymbol{Y} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \ \textit{and} \ \boldsymbol{Z} = \begin{bmatrix} -8 \\ -10 \end{bmatrix}$$

We see $\boldsymbol{Z} = -2\,\boldsymbol{Y}$ *and we have the system*

$$\begin{aligned} 4\,x + 5\,y &= 11 = D_1 \\ -2\,(4\,x + 5\,y) &= -22 = D_2 \end{aligned}$$

This system would be consistent *if the bottom equation was exactly minus two times the top equation. For this to happen, we need* $D_2 = -2\, D_1$*; i.e., we need* $-22 = -2 \times 11$ *which is true. So these equations are* consistent.

Example 2.6.2 Consider the system

$$\begin{aligned} 6\,x + 8\,y &= 14 \\ 18\,x + 24\,y &= 48 \end{aligned}$$

Determine if this system is consistent or inconsistent.

Solution *We see immediately that the determinant of the coefficient matrix* $\boldsymbol{A}$ *is zero. So again, the question of consistency is reasonable to ask. Here, the column vectors of* $\boldsymbol{A^T}$ *are*

$$\boldsymbol{Y} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \; and \; \boldsymbol{Z} = \begin{bmatrix} 18 \\ 24 \end{bmatrix}$$

We see $\boldsymbol{Z} = 3\,\boldsymbol{Y}$ *and we have the system*

$$\begin{aligned} 6\,x + 8\,y &= 14 = D_1 \\ 3\;(6\,x + 8\,y) &= 48 = D_2 \end{aligned}$$

This system would be consistent *if the bottom equation was exactly three times the top equation. For this to happen, we need* $D_2 = 3\,D_1$*; i.e., we need* $-48 = 3 \times 14$ *which is not true. So these equations are* inconsistent.

2.6.2 Homework

Exercise 2.6.1 *Consider the system*

$$\begin{aligned} 2\,x + 5\,y &= 1 \\ 8\,x + 20\,y &= 4 \end{aligned}$$

Determine if this system is consistent or inconsistent.

Exercise 2.6.2 *Consider the system*

$$\begin{aligned} 60\,x + 80\,y &= 120 \\ 6\,x + 8\,y &= 13 \end{aligned}$$

Determine if this system is consistent or inconsistent.

Exercise 2.6.3 *Consider the system*

$$\begin{aligned} -2\,x + 7\,y &= 10 \\ 20\,x - 70\,y &= 4 \end{aligned}$$

Determine if this system is consistent or inconsistent.

Exercise 2.6.4 *Consider the system*

$$\begin{aligned} x + y &= 1 \\ 2\,x + 2\,y &= 3 \end{aligned}$$

Determine if this system is consistent or inconsistent.

Exercise 2.6.5 *Consider the system*

$$\begin{aligned} -11\,x - 3\,y &= -2 \\ 33\,x + 9\,y &= 6 \end{aligned}$$

Determine if this system is consistent or inconsistent.

2.7 Specializing to Zero Data

If the system we want to solve has zero data, then we must solve a system of equations like

$$\begin{aligned} a\,x + b\,y &= 0 \\ c\,x + d\,y &= 0. \end{aligned}$$

Define the vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ as usual. Note $\boldsymbol{D}$ is now the zero vector

$$\boldsymbol{V} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} b \\ d \end{bmatrix}, \quad \text{and} \quad \boldsymbol{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Also, define the matrix $\boldsymbol{A}$ by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}$$

Then, if $det(\boldsymbol{A}) \neq 0$, the unique solution to this system of equations is given by

$$\begin{aligned} x &= \frac{det\left(\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}\right)}{det\ (\boldsymbol{A})} \\ &= \frac{0}{ad - bc} = 0 \end{aligned}$$

$$\begin{aligned} y &= \frac{det\left(\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\right)}{det\ (\boldsymbol{A})} \\ &= \frac{0}{ad - bc} = 0 \end{aligned}$$

Hence, the unique solution to a system of the form $\boldsymbol{A}\ \boldsymbol{X} = \boldsymbol{0}$ is $x = 0$ and $y = 0$. But what happens if the determinant of $\boldsymbol{A}$ is zero? In this case, we know the column vectors of $\boldsymbol{A}^{\boldsymbol{T}}$ are collinear: i.e.

$$\boldsymbol{Y} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \boldsymbol{Z} = \begin{bmatrix} c \\ d \end{bmatrix},$$

are collinear and so there is a non zero constant r so that $a = rc$ and $b = rd$. This gives the system

$$\begin{aligned} r\ (c\ x + b\ y) &= 0 \\ c\ x + d\ y &= 0. \end{aligned}$$

Now if you multiply the bottom equation by r and subtract from the top equation, you get 0. This tells us the system is *consistent*. The original system of two equations is thus only one equation. We can choose to use either the original top or bottom equation to solve. Say we choose the original top equation. Then we need to find x and y choices so that

$$a\ x + b\ y = 0$$

There are in finitely many solutions here! It is easiest to see how to solve this kind of problem using some examples.

2.7.1 Worked Out Examples

Example 2.7.1 Find all solutions to the consistent system

$$\begin{aligned} -2\,x + 7\,y &= 0 \\ 20\,x - 70\,y &= 0 \end{aligned}$$

Solution *First, note the determinant of the coefficient matrix of the system is zero. Also, since the bottom equation is* -10 *times the top equation, we see the system is also consistent. We solve using the top equation:*

$$-2\,x + 7\,y = 0$$

Thus,

$$\begin{aligned} 7\,y &= 2\,x \\ y &= (2/7)\,x \end{aligned}$$

We see a solution vector of the form

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (2/7)\,x \end{bmatrix} = x\,\begin{bmatrix} 1 \\ (2/7) \end{bmatrix}$$

will always work. There is a lot of ambiguity here as the multiplier x *is completely arbitrary. For example,if we let* $x = 7\,c$ *for an arbitrary* c*, then solving for* y*, we find* $y = 2\,c$*. We can then rewrite the solution vector as*

$$c\,\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

in terms of the arbitrary multiplier c*. It does not really matter what form we pick, however we often try to pick a form which has integers as components.*

Example 2.7.2 Find all solutions to the consistent system

$$\begin{aligned} 4\,x + 5\,y &= 0 \\ 8\,x + 10\,y &= 0 \end{aligned}$$

Solution *First, note the determinant of the coefficient matrix of the system is zero. Also, since the bottom equation is* 2 *times the top equation, we see the system is also consistent. We solve using the bottom equation this time:*

$$8\,x + 10\,y = 0$$

Thus,

$$\begin{aligned} 10\,y &= -8\,x \\ y &= -(4/5)\,x \end{aligned}$$

We see a solution vector of the form

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -(4/5)\,x \end{bmatrix} = x \begin{bmatrix} 1 \\ -(4/5) \end{bmatrix}$$

will always work. Again, there is a lot of ambiguity here as the multiplier x is completely arbitrary. For example,if we let $x = 10\,d$ for an arbitrary d, then solving for y, we find $y = -8\,d$. We can then rewrite the solution vector as

$$d \begin{bmatrix} 10 \\ -8 \end{bmatrix}$$

in terms of the arbitrary multiplier d. Again, it is important to note that it does not really matter what form we pick, however we often try to pick a form which has integers as components.

2.7.2 Homework

Exercise 2.7.1 *Find all solutions to the consistent system*

$$\begin{aligned} x + 3\,y &= 0 \\ 6\,x + 18\,y &= 0 \end{aligned}$$

Exercise 2.7.2 *Find all solutions to the consistent system*

$$\begin{aligned} -3\,x + 4\,y &= 0 \\ 9\,x - 12\,y &= 0 \end{aligned}$$

Exercise 2.7.3 *Find all solutions to the consistent system*

$$\begin{aligned} 2\,x + 7\,y &= 0 \\ 1\,x + (3/2)\,y &= 0 \end{aligned}$$

Exercise 2.7.4 *Find all solutions to the consistent system*

$$\begin{aligned} -10\,x + 5\,y &= 0 \\ 20\,x - 10\,y &= 0 \end{aligned}$$

Exercise 2.7.5 *Find all solutions to the consistent system*

$$\begin{aligned} -12\,x + 5\,y &= 0 \\ 4\,x - (5/3)\,y &= 0 \end{aligned}$$

2.8 Matrix Inverses

If a matrix $\boldsymbol{A}$ has a non zero determinant, we know the system

$$\boldsymbol{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

has a unique solution for each right hand side vector

$$\boldsymbol{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

If we could find another matrix $\boldsymbol{B}$ which satisfied

$$\boldsymbol{B}\,\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{I}$$

we could multiply both sides of our system by $\boldsymbol{B}$ to find

$$\begin{aligned} \boldsymbol{B}\,\boldsymbol{A} \begin{bmatrix} x \\ y \end{bmatrix} &= \boldsymbol{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \\ \boldsymbol{I} \begin{bmatrix} x \\ y \end{bmatrix} &= \boldsymbol{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \boldsymbol{B} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \end{aligned}$$

which is the solution to our system! The matrix $\boldsymbol{B}$ is, of course, special and clearly plays the role of an inverse for the matrix $\boldsymbol{A}$. When such a matrix $\boldsymbol{B}$ exists, it is called the inverse of $\boldsymbol{A}$ and is denoted by $\boldsymbol{A}^{-1}$.

Definition 2.8.1 (*The Inverse of the matrix* $\boldsymbol{A}$)
If there is a matrix $\boldsymbol{B}$ of the same size as the square matrix $\boldsymbol{A}$, $\boldsymbol{B}$ is said to be inverse of $\boldsymbol{A}$ is

$$\boldsymbol{B}\,\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{I}$$

In this case, we denote the inverse of $\boldsymbol{A}$ by $\boldsymbol{A}^{-1}$.

We can show that the inverse of $\boldsymbol{A}$ exists if and only if $det(\boldsymbol{A}) \neq 0$. In general, it is very hard to find the inverse of a matrix, but in the case of a 2×2 matrix, it is very easy.

Definition 2.8.2 (*The Inverse of the* 2×2 *matrix* $\boldsymbol{A}$)
Let

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and assume $det(\boldsymbol{A}) \neq 0$. Then, the inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-1} = \frac{1}{det(\boldsymbol{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2.8.1 Worked Out Examples

Example 2.8.1 For

$$\boldsymbol{A} = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$$

find $\boldsymbol{A}^{-1}$.

Solution *Since* $det(\boldsymbol{A}) = 18$, *we see*

$$\boldsymbol{A}^{-1} = \frac{1}{18} \begin{bmatrix} 4 & -2 \\ -3 & 6 \end{bmatrix}$$

Example 2.8.2 For the system

$$\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

find the unique solution.

Solution *The coefficient matrix here is*

$$\boldsymbol{A} = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$$

Since $det(\boldsymbol{A}) = -22$, *we see*

$$\boldsymbol{A}^{-1} = \frac{-1}{22} \begin{bmatrix} 5 & -4 \\ -3 & -2 \end{bmatrix}$$

and hence,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{A}^{-1} \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$
$$= \frac{-1}{22} \begin{bmatrix} 40 - 28 \\ -24 - 14 \end{bmatrix} = \begin{bmatrix} \frac{-12}{22} \\ \frac{38}{22} \end{bmatrix}$$

2.8.2 Homework

Exercise 2.8.1 *For*

$$\boldsymbol{A} = \begin{bmatrix} 6 & 8 \\ 3 & 4 \end{bmatrix}$$

find $\boldsymbol{A}^{-1}$ *if it exists.*

Exercise 2.8.2 *For*

$$\boldsymbol{A} = \begin{bmatrix} 6 & 8 \\ 3 & 5 \end{bmatrix}$$

find $\boldsymbol{A}^{-1}$ *if it exists.*

Exercise 2.8.3 *For*

$$\boldsymbol{A} = \begin{bmatrix} -3 & 2 \\ 3 & 5 \end{bmatrix}$$

find $\boldsymbol{A}^{-1}$ *if it exists.*

Exercise 2.8.4 *For the system*

$$\begin{bmatrix} 4 & 3 \\ 11 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

find the unique solution if it exists.

Exercise 2.8.5 *For the system*

$$\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$$

find the unique solution if it exists.

Exercise 2.8.6 *For the system*

$$\begin{bmatrix} 40 & 30 \\ 16 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

find the unique solution if it exists.

2.9 Computational Linear Algebra

Let's look at how we can use MatLab/Octave to solve the general linear system of equations

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$$

where $\boldsymbol{A}$ is a $n \times n$ matrix, $\boldsymbol{x}$ is a column vector with n rows whose components are the unknowns we wish to solve for and $\boldsymbol{b}$ is the data vector.

2.9.1 A Simple Lower Triangular System

We will start by writing a function to solve a special system of equations; we begin with a lower triangular matrix system $Lx = b$. For example, if the system we wanted to solve was

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 2 \end{bmatrix}$$

this is easily solve by starting at the last equation and working backwards. This is called *backsolving*. Here, we have

$$\begin{aligned} z &= \frac{2}{6} \\ 4y &= 8 - z = 8 - \frac{1}{3} = \frac{23}{3} \\ y &= \frac{23}{12} \\ x &= 9 + 2y - 3z = 9 + \frac{23}{6} - 1 = \frac{71}{6} \end{aligned}$$

It is easy to write code to do this in MatLab as we do below.

2.9.2 A Lower Triangular Solver

Here is a simple function to solve such a system.

Listing 2.1: Lower Triangular Solver

```
function x = LTriSol(L,b)
%
% L is n x n Lower Triangular Matrix
% b is nx1 data vector
% Obtain x by forward substitution
%
n = length(b);
x = zeros(n,1);
for j=1:n-1
  x(j) = b(j)/L(j,j);
  b(j+1:n) = b(j+1:n) - x(j)*L(j+1:n,j);
end
x(n) = b(n)/L(n,n);
end
```

To use this function, we would enter the following commands at the Matlab prompt. For now, we are assuming that you are running Matlab in a local directory which contains your Matlab code **`LTriSol.m`**. So we fire up Matlab and enter these commands:

Listing 2.2: Sample Solution with LTriSol

```
A = [2 0 0; 1 5 0; 7 9 8]
A =
     2     0     0
     1     5     0
     7     9     8
b = [6; 2; 5]
b =
     6
     2
     5
x = LTriSol(A,b)
x =
    3.0000
   -0.2000
   -1.7750
```

which solves the system as we wanted.

2.9.3 An Upper Triangular Solver

Here is a simple function to solve a similar system where this time A is upper triangular. The code is essentially the same although the solution process starts at the *top* and sweeps *down*.

Listing 2.3: Upper Triangular Solver

```
function x = UTriSol(U,b)
%
% U is nxn nonsingular Upper Triangular matrix
% b is nx1 data vector
% x is solved by back substitution
%
n = length(b);
x = zeros(n,1);
for j = n:-1:2
  x(j) = b(j)/U(j,j);
  b(1:j-1) = b(1:j-1) - x(j)*U(1:j-1,j);
end
x(1) = b(1)/U(1,1);
end
```

As usual, to use this function, we would enter the following commands at the Matlab prompt. We are still assuming that you are running Matlab in a local directory and that your Matlab code **`UTriSol.m`** is also in this directory.

So we enter these commands in Matlab.

Listing 2.4: Sample Solution with UTriSol

```
C = [7 9 8; 0 1 5; 0 0 2]
C =
     7     9     8
     0     1     5
     0     0     2
b = [6; 2; 5]
b =
     6
     2
     5
x = UTriSol(C,b)
x =
   11.5000
  -10.5000
    2.5000
```

which again solves the system as we wanted.

2.9.4 The LU Decomposition of A Without Pivoting

It is possible to take a general matrix $\boldsymbol{A}$ and rewrite it as the product of a lower triangular matrix $\boldsymbol{L}$ and an upper triangular matrix $\boldsymbol{U}$. Here is a simple function to solve a system using the LU decomposition of A. First, it finds the LU decomposition and then it uses the lower triangular and upper triangular solvers we wrote earlier. To do this, we add and subtract multiples of rows together to remake the original matrix A into an upper triangular matrix. Let's do a simple example. Let the matrix A be given by

$$A = \begin{bmatrix} 8 & 2 & 3 \\ -4 & 3 & 2 \\ 7 & 8 & 9 \end{bmatrix}$$

Start in the row 1 and column 1 position in A. The entry there is the *pivoting element*. Divide the entries below it by the 8 and store them in the rest of column 1. This gives the new matrix A^*

$$A^* = \begin{bmatrix} 8 & 2 & 3 \\ -\frac{4}{8} & 3 & 2 \\ \frac{7}{8} & 8 & 9 \end{bmatrix}$$

If we took the original row 1 and multiplied it by the $-\frac{4}{8}$ and subtracted it from row 2, we would have the new second row

$$\begin{bmatrix} 0 & 4 & 3.5 \end{bmatrix}$$

If we took the $\frac{7}{8}$, multiplied the original row 1 by it and subtracted it from the original row 3, we would have

$$\begin{bmatrix} 0 & \frac{50}{8} & \frac{51}{8} \end{bmatrix}$$

With these operations done, we have the matrix A^* taking the form

$$A^* = \begin{bmatrix} 8 & 2 & 3 \\ -\frac{4}{8} & 4 & 3.5 \\ \frac{7}{8} & \frac{25}{4} & \frac{51}{8} \end{bmatrix}$$

The *multipliers* in the lower part of column 1 are important to what we are doing, so we are saving them in the parts of column 1 we have made zero. In MatLab, what we have just done could be written like this

Listing 2.5: Storing multipliers

```
% this is a 3x3 matrix
n = 3
% store multipliers in the rest of column 1
A(2:n,1) = A(2:n,1)/A(1,1);
% compute the new the 2x2 block which
% removes column 1 and row 1
A(2:n,2:n) = A(2:n,2:n) - A(2:n,1)*A(1,2:n);
```

The code above does what we just did by hand. Now do the same thing again, but start in the column 2 and row 2 position in the new matrix A^*. The new *pivoting element* is 4, so below it in column 2, we divide the rest of the elements of column 2 by 4 and store the results. This gives

$$A^* = \begin{bmatrix} 8 & 2 & 3 \\ -\frac{4}{8} & 4 & 3.5 \\ \frac{7}{8} & \frac{25}{16} & \frac{51}{8} \end{bmatrix}$$

We are not done. We now calculate the multiplier $\frac{25}{16}$ times the part of this row 2 past the pivot position and subtract it from the rest of row 3. We actually have a 0 then in both column 1 and column 2 of row 3 now. So, the calculations give

$$\begin{bmatrix} 0 \; 0 \; \frac{29}{32} \end{bmatrix}$$

although the row we store in A^* is

$$\begin{bmatrix} \frac{7}{8} \; \frac{25}{16} \; \frac{29}{32} \end{bmatrix}$$

We are now done. We have converted A into the form

$$A^* = \begin{bmatrix} 8 & 2 & 3 \\ -\frac{4}{8} & 4 & 3.5 \\ \frac{7}{8} & \frac{25}{16} & \frac{29}{32} \end{bmatrix}$$

Let this final matrix be called B. We can extract the lower triangular part of B using the MatLab command **`tril(B,-1)`** and the lower triangular matrix L formed from A is then made by adding a main diagonal of 1's to this. The upper triangular part of A is then U which we find by using **`triu(B)`**. In code this is

Listing 2.6: Extracting Lower and Upper Parts of a matrix

```
L = eye(n,n) + tril(A,-1);
U = triu(A);
```

In our example, we find

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{4}{8} & 1 & 0 \\ \frac{7}{8} & \frac{25}{16} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 8 & 2 & 3 \\ 0 & 4 & \frac{7}{2} \\ 0 & 0 & \frac{29}{32} \end{bmatrix}$$

The full code is listed below.

Listing 2.7: LU Decomposition of A Without Pivoting

```
function [L,U] = GE(A)
%
% A is nxn matrix
% L is nxn lower triangular
% U is nxn upper triangular
%
% We compute the LU decomposition of A using
% Gaussian Elimination
%

[n,n] = size(A);
for k=1:n-1
  % find multiplier
  A(k+1:n,k) = A(k+1:n,k)/A(k,k);
  % zero out column
  A(k+1:n,k+1:n) = A(k+1:n,k+1:n) - A(k+1:n,k)*A(k,k+1:n);
end
L = eye(n,n) + tril(A,-1);
U = triu(A);
end
```

Now in MatLab, to see it work, we enter these commands:

Listing 2.8: Solution using LU Decomposition

```
A = [17 24 1 8 15; 23 5 7 14 16; 4 6 13 20 22;...
        10 12 19 21 3;11 18 25 2 9]
A =
    17    24     1     8    15
    23     5     7    14    16
     4     6    13    20    22
    10    12    19    21     3
    11    18    25     2     9
[L,U] = GE(A);
L
L =
    1.0000         0         0         0         0
    1.3529    1.0000         0         0         0
    0.2353   -0.0128    1.0000         0         0
    0.5882    0.0771    1.4003    1.0000         0
    0.6471   -0.0899    1.9366    4.0578    1.0000
U
U =

   17.0000   24.0000    1.0000    8.0000   15.0000
         0  -27.4706    5.6471    3.1765   -4.2941
         0         0   12.8373   18.1585   18.4154
         0         0         0   -9.3786  -31.2802
         0         0         0         0   90.1734
b = [1; 3; 5; 7; 9]
b =
     1
     3
     5
     7
     9
y = LTriSol(L,b)
y =
    1.0000
```

```
      1.6471
      4.7859
     -0.4170
      0.9249
x = UTriSol(U,y)
x =

      0.0103
      0.0103
      0.3436
      0.0103
      0.0103
c = A*x
c =
      1.0000
      3.0000
      5.0000
      7.0000
      9.0000
```

which solves the system as we wanted.

2.9.5 The LU Decomposition of A with Pivoting

Here is a simple function to solve a system using the LU decomposition of A with what is called pivoting. This means we find the largest absolute value entry in the column we are trying to zero out and perform row interchanges to bring that one to the pivot position. The MatLab code changes a bit; see if you can see what we are doing and why! Note that this pivoting step is needed if a pivot element in a column k, row k position is very small. Using it as a divisor would then cause a lot of numerical problems because we would be multiplying by very large numbers.

Listing 2.9: LU Decomposition of A With Pivoting

```
function [L,U,piv] = GePiv(A);
%
% A is nxn matrix
% L is nxn lower triangular matrix
% U is nxn upper triangular matrix
% piv is a nx1 integer vector to hold variable order
% permutations
%
[n,n] = size(A);
piv = 1:n;
for k=1:n-1
  [maxc,r] = max(abs(A(k:n,k)));
  q = r+k-1;
   piv([k q]) = piv([q k]);
   A([k q],:) = A([q k],:);
   if A(k,k) ~=0
     A(k+1:n,k) = A(k+1:n,k)/A(k,k);
     A(k+1:n,k+1:n) = A(k+1:n,k+1:n) - A(k+1:n,k)*A(k,k+1:n);
   end
end
L = eye(n,n) + tril(A,-1);
U = triu(A);
end
```

We use this code to solve a system as follows:

Listing 2.10: Solving a System with pivoting

```
A = [17 24  1  8 15; 23  5  7 14  16; 4 6 13 20 22; ...
         10 12 19 21  3; 11 18 25  2  9]
A =
     17     24      1      8     15
     23      5      7     14     16
      4      6     13     20     22
     10     12     19     21      3
     11     18     25      2      9
b = [1; 3; 5; 7; 9]
b =
      1
      3
      5
      7
      9
[L,U,piv] = GePiv(A);
L
L =
    1.0000         0         0         0         0
    0.7391    1.0000         0         0         0
    0.4783    0.7687    1.0000         0         0
    0.1739    0.2527    0.5164    1.0000         0
    0.4348    0.4839    0.7231    0.9231    1.0000
U
U =
   23.0000    5.0000    7.0000   14.0000   16.0000
         0   20.3043   -4.1739   -2.3478    3.1739
         0         0   24.8608   -2.8908   -1.0921
         0         0         0   19.6512   18.9793
         0         0         0         0  -22.2222
piv
piv =

      2      1      5      3      4
y = LTriSol(L,b(piv));
y
y =
    3.0000
   -1.2174
    8.5011
    0.3962
   -0.2279
x = UTriSol(U,y);
x
x =
    0.0103
    0.0103
    0.3436
    0.0103
    0.0103
c = A*x
c =
    1.0000
    3.0000
    5.0000
    7.0000
    9.0000
```

which solves the system as we wanted.

2.10 Eigenvalues and Eigenvectors

Another important aspect of matrices is called the *eigenvalues* and *eigenvectors* of a matrix. We will motivate this in the context of 2×2 matrices of real numbers, and then note it can also be done for the general square $n \times n$ matrix. Consider the general 2×2 matrix $\boldsymbol{A}$ given by

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Is it possible to find a non zero vector $\boldsymbol{v}$ and a number r so that

$$\boldsymbol{A}\ \boldsymbol{v} = r\ \boldsymbol{v}? \tag{2.5}$$

There are many ways to interpret what such a number and vector pair means, but for the moment, we will concentrate on finding such a pair $(r, \boldsymbol{v})$. Now, if this was true, we could rewrite the equation as

$$r\ \boldsymbol{v} - \boldsymbol{A}\ \boldsymbol{v} = \boldsymbol{0} \tag{2.6}$$

where $\boldsymbol{0}$ denotes the vector of all zeros

$$\boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Next, recall that the two by two identity matrix $\boldsymbol{I}$ is given by

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and it acts like multiplying by one with numbers; i.e. $\boldsymbol{I}\ \boldsymbol{v} = \boldsymbol{v}$ for any vector $\boldsymbol{v}$. Thus, instead of saying $r\ \boldsymbol{v}$, we could say $r\ \boldsymbol{I}\ \boldsymbol{v}$. We can therefore write Eq. 2.6 as

$$r\ \boldsymbol{I}\ \boldsymbol{v} - \boldsymbol{A}\ \boldsymbol{v} = \boldsymbol{0} \tag{2.7}$$

We know that we can factor the vector $\boldsymbol{v}$ out of the left hand side and rewrite again as Eq. 2.8.

$$\left(r\ \boldsymbol{I} - \boldsymbol{A}\right)\ \boldsymbol{v} = \boldsymbol{0} \tag{2.8}$$

Now recall that we want the vector $\boldsymbol{v}$ to be non zero. Note, in solving this system, there are two possibilities:

(i): the determinant of $\boldsymbol{B}$ is non zero which implies the only solution is $\boldsymbol{v} = \boldsymbol{0}$.
(ii): the determinant of $\boldsymbol{B}$ is zero which implies the there are infinitely many solutions for $\boldsymbol{v}$ all of the form a constant c times some non zero vector $\boldsymbol{E}$.

Here the matrix $\boldsymbol{B} = \boldsymbol{r}\boldsymbol{I} - \boldsymbol{A}$. Hence, if we want a non zero solution $\boldsymbol{v}$, we must look for the values of r that force $det(\boldsymbol{r}\boldsymbol{I} - \boldsymbol{A}) = 0$. Thus, we want

$$\begin{aligned} 0 &= det(\boldsymbol{r}\boldsymbol{I} - \boldsymbol{A}) \\ &= det \begin{bmatrix} r-a & -b \\ -c & r-d \end{bmatrix} \\ &= (r-a)\,(r-d) - b\,c \\ &= r^2 - (a+d)\,r + ad - bc. \end{aligned}$$

This important quadratic equation in the variable r determines what values of r will allow us to find non zero vectors $\boldsymbol{v}$ so that $\boldsymbol{A}\,\boldsymbol{v} = r\,\boldsymbol{v}$. Note that although we started out in our minds thinking that r would be a real number, what we have done above shows us that it is possible that r could be complex.

Definition 2.10.1 (*The Eigenvalues and Eigenvectors of a 2 by 2 Matrix*)
Let $\boldsymbol{A}$ be the 2×2 matrix

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then an eigenvalue r of the matrix $\boldsymbol{A}$ is a solution to the quadratic equation defined by

$$det(\boldsymbol{r}\boldsymbol{I} - \boldsymbol{A}) = 0.$$

Any non zero vector that satisfies the equation

$$\boldsymbol{A}\,\boldsymbol{v} = r\,\boldsymbol{v}$$

for the eigenvalue r is then called an eigenvector associated with the eigenvalue r for the matrix $\boldsymbol{A}$.

Comment 2.10.1 *Since this is a quadratic equation, there are always two roots which take the forms below:*

(i): the roots r_1 and r_2 are real and distinct,
(ii): the roots are repeated $r_1 = r_2 = c$ for some real number c,
(iii): the roots are complex conjugate pairs; i.e. there are real numbers α and β so that $r_1 = \alpha + \beta\,\boldsymbol{i}$ and $r_2 = \alpha - \beta\,\boldsymbol{i}$.

Let's look at some examples:

Example 2.10.1 Find the eigenvalues and eigenvectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}$$

Solution *The characteristic equation is*

$$det\left(r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}\right) = 0$$

or

$$\begin{aligned} 0 &= det\left(\begin{bmatrix} r+3 & -4 \\ 1 & r-2 \end{bmatrix}\right) \\ &= (r+3)(r-2)+4 \\ &= r^2 + r - 2 \\ &= (r+2)(r-1) \end{aligned}$$

Hence, the roots, or **eigenvalues**, *of the characteristic equation are* $r_1 = -2$ *and* $r_2 = 1$. *Next, we find the* **eigenvectors** *associated with these eigenvalues.*

1. *For eigenvalue* $r_1 = -2$, *substitute the value of this eigenvalue into*

$$\begin{bmatrix} r+3 & -4 \\ 1 & r-2 \end{bmatrix}$$

This gives

$$\begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$$

The two rows of this matrix should be multiples of one another. If not, we made a mistake and we have to go back and find it. Our rows are indeed multiples, so pick one row to solve for the eigenvector. We need to solve

$$\begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Picking the top row, we get

$$\begin{aligned} v_1 - 4\,v_2 &= 0 \\ v_2 &= \frac{1}{4}v_1 \end{aligned}$$

Letting $v_1 = A$*, we find the solutions have the form*

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}$$

The vector

$$\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$$

is our choice for an eigenvector corresponding to eigenvalue $r_1 = -2$*.*

2. *For eigenvalue* $r_2 = 1$*, substitute the value of this eigenvalue into*

$$\begin{bmatrix} r+3 & -4 \\ 1 & r-2 \end{bmatrix}$$

This gives

$$\begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix}$$

Again, the two rows of this matrix should be multiples of one another. If not, we made a mistake and we have to go back and find it. Our rows are indeed multiples, so pick one row to solve for the eigenvector. We need to solve

$$\begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Picking the bottom row, we get

$$\begin{aligned} v_1 - v_2 &= 0 \\ v_2 &= v_1 \end{aligned}$$

Letting $v_1 = B$*, we find the solutions have the form*

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = B \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The vector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is our choice for an eigenvector corresponding to eigenvalue $r_2 = 1$*.*

Example 2.10.2 Find the eigenvalues and eigenvectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 4 & 9 \\ -1 & -6 \end{bmatrix}$$

Solution *The characteristic equation is*

$$det\left(r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ -1 & -6 \end{bmatrix}\right) = 0$$

or

$$\begin{aligned} 0 &= det\left(\begin{bmatrix} r-4 & -9 \\ 1 & r+6 \end{bmatrix}\right) \\ &= (r-4)(r+6)+9 \\ &= r^2 + 2\,r - 15 \\ &= (r+5)(r-3) \end{aligned}$$

Hence, the roots, or **eigenvalues***, of the characteristic equation are* $r_1 = -5$ *and* $r_2 = 3$*. Next, we find the* **eigenvectors** *associated with these eigenvalues.*

1. *For eigenvalue* $r_1 = -5$*, substitute the value of this eigenvalue into*

$$\begin{bmatrix} r-4 & -9 \\ 1 & r+6 \end{bmatrix}$$

This gives

$$\begin{bmatrix} -9 & -9 \\ 1 & 1 \end{bmatrix}$$

The two rows of this matrix should be multiples of one another. If not, we made a mistake and we have to go back and find it. Our rows are indeed multiples, so pick one row to solve for the eigenvector. We need to solve

$$\begin{bmatrix} -9 & -9 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Picking the bottom row, we get

$$\begin{aligned} v_1 + v_2 &= 0 \\ v_2 &= -\,v_1 \end{aligned}$$

Letting $v_1 = A$, we find the solutions have the form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The vector

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is our choice for an eigenvector corresponding to eigenvalue $r_1 = -5$.

2. *For eigenvalue $r_2 = 3$, substitute the value of this eigenvalue into*

$$\begin{bmatrix} r-4 & -9 \\ 1 & r+6 \end{bmatrix}$$

This gives

$$\begin{bmatrix} -1 & -9 \\ 1 & 9 \end{bmatrix}$$

Again, the two rows of this matrix should be multiples of one another. If not, we made a mistake and we have to go back and find it. Our rows are indeed multiples, so pick one row to solve for the eigenvector. We need to solve

$$\begin{bmatrix} -1 & -9 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Picking the bottom row, we get

$$\begin{aligned} v_1 \; + 9\, v_2 &= 0 \\ v_2 &= \frac{-1}{9}\, v_1 \end{aligned}$$

Letting $v_1 = B$, we find the solutions have the form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = B \begin{bmatrix} 1 \\ \frac{-1}{9} \end{bmatrix}$$

The vector

$$\begin{bmatrix} 1 \\ \frac{-1}{9} \end{bmatrix}$$

is our choice for an eigenvector corresponding to eigenvalue $r_2 = 3$.

2.10.1 Homework

Exercise 2.10.1 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} 6 & 3 \\ -11 & -8 \end{bmatrix}$$

Exercise 2.10.2 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -3 \end{bmatrix}$$

Exercise 2.10.3 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} -2 & -1 \\ 8 & 7 \end{bmatrix}$$

Exercise 2.10.4 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} -6 & -3 \\ 4 & 1 \end{bmatrix}$$

Exercise 2.10.5 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{bmatrix} -4 & -2 \\ 13 & 11 \end{bmatrix}$$

2.10.2 The General Case

For a general $n \times n$ matrix $\boldsymbol{A}$, we have the following:

Definition 2.10.2 (*The Eigenvalues and Eigenvectors of a n by n Matrix*)
Let $\boldsymbol{A}$ be the $n \times n$ matrix.

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{n1} \\ A_{21} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}.$$

Then an eigenvalue r of the matrix $\boldsymbol{A}$ is a solution to the polynomial defined by

$$det(\boldsymbol{rI} - \boldsymbol{A}) = 0.$$

Any non zero vector that satisfies the equation

$$\boldsymbol{A}\, \boldsymbol{v} = r\, \boldsymbol{v}$$

for the eigenvalue r is then called an eigenvector associated with the eigenvalue r for the matrix $\boldsymbol{A}$.

Comment 2.10.2 *Since this is a polynomial equation, there are always n roots some of which are real numbers which are distinct, some might be repeated and some might be complex conjugate pairs (and they can be repeated also!). An example will help. Suppose we started with a* 5×5 *matrix. Then, the roots could be*

1. *All the roots are real and distinct; for example,* 1, 2, 3, 4 *and* 5.
2. *Two roots are the same and three roots are distinct; for examples,* 1, 1, 3, 4 *and* 5.
3. *Three roots are the same and two roots are distinct; for examples,* 1, 1, 1, 4 *and* 5.
4. *Four roots are the same and one roots is distinct from that; for examples,* 1, 1, 1, 1 *and* 5.
5. *Five roots are the same; for examples,* 1, 1, 1, 1 *and* 1.
6. *Two pairs of roots are the same and one roots is different from them; for examples,* 1, 1, 3, 3 *and* 5.
7. *One triple root and one pair of real roots; for examples,* 1, 1, 1, 3 *and* 3.
8. *One triple root and one complex conjugate pair of roots; for examples,* 1, 1, 1, $3 + 4i$ *and* $3 - 4i$.
9. *One double root and one complex conjugate pair of roots and one different real root; for examples,* 1, 1, 2, $3 + 4i$ *and* $3 - 4i$.
10. *Two complex conjugate pair of roots and one different real root; for examples,* -2, $1 + 6i$, $1 - 6i$, $3 + 4i$ *and* $3 - 4i$.

2.10.3 The MatLab Approach

We will now discuss certain ways to compute eigenvalues and eigenvectors for a square matrix in MatLab. For a given $\boldsymbol{A}$, we can compute its eigenvalues as follows:

Listing 2.11: Eigenvalues in Matlab

```
A = [1 2 3; 4 5 6; 7 8 -1]
A =
     1     2     3
     4     5     6
     7     8    -1
E = eig(A)
E =
   -0.3954
   11.8161
   -6.4206
```

So we have found the eigenvalues of this small 3×3 matrix. To get the eigenvectors, we do this:

Listing 2.12: Eigenvectors in Matlab

```
 [V,D] = eig(A)
V =
    0.7530   -0.3054   -0.2580
   -0.6525   -0.7238   -0.3770
    0.0847   -0.6187    0.8896
D =
   -0.3954         0         0
         0   11.8161         0
         0         0   -6.4206
```

Note the eigenvalues are not returned in ranked order. The eigenvalue/eigenvector pairs are thus

$$\begin{aligned} \lambda_1 &= -0.3954 \\ V_1 &= \begin{bmatrix} 0.7530 \\ -0.6525 \\ 0.0847 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \lambda_2 &= 11.8161 \\ V_2 &= \begin{bmatrix} -0.3054 \\ -0.7238 \\ -0.6187 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \lambda_3 &= -6.4206 \\ V_3 &= \begin{bmatrix} -0.2580 \\ -0.3770 \\ 0.8896 \end{bmatrix} \end{aligned}$$

Now let's try a nice 5×5 array that is symmetric:

Listing 2.13: Eigenvalues and Eigenvectors Example

```
B = [1  2  3  4  5;
         2  5  6  7  9;
         3  6  1  2  3;
         4  7  2  8  9;
         5  9  3  9  6]
B =
     1     2     3     4     5
     2     5     6     7     9
     3     6     1     2     3
     4     7     2     8     9
     5     9     3     9     6
 [W,Z] = eig(B)
W =

    0.8757    0.0181   -0.0389    0.4023    0.2637
   -0.4289   -0.4216   -0.0846    0.6134    0.5049
    0.1804   -0.6752    0.4567   -0.4866    0.2571
   -0.1283    0.5964    0.5736   -0.0489    0.5445
    0.0163    0.1019   -0.6736   -0.4720    0.5594
Z =
    0.1454         0         0         0         0
         0    2.4465         0         0         0
         0         0   -2.2795         0         0
         0         0         0   -5.9321         0
         0         0         0         0   26.6197
```

It is possible to show that the eigenvalues of a symmetric matrix will be real and eigenvectors corresponding to distinct eigenvalues will be 90° apart. Such vectors are called **orthogonal** and recall this means their inner product is 0. Let's check it out. The eigenvectors of our matrix are the columns of W above. So their dot product should be 0!

Listing 2.14: Checking orthogonality

```
C = dot(W(1:5,1),W(1:5,2))
C =
    1.3336e-16
```

Well, the dot product is not actually 0 because we are dealing with floating point numbers here, but as you can see it is close to machine zero (the smallest number our computer chip can *detect*). Welcome to the world of computing!

Reference

J. Peterson, *Calculus for Cognitive Scientists: Derivatives, Integration and Modeling*, Springer Series on Cognitive Science and Technology (Springer Science + Business Media Singapore Pte Ltd., Singapore, 2015 in press)

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