Chapter 2
Consensus with Utility Preferences

The basic idea of the consensus with minimum adjustments (or cost) is clarified. Meanwhile, the consensus model with minimum adjustments (or cost) is investigated under the utility preferences and aggregation functions.

2.1 Basic Idea of the Consensus with Minimum Adjustments

This section introduces the basic idea of the consensus with minimum adjustments (or cost). As shown in Eq. (1.8), a core problem in consensus building is to search the consensus path with minimum adjustments. In existing researches, Dong et al. [10] proposed the consensus model with minimum adjustments. Meanwhile, Ben-Arieh and Easton [4] presented the consensus model with minimum cost.

2.1.1 Consensus with Minimum Adjustments or Cost

The basic models of the consensus with minimum adjustments or cost are presented below.

(1) Consensus with minimum adjustments

For a GDM problem, let $E = \{E_1, E_2, ..., E_n\}$ denote a set of n experts, and let $o_i \in R$ represent the original opinion of the expert E_i. Furthermore, the original collective opinion is denoted as o.

Several methods exist for measuring the consensus level [3, 7, 8, 13, 16, 17]. The most commonly used method is the Manhattan distance measure [4, 5]. By calculating the Manhattan distance between o_i and o, $|o_i - o|, i = 1, 2..., n$, the *consensus level* of the expert $E_i (i = 1, 2, ..., n)$ can be measured, i.e.,

$$CL(E_i) = |o_i - o|. \tag{2.1}$$

Y. Dong and J. Xu, *Consensus Building in Group Decision Making*,
DOI 10.1007/978-981-287-892-2_2

Let ε denote the established consensus threshold. If $CL(E_i) \leq \varepsilon$, then the expert E_i is of acceptable consensus. Otherwise, we suggest the expert E_i adjust his/her original opinion to reach the established consensus level.

Let $\overline{o_i} \in R$ denote the adjusted individual opinion of expert E_i, and $\overline{o}$ represent the adjusted collective opinion. Naturally, we hope to minimize the adjustment amounts (in the sense of Manhattan distance) in consensus building, i.e.,

$$\min_{\overline{o_i}} \sum_{i=1}^{n} |o_i - \overline{o_i}|. \tag{2.2}$$

Meanwhile, the adjusted individual opinions $\overline{o_i}(i = 1, 2, ..., n)$ are of acceptable consensus, i.e.,

$$CL(E_i) = |\overline{o_i} - \overline{o}| \leq \varepsilon, \quad i = 1, 2, ..., n, \tag{2.3}$$

where, the adjusted collective opinion $\overline{o}$ is obtained by using an aggregation function F, i.e.,

$$\overline{o} = F(o_1, o_2, ..., o_n). \tag{2.4}$$

Based on Eqs. (2.2)–(2.4), the consensus model with minimum adjustments is proposed as follows:

$$\begin{cases} \min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} |o_i - \overline{o_i}| \\ s.t. \begin{cases} |\overline{o_i} - \overline{o}| \leq \varepsilon, \quad i = 1, ..., n \\ \overline{o} = F\left(\overline{o_1}, \overline{o_2}, ..., \overline{o_n}\right) \end{cases} \end{cases}. \tag{2.5}$$

(2) Consensus with minimum cost

Ben-Arieh and Easton [4], and Ben-Arieh et al. [5] defined c_i as the cost of moving expert E_i's opinion 1 unit. Then, Gong et al. [11, 12] pursued the consensus model with minimum cost. Ben-Arieh and Easton [4] defined the *linear consensus cost* of moving expert E_i's opinion from o_i to $\overline{o_i}$:

$$f_i(o_i, \overline{o_i}) = c_i|\overline{o_i} - o_i|. \tag{2.6}$$

The total consensus cost is computed by

$$\sum_{i=1}^{n} c_i|\overline{o_i} - o_i|. \tag{2.7}$$

The consensus model with minimum cost can be presented using an optimization model, i.e.,

$$\begin{cases} \min \ \sum_{i=1}^{n} c_i \left| \overline{o_i} - o_i \right| \\ s.t. \ \left| \overline{o_i} - \overline{o} \right| \le \varepsilon, \ \ i = 1, 2, ...n \end{cases}. \tag{2.8}$$

Denote model (2.8) as $P_{2.1}$. Let $\Omega_{2.1}$ represent the feasible set corresponding to model $P_{2.1}$. Solving model $P_{2.1}$ yields the optimal adjusted individual opinions $\overline{o_i^*}$ $(i = 1, 2, ..., n)$ and the optimal adjusted collective opinion $\overline{o^*}$.

For $P_{2.1}$, the expert opinion does not need to be changed if it is within ε of $\overline{o^*}$. Furthermore, any original expert opinion that is further than ε from $\overline{o^*}$ should only be adjusted until it is exactly ε from $\overline{o^*}$. Thus Lemma 2.1 is obtained.

Lemma 2.1 *Let* $\left\{\overline{o_1^*}, \overline{o_2^*}, ..., \overline{o_n^*}, \overline{o^*}\right\}$ *denote the optimal solution to* $P_{2.1}$. *Then the following is obtained:*

$$\overline{o_i^*} = \begin{cases} \overline{o^*} - \varepsilon, & i \in \{i : o_i \le \overline{o^*} - \varepsilon\} \\ \overline{o^*} + \varepsilon, & i \in \{i : o_i \ge \overline{o^*} + \varepsilon\} \\ o_i, & i \in \{i : \overline{o^*} - \varepsilon < o_i < \overline{o^*} + \varepsilon\} \end{cases}. \tag{2.9}$$

Proof It is obvious and the proof is omitted.

Based on Lemma 2.1, $P_{2.1}$ can be expressed using an equivalent model when seeking the optimal adjusted collective opinion $\overline{o^*}$:

$$\min_{\overline{o}} \left\{ \sum\nolimits_{i:\ o_i < \overline{o} - \varepsilon} c_i \left| \overline{o} - \varepsilon - o_i \right| + \sum\nolimits_{i:\ o_i > \overline{o} + \varepsilon} c_i \left| \overline{o} + \varepsilon - o_i \right| \right\}. \tag{2.10}$$

2.1.2 Internal Aggregation Function

In the consensus model, $P_{2.1}$, the aggregation function is not considered. By taking into account the consensus cost and the aggregation function, Zhang et al. [20] proposed a general consensus model to connect model (2.5) and $P_{2.1}$. The consensus model presented in Zhang et al. [20] is described as follows:

$$\begin{cases} \min\limits_{\overline{o_i}} \sum_{i=1}^{n} c_i \left| o_i - \overline{o_i} \right| \\ s.t. \begin{cases} \left| \overline{o_i} - \overline{o} \right| \le \varepsilon, \ \ i = 1, ..., n \\ \overline{o} = F\left(\overline{o_1}, \overline{o_2}, ..., \overline{o_n}\right) \end{cases} \end{cases}. \tag{2.11}$$

Denote model (2.11) as $P_{2.2}$. Let $\Omega_{2.2}$ represent the feasible set corresponding to $P_{2.2}$.

Let $w^* = (\frac{1}{2}, 0, ..., 0, \frac{1}{2})^T$. Using the OWA operator, $F_{w^*}^{OWA}$, to aggregate the expert opinions in $P_{2.2}$ obtains the following model:

$$\begin{cases} \min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} c_i \left| o_i - \overline{o_i} \right| \\ s.t. \begin{cases} |\overline{o_i} - \overline{o}| \leq \varepsilon, \quad i = 1, \ldots, n \\ \overline{o} = \frac{\max\{\overline{o_i}\} + \min\{\overline{o_i}\}}{2} \end{cases} \end{cases}. \tag{2.12}$$

Denote model (2.12) as $P_{2.3}$. Let $\Omega_{2.3}$ represent the feasible set corresponding to $P_{2.3}$.

Theorem 2.1 *Let $\overline{o_i^*}$ $(i = 1, 2, \ldots, n)$ and $\overline{o^*}$ denote the optimal adjusted individual opinions and adjusted collective opinion obtained using $P_{2.3}$, respectively. Then, $\{\overline{o_1^*}, \overline{o_2^*}, \ldots, \overline{o_n^*}, \overline{o^*}\}$ is the optimal solution to $P_{2.1}$.*

Proof Let $\{\overline{\overline{o_1}}, \overline{\overline{o_2}}, \ldots, \overline{\overline{o_n}}, \overline{\overline{o}}\}$ be the optimal solution to $P_{2.1}$. Since $\{\overline{o_1^*}, \overline{o_2^*}, \ldots, \overline{o_n^*}, \overline{o^*}\}$ is the optimal solution to $P_{2.3}$ and $\Omega_{2.3} \subseteq \Omega_{2.1}$, the following can be obtained:

$$\begin{aligned} \sum_{i=1}^{n} c_i |o_i - \overline{\overline{o_i}}| &= \min_{\{\overline{o_1}, \overline{o_2}, \ldots, \overline{o_n}, \overline{o}\} \in \Omega_{2.1}} \sum_{i=1}^{n} c_i |o_i - \overline{o_i}| \\ &\leq \min_{\{\overline{o_1}, \overline{o_2}, \ldots, \overline{o_n}, \overline{o}\} \in \Omega_{2.3}} \sum_{i=1}^{n} c_i |o_i - \overline{o_i}| \\ &= \sum_{i=1}^{n} c_i |o_i - \overline{o_i^*}|. \end{aligned} \tag{2.13}$$

Moreover, the following relationship between $\overline{\overline{o}}$ and $\overline{\overline{o_i}}$ can be obtained based on Lemma 2.1:

$$\overline{\overline{o_i}} = \begin{cases} \overline{\overline{o}} - \varepsilon, & i \in \{i : o_i \leq \overline{\overline{o}} - \varepsilon\} \\ \overline{\overline{o}} + \varepsilon, & i \in \{i : o_i \geq \overline{\overline{o}} + \varepsilon\} \\ o_i, & i \in \{i : \overline{\overline{o}} - \varepsilon < o_i < \overline{\overline{o}} + \varepsilon\}, \end{cases} \tag{2.14}$$

which satisfies $\max\{\overline{\overline{o_i}}\} - \min\{\overline{\overline{o_i}}\} \leq 2\varepsilon$. Furthermore, we easily obtain

$$|\overline{\overline{o_i}} - \frac{\max\{\overline{\overline{o_i}}\} + \min\{\overline{\overline{o_i}}\}}{2}| \leq \varepsilon, \quad i = 1, 2, \ldots, n.$$

Thus, $\{\overline{\overline{o_1}}, \overline{\overline{o_2}}, \ldots, \overline{\overline{o_n}}, \frac{\max\{\overline{\overline{o_i}}\} + \min\{\overline{\overline{o_i}}\}}{2}\} \in \Omega_{2.3}$. Consequently,

$$\sum_{i=1}^{n} c_i |o_i - \overline{\overline{o_i}}| \geq \min_{\{\overline{o_1}, \overline{o_2}, \ldots, \overline{o_n}, \overline{o}\} \in \Omega_{2.3}} \sum_{i=1}^{n} c_i |o_i - \overline{o_i}| = \sum_{i=1}^{n} c_i |o_i - \overline{o_i^*}|. \tag{2.15}$$

The following can be obtained based on (2.13) and (2.15)

$$\min_{\{\overline{o_1}, \overline{o_2}, \ldots, \overline{o_n}, \overline{o}\} \in \Omega_{2.1}} \sum_{i=1}^{n} c_i |o_i - \overline{o_i}| = \sum_{i=1}^{n} c_i |o_i - \overline{o_i^*}|. \tag{2.16}$$

Thus, $\{\overline{o_1^*}, \overline{o_2^*}, ..., \overline{o_n^*}, \overline{o^*}\}$ is the optimal solution to $P_{2.1}$. This completes the proof of Theorem 2.1.

Based on Theorem 2.1, the consensus model, $P_{2.2}$, reduces to $P_{2.1}$, and the internal aggregation function in $P_{2.1}$ is the OWA operator, $F_{w^*}^{OWA}$, where $w^* = (\frac{1}{2}, ..., 0, ..., \frac{1}{2})^T$. In general, $F_{w^*}^{OWA}$ is not employed as the aggregation function in GDM problems.

$P_{2.1}$ and $P_{2.2}$ are called minimum cost consensus models (MCCMs). However, the adjustment cost c_i is difficult to evaluate in practice. So, we only take into account the adjustment cost in the GDM with utility preferences, and in the later chapters the adjustment cost is not considered.

2.2 Consensus Under Aggregation Function

In the section, we investigate the MCCM under the utility preferences and aggregation functions. Moreover, we extend the MCCM into the maximum expert consensus model (MECM).

2.2.1 Minimum Cost Consensus Model

Considering different aggregation functions in $P_{2.2}$ yields different MCCMs. Here, we examine the MCCMs under the WA operator and OWA operator in detail.

(1) Minimum cost consensus model with WA operator

Selecting WA with the weighting vector w as the operator in $P_{2.2}$ yields the following model:

$$\begin{cases} \min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} c_i \left| o_i - \overline{o_i} \right| \\ s.t. \begin{cases} |\overline{o_i} - \overline{o}| \leq \varepsilon, \quad i = 1, ..., n \\ \overline{o} = \sum_{i=1}^{n} w_i \overline{o_i} \end{cases} \end{cases}. \tag{2.17}$$

Denote model (2.17) as $P_{2.4}$. Further, let $\Omega_{2.4}$ represent the feasible set corresponding to $P_{2.4}$.

Theorem 2.2 *$P_{2.4}$ can be equivalently transformed into the following linear programming model:*

$$\begin{cases} \min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} c_i d_i \\ s.t. \begin{cases} \overline{o_i} - \overline{o} \leq \varepsilon, \quad i = 1, ..., n \\ \overline{o} - \overline{o_i} \leq \varepsilon, \quad i = 1, ..., n \\ \overline{o} = \sum\limits_{i=1}^{n} w_i \overline{o_i} \\ o_i - \overline{o_i} = x_i, \quad i = 1, ..., n \\ x_i \leq d_i, \quad i = 1, ..., n \\ -x_i \leq d_i, \quad i = 1, ..., n \end{cases} \end{cases}, \tag{2.18}$$

where $x_i = o_i - \overline{o_i}$ and $d_i \geq |x_i|$.

Proof In the linear programming model (2.18), constraints $\overline{o_i} - \overline{o} \leq \varepsilon$ and $\overline{o} - \overline{o_i} \leq \varepsilon (i = 1, ..., n)$ guarantee that $|\overline{o_i} - \overline{o}| \leq \varepsilon$, and constraints $o_i - \overline{o_i} = x_i$, $x_i \leq d_i$ and $-x_i \leq d_i$ $(i = 1, ..., n)$ guarantee that $d_i \geq |x_i| = |o_i - \overline{o_i}|$. According to the objective function, $\min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} c_i d_i$, any feasible solutions with $d_i > |x_i|$ are not the optimal solution to model (2.18). Thus, the six constraints of model (2.18) can guarantee that $d_i = |x_i| = |o_i - \overline{o_i}|$. Therefore, $P_{2.4}$ can be transformed into model (2.18). This completes the proof of Theorem 2.2.

According to Theorem 2.2, the optimal solution to $P_{2.4}$ can be obtained by solving the linear programming model (2.18).

(2) Minimum cost consensus model with OWA operator

Selecting OWA with the weighting vector w as the operator in $P_{2.2}$ yields the following model:

$$\begin{cases} \min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} c_i |o_i - \overline{o_i}| \\ s.t. \begin{cases} |\overline{o_i} - \overline{o}| \leq \varepsilon, \quad i = 1, ..., n \\ \overline{o} = F_w^{OWA} (\overline{o_1}, \overline{o_2}, ..., \overline{o_n}) \end{cases} \end{cases} \tag{2.19}$$

Obviously, model (2.19) is a nonlinear optimization model which is difficult to solve. In the following, we consider two cases of model (2.19):

Case A: Homogeneous unit consensus cost. In this case, the cost of adjusting the opinion 1 unit is the same for each expert (i.e., $c_i = c_j$ for $i, j = 1, 2, ..., n$). In this case, the corresponding consensus model (see model (2.20)) has a definite physical implication and can search the consensus path with minimum adjustments (in the sense of Manhattan distance).

Case B: Heterogeneous unit consensus cost. In this case, there exists $c_i \neq c_j$ for $i, j = 1, 2, ..., n$.

These two cases are discussed in more detail below.

(i) Case A

In this case, model (2.19) is transformed into model (2.20):

$$\begin{cases} \min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} |o_i - \overline{o_i}| \\ s.t. \begin{cases} |\overline{o_i} - \overline{o}| \leq \varepsilon, \quad i = 1, ..., n \\ \overline{o} = F_w^{OWA}(\overline{o_1}, \overline{o_2}, ..., \overline{o_n}) \end{cases} \end{cases}. \tag{2.20}$$

Denote $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ as the permutation of $\{1, 2, ..., n\}$ such that $o_{\sigma(i-1)} \geq o_{\sigma(i)}$ (i.e., $o_{\sigma(i)}$ is the ith largest variable in $\{o_1, o_2, ..., o_n\}$). Similarly, $\{\delta(1), ..., \delta(i), ..., \delta(n)\}$ is denoted as the permutation of $\{1, 2, ..., n\}$ such that $\overline{o_{\delta(i-1)}} \geq \overline{o_{\delta(i)}}$ (i.e., $\overline{o_{\delta(i)}}$ is the ith largest variable in $\{\overline{o_1}, \overline{o_2}, ..., \overline{o_n}\}$).

Before presenting the method of solving model (2.20), a new model is introduced, as follows:

$$\begin{cases} \min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} |o_i - \overline{o_i}| \\ s.t. \begin{cases} |\overline{o_i} - \overline{o}| \leq \varepsilon, \quad i = 1, ..., n \\ \overline{o} = F_w^{OWA}(\overline{o_1}, \overline{o_2}, ..., \overline{o_n}) \\ \overline{o_{\sigma(i)}} - \overline{o_{\sigma(i-1)}} \leq 0, \quad i = 2, ..., n \end{cases} \end{cases}. \tag{2.21}$$

The constraint conditions $\overline{o_{\sigma(i)}} - \overline{o_{\sigma(i-1)}} \leq 0$ $(i = 2, ..., n)$ guarantee that $\sigma(i) = \delta(i)$ $(i = 1, ..., n)$. Consequently,

$$F_w^{OWA}(\overline{o_1}, ..., \overline{o_n}) = \sum_{i=1}^{n} w_i \overline{o_{\delta(i)}} = \sum_{i=1}^{n} w_i \overline{o_{\sigma(i)}}. \tag{2.22}$$

Based on (2.22), model (2.21) can be reorganized as follows:

$$\begin{cases} \min\limits_{\overline{o_i}} \sum\limits_{i=1}^{n} |o_i - \overline{o_i}| \\ s.t. \begin{cases} |\overline{o_i} - \overline{o}| \leq \varepsilon, \quad i = 1, ..., n \\ \overline{o} = \sum\limits_{i=1}^{n} w_i \overline{o_{\sigma(i)}} \\ \overline{o_{\sigma(i)}} - \overline{o_{\sigma(i-1)}} \leq 0, \quad i = 2, ..., n \end{cases} \end{cases}. \tag{2.23}$$

Denote models (2.20) and (2.23) as $P_{2.5}$ and $P_{2.6}$, respectively. Moreover, let $\Omega_{2.5}$ and $\Omega_{2.6}$ represent the feasible sets corresponding to $P_{2.5}$ and $P_{2.6}$, respectively. To facilitate the construction of a good linkage of $P_{2.6}$ to $P_{2.5}$, Lemmas 2.2 and 2.3 are introduced.

Lemma 2.2 *For any real numbers* x_1, x_2, y_1 *and* y_2, $|x_1 - y_1| + |x_2 - y_2| \leq |x_2 - y_1| + |x_1 - y_2|$ *if* $x_1 \leq x_2$ *and* $y_1 \leq y_2$.

Proof It is obvious and the proof is omitted.

Lemma 2.3 *Let* $\{\overline{o_1^*}, ..., \overline{o_n^*}, \overline{o^*}\}$ *be the optimal solution to* $P_{2.5}$*. If* $\overline{o_p^*} < \overline{o_q^*}$ *and* $o_p > o_q$*, then* $\{\overline{\overline{o_1}}, ..., \overline{\overline{o_n}}, \overline{o^*}\}$ *is the optimal solution to model* $P_{2.5}$*, where*

$$\overline{\overline{o_i}} = \begin{cases} \overline{o_q^*}, \ for\ i = p \\ \overline{o_p^*}, \ for\ i = q \\ \overline{o_i^*}, \ for\ i \neq p, q \end{cases}.$$

Proof Since

$$\overline{\overline{o_i}} = \begin{cases} \overline{o_q^*}, \ for\ i = p \\ \overline{o_p^*}, \ for\ i = q \\ \overline{o_i^*}, \ for\ i \neq p, q \end{cases}, \tag{2.24}$$

thus

$$\overline{o^*} = F_w^{OWA}(\overline{o_1^*}, ..., \overline{o_n^*}) = F_w^{OWA}(\overline{\overline{o_1}}, ..., \overline{\overline{o_n}}) \tag{2.25}$$

and

$$\max_i |\overline{o_i^*} - \overline{o^*}| = \max_i |\overline{\overline{o_i}} - \overline{o^*}| \leq \varepsilon. \tag{2.26}$$

It is shown by (2.25) and (2.26) that $\{\overline{\overline{o_1}}, ..., \overline{\overline{o_n}}, \overline{o^*}\} \in \Omega_{2.5}$. Consequently,

$$\sum_{i=1}^{n} |o_i - \overline{\overline{o_i}}| \geq \min_{\{\overline{o_1}, ..., \overline{o_n}, \overline{o}\} \in \Omega_{2.5}} \sum_{i=1}^{n} |o_i - \overline{o_i}|. \tag{2.27}$$

Since $\{\overline{o_1^*}, ..., \overline{o_n^*}, \overline{o^*}\}$ is the optimal solution to $P_{2.5}$, thus

$$\sum_{i=1}^{n} |o_i - \overline{o_i^*}| = \min_{\{\overline{o_1}, ..., \overline{o_n}, \overline{o}\} \in \Omega_{2.5}} \sum_{i=1}^{n} |o_i - \overline{o_i}|. \tag{2.28}$$

The following can be obtained based on (2.24) and (2.28):

$$\begin{aligned} &\sum_{i=1}^{n} |o_i - \overline{\overline{o_i}}| - \min_{\{\overline{o_1}, ..., \overline{o_n}, \overline{o}\} \in \Omega_{2.5}} \sum_{i=1}^{n} |o_i - \overline{o_i}| \\ &= \sum_{i=1}^{n} |o_i - \overline{\overline{o_i}}| - \sum_{i=1}^{n} |o_i - \overline{o_i^*}| \\ &= |o_p - \overline{\overline{o_p}}| + |o_q - \overline{\overline{o_q}}| - |o_p - \overline{o_p^*}| - |o_q - \overline{o_q^*}| \\ &= |o_p - \overline{o_q^*}| + |o_q - \overline{o_p^*}| - |o_p - \overline{o_p^*}| - |o_q - \overline{o_q^*}|. \end{aligned} \tag{2.29}$$

Since $\overline{o_p^*} < \overline{o_q^*}$ and $o_q < o_p$, according to Lemma 2.2, the following can be obtained

$$\textstyle\sum_{i=1}^{n}|o_i-\overline{\overline{o_i}}| - \min_{\{\overline{o_1},\ldots,\overline{o_n},\overline{o}\}\in\Omega_{2.5}} \sum_{i=1}^{n}|o_i-\overline{o_i}| \leq 0. \tag{2.30}$$

Consequently,

$$\textstyle\sum_{i=1}^{n}|o_i-\overline{\overline{o_i}}| \leq \min_{\{\overline{o_1},\ldots,\overline{o_n},\overline{o}\}\in\Omega_{2.5}} \sum_{i=1}^{n}|o_i-\overline{o_i}|. \tag{2.31}$$

Equations (2.27) and (2.31) show that $\sum_{i=1}^{n}|o_i-\overline{\overline{o_i}}| = \min_{\{\overline{o_1},\overline{o_2},\ldots,\overline{o_n},\overline{o}\}\in\Omega_{2.5}} \sum_{i=1}^{n}|o_i-\overline{o_i}|$. So, $\{\overline{\overline{o_1}}, \ldots, \overline{\overline{o_n}}, \overline{o^*}\}$ is the optimal solution to $P_{2.5}$. This completes the proof of Lemma 2.3.

Theorem 2.3 is obtained based on Lemmas 2.2 and 2.3.

Theorem 2.3 *If* $\{\overline{o_1^*}, \ldots, \overline{o_n^*}, \overline{o^*}\}$ *is the optimal solution to* $P_{2.6}$, *then* $\{\overline{o_1^*}, \ldots, \overline{o_n^*}, \overline{o^*}\}$ *is the optimal solution to* $P_{2.5}$.

Proof Without loss of generality, assume that $o_1 \geq o_2 \ldots \geq o_n$. Let $\{\overline{\overline{o_1}}, \ldots, \overline{\overline{o_n}}, \overline{\overline{o}}\}$ be the optimal solution to $P_{2.5}$. Denote $\{\rho(1), \ldots, \rho(n)\}$ as the permutation of $\{1, 2, \ldots, n\}$ such that $\overline{\overline{o_{\rho(i-1)}}} \geq \overline{\overline{o_{\rho(i)}}}$ (i.e., $\overline{\overline{o_{\rho(i)}}}$ is the ith largest variable in $\{\overline{\overline{o_1}}, \ldots, \overline{\overline{o_n}}\}$). Based on Lemma 2.3, $\{\overline{\overline{o_{\rho(1)}}}, \ldots, \overline{\overline{o_{\rho(n)}}}, \overline{\overline{o}}\}$ is also the optimal solution to $P_{2.5}$. Since $\Omega_{2.6} \subseteq \Omega_{2.5}$, we have that

$$\begin{aligned}\sum_{i=1}^{n}|o_i-\overline{\overline{o_{\rho(i)}}}| &= \min_{\{\overline{o_1},\ldots,\overline{o_n},\overline{o}\}\in\Omega_{2.5}} \sum_{i=1}^{n}|o_i-\overline{o_i}| \\ &\leq \min_{\{\overline{o_1},\ldots,\overline{o_n},\overline{o}\}\in\Omega_{2.6}} \sum_{i=1}^{n}|o_i-\overline{o_i}| = \sum_{i=1}^{n}|o_i-\overline{o_i^*}|.\end{aligned} \tag{2.32}$$

Since $\overline{\overline{o_{\rho(i)}}} - \overline{\overline{o_{\rho(i-1)}}} \leq 0$, it follows $\{\overline{\overline{o_{\rho(1)}}}, \ldots, \overline{\overline{o_{\rho(n)}}}, \overline{\overline{o}}\} \in \Omega_{2.6}$. Consequently,

$$\sum_{i=1}^{n}|o_i-\overline{\overline{o_{\rho(i)}}}| \geq \min_{\{\overline{o_1},\ldots,\overline{o_n},\overline{o}\}\in\Omega_{2.6}} \sum_{i=1}^{n}|o_i-\overline{o_i}| = \sum_{i=1}^{n}|o_i-\overline{o_i^*}|. \tag{2.33}$$

Equations (2.32) and (2.33) show that

$$\min_{\{\overline{o_1},\ldots,\overline{o_n},\overline{o}\}\in\Omega_{2.5}} \sum_{i=1}^{n}|o_i-\overline{o_i}| = \sum_{i=1}^{n}|o_i-\overline{o_i^*}|. \tag{2.34}$$

Thus, $\{\overline{o_1^*}, \ldots, \overline{o_n^*}, \overline{o^*}\}$ is the optimal solution to $P_{2.5}$. This completes the proof of Theorem 2.3.

Based on Theorem 2.3, the optimal solution to $P_{2.6}$ is the optimal solution to $P_{2.5}$. Similar to Theorem 2.2, when using two transformed decision variables: $x_i = o_i - \overline{o_i}$ and $d_i = |x_i|$, $P_{2.6}$ can be transformed into a linear programming model. Therefore, solving the linear programming model yields the optimal solution to $P_{2.5}$.

(ii) Case B

In this case, to facilitate the solving process of model (2.19), Lemma 2.4 is introduced.

Lemma 2.4 *Let* $\overline{o} = F_w^{OWA}(\overline{o_1}, \overline{o_2}, ..., \overline{o_n})$ *and* $w = (w_1, w_2, ..., w_n)^T$. *Then,* $\overline{o} = \sum_{k=1}^{n} w_k r_k$ *if and only if the following constraints are satisfied.*

$$\begin{cases} r_k \le \overline{o_i} + MA_{ki}, \quad k, i = 1, 2, ..., n \\ r_k \ge \overline{o_i} - MB_{ki}, \quad k, i = 1, 2, ..., n \\ \sum\limits_{i=1}^{n} A_{ki} \le k - 1, \quad k = 1, 2, ..., n \\ \sum\limits_{i=1}^{n} B_{ki} \le n - k, \quad k = 1, 2, ..., n \\ A_{ki}, B_{ki} \in \{0, 1\}, \quad k, i = 1, 2, ..., n \end{cases} \tag{2.35}$$

where M *is* $+\infty$.

Proof Let $\overline{o_{(k)}}$ be the kth smallest number of $(\overline{o_1}, \overline{o_2}, ..., \overline{o_n})$. Consider two 0-1 mixed programming models:

P_A:

$$\begin{cases} \max r_k \\ s.t. \begin{cases} r_k \le \overline{o_i} + MA_{ki}, \quad k = 1, ..., n \\ \sum\limits_{i=1}^{n} A_{ki} \le k - 1, \quad k = 1, ..., n \\ A_{ki} \in \{0, 1\}, \quad k, i = 1, ..., n \end{cases} \end{cases} \tag{2.36}$$

P_B:

$$\begin{cases} \min r_k \\ s.t. \begin{cases} r_k \ge \overline{o_i} - MB_{ki}, \quad k = 1, ..., n \\ \sum\limits_{i=1}^{n} B_{ki} \le n - k, \quad k = 1, ..., n \\ B_{ki} \in \{0, 1\}, \quad k, i = 1, ..., n \end{cases} \end{cases} \tag{2.37}$$

where M is $+\infty$.

Based on Ogryczak and Śliwiński [18], it is obvious that both of the optimal values of P_A and P_B are $r_k = \overline{o_{(k)}}$, so the solution of constraints of P_A and P_B is $r_k = \overline{o_{(k)}}$. Thus, the constraints in Lemma 2.4 can ensure that $r_k = \overline{o_{(k)}}$. This completes the proof of Lemma 2.4.

Then, based on Lemma 2.4, Theorem 2.4 can be obtained to transform the MCCM under the OWA operator with any weights into mixed 0-1 linear programming problems.

Theorem 2.4 *Model (2.19) can be equivalently transformed into the following mixed 0-1 linear programming model:*

$$\begin{cases} \min \sum_{i=1}^{n} c_i d_i \\ s.t. \begin{cases} \overline{o} - \overline{o_i} \le \varepsilon, \quad i = 1, ..., n \\ -\overline{o} + \overline{o_i} \le \varepsilon, \quad i = 1, ..., n \\ o_i - \overline{o_i} \le x_i, \quad i = 1, ..., n \\ x_i \le d_i, \quad i = 1, ..., n \\ -x_i \le d_i, \quad i = 1, ..., n \\ \overline{o} = \sum_{i=1}^{n} w_i r_i \\ r_k \le \overline{o_i} + MA_{ki}, \quad k, i = 1, ..., n \\ r_k \ge \overline{o_i} - MB_{ki}, \quad k, i = 1, ..., n \\ \sum_{i=1}^{n} A_{ki} \le k - 1, \quad k, i = 1, ..., n \\ \sum_{i=1}^{n} B_{ki} \le n - k, \quad k, i = 1, ..., n \\ A_{ki}, B_{ki} \in \{0, 1\}, \quad k, i = 1, ..., n \end{cases} \end{cases}, \tag{2.38}$$

where $x_i = o_i - \overline{o_i}$ and $d_i \ge |x_i|$

Proof In model (2.38), $\overline{o_i} - \overline{o} \le \varepsilon$ and $\overline{o} - \overline{o_i} \le \varepsilon$ can guarantee that $|\overline{o_i} - \overline{o}| \le \varepsilon$. $o_i - \overline{o_i} = x_i$, $x_i \le d_i$ and $-x_i \le d_i$ can guarantee that $d_i \ge |x_i| = |o_i - \overline{o_i}|$. According to the objective function of model (2.38), any feasible solutions with $d_i > |x_i|$ are not the optimal solutions to model (2.38). Thus, $o_i - \overline{o_i} = x_i$, $x_i \le d_i$ and $-x_i \le d_i$ can guarantee that $d_i = |x_i| = |o_i - \overline{o_i}|$. So the following inequalities can guarantee that $|\overline{o_i} - \overline{o}| \le \varepsilon$ and the transformation of objective function from model (2.19) to model (2.38):

$$\begin{cases} \overline{o_i} - \overline{o} \le \varepsilon, \quad i = 1, ..., n \\ \overline{o} - \overline{o_i} \le \varepsilon, \quad i = 1, ..., n \\ o_i - \overline{o_i} = x_i, \quad i = 1, ..., n\,. \\ -x_i \le d_i, \quad i = 1, ..., n \\ x_i \le d_i, \quad i = 1, ..., n \end{cases} \tag{2.39}$$

Then, based on Lemma 2.4, other constraints in model (2.38) can guarantee that $\overline{o} = F_w^{OWA}(\overline{o_1}, \overline{o_2}, ..., \overline{o_n})$. Therefore, model (2.19) can be transformed into model (2.38). This completes the proof of Theorem 2.4.

Denote model (2.38) as $P_{2.7}$. Based on Theorem 2.4, the optimal solution to model (2.19) can be obtained by solving the mixed 0-1 linear programming model $P_{2.7}$. Generally, cutting plane is used to solve mixed 0-1 linear programming models [2], and Balas et al. [1] proposed the Specialized Cutting Plane Algorithm to solve mixed 0-1 programs in finitely much iteration. The algorithm is very efficient and valid. Several software packages such as CPLEX and MATLAB also provide efficient algorithm and solver to solve the mixed 0-1 linear programming problems.

(3) Numerical examples

Here, we provide three numerical examples to demonstrate how the MCCMs work in practice.

(i) Example 2.1

Consider the example used in Ben-Arieh and Easton [4]. This example involves a GDM problem evaluated by five experts $\{E_1, E_2, ..., E_5\}$. Let the expert opinions be represented by real numbers, as follows:

$$\{o_1, o_2, o_3, o_4, o_5\} = \{0.5, 1.0, 2.5, 3.0, 6.0\}.$$

Let $\varepsilon = 0.8$, and let the associated weight vector of the weighted averaging operator be $w = (0.375, \ 0.1875, \ 0.25, 0.0625, \ 0.125)^T$. We use model $P_{2.4}$ to reach consensus among $\{o_1, o_2, o_3, o_4, o_5\}$. Table 2.1 lists the optimal adjusted opinions obtained using $P_{2.4}$ when setting different values of c_i $(i = 1, 2, ..., 5)$.

Let the associated weight vector of the selected OWA operator be $w = (0.375, 0.1875, 0.25, 0.0625, 0.125)^T$. Model $P_{2.5}$ is used to aggregate expert opinions. Table 2.2 lists the optimal adjusted opinions when setting different values for parameter ε.

Let the associated weight vector of the selected OWA operator be $w = (0.375, 0.1875, 0.25, 0.0625, 0.125)^T$. Let the consensus threshold $\varepsilon = 0.8$. The consensus model $P_{2.7}$ is used to help experts to reach a consensus. Table 2.3 displays the adjusted individual opinions, the collective group opinion and the total consensus cost under different cost vectors $(c_1, c_2, c_3, c_4, c_5)^T$.

Table 2.1 $P_{2.4}$ under different cost vectors in Example 2.1

$(c_1, c_2, c_3, c_4, c_5)$	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$
(1, 4, 3, 5, 2)	2	1.4	2.5	3	3	2.2
(6, 3, 4, 1, 2)	0.5	1.0	2.1	2.1	2.1	1.3
(3, 4, 1, 6, 2)	1.6	1.32	2.9	2.92	2.92	2.12

Table 2.2 $P_{2.5}$ under different consensus thresholds in Example 2.1

ε	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$
0.5	2.43	2.43	2.5	3.0	3.43	2.93
0.6	2.37	2.37	2.5	3.0	3.57	2.97
0.7	2.3	2.3	2.5	3.0	3.7	3.0
0.8	2.25	2.25	2.5	3.0	3.85	3.05

Table 2.3 $P_{2.7}$ under different cost vectors in Example 2.1

$(c_1, c_2, c_3, c_4, c_5)$	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$	Total cost
(1, 4, 3, 5, 2)	2.575	1.4	2.5	3	3	2.2	9.675
(2, 4, 3, 1, 2.5)	1.078	1.078	2.5	2.5	2.5	1.7	10.72
(4, 1, 4, 2, 5)	1.4	2.575	2.5	3	3	2.2	20.175

Let the associated weight vector of the selected OWA operator be $w = (0.375, 0.1875, 0.25, 0.0625, 0.125)^T$. Then, let the cost vector $(c_1, c_2, c_3, c_4, c_5)^T = (1, 4, 3, 5, 2)^T$. Table 2.4 displays the adjusted individual opinions, the collective group opinion and the total consensus cost under different consensus thresholds ε, using $P_{2.7}$.

(ii) Example 2.2

Consider another example from Ben-Arieh et al. [5]. This example contains four experts $\{E_1, E_2, E_3, E_4\}$, providing the following opinions:

$$\{o_1, o_2, o_3, o_4\} = \{0, 3, 6, 10\}.$$

Let $\varepsilon = 0.8$, and let the associated weight vector of the weighted averaging operator be $w = (0.3, 0.1, 0.4, 0.2)^T$. Model $P_{2.4}$ is used to reach a consensus among $\{o_1, o_2, o_3, o_4\}$. Table 2.5 lists the optimal adjusted opinions obtained using $P_{2.4}$ when setting different values for c_i $(i = 1, 2, ..., 4)$.

Let the associated weight vector of the selected OWA operator be $w = (0.3, 0.1, 0.4, 0.2)^T$. Model $P_{2.5}$ is used to aggregate expert opinions. Table 2.6 lists the optimal adjusted opinions when considering different consensus thresholds ε.

Table 2.4 $P_{2.7}$ under different consensus thresholds in Example 2.1

ε	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$	Total cost
0.5	2.875	2	2.5	3	3	2.5	12.375
0.6	2.775	1.8	2.5	3	3	2.4	11.475
0.7	2.675	1.6	2.5	3	3	2.3	10.575
0.8	2.575	1.4	2.5	3	3	2.2	9.675

Table 2.5 $P_{2.4}$ under different cost vectors in Example 2.2

(c_1, c_2, c_3, c_4)	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o}$
(1, 4, 3, 5)	4.8	4.8	6	6.4	5.6
(6, 3, 4, 1)	1.4	3	3	1.4	2.2
(3, 6, 4, 1)	3	3	4.4	4.2	3.8

Table 2.6 $P_{2.5}$ under different consensus thresholds in Example 2.2

ε	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o}$
0.5	2.75	3.0	3.75	3.75	3.25
0.6	2.7	3.0	3.9	3.9	3.3
0.7	2.65	3.0	4.05	4.05	3.35
0.8	2.6	3.0	4.2	4.2	3.4

(iii) Example 2.3

Example 2.3 involves an apartment buyer. The family of buyer comprises four members $\{E_1, E_2, E_3, E_4\}$, and five alternative flats $\{X_1, X_2, ..., X_5\}$ are available for consideration. The family members express their opinions regarding the alternatives using real numbers in [1, 5], and o_{ij} estimates the opinion of E_j on X_i. Larger o_{ij} indicates stronger preference of member E_j for alternative X_i. Table 2.7 lists the values of o_{ij}.

Table 2.7 shows that the preferred alternatives of E_1, E_2, E_3 and E_4 are X_3, X_2, X_4 and X_1, respectively. The following uses MCCMs to reach consensus.

Let $\{c_1, c_2, c_3, c_4\} = \{1, 2, 1, 1\}$, and $\varepsilon = 1$. We use $P_{2.1}$ to aggregate the opinions of the family members. Let $\overline{o_{ij}}$ denote the optimal adjusted individual opinions of E_j on X_i. Let $\overline{o_i}$ represent the optimal adjusted collective opinions on X_i. Table 2.8 lists the values of $\overline{o_{ij}}$ and $\overline{o_i}$.

Table 2.8 reveals that $\overline{o_i} = \frac{\min_j\{\overline{o_{ij}}\}+\max_j\{\overline{o_{ij}}\}}{2}$, which is consistent with Theorem 2.1.

Table 2.7 The values of o_{ij} in Example 2.3

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	1	3	1	5
$i = 2$	3	5	3	2
$i = 3$	4	1	2	1
$i = 4$	2	3	5	1
$i = 5$	2	4	2	3

Table 2.8 The values of $\overline{o_{ij}}$ and $\overline{o_i}$ obtained from $P_{2.1}$ in Example 2.3

	$\overline{o_{i,1}}$	$\overline{o_{i,2}}$	$\overline{o_{i,3}}$	$\overline{o_{i,4}}$	$\overline{o_i}$
$i = 1$	1	3	1	3	2
$i = 2$	3	5	3	3	4
$i = 3$	3	1	2	1	2
$i = 4$	2	3	3	1	2
$i = 5$	2	4	2	3	3

Table 2.9 The values of $\overline{o_{ij}}$ and $\overline{o_i}$ obtained from $P_{2.4}$ in Example 2.3

	$\overline{o_{i,1}}$	$\overline{o_{i,2}}$	$\overline{o_{i,3}}$	$\overline{o_{i,4}}$	$\overline{o_i}$
$i = 1$	1.12	3	1.12	2.86	2.12
$i = 2$	3	4.12	3	2.12	3.12
$i = 3$	2.56	1	2	1	1.56
$i = 4$	2	3	3.59	1.6	2.59
$i = 5$	2	3.8	2	3	2.8

Table 2.10 The values of $\overline{o_{ij}}$ and $\overline{o_i}$ obtained from $P_{2.5}$ in Example 2.3

	$\overline{o_{i,1}}$	$\overline{o_{i,2}}$	$\overline{o_{i,3}}$	$\overline{o_{i,4}}$	$\overline{o_i}$
$i = 1$	1	2.96	1	2.9	1.96
$i = 2$	3	3.9	3	2	2.9
$i = 3$	2.62	1	2	1	1.62
$i = 4$	2	3	3.4	1.5	2.5
$i = 5$	2	3.62	2	3	2.62

Let $\{c_1, c_2, c_3, c_4\} = \{1, 2, 1, 1\}$, and $\varepsilon = 1$. Moreover, let the associated weight vector of the weighted averaging operator be $w = (0.2, 0.3, 0.25, 0.25)^T$. We use model $P_{2.4}$ to aggregate the opinions of the family members. Table 2.9 lists the optimal adjusted opinions.

Let $\varepsilon = 1$, and let the associated weight vector of the selected OWA operator be $w = (0.2, 0.3, 0.25, 0.25)^T$. We use model $P_{2.5}$ to aggregate the opinions of the family members. Table 2.10 lists the optimal adjusted opinions.

Tables 2.8, 2.9 and 2.10 reveal that $\overline{o_2} = \max\{\overline{o_i}\}$. Therefore, $P_{2.1}$, $P_{2.4}$, $P_{2.5}$ and $P_{2.7}$ all indicate that the best alternative is X_2.

2.2.2 Maximum Expert Consensus Model

Zhang et al. [19] further develop the the maximum expert consensus model (MECM) with aggregation functions based on the MCCM.

Given a special cost budget, the MECM seeks to find the maximum number of experts that can fit within the consensus.

Let B denote a specified consensus cost budget. In consensus building, the total consensus cost cannot exceed the cost budget B, i.e.,

$$\sum_{i=1}^{n} c_i \left|\overline{o_i} - o_i\right| \leq B. \tag{2.40}$$

We consider that the adjusted collective opinion $\overline{o}$ is obtained by aggregating the adjusted individual opinions $\overline{o_i}(i = 1, 2, ..., n)$, i.e.,

$$\overline{o} = F(\overline{o_1}, \overline{o_2}, ..., \overline{o_n}). \tag{2.41}$$

Meanwhile, we hope to maximize the number of experts with consensus, i.e.,

$$\max \sum_{i=1}^{n} x_i, \tag{2.42}$$

where, x_i is a 0-1 variable, and x_i is defined as:

$$x_i = \begin{cases} 1, \; if \; |\overline{o_i} - \overline{o}| \le \varepsilon \\ 0, \; else \end{cases}. \tag{2.43}$$

Based on Eqs. (2.40)–(2.43), the MECM can be presented using an optimization model as follows:

$$\begin{cases} \max \sum\limits_{i=1}^{n} x_i \\ s.t. \begin{cases} \sum\limits_{i=1}^{n} c_i \, |\overline{o_i} - o_i| \le B \\ \overline{o} = F\,(\overline{o_1}, \overline{o_2}, ..., \overline{o_n}) \\ x_i = \begin{cases} 1, if \; |\overline{o_i} - \overline{o}| \le \varepsilon \\ 0, else \end{cases}, \quad i = 1, ..., n \end{cases} \end{cases}. \tag{2.44}$$

Denote model (2.44) as $P_{2.8}$. Notably, we use two aggregation functions: the WA operator and the OWA operator, in $P_{2.8}$.

(1) Maximum expert consensus model with WA operator

Selecting the WA operator to aggregate individuals' opinions in $P_{2.8}$ can yield the following model:

$$\begin{cases} \max \sum\limits_{i=1}^{n} x_i \\ s.t. \begin{cases} \sum\limits_{i=1}^{n} c_i \, |\overline{o_i} - o_i| \le B \\ \overline{o} = F_w^{WA}\,(\overline{o_1}, \overline{o_2}, ..., \overline{o_n}) \\ x_i = \begin{cases} 1, if \; |\overline{o_i} - \overline{o}| \le \varepsilon \\ 0, else \end{cases}, \quad i = 1, ..., n \end{cases} \end{cases}. \tag{2.45}$$

Denote model (2.45) as $P_{2.9}$. In $P_{2.9}$, $w = (w_1, w_2, ..., w_n)^T$ is the associated weight vector of the WA operator. Let $\Omega_{2.9}$ denote the feasible region of $P_{2.9}$. In order to solve $P_{2.9}$, we propose several equivalent transformations.

Lemma 2.5 *$P_{2.9}$ can be equivalently transformed into the nonlinear mixed 0-1 programming model $P_{2.10}$:*

$$\begin{cases} \max \sum_{i=1}^{n} x_i \\ s.t. \begin{cases} \sum_{i=1}^{n} c_i \left|\overline{o_i} - o_i\right| \leq B \\ \overline{o} = \sum_{i=1}^{n} w_i \overline{o_i} \\ x_i \left|\overline{o_i} - \overline{o}\right| \leq \varepsilon, \quad i = 1, ..., n \\ x_i \in \{0, 1\}, \quad i = 1, ..., n \end{cases} \end{cases}. \tag{2.46}$$

Proof Let $\Omega_{2.10}$ denote the feasible region of $P_{2.10}$. Let $\left\{\overline{o_1^*}, \overline{o_2^*}, ..., \overline{o_n^*}, \overline{o^*}, \overline{x_1^*}, \overline{x_2^*}, ..., \overline{x_n^*}\right\}$ be the optimal solution to $P_{2.9}$ and let $\left\{\overline{\overline{o_1^*}}, \overline{\overline{o_2^*}}, ..., \overline{\overline{o_n^*}}, \overline{\overline{o^*}}, \overline{\overline{x_1^*}}, \overline{\overline{x_2^*}}, ..., \overline{\overline{x_n^*}}\right\}$ be the optimal solution to $P_{2.10}$. For any feasible solution to $P_{2.9}$ $\{\overline{o_1}, \overline{o_2}, ..., \overline{o_n}, \overline{o}, \overline{x_1}, \overline{x_2}, ..., \overline{x_n}\} \in \Omega_{2.9}$, the following two cases can be obtained:

Case 1: $\left|\overline{o_i} - \overline{o}\right| \leq \varepsilon$

Based on the constraints of $P_{2.9}$, if $\left|\overline{o_i} - \overline{o}\right| \leq \varepsilon$, the following can be obtained: $x_i = 1$;

Consequently

$$x_i \left|\overline{o_i} - \overline{o}\right| = \left|\overline{o_i} - \overline{o}\right| \leq \varepsilon\,; \tag{2.47}$$

Case 2: $\left|\overline{o_i} - \overline{o}\right| > \varepsilon$

Based on the constraints of $P_{2.9}$, if $\left|\overline{o_i} - \overline{o}\right| > \varepsilon$, the following can be obtained:

$$x_i = 0;$$

Consequently

$$x_i \left|\overline{o_i} - \overline{o}\right| = 0 \leq \varepsilon\,. \tag{2.48}$$

Then, based on Case 1 and Case 2, we can obtain the following inequality:

$$x_i \left|\overline{o_i} - \overline{o}\right| \leq \varepsilon. \tag{2.49}$$

Since $\{\overline{o_1}, \overline{o_2}, ..., \overline{o_n}, \overline{o}, \overline{x_1}, \overline{x_2}, ..., \overline{x_n}\} \in \Omega_{2.9}$, we have the following constraints:

$$\sum_{i=1}^{n} c_i \left|\overline{o_i} - o_i\right| \leq B, \tag{2.50}$$

$$\overline{o} = F_w^{WA}\left(\overline{o_1}, \overline{o_2} ..., \overline{o_n}\right). \tag{2.51}$$

Based on (2.49)–(2.51), it follows $\{\overline{o_1}, \overline{o_2}, ..., \overline{o_n}, \overline{o}, \overline{x_1}, \overline{x_2}, ..., \overline{x_n}\} \in \Omega_{2.10}$. As a result, we can obtain that $\Omega_{2.9} \subseteq \Omega_{2.10}$. Then, based on the objective functions of $P_{2.9}$ and $P_{2.10}$, we have that

$$\sum_{i=1}^{n} \overline{x_i^*} \leq \sum_{i=1}^{n} \overline{\overline{x_i^*}}. \tag{2.52}$$

Since $\left\{\overline{\overline{o_1^*}}, \overline{\overline{o_2^*}}, ..., \overline{\overline{o_n^*}}, \overline{\overline{o^*}}, \overline{\overline{x_1^*}}, \overline{\overline{x_2^*}}, ..., \overline{\overline{x_n^*}}\right\}$ is the optimal solution to $P_{2.10}$, the following two cases can be obtained:

Case A: $\left|\overline{\overline{o_k^*}} - \overline{\overline{o^*}}\right| \leq \varepsilon$

Based on the constraints of $P_{2.10}$, if $\left|\overline{\overline{o_k^*}} - \overline{\overline{o^*}}\right| \leq \varepsilon$, the following can be obtained: $\overline{\overline{x_k^*}} = 0 \ or \ 1$;

Since $\sum\limits_{i=1, i\neq k}^{n} \overline{\overline{x_i^*}} + 1 \geq \sum\limits_{i=1, i\neq k}^{n} \overline{\overline{x_i^*}} + 0$

Thus $\overline{\overline{x_k^*}} = 1$;

Case B: $\left|\overline{\overline{o_k^*}} - \overline{\overline{o^*}}\right| > \varepsilon$

Based on the constraints of $P_{2.10}$, if $\left|\overline{\overline{o_k^*}} - \overline{\overline{o^*}}\right| > \varepsilon$, we can obtain that: $\overline{\overline{x_k^*}} = 0$;

Based on Case A and Case B, the following can be obtained:

$$\overline{\overline{x_i^*}} = \begin{cases} 1, & if \ \left|\overline{\overline{o_k^*}} - \overline{\overline{o^*}}\right| \leq \varepsilon \\ 0, & else \end{cases}, \ i = 1, 2, ..., n. \tag{2.53}$$

Since $\left\{\overline{\overline{o_1^*}}, \overline{\overline{o_2^*}}, ..., \overline{\overline{o_n^*}}, \overline{\overline{o^*}}, \overline{\overline{x_1^*}}, \overline{\overline{x_2^*}}, ..., \overline{\overline{x_n^*}}\right\} \in \Omega_{2.10}$, we have that:

$$\sum_{i=1}^{n} c_i \left|\overline{\overline{o_i^*}} - o_i\right| \leq B, \tag{2.54}$$

$$\overline{\overline{o^*}} = F_w^{WA}\left(\overline{\overline{o_1^*}}, \overline{\overline{o_2^*}} ..., \overline{\overline{o_n^*}}\right). \tag{2.55}$$

Based on (2.53)–(2.55), the following can be obtained:

$$\left\{\overline{\overline{o_1^*}}, \overline{\overline{o_2^*}}, ..., \overline{\overline{o_n^*}}, \overline{\overline{o^*}}, \overline{\overline{x_1^*}}, \overline{\overline{x_2^*}}, ..., \overline{\overline{x_n^*}}\right\} \in \Omega_{2.9}. \tag{2.56}$$

Then, since $\left\{\overline{o_1^*}, \overline{o_2^*}, ..., \overline{o_n^*}, \overline{o^*}, \overline{x_1^*}, \overline{x_2^*}, ..., \overline{x_n^*}\right\}$ is the optimal solution to $P_{2.9}$, the following can be obtained:

$$\sum_{i=1}^{n} \overline{\overline{x_i^*}} \leq \sum_{i=1}^{n} \overline{x_i^*}. \tag{2.57}$$

Based on (2.52) and (2.57), we have that:

$$\sum_{i=1}^{n} \overline{x_i^*} = \sum_{i=1}^{n} \overline{\overline{x_i^*}}. \tag{2.58}$$

Thus, the optimal solutions to $P_{2.9}$ and $P_{2.10}$ are equal. This completes the proof of Lemma 2.5.

According to Lemma 2.5, the optimal solution to $P_{2.9}$ can be obtained by solving model (2.46). In order to solve model (2.46), Lemma 2.6 is introduced.

Lemma 2.6 (Berthold et al. [6]) *If a constraint in a mixed 0-1 programming contains a product of a binary variable x with a linear term $\sum_{i=1}^{n} a_i y_i$, where $y_i (i = 1, 2, ..., n)$ are variables with finite bounds, this product can be replaced by a new variable z and the following linear constraints:*

$$\begin{cases} xy^L \le z \le xy^U \\ \sum\limits_{i=1}^{n} a_i y_i - (1-x)y^U \le z \le \sum\limits_{i=1}^{n} a_i y_i - (1-x)y^L \\ y^L = min \sum\limits_{i=1}^{n} a_i y_i, \quad y^U = max \sum\limits_{i=1}^{n} a_i y_i \end{cases}. \tag{2.59}$$

Theorem 2.5 is obtained based on Lemma 2.6.

Theorem 2.5 *$P_{2.9}$ can be equivalently transformed into the mixed 0-1 linear programming model $P_{2.11}$:*

$$\begin{cases} \max \sum\limits_{i=1}^{n} x_i \\ s.t. \begin{cases} \sum\limits_{i=1}^{n} c_i y_i \le B \\ \overline{o_i} - o_i \le y_i, \quad i = 1, ..., n \\ -\overline{o_i} + o_i \le y_i, \quad i = 1, ..., n \\ \overline{o} = \sum\limits_{i=1}^{n} w_i \overline{o_i} \\ z_i \le \varepsilon, \quad i = 1, ..., n \\ -z_i \le \varepsilon, \quad i = 1, ..., n \\ (1-w_i)(a-b)x_i \le z_i \le (1-w_i)(b-a)x_i, \quad i = 1, ..., n \\ \overline{o_i} - \overline{o} - (1-w_i)(b-a)(1-x_i) \le z_i \le \overline{o_i} - \overline{o} - \\ \quad (1-w_i)(a-b)(1-x_i), \quad i = 1, ..., n \\ a \le \overline{o_i} \le b, \quad i = 1, ..., n \\ x_i \in \{0, 1\}, \quad i = 1, ..., n \end{cases} \end{cases}. \tag{2.60}$$

Proof The proof of Theorem 2.5 can be divided into two parts:

Part 1: Proving that constraint $x_i \left|\overline{o_i} - \overline{o}\right| \le \varepsilon$ in $P_{2.9}$ can be transformed into the following constraints in $P_{2.11}$:

$$\begin{cases} z_i \le \varepsilon, \quad i = 1, ..., n \\ -z_i \le \varepsilon, \quad i = 1, ..., n \\ (1 - w_i)(a - b)x_i \le z_i \le (1 - w_i)(b - a)x_i, \quad i = 1, ..., n \\ \overline{o_i} - \overline{o} - (1 - w_i)(b - a)(1 - x_i) \le z_i \le \\ \overline{o_i} - \overline{o} - (1 - w_i)(a - b)(1 - x_i), \quad i = 1, ..., n \end{cases}. \tag{2.61}$$

We can get the following constraints from $x_i \, |\overline{o_i} - \overline{o}| \le \varepsilon$:

$$\begin{cases} x_i \, (\overline{o_i} - \overline{o}) \le \varepsilon \\ -x_i \, (\overline{o_i} - \overline{o}) \le \varepsilon \end{cases}. \tag{2.62}$$

Obviously, $\overline{o_i}$ is bounded in GDM problems. Without loss of generality, assume that $\overline{o_i}$ ranges from a to b. Since $\overline{o} = \sum_{i=1}^{n} w_i \overline{o_i}$, $(\overline{o_i} - \overline{o})$ is a linear term.

$$\begin{cases} \overline{o_i} - \overline{o} = \overline{o_i} - \sum_{j=1}^{n} w_j \overline{o_j} = (1 - w_i)\overline{o_i} - \sum_{j=1, j \ne i}^{n} w_j \overline{o_j} \\ \max \{\overline{o_i} - \overline{o}\} = (1 - w_i)b - a \sum_{j=1, j \ne i}^{n} w_j = (b - a)(1 - w_i) \\ \min \{\overline{o_i} - \overline{o}\} = (1 - w_i)a - b \sum_{j=1, j \ne i}^{n} w_j = (a - b)(1 - w_i) \end{cases}. \tag{2.63}$$

Since x_i is a binary variable. Then, let

$$z_i = x_i \, (\overline{o_i} - \overline{o}) \, . \tag{2.64}$$

The following can be obtained based on Lemma 2.6:

$$\begin{cases} z_i \le \varepsilon, \quad i = 1, ..., n \\ -z_i \le \varepsilon, \quad i = 1, ..., n \\ (1 - w_i)(a - b)x_i \le z_i \le (1 - w_i)(b - a)x_i, \quad i = 1, ..., n \\ \overline{o_i} - \overline{o} - (1 - w_i)(b - a)(1 - x_i) \le z_i \le \\ \quad \overline{o_i} - \overline{o} - (1 - w_i)(a - b)(1 - x_i), \quad i = 1, ..., n \end{cases}. \tag{2.65}$$

So constraint $x_i \, |\overline{o_i} - \overline{o}| \le \varepsilon$ in $P_{2.9}$ can be equivalently transformed into these constraints in model $P_{2.11}$.

Part 2: Proving that constraint $\sum_{i=1}^{n} c_i \, |\overline{o_i} - o_i| \le B$ in model $P_{2.9}$ can be transformed into the following constraints in model $P_{2.11}$:

$$\begin{cases} \sum_{i=1}^{n} c_i y_i \le B \\ \overline{o_i} - o_i \le y_i \\ -\overline{o_i} + o_i \le y_i \end{cases}. \tag{2.66}$$

Let $\Omega_{2.10}$ denote the feasible region of $P_{2.10}$ and let $\Omega_{2.11}$ denote the feasible region of model $P_{2.11}$.

Let $\left\{\overline{\overline{o_1^*}},\ \overline{\overline{o_2^*}}, ..., \overline{\overline{o_n^*}}, \overline{\overline{o^*}}, \overline{\overline{x_1^*}}, \overline{\overline{x_2^*}}, ..., \overline{\overline{x_n^*}}\right\}$ be the optimal solution to $P_{2.10}$ and let $\left\{\widetilde{o_1^*},\ \widetilde{o_2^*}, ..., \widetilde{o_n^*}, \widetilde{o^*}, \widetilde{x_1^*}, \widetilde{x_2^*}, ..., \widetilde{x_n^*}, \widetilde{y_1^*}, \widetilde{y_2^*}, ...,\ \widetilde{y_n^*}\right\}$ be the optimal solution to $P_{2.11}$.

Since $\left\{\widetilde{o_1^*},\ \widetilde{o_2^*}, ..., \widetilde{o_n^*}, \widetilde{o^*}, \widetilde{x_1^*}, \widetilde{x_2^*}, ..., \widetilde{x_n^*}, \widetilde{y_1^*}, \widetilde{y_2^*}, ...,\ \widetilde{y_n^*}\right\} \in \Omega_{2.11}$, we can obtain that:

$$\begin{cases} \widetilde{o_i^*} - o_i \le \widetilde{y_i^*} \\ -\widetilde{o_i^*} + o_i \le \widetilde{y_i^*} \end{cases}. \tag{2.67}$$

Thus

$$\sum_{i=1}^{n} c_i \left|\widetilde{o_i^*} - o_i\right| \le \sum_{i=1}^{n} c_i \widetilde{y_i^*} \le B. \tag{2.68}$$

Since Part 1 guarantees the transformation of other constraints from $P_{2.10}$ to $P_{2.11}$, we can obtain that:

$$\left\{\widetilde{o_1^*}, \widetilde{o_2^*}, ..., \widetilde{o_n^*}, \widetilde{o^*}, \widetilde{x_1^*}, \widetilde{x_2^*}, ..., \widetilde{x_n^*}, \widetilde{y_1^*}, \widetilde{y_2^*}, ..., \widetilde{y_n^*}\right\} \in \Omega_{2.10}.$$

Then, since $\left\{\overline{\overline{o_1^*}}, \overline{\overline{o_2^*}}, ..., \overline{\overline{o_n^*}}, \overline{\overline{o^*}}, \overline{\overline{x_1^*}}, \overline{\overline{x_2^*}}, ..., \overline{\overline{x_n^*}}\right\}$ is the optimal solution to $P_{2.10}$, based on the objective functions of $P_{2.10}$ and $P_{2.11}$, the following can be obtained:

$$\sum_{i=1}^{n} \widetilde{x_i^*} \le \sum_{i=1}^{n} \overline{\overline{x_i^*}}. \tag{2.69}$$

Since $\left\{\overline{\overline{o_1^*}}, \overline{\overline{o_2^*}}, ..., \overline{\overline{o_n^*}}, \overline{\overline{o^*}}, \overline{\overline{x_1^*}}, \overline{\overline{x_2^*}}, ..., \overline{\overline{x_n^*}}\right\}$ is the optimal solution to $P_{2.10}$, the following inequality can be obtained:

$$\sum_{i=1}^{n} c_i |\overline{\overline{o_i^*}} - o_i| \le B. \tag{2.70}$$

Let

$$y_i = |\overline{\overline{o_i^*}} - o_i|. \tag{2.71}$$

Then, we have the following inequalities:

$$\overline{\overline{o_i^*}} - o_i \le y_i, \tag{2.72}$$

$$-\overline{\overline{o_i^*}} + o_i \le y_i. \tag{2.73}$$

Thus, since Part 1 guarantees the transformation of other constraints from model $P_{2.10}$ to $P_{2.11}$, the following can be obtained based on (2.70), (2.72) and (2.73):

$$\left\{\overline{\overline{o_1^*}}, \overline{\overline{o_2^*}}, ..., \overline{\overline{o_n^*}}, \overline{\overline{o^*}}, \overline{\overline{x_1^*}}, \overline{\overline{x_2^*}}, ..., \overline{\overline{x_n^*}}, y_1, y_2, ..., y_n\right\} \in \Omega_{2.11}.$$

Since $\left\{\widetilde{o_1^*}, \widetilde{o_2^*}, ..., \widetilde{o_n^*}, \widetilde{o^*}, \widetilde{x_1^*}, \widetilde{x_2^*}, ..., \widetilde{x_n^*}, \widetilde{y_1^*}, \widetilde{y_2^*}, ..., \widetilde{y_n^*}\right\}$ is the optimal solution to $P_{2.11}$, based on the objective functions of $P_{2.10}$ and $P_{2.11}$, we can obtain:

$$\sum_{i=1}^{n} \overline{\overline{x_i^*}} \leq \sum_{i=1}^{n} \widetilde{x_i^*}. \tag{2.74}$$

Then, the following can be obtained based on (2.69) and (2.74):

$$\sum_{i=1}^{n} \overline{\overline{x_i^*}} = \sum_{i=1}^{n} \widetilde{x_i^*}. \tag{2.75}$$

So constraint $\sum_{i=1}^{n} c_i \left|\overline{o_i} - o_i\right| \leq B$ in $P_{2.10}$ can be transformed into the corresponding constraints in $P_{2.11}$.

Thus, based on Part 1 and Part 2, $P_{2.10}$ can be equivalently transformed into $P_{2.11}$. This completes the proof of Theorem 2.5.

From model (2.59) to model (2.60), all nonlinear constraints are transformed into linear constraints. Consequently, the MECM with WA operator is equivalently transformed into a mixed 0-1 linear programming model.

(2) Maximum expert consensus model with OWA operator

Selecting the OWA operator to aggregate individuals' opinions in $P_{2.8}$ can yield the following model:

$$\begin{cases} \max \sum_{i=1}^{n} x_i \\ s.t. \begin{cases} \sum_{i=1}^{n} c_i \left|\overline{o_i} - o_i\right| \leq B \\ \overline{o} = F_w^{OWA}\left(\overline{o_1}, \overline{o_2}, ..., \overline{o_n}\right) \\ x_i = \begin{cases} 1, if \left|\overline{o_i} - \overline{o}\right| \leq \varepsilon \\ 0, else \end{cases}, \quad i = 1, ..., n \end{cases} \end{cases}. \tag{2.76}$$

Denote model (2.76) as $P_{2.12}$.

Then, Theorem 2.6 can be obtained based on Lemma 2.4 and Theorem 2.5.

Theorem 2.6 *$P_{2.12}$ can be equivalently transformed into the mixed 0-1 linear programming model $P_{2.13}$:*

$$\begin{cases} \max \sum_{i=1}^{n} x_i \\ s.t. \begin{cases} \sum_{i=1}^{n} c_i y_i \le B \\ \overline{o_i} - o_i \le y_i, \quad i = 1, ..., n \\ -\overline{o_i} + o_i \le y_i, \quad i = 1, ..., n \\ \overline{o} = \sum_{i=1}^{n} w_i r_i \\ r_k \le \overline{o_i} + M A_{ki}, \quad k, i = 1, ..., n \\ r_k \ge \overline{o_i} - M B_{ki}, \quad k, i = 1, ..., n \\ \sum_{i=1}^{n} A_{ki} \le k - 1, \quad k = 1, ..., n \\ \sum_{i=1}^{n} B_{ki} \le n - k, \quad k = 1, ..., n \\ z_i \le \varepsilon, \quad i = 1, ..., n \\ -z_i \le \varepsilon, \quad i = 1, ..., n \\ (1 - w_i)(a - b)x_i \le z_i \le (1 - w_i)(b - a)x_i, \quad i = 1, ..., n \\ \overline{o_i} - \overline{o} - (1 - w_i)(b - a)(1 - x_i) \le z_i \\ \quad \le \overline{o_i} - \overline{o} - (1 - w_i)(a - b)(1 - x_i), \quad i = 1, ..., n \\ a \le \overline{o_i} \le b, \quad i = 1, ..., n \\ x_i, A_{ki}, B_{ki} \in \{0, 1\}, \quad k, i = 1, ..., n \end{cases} \end{cases}. \tag{2.77}$$

Proof Based on the proof of Theorem 2.5, constraint $\sum_{i=1}^{n} c_i \left|\overline{o_i} - o_i\right| \le B$ in model $P_{2.12}$ can be transformed into the following constraints in $P_{2.13}$:

$$\begin{cases} \sum_{i=1}^{n} c_i y_i \le B \\ \overline{o_i} - o_i \le y_i, \quad i = 1, ..., n \\ -\overline{o_i} + o_i \le y_i, \quad i = 1, ..., n \end{cases}. \tag{2.78}$$

Then, based on Lemma 2.4, constraint $\overline{o} = F_w^{OWA}(\overline{o_1}, \overline{o_2}, ..., \overline{o_n})$ in $P_{2.12}$ can be equivalently transformed into the following constraints in model $P_{2.13}$:

$$\begin{cases} \overline{o} = \sum_{i=1}^{n} w_i r_i \\ r_k \le \overline{o_i} + M A_{ki}, \quad k, i = 1, ..., n \\ r_k \ge \overline{o_i} - M B_{ki}, \quad k, i = 1, ..., n \\ \sum_{i=1}^{n} A_{ki} \le k - 1, \quad k = 1, ..., n \\ \sum_{i=1}^{n} B_{ki} \le n - k, \quad k = 1, ..., n \\ A_{ki}, B_{ki} \in \{0, 1\}, \quad k, i = 1, ..., n \end{cases}. \tag{2.79}$$

Based on the proof of Lemma 2.4, the definition of x_i in model $P_{2.12}$ can be transformed into the following constraints:

$$\begin{cases} x_i \left|\overline{o_i} - \overline{o}\right| \le \varepsilon, \quad i = 1, ..., n \\ x_i \in \{0, 1\}, \quad i = 1, ..., n \end{cases}. \tag{2.80}$$

Then, based on the proof of Theorem 2.5, constraint $x_i|\overline{o_i} - \overline{o}| \le \varepsilon$ can be transformed into the following constraints in model $P_{2.13}$:

$$\begin{cases} z_i \le \varepsilon, \quad i = 1, ..., n \\ -z_i \le \varepsilon, \quad i = 1, ..., n \\ (1 - w_i)(a - b)x_i \le z_i \le (1 - w_i)(b - a)x_i, \quad i = 1, ..., n \\ \overline{o_i} - \overline{o} - (1 - w_i)(b - a)(1 - x_i) \le z_i \le \overline{o_i} - \overline{o} \\ \quad -(1 - w_i)(a - b)(1 - x_i), \quad i = 1, ..., n \\ a \le \overline{o_i} \le b, \quad i = 1, ..., n \end{cases}. \tag{2.81}$$

Based on (2.78), (2.79) and (2.81), all the constraints in model $P_{2.12}$ can be equivalently transformed into the constraints in model $P_{2.13}$. This completes the proof of Theorem 2.6.

According to Theorem 2.6, the MECM with OWA operator can be equivalently transformed into a mixed 0-1 linear programming model.

(3) Example 2.4

We continue with Example 2.1, and use MECMs to help experts to reach a consensus.

Let the associated weight vector of the aggregation operator be $w = (0.375, 0.1875, 0.25, 0.0625, 0.125)^T$.

(i) The use of the MECM with WA operator

We use the MECM with WA operator (i.e., $P_{2.11}$) to obtain the adjusted individual opinions, the collective group opinion, the experts within consensus and the total consensus cost.

Let the consensus threshold $\varepsilon = 0.8$, and let the consensus cost budget $B = 5$. Table 2.11 explores the adjusted individual opinions, the collective group opinion, the experts within consensus and the total consensus cost under different cost

Table 2.11 $P_{2.11}$ under different cost vectors in Example 2.4

$(c_1, c_2, c_3, c_4, c_5)$	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$	Experts within consensus	Total cost
(1, 4, 3, 5, 2)	1.743	1.743	2.5	3	6	2.543	$\{E_1, E_2, E_3, E_4\}$	4.215
(2, 4, 3, 1, 2.5)	1	1	1.3	2.6	6	1.8	$\{E_1, E_2, E_3, E_4\}$	5
(4, 1, 4, 2, 5)	0.5	1.15	2.5	2.75	6	1.95	$\{E_2, E_3, E_4\}$	0.65

Table 2.12 $P_{2.11}$ under different consensus thresholds in Example 2.4

ε	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$	Experts within consensus	Total cost
0.5	2	1	2.5	3	6	2.5	$\{E_1, E_3, E_4\}$	1.5
0.6	1.84	1	2.5	3	6	2.44	$\{E_1, E_3, E_4\}$	1.34
0.7	1.68	1	2.5	3	6	2.38	$\{E_1, E_3, E_4\}$	1.18
0.8	1.743	1.743	2.5	3	6	2.543	$\{E_1, E_2, E_3, E_4\}$	4.215

Table 2.13 $P_{2.11}$ under different cost budgets in Example 2.4

B	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$	Experts within consensus	Total cost
5	1.743	1.743	2.5	3	6	2.543	$\{E_1, E_2, E_3, E_4\}$	4.215
4	1.52	1	2.5	3	6	2.32	$\{E_1, E_3, E_4\}$	1.02
3	1.52	1	2.5	3	6	2.32	$\{E_1, E_3, E_4\}$	1.02
2	1.52	1	2.5	3	6	2.32	$\{E_1, E_3, E_4\}$	1.02

vectors $(c_1, c_2, c_3, c_4, c_5)^T$. Moreover, let the cost vector $(c_1, c_2, c_3, c_4, c_5)^T = (1, 4, 3, 5, 2)^T$, and let the consensus cost budget $B = 5$. Table 2.12 displays the adjusted individual opinions, the collective group opinion, the experts within consensus and the total consensus cost under different consensus thresholds ε. Finally, let the consensus threshold $\varepsilon = 0.8$, and let the cost vector $(c_1, c_2, c_3, c_5)^T = (1, 4, 3, 5, 2)^T$. Table 2.13 displays the adjusted individual opinions, the collective group opinion, the experts within consensus and the total consensus cost under different consensus cost budgets B.

(ii) The use of the MECM with OWA operator

We use the MECM with OWA operator (i.e., $P_{2.13}$) to obtain the adjusted individual opinions, the collective group opinion, the experts within consensus and the total consensus cost.

Let the consensus threshold $\varepsilon = 0.8$, and let the consensus cost budget $B = 5$. Table 2.14 displays the adjusted individual opinions, the collective group opinion, the experts within consensus and the total consensus cost under different cost

Table 2.14 $P_{2.13}$ under different cost vectors in Example 2.4

$(c_1, c_2, c_3, c_4, c_5)$	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$	Experts within consensus	Total cost
(1, 4, 3, 5, 2)	1.743	1.743	2.5	3	6	2.543	$\{E_1, E_2, E_3, E_4\}$	4.215
(2, 4, 3, 1, 2.5)	1	1	2.5	1.325	6	1.8	$\{E_1, E_2, E_3, E_4\}$	2.675
(4, 1, 4, 2, 5)	0.5	1.15	2.5	2.75	6	1.95	$\{E_2, E_3, E_4\}$	0.65

Table 2.15 $P_{2.13}$ under different consensus thresholds in Example 2.4

ε	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$	Experts within consensus	Total cost
0.5	2.875	1	2.5	3	6	2.5	$\{E_1, E_3, E_4\}$	2.375
0.6	2.467	1	2.5	3	6	2.4	$\{E_1, E_3, E_4\}$	1.967
0.7	1.933	1	2.5	3	6	2.3	$\{E_1, E_3, E_4\}$	1.433
0.8	1.743	1.743	2.5	3	6	2.543	$\{E_1, E_2, E_3, E_4\}$	4.214

Table 2.16 $P_{2.13}$ under different cost budgets in Example 2.4

B	$\overline{o_1}$	$\overline{o_2}$	$\overline{o_3}$	$\overline{o_4}$	$\overline{o_5}$	$\overline{o}$	Experts within consensus	Total cost
5	1.743	1.743	2.5	3	6	2.543	$\{E_1, E_2, E_3, E_4\}$	4.214
4	1.4	1	2.5	3	6	2.2	$\{E_1, E_3, E_4\}$	0.9
3	1.4	1	2.5	3	6	2.2	$\{E_1, E_3, E_4\}$	0.9
2	1.4	1	2.5	3	6	2.2	$\{E_1, E_3, E_4\}$	0.9

vectors $(c_1, c_2, c_3, c_4, c_5)^T$. Moreover, let the cost vector $(c_1, c_2, c_3, c_4, c_5)^T = (1, 4, 3, 5, 2)^T$, and let the consensus cost budget $B = 5$. Table 2.15 displays the adjusted individual opinions, the collective opinion, the experts within consensus and the total consensus cost under different consensus thresholds ε. Finally, let the consensus threshold $\varepsilon = 0.8$, and let the cost vector $(c_1, c_2, c_3, c_4, c_5)^T = (1, 4, 3, 5, 2)^T$. Table 2.16 displays the adjusted individual opinions, the collective group opinion, the experts within consensus and the total consensus cost under different consensus cost budgets B.

2.3 Comparison Analysis

In this section, a comparison analysis is performed to show the advantage of the proposed consensus models. We only compare the MECM with the consensus model based on IR and DR rules (IR-DR consensus model). Without loss of generality, the OWA operator is used to aggregate experts' opinions in the comparison analysis. When using the WA operator, the comparison results are similar. Meanwhile, when comparing the MCCM with the IR-DR consensus model, we can obtain the similar results.

2.3.1 Consensus Based on IR and DR Rules

Based on IR and DR [9, 14, 15] introduced in Chap. 1, we display an IR-DR consensus model. Based on IR, we identify the expert whose opinion has the biggest difference from the collective opinion and should change his/her opinion. Based on DR, we adjust the opinion, which needs to be changed based on IR, to make it reach the given consensus level with minimum cost. Moreover, the OWA operator is used to aggregate individual opinions into a collective opinion. This IR-DR consensus model is described as follows.

IR-DR consensus model

Input: The original opinion of the expert o_i $(i = 1, 2, ..., n)$, the associated weight vector of the aggregation operator $w = (w_1, w_2, ..., w_n)^T$, the consensus threshold ε, the cost vector $c = (c_1, c_2, ..., c_n)^T$, cost budget B, the maximum number of iterations $max\_rounds \geq 1$.

Output: the number of experts within consensus X

Step 1: Set $z = 0$, $o_i^z = o_i (i = 1, 2, ..., n)$ and $B^z = B$.

Step 2: Calculate the collective opinion $o_c^z = F_w^{OWA}(o_1^z, o_2^z, ..., o_n^z)$ and the consensus index $CL^z(E_i) = \left|o_i^z - o_c^z\right|$, $(i = 1, 2, ..., n)$. $CL^z(E_k) = \max\limits_i \{CL^z(E_i)\}$. Based on IR, the opinion of expert E_k has the biggest difference from the collective opinion. If $CL^z(E_k) \leq \varepsilon$ or $z \geq max\_rounds$, go to Step 4. Otherwise, go to the next step.

Step 3: Based on DR, we adjust the opinion of expert E_k to make it reach the given consensus threshold ε with minimum cost. $B^{z+1} = B^z - c_k \cdot |CL^z(E_k) - \varepsilon|$. If $B^{z+1} < 0$, go to Step 4. Otherwise,

$$o_k^z = \begin{cases} o_c^z + \varepsilon, & if\ o_k^z > o_c^z \\ o_c^z - \varepsilon, & if\ o_k^z \leq o_c^z \end{cases}$$

$o_i^{z+1} = o_i^z$, $z = z + 1$. Go to Step 2.

Step 4: Output the number of experts within consensus X.

2.3.2 Comparison Results

We consider three examples to compare the MECM with the IR-DR consensus model. The three examples are drawn from Examples 2.1–2.3, respectively.

(1) Comparison analysis 2.1

There are five experts in Example 2.1 and their initial opinions are

$$\{o_1, o_2, o_3, o_4, o_5\} = \{0.5, 1, 2.5, 3, 6\}.$$

Let the associated weight vector of the aggregation operator $w = (0.375, 0.1875, 0.25, 0.0625, 0.125)^T$. Firstly, let the cost budget $B = 5$, and let the cost vector $c = (1, 1, 1, 1, 1)^T$. Figure 2.1 shows the comparison results between the MECM and the IR-DR consensus model under different consensus thresholds ε. Then, let the cost vector $c = (1, 4, 3, 5, 2)^T$, and the consensus threshold $\varepsilon = 0.8$. Figure 2.2

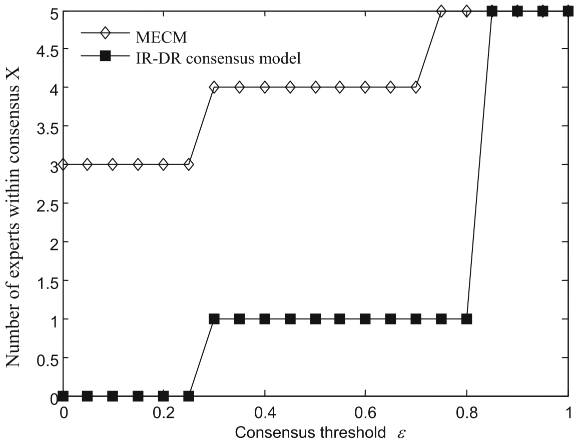

Fig. 2.1 Comparison between the MECM and the IR-DR consensus model under different consensus thresholds in comparison analysis 2.1

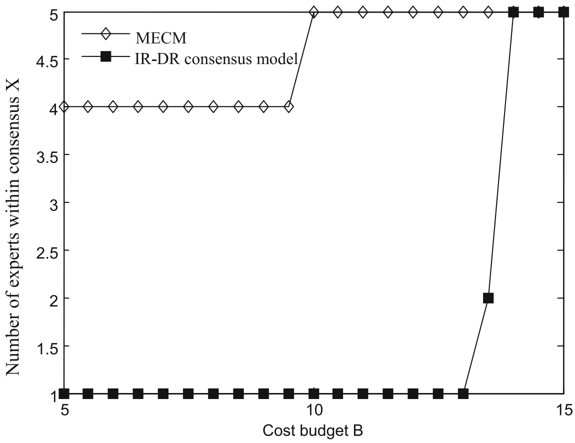

Fig. 2.2 Comparison between the MECM and the IR-DR consensus model under different cost budgets in comparison analysis 2.1

shows the comparison results between the MECM and the IR-DR consensus model under different cost budgets B.

(2) Comparison analysis 2.2

There are four experts in Example 2.2 and their initial opinions are

$$\{o_1, o_2, o_3, o_4\} = \{0, 3, 6, 10\}.$$

Let the associated weight vector of the aggregation operator $w = (0.3, 0.1, 0.4, 0.2)^T$. Firstly, let the cost budget $B = 5$, and let the cost vector $c = (1, 1, 1, 1)^T$. Figure 2.3 shows the comparison results between the MECM and the IR-DR consensus model under different consensus thresholds ε. Then, let the cost vector $c = (1, 4, 3, 5)^T$, and the consensus threshold $\varepsilon = 0.8$. Figure 2.4 shows the com-

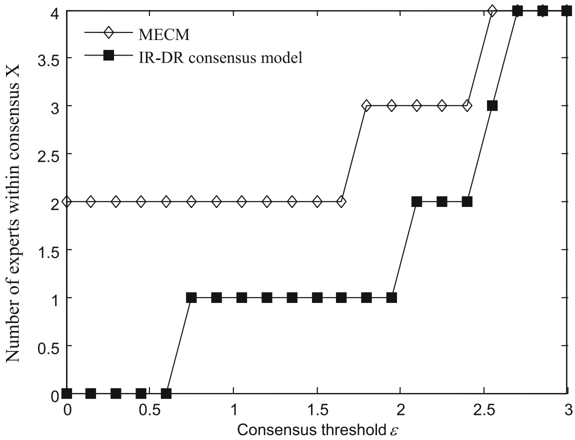

Fig. 2.3 Comparison between the MECM and the IR-DR consensus model under different consensus thresholds in comparison analysis 2.2

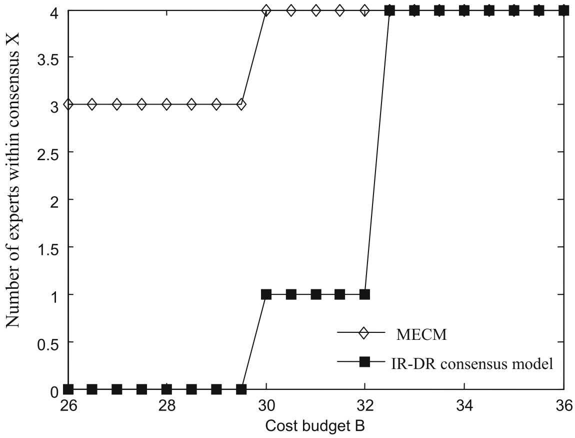

Fig. 2.4 Comparison between the MECM and the IR-DR consensus model under different cost budgets in comparison analysis 2.2

parison results between the MECM and the IR-DR consensus model under different cost budgets B.

(3) Comparison analysis 2.3

There are four experts in Example 2.3 and their initial opinions are

$$\{o_1, o_2, o_3, o_4\} = \{1, 3, 1, 3\}.$$

Let the associated weight vector of the aggregation operator $w = (0.2, 0.3, 0.25, 0.25)^T$. Firstly, let the cost budget $B = 2$, and let the cost vector $c = (1, 1, 1, 1)^T$. Figure 2.5 shows the comparison results between the MECM and the IR-DR consensus model under different consensus thresholds ε. Then, let the cost vector $c = (1, 2, 1, 1)^T$, and the consensus threshold $\varepsilon = 0.8$, and Fig. 2.6 shows the com-

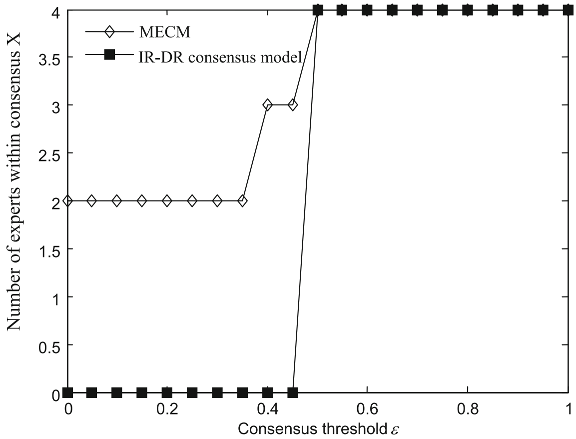

Fig. 2.5 Comparison between the MECM and the IR-DR consensus model under different consensus thresholds in comparison analysis 2.3

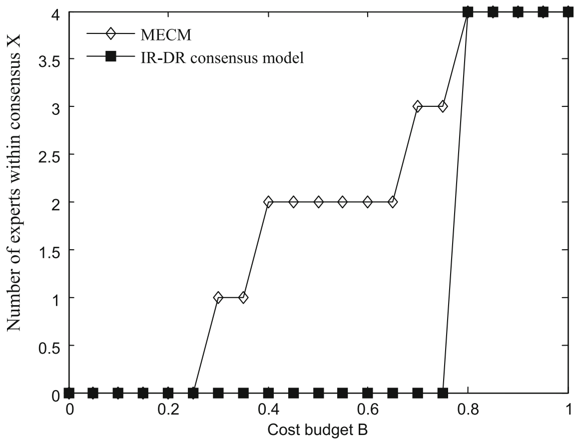

Fig. 2.6 Comparison between the MECM and the IR-DR consensus model under different cost budgets in comparison analysis 2.3

parison result between the MECM and the IR-DR consensus model under different cost budgets B.

In the above three comparison examples, the MECM can make more experts reach the consensus level under the given cost budget. As a result, when the cost budget is not enough to make all the experts reach the consensus level, the MECM can give a more effective aid for GDM problems.

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http://www.springer.com/978-981-287-890-8

Consensus Building in Group Decision Making
Searching the Consensus Path with Minimum Adjustments
Dong, Y.; Xu, J.
2016, XI, 201 p., Hardcover
ISBN: 978-981-287-890-8