

Chapter 2

Topological Spaces

Metric spaces are convenient for many applications in geometry and physics, but sometimes we need a broader framework: that of topological spaces. In this chapter we give a short introduction to this theory.

The results of the preceding chapter on continuous functions remain valid. On the other hand, the reader may observe the absence of *sequences*: they retain their usefulness only in special topological spaces.¹

Topological spaces are not suitable for the study of *uniform* continuity either.²

2.1 Definitions and Examples

Definitions By a *topology* on a non-empty set X we mean a family \mathcal{T} of subsets of X satisfying the following conditions:

- (a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
- (b) the intersection of *finitely many* sets of \mathcal{T} still belongs to \mathcal{T} ;
- (c) the union of an *arbitrary* subfamily of \mathcal{T} still belongs to \mathcal{T} .

If \mathcal{T} is a topology on X , then the pair (X, \mathcal{T}) is called a *topological space*, and the elements of \mathcal{T} are called the *open sets* of (X, \mathcal{T}) . When the topology is evident from the context, we speak simply of the open sets of X . The elements of X are also called *points*.

¹In the last, optional section of this chapter we introduce a generalization of sequences that is well suited for all topological spaces.

²The convenient framework for such studies is provided by the *uniform spaces*, see, e.g., Császár [118] or Kelley [273].

If \mathcal{T}, \mathcal{S} are two topologies on the same set X and $\mathcal{T} \subset \mathcal{S}$, then we say that \mathcal{T} is *coarser* or *weaker* than \mathcal{S} , or that \mathcal{S} is *finer* or *stronger* than \mathcal{T} .³

Examples Let X be a non-empty set.

- The family of *all* subsets of X is called the *discrete topology* on X : every subset of X is open. This is the finest topology on X .
- The family $\mathcal{T} = \{\emptyset, X\}$ is called the *antidiscrete topology* on X : there are only two open sets. This is the coarsest topology on X .

We introduce an important special class of topological spaces:

Definition A topological space X is *separated* or is a *Hausdorff space* if any two points $x, y \in X$ may be separated by disjoint open sets U and V :

$$x \in U, \quad y \in V \quad \text{and} \quad U \cap V = \emptyset.$$

Examples

- Every discrete topological space is a Hausdorff space.
- The antidiscrete topology on a set of at least two points is not separated.

Proposition 1.1 (p. 5) shows that the open sets of a metric space form a separated topology. Henceforth we associate this topology with the metric. Then we have the

Proposition 2.1 *Every metric space is a Hausdorff space.*

***Remarks**

- The topology associated with a discrete metric is the discrete topology, so that our terminology is consistent.
- Two metrics d_1 and d_2 on the same set X are called *equivalent* if they are associated with the same topology. This is the case, for example, if there exist two positive constants c_1, c_2 such that

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y) \tag{2.1}$$

for all $x, y \in X$.

- The condition (2.1) is sufficient, but not necessary for the equivalence. For example, on the set of positive integers the two metrics

$$d_1(x, y) = |x - y| \quad \text{and} \quad d_2(x, y) = |x^{-1} - y^{-1}|$$

define the same (discrete) topology, although (2.1) is not satisfied.

- Although the above two metrics define the same topology, d_1 is complete, while d_2 is not. This shows that completeness is not a topological property.

³We do not exclude equality.

- Not all Hausdorff spaces are metrizable⁴: see, e.g., the second example on p. 45 and the examples on p. 58. Other, natural examples appear in the study of weak topologies in *Functional Analysis*.

Next we define products and subspaces of topological spaces.

Proposition 2.2 *If \mathcal{T} is a topology on X and $Y \subset X$ is a non-empty set, then*

$$\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}$$

is a topology on Y .

Proof We check the three properties of topologies.

- (a) $\emptyset = \emptyset \cap Y \in \mathcal{T}_Y$ and $Y = X \cap Y \in \mathcal{T}_Y$.
- (b) If $U_i \in \mathcal{T}$ for $i = 1, \dots, n$, then

$$\bigcap_{i=1}^n (U_i \cap Y) = \left(\bigcap_{i=1}^n U_i \right) \cap Y \in \mathcal{T}_Y.$$

- (c) If $\{U_i\}_{i \in I}$ is an arbitrary subfamily of \mathcal{T} , then

$$\bigcup_{i \in I} (U_i \cap Y) = \left(\bigcup_{i \in I} U_i \right) \cap Y \in \mathcal{T}_Y.$$

□

Definition By a (*topological*) *subspace* of a topological space X we mean a non-empty subset $Y \subset X$ endowed with the topology \mathcal{T}_Y .⁵

If \mathcal{T}_i is a topology on X_i for $i = 1, \dots, m$, then we denote by \mathcal{B} the family of *base sets*

$$B = U_1 \times \cdots \times U_m$$

in $X := X_1 \times \cdots \times X_m$, where U_i runs over \mathcal{T}_i for each i .

Furthermore, we denote by \mathcal{T} the family of arbitrary (finite or infinite) unions of base sets:

$$\mathcal{T} = \{ \bigcup_{\alpha \in A} B^\alpha : B^\alpha \in \mathcal{B} \text{ for all } \alpha \in A \}.$$

Proposition 2.3 *\mathcal{T} is a topology on X .*

Proof We check again the three properties of topologies.

- (a) $\emptyset = \emptyset \times \cdots \times \emptyset \in \mathcal{T}$ and $X = X_1 \times \cdots \times X_m \in \mathcal{T}$.

⁴See, e.g., Császár [118] or Kelley [273] for the characterization of metrizable topological spaces.

⁵This is consistent with our terminology on metric spaces: the topology of a metric subspace coincides with the subspace topology.

(b) If $U_1^i \times \cdots \times U_m^i \in \mathcal{B}$ for $i = 1, \dots, n$, then

$$\bigcap_{i=1}^n (U_1^i \times \cdots \times U_m^i) = \left(\bigcap_{i=1}^n U_1^i \right) \times \cdots \times \left(\bigcap_{i=1}^n U_m^i \right) \in \mathcal{B}.$$

Next, if $U^1, \dots, U^n \in \mathcal{T}$, then each U^i has the form

$$U^i = \bigcup_{\alpha_i \in A_i} B^{\alpha_i}$$

with suitable sets $B^{\alpha_i} \in \mathcal{B}$. Hence

$$\bigcap_{i=1}^n U^i = \bigcup_{\alpha_1 \in A_1} \cdots \bigcup_{\alpha_n \in A_n} (B^{\alpha_1} \cap \cdots \cap B^{\alpha_n}) \in \mathcal{T}.$$

(c) Each U^i again has the form

$$U^i = \bigcup_{\alpha_i \in A_i} B^{\alpha_i}$$

with suitable sets $B^{\alpha_i} \in \mathcal{B}$. Hence

$$\bigcup_{i \in I} U^i = \bigcup_{i \in I} \bigcup_{\alpha_i \in A_i} B^{\alpha_i} \in \mathcal{T}.$$

□

Definition The pair (X, \mathcal{T}) is called the *(topological) product* of $(X_1, \mathcal{T}_1), \dots, (X_m, \mathcal{T}_m)$.⁶

Different topological spaces may have the same structure:

Definition X and Y are *homeomorphic* if there exists a bijection f between X and Y such that

$$U \text{ is open in } X \iff f(U) \text{ is open in } Y.$$

The map f is then called a *homeomorphism*.

It is clear that homeomorphism is an equivalence relation between topological spaces. Homeomorphic topological spaces have the same topological properties. For example, if X is compact, separable or separated, then every homeomorphic topological space is also compact, separable or separated.

⁶The topology of a product of metric spaces coincides with the product of the corresponding topologies. We will also define the product of infinitely many spaces in Sect. 2.4, p. 53.

The following result asserts that the topological product is essentially commutative, associative, and the projections are homeomorphisms.

Proposition 2.4 *Let X_1, X_2, X_3 be topological spaces.*

- (a) $X_1 \times X_2$ is homeomorphic to $X_2 \times X_1$.
- (b) $(X_1 \times X_2) \times X_3$ is homeomorphic to $X_1 \times X_2 \times X_3$.
- (c) $X_1 \times \{a_2\}$ is homeomorphic to X_1 for each fixed $a_2 \in X_2$.

Proof It follows from the definition of the product topology that the maps $(x_1, x_2) \mapsto (x_2, x_1)$, $((x_1, x_2), x_3) \mapsto (x_1, x_2, x_3)$ and $(x_1, a_2) \mapsto x_1$ are suitable homeomorphisms. \square

***Proposition 2.5**

- (a) *All subspaces of Hausdorff spaces are Hausdorff spaces.*
- (b) *The products of Hausdorff spaces are Hausdorff spaces.*

Proof

- (a) Let Y be a subspace of X , and $a, b \in Y$ two distinct points. Since X is separated, there exist two disjoint open sets U, V in X satisfying $a \in U$ and $b \in V$. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y , and $a \in U \cap Y, b \in V \cap Y$.
- (b) Let (X, \mathcal{T}) be the product of the Hausdorff spaces

$$(X_1, \mathcal{T}_1), \dots, (X_m, \mathcal{T}_m),$$

and $a = (a_1, \dots, a_m), b = (b_1, \dots, b_m)$ be two distinct points in X . There exists an index j such that $a_j \neq b_j$, and then there exist two disjoint open sets U_j, V_j in X_j such that $a_j \in U_j$ and $b_j \in V_j$. Then

$$U := \{(x_1, \dots, x_m) \in X : x_j \in U_j\}$$

and

$$V := \{(x_1, \dots, x_m) \in X : x_j \in V_j\}$$

are disjoint open sets in X with $a \in U$ and $b \in V$. \square

We define the closed sets in the same way as in metric spaces:

Definition A set in a topological space is *closed* if its complement is open.

The definition yields at once the following

Proposition 2.6 *Let X be a topological space.*

- (a) \emptyset and X are closed sets.
- (b) The union of finitely many closed sets is closed.
- (c) The intersection of any (finite or infinite) family of closed sets is closed.

The closed sets of (topological) subspaces are easily characterized:

Proposition 2.7 *Let Y be a subspace of a topological space X . The closed sets of Y are the sets $F \cap Y$ where F runs over the closed sets of X .*

Proof The closed sets of Y are the complements of the open sets of Y , i.e. the sets $Y \setminus (Y \cap U)$, where U runs over the open sets of X . Since

$$Y \setminus (Y \cap U) = Y \cap (X \setminus U),$$

we conclude by observing that $X \setminus U$ runs over the closed sets of X . \square

Definitions Let A be a set in a topological space X .

- The union of all open subsets $A \subset X$ is the largest open subset of A . It is called the *interior* of A , and is denoted by $\text{int} A$. Its elements are called the *interior points* of A .
- The intersection of all closed sets $F \supset A$ is the smallest closed set containing A as a subset. It is called the *closure* of A , and is denoted by \overline{A} .
- We have

$$\text{int} A \subset A \subset \overline{A}$$

by definition. The set $\partial A := \overline{A} \setminus \text{int} A$ is called the *boundary* of A . Its elements are called the *boundary points* of A .

- The union of all open sets disjoint from A is the largest open set disjoint from A . It is called the *exterior* of A , and is denoted by $\text{ext} A$. Its elements are called the *exterior points* of A .

Remarks

- It follows from the definitions that the sets $\text{ext} A$, $\text{int} A$, ∂A form a (disjoint) partition of X .
- Since $\overline{A} = X \setminus \text{ext} A$,

$$a \in \overline{A} \iff \text{each ball } B_r(a) \text{ meets } A.$$

By Lemma 1.3 there is a sequential characterization of \overline{A} in *metric spaces*:

$$a \in \overline{A} \iff A \text{ contains a sequence converging to } a.$$

- Since $\partial A = \overline{A} \cap (X \setminus \text{int} A)$ is the intersection of two closed sets, the boundary of a set is always closed.

In the rest of this chapter the letters X , Y , Z will always denote topological spaces.

2.2 Neighborhoods: Continuous Functions

In order to define continuity we introduce the neighborhoods of a point.

Definition Let $a \in X$ and $V \subset X$. We say that V is a *neighborhood* of a if there is an open set U in X satisfying $a \in U \subset V$.

Lemma 2.8 Let (X, d) and (Y, d') be two metric spaces, $f : X \rightarrow Y$ and $a \in X$. The following two properties are equivalent:

(a) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(B_\delta(a)) \subset B_\varepsilon(f(a))$, i.e.,

$$x \in X \quad \text{and} \quad d(x, a) < \delta \quad \implies \quad d'(f(x), f(a)) < \varepsilon;$$

(b) for each neighborhood V of $f(a)$, $f^{-1}(V)$ is a neighborhood of a .

Proof

(a) \implies (b). Since V is a neighborhood of $f(a)$, there exists an open set U in Y such that $f(a) \in U \subset V$. By the definition of open sets in a metric space there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subset U$. Choose δ according to property (a), then $f(B_\delta(a)) \subset B_\varepsilon(f(a))$, so that

$$a \in B_\delta(a) \subset f^{-1}(B_\varepsilon(f(a))) \subset f^{-1}(U) \subset f^{-1}(V).$$

Since the ball $B_\delta(a)$ is open, $f^{-1}(V)$ is a neighborhood of a in X by definition.

(b) \implies (a). For any fixed $\varepsilon > 0$ the open ball $B_\varepsilon(f(a))$ is a neighborhood of $f(a)$ in Y by definition. Then $f^{-1}(B_\varepsilon(f(a)))$ is a neighborhood of a in X by property (b). There exists therefore an open set U satisfying $a \in U \subset f^{-1}(B_\varepsilon(f(a)))$. Using the definition of open sets in a metric space, there exists a $\delta > 0$ such that $B_\delta(a) \subset U$. Then $B_\delta(a) \subset f^{-1}(B_\varepsilon(f(a)))$, whence $f(B_\delta(a)) \subset B_\varepsilon(f(a))$. \square

Comparing Lemmas 1.5 and 2.8 we see that the following definitions are consistent with those given earlier for metric spaces:

Definitions A function $f : X \rightarrow Y$ is *continuous* at $a \in X$ if for each neighborhood V of $f(a)$ (in Y), its inverse image $f^{-1}(V)$ is a neighborhood of a (in X).

The function f is *continuous* if it is continuous at every $a \in X$.

The earlier results on the continuity of composite functions remain valid:

Proposition 2.9 Consider two functions $g : X \rightarrow Y$ and $f : Y \rightarrow Z$.

- (a) If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a .
- (b) If f and g are continuous, then $f \circ g$ is continuous.

Proof

- (a) If V is a neighborhood of $(f \circ g)(a) = f(g(a))$ in Z , then $f^{-1}(V)$ is a neighborhood of $g(a)$ in Y because f is continuous at $g(a)$, and then $g^{-1}(f^{-1}(V))$ is a neighborhood of a in X because g is continuous at a . This proves our claim because

$$(f \circ g)^{-1}(V) = g^{-1}(f^{-1}(V)).$$

- (b) Apply (a) to each $a \in X$. □

There is an elegant characterization of continuity in terms of open or closed sets:

Proposition 2.10 (Hausdorff) *Given a function $f : X \rightarrow Y$, the following properties are equivalent:*

- (a) f is continuous;
- (b) the inverse image $f^{-1}(U) \subset X$ of each open set $U \subset Y$ is open;
- (c) the inverse image $f^{-1}(F) \subset X$ of each closed set $F \subset Y$ is closed.

As a consequence, a bijection f between X and Y is a homeomorphism if and only if both f and f^{-1} are continuous.

Proof

(a) \Rightarrow (b). Let U be an open set in Y . We have to show that $f^{-1}(U)$ is open in X , i.e., it is a neighborhood of each $a \in f^{-1}(U)$. Since U is open and $f(a) \in f(f^{-1}(U)) \subset U$, U is a neighborhood of $f(a)$. Using our assumption we conclude that $f^{-1}(U)$ is a neighborhood of a .

(b) \Rightarrow (a). If V is a neighborhood of some point $f(a) \in Y$, then there exists an open set U in Y such that $f(a) \in U \subset V$. It follows that $a \in f^{-1}(U) \subset f^{-1}(V)$. Since $f^{-1}(U)$ is open in X by (b), $f^{-1}(V)$ is a neighborhood of a in X .

(b) \Rightarrow (c). If F is closed in Y , then $Y \setminus F$ is open in Y , and therefore $f^{-1}(Y \setminus F)$ is open in X by assumption. Using the equality

$$f^{-1}(F) = X \setminus f^{-1}(Y \setminus F)$$

we conclude that $f^{-1}(F)$ is closed.

(c) \Rightarrow (b). If U is open in Y , then $Y \setminus U$ is closed in Y , and therefore $f^{-1}(Y \setminus U)$ is closed in X by assumption. But then its complement

$$f^{-1}(U) = X \setminus f^{-1}(Y \setminus U)$$

is open.

The last assertion follows from the equivalence (a) \iff (b). □

Next we generalize two more notions from the theory of metric spaces.

Definitions Let X be a topological space.

- A set $D \subset X$ is *dense* if it meets every non-empty open set, i.e., if $\overline{D} = X$.⁷
- X is *separable* if it contains a countable dense set.

Two properties in Proposition 1.30 (p. 29) remain valid:

Corollary 2.11 *Let X, Y be topological spaces and $f : X \rightarrow Y$ a continuous map onto Y .*

- (a) *If D is dense in X , then $f(D)$ is dense in Y .*
- (b) *If X is separable, then Y is also separable.*

Proof

- (a) Given any non-empty open set V in Y , its preimage $f^{-1}(V)$ is a *non-empty* open set in X . By our assumption $f^{-1}(V) \cap D$ contains at least one point x , and then $f(x) \in V \cap f(D)$.
- (b) If D is a countable dense set in X , then $f(D)$ is a countable dense set in Y by (a). \square

The two other properties in Proposition 1.30 may fail in topological spaces:

Examples

- The empty set and the sets containing 0 form a separable topology on \mathbb{R} because the one-point set $\{0\}$ is dense. But its subspace $\mathbb{R} \setminus \{0\}$ is not separable because it is an uncountable set endowed with the discrete topology, so that no countable set is dense.⁸
- A *compact Hausdorff* topology is defined on \mathbb{R} by the complements of the finite sets and the subsets of $\mathbb{R} \setminus \{0\}$. Then the closed sets are the finite sets and the sets containing 0. Hence $\overline{D} \subset D \cup \{0\}$ for all sets, and therefore the closure of any countable set is also countable. Since \mathbb{R} is uncountable, no countable set is dense.

Corollary 2.12 *Let $f : X \rightarrow \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$.*

- (a) *If U is an open set in X , then the sets*

$$\{x \in U : f(x) < c\}, \quad \{x \in U : f(x) > c\}, \quad \{x \in U : f(x) \neq c\}$$

are open in X .

- (b) *If F is a closed set in X , then the sets*

$$\{x \in F : f(x) \leq c\}, \quad \{x \in F : f(x) \geq c\}, \quad \{x \in F : f(x) = c\}$$

are closed in X .

⁷This definition is consistent with the definition of density in metric spaces; see pp. 18 and 42.

⁸All sets are closed in the discrete topology.

Proof Introducing the open interval $I := (-\infty, c)$, the set

$$\{x \in U : f(x) < c\}$$

is the intersection of the open sets $f^{-1}(I)$ and U , hence it is also open.

The other proofs are similar. \square

Before giving an important example we generalize a classical theorem stating that the *uniform limit of a sequence of continuous functions is also continuous*:

Proposition 2.13 *Let K be a topological space, (X, d) a metric space, and (f_n) a sequence of functions $f_n : K \rightarrow X$. Assume that (f_n) converges uniformly to some function $f : K \rightarrow X$.*

If each f_n is continuous at some point $a \in K$, then f is also continuous at a .

Proof For any given $\varepsilon > 0$ we choose n such that

$$d(f(t), f_n(t)) < \varepsilon/3 \quad \text{for all } t \in K,$$

and then we choose a small neighborhood U of a such that

$$d(f_n(t), f_n(a)) < \varepsilon/3 \quad \text{for all } t \in U.$$

Then

$$\begin{aligned} d(f(t), f(a)) &\leq d(f(t), f_n(t)) + d(f_n(t), f_n(a)) + d(f_n(a), f(a)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for every point $t \in U$. \square

Example Let K be a topological space and X a metric space. By the preceding proposition the *continuous* and *bounded* functions $f : K \rightarrow X$ form a *closed* subspace $C_b(K, X)$ of the metric space $\mathcal{B}(K, X)$. Consequently, if X is complete, then $C_b(K, X)$ is also complete.

When $X = \mathbb{R}$ we often write $C_b(K)$ instead of $C_b(K, X)$ for brevity.

2.3 Connectedness

The intervals C of the real line are connected in the sense that whenever $x, z \in C$ and $x < y < z$, we also have $y \in C$. There is also a topological characterization:

Proposition 2.14 *A non-empty set $A \subset \mathbb{R}$ is an interval if and only if in its subspace topology only A and \emptyset are both open and closed.*

Proof If A is not an interval, then there exist points $x < y < z$ such that $x, z \in A$, but $y \notin A$. Then $B := \{a \in A : a < y\}$ is a non-empty proper subset of A . Since the relation $y \notin A$ implies that

$$B = A \cap (-\infty, y) = A \cap (-\infty, y],$$

B is both open and closed by the definition of the subspace topology and by Proposition 2.7 (p. 42).

Now let B be a non-empty proper subset of an interval A . Fix two points $x \in B$ and $z \in A \setminus B$. Assume that $x < z$ (the other case is similar), and consider the point $y = \sup \{a \in B : a < z\}$. Then $x \leq y \leq z$, so that $y \in A$. If $y \notin B$, then B is not closed. If $y \in B$, then $y < z$ and $(y, z] \cap B = \emptyset$, so that B is not open. \square

The preceding proposition suggests the following

Definition A topological space X (and its topology) is *connected* if only X and \emptyset are both open and closed.

Equivalently, X is connected if every non-empty proper subset of X has a boundary point.

A subset of X is *connected* if it is empty, or if its subspace topology is connected.

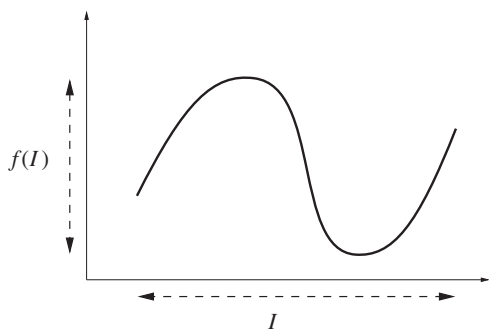
Examples

- Every antidiscrete topological space is connected.
- The discrete topological spaces having at least two points are not connected.

A well-known theorem of Bolzano states that continuous images of intervals are also intervals: see Fig. 2.1. More generally, *continuous images of connected sets are also connected*:

Theorem 2.15 Let $f : X \rightarrow Y$ be a continuous function. If X is connected, then $f(X)$ is also connected.

Fig. 2.1 Bolzano's theorem



Proof We have to show that if B is a non-empty, open and closed subset of the subspace $f(X)$, then $B = f(X)$. Since f is continuous, $f^{-1}(B)$ is a non-empty, open and closed subset of X . Since X is connected, this implies that $f^{-1}(B) = X$. We conclude by using the relations

$$f(X) = f(f^{-1}(B)) \subset B \subset f(X).$$

□

Let us prove some basic properties of connected sets:

Proposition 2.16

- (a) Let $\{A_\alpha\}_{\alpha \in I}$ be a (finite or infinite) family of connected sets in a topological space. If $\cap A_\alpha \neq \emptyset$, then $\cup A_\alpha$ is connected.
- (b) The product of finitely many connected topological spaces is connected.
- (c) The closure of a connected set is connected.

Proof

- (a) Let C be a non-empty, open and closed set in $\cup A_\alpha$. We have to show that $A_\alpha \subset C$ for each α .

If C meets one of the sets A_α , then $A_\alpha \subset C$. Indeed, $C \cap A_\alpha$ is a non-empty, open and closed set in the subspace topology of A_α . Since A_α is connected, $C \cap A_\alpha = A_\alpha$, i.e., $A_\alpha \subset C$.

Since C is non-empty, it meets at least one of the sets A_β . Then it contains this set, and hence it contains $\cap A_\alpha \neq \emptyset$ as well. Therefore C meets each A_α , and thus $A_\alpha \subset C$ for each α .

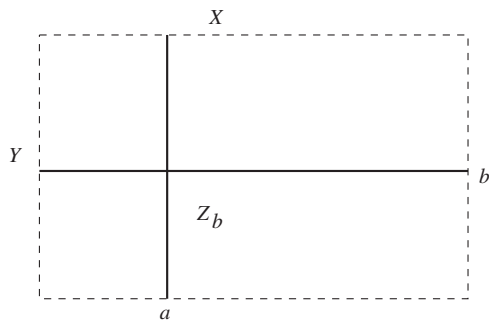
- (b) Using induction on the number of factors, by Proposition 2.4 (b) (p. 41) it suffices to prove that the product Z of two connected sets X and Y is connected.

Fix $a \in X$ arbitrarily, and consider the sets

$$Z_b = (\{a\} \times Y) \cup (X \times \{b\}), \quad b \in Y;$$

see Fig. 2.2.

Fig. 2.2 A product of connected spaces



By Proposition 2.4 (c) the subspaces $\{a\} \times Y$ and $X \times \{b\}$ of Z are homeomorphic to Y and X , respectively, and hence connected. Since they have a common point (a, b) , Z_b is connected for all b by (a).

The intersection of the sets Z_b contains $\{a\} \times Y$, hence it is non-empty. Applying (a) again we conclude that $Z = \cup_{b \in Y} Z_b$ is connected.

- (c) Let C be a non-empty, open and closed set in \bar{A} . We have to show that $C = \bar{A}$. Since \bar{A} is the smallest closed set containing A as a subset, it suffices to show that $A \subset C$.

C is a non-empty open set in \bar{A} , hence it meets A by the definition of the closure. Therefore $C \cap A$ is non-empty, open and closed in A . Since A is connected, we conclude that $C \cap A = A$, i.e., $A \subset C$. \square

Remarks

- We say that two points of a topological space X *may be connected* if they belong to some connected set. By the preceding proposition this is an equivalence relation, and the corresponding equivalence classes are connected and closed. They are the maximal connected sets of X , called its *connected components*.
- Each open and closed subset A of X is a union of connected components. Indeed, if A meets a component C , then $A \cap C$ is a non-empty open and closed subset of the subspace C , and hence equal to C by its connectedness. Equivalently, $C \subset A$.
- A connected component is not necessarily open. To see this we consider the subspace of \mathbb{R} formed by 0 and the numbers $1/n$, $n = 1, 2, \dots$. The one-point sets $\{1/n\}$ are both open and closed, hence connected components by our preceding remark. Then the remaining set $\{0\}$ is also a connected component, and it is not open.

We end this section by giving a useful and intuitive *sufficient* condition for connectedness.

Definition

- Two points a and b in a topological space X *may be connected by a path* if there exists a continuous function $f : [0, 1] \rightarrow X$ satisfying $f(0) = a$ and $f(1) = b$. The function f is also called a *path*.
- X is *pathwise connected* if any two points of X may be connected by a path.

Proposition 2.17 *Every pathwise connected space is connected.*

Proof Fix $a \in X$ arbitrarily, and choose for each $x \in X$ a path f_x connecting a and x . Their ranges have a common point a , hence their union, i.e., the space X , is connected by Theorem 2.15 and Proposition 2.16 (a). \square

2.4 * Compactness

Compactness may be defined in every topological space. Moreover, all results of Sect. 1.4 remain valid except those using the notions of metric, completeness, boundedness or complete boundedness. However, new proofs are needed that avoid the use of sequences.

Definition A set K in a topological space X is *compact* if every open cover of K has a finite subcover.

If X itself is compact, then X is called a *compact topological space*.

Remark Theorem 1.28 (p. 28) shows that the above definition is consistent with the earlier one for metric spaces. The other two properties of Theorem 1.28 cannot be used here. Indeed, as we observed on page 38, completeness is not a topological property. Furthermore, we should modify the definition of cluster points to avoid the countable nature of sequences.⁹

Proposition 2.18

- (a) *In a compact topological space all closed subsets are compact.*
- (b) *The product of finitely many compact topological spaces is compact.*

Proof

- (a) Let F be a closed subset of a compact topological space X , and $\{U_i\}$ an open cover of F . Then adding $X \setminus F$ we obtain an open cover of X . By assumption the latter has a finite subcover:

$$X = U_{i_1} \cup \dots \cup U_{i_n}$$

or

$$X = (X \setminus F) \cup U_{i_1} \cup \dots \cup U_{i_n}.$$

In both cases we have

$$F \subset U_{i_1} \cup \dots \cup U_{i_n},$$

so that $\{U_i\}$ has a finite subcover of F .

- (b) Using induction on the number of factors, by Proposition 2.4 (b) (p. 41) it suffices to consider the product $Z = X \times Y$ of two compact topological spaces. Let $\{W_\alpha\}$ be an open cover of Z .

⁹See Sect. 2.5 for such a modification.

By the definition of the product topology each point $z = (x, y) \in Z$ has a neighborhood $W'_z = U_z \times V_z$ such that $W'_z \subset W_\alpha$ for some $\alpha = \alpha(z)$. Then $\{W'_z : z \in Z\}$ is an open cover of Z . It suffices to find a finite subcover

$$Z = W'_{z_1} \cup \dots \cup W'_{z_n}$$

of the latter because then we will also have

$$Z = W_{\alpha(z_1)} \cup \dots \cup W_{\alpha(z_n)}.$$

Fix $a \in X$ arbitrarily and consider the sets $W'_z = U_z \times V_z$ meeting $\{a\} \times Y$. The corresponding open sets V_z cover the compact set Y , hence there exists a finite subcover: $Y = \bigcup_{i=1}^{k(a)} V_{a, z_i}$. Setting $U'_a := \bigcap_{i=1}^{k(a)} U_{a, z_i}$, we also have $U'_a \times Y \subset \bigcup_{i=1}^{k(a)} W'_{a, z_i}$.

The sets U'_a form an open cover of the compact set X , hence there exists a finite subcover $X = \bigcup_{j=1}^m U_{a_j}$. Then

$$Z = \bigcup_{j=1}^m \bigcup_{i=1}^{k(a_j)} W'_{a_j, z_i}$$

as required. □

Proposition 2.19 *In Hausdorff spaces every compact set is closed.*

Example The Hausdorff property cannot be omitted: for example, in an antidiscrete topological space X all sets are compact, but only \emptyset and X are closed.

Proof Let K be a compact set in a Hausdorff space X . For any fixed point $a \in X \setminus K$ we have to find an open set U satisfying $a \in U \subset X \setminus K$: this will prove that $X \setminus K$ is open, i.e., K is closed.

By the separation hypothesis, for each point $b \in K$ there exist open sets U_b and V_b satisfying $a \in U_b$, $b \in V_b$ and $U_b \cap V_b = \emptyset$. Since K is compact, the open cover $K \subset \bigcup_{b \in K} V_b$ has a finite subcover, say

$$K \subset V_{b_1} \cup \dots \cup V_{b_n}.$$

Then $U := U_{b_1} \cap \dots \cap U_{b_n}$ is an open neighborhood of a , and $U \cap K = \emptyset$ because U does not meet any of the sets V_{b_i} . □

Proposition 2.20 (Cantor's Intersection Theorem) *Let (F_n) be a non-increasing sequence of non-empty compact sets in a topological space X . Then the sets F_n have at least one common point.*¹⁰

¹⁰By Proposition 2.18 (a) the hypotheses are satisfied if (F_n) is a non-increasing sequence of non-empty *closed* sets in a *compact* topological space.

Proof Assume on the contrary that $\cap F_n = \emptyset$. Then their complements form an open cover of X , and hence also of F_1 . Since F_1 is compact, it has a finite subcover. Furthermore, since the complements form a non-decreasing sequence, we have $F_1 \subset X \setminus F_m$ for a sufficiently large index m . Since $F_m \subset F_1$, this implies that F_m is empty, contradicting our assumption. \square

Theorem 2.21 (Hausdorff) *Let $f : X \rightarrow Y$ be a continuous function. If K is a compact set in X , then $f(K)$ is compact in Y .*

Proof If $\{U_i\}$ is an open cover of $f(K)$ in Y , then $\{f^{-1}(U_i)\}$ is an open cover of K in X by the continuity of f . Since K is compact, there exists a finite subcover:

$$K \subset f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_n}).$$

Hence

$$f(K) \subset U_{i_1} \cup \dots \cup U_{i_n},$$

i.e., $\{U_i\}$ contains a finite subcover of $f(K)$. \square

Theorem 2.22 (Weierstrass) *If $f : K \rightarrow \mathbb{R}$ is a continuous function on a compact topological space K , then f is bounded; moreover, it has maximal and minimal values.*

The result may be deduced from the preceding theorem: the set $f(K) \subset \mathbb{R}$ is compact, hence bounded and closed. Since K and hence $f(K)$ is non-empty, it has a maximal and a minimal element.

Let us also give a direct proof:

Proof Assume on the contrary that $\alpha := \inf_K f$ is not attained (the proof for $\sup_K f$ is similar).

Then the open sets

$$U_n := \left\{ x \in K : f(x) > \alpha + \frac{1}{n} \right\}, \quad n = 1, 2, \dots$$

cover K . By the compactness of K there exists a finite subcover. Since (U_n) is a non-decreasing set sequence, we have $K = U_N$ for some N . This implies that $f(x) > \alpha + N^{-1}$ for all $x \in K$, contradicting the definition of α . \square

Remark Consider the spaces $C_b(K, X)$ (p. 46) where K is compact. Then every continuous function $f : K \rightarrow X$ is bounded, so that we often write $C(K, X)$ and $C(K)$ instead of $C_b(K, X)$ and $C_b(K)$. Furthermore, we may write

$$d_\infty(f, g) = \max_{t \in K} d_\infty(f(t), g(t))$$

instead of \sup for all $f, g \in C(K, X)$.

The following sufficient condition is often used to check that a given map is a homeomorphism:

Proposition 2.23 (Hausdorff) *Let $f : X \rightarrow Y$ be a continuous bijection of a compact topological space onto a Hausdorff space. Then f is a homeomorphism.*

Proof We have to prove that the inverse map $f^{-1} : Y \rightarrow X$ is continuous. Equivalently, we prove that if A is a closed set in X , then $f(A)$ is closed in Y .

If A is closed in X , then it is also compact because X is compact. Then the continuous image $f(A)$ of A is also compact. Finally, since Y is a Hausdorff space, $f(A)$ is closed in Y . \square

Example By a *simple plane curve* we mean the range of a continuous one-to-one function $f : [0, 1] \rightarrow \mathbb{R}^2$. Such curves are homeomorphic to the interval $[0, 1]$. Indeed, $[0, 1]$ is compact, and a simple plane curve, as a subspace of the Hausdorff space \mathbb{R}^2 is also a Hausdorff space.

Remark In view of the importance of compactness we remark that every topological space may be considered as a dense subspace of a compact topological space. There are usually many ways to do this.¹¹

We may also define infinite products of topological spaces:

Definition Let $(X_i)_{i \in I}$ be an arbitrary family of topological spaces. Let us denote by X the set of points of the form $x = (x_i)_{i \in I}$ where $x_i \in X_i$ for each $i \in I$.¹² We endow X with the following topology: a set $U \subset X$ belongs to \mathcal{T} if for each $a \in U$ there exists a *finite* index set $J \subset I$ and for each $j \in J$ an open set U_j in X_j such that

$$a \in \{x \in X : x_j \in U_j \text{ for all } j \in J\} \subset U.$$

One can readily check that \mathcal{T} is indeed a topology on X , and that for finite index sets I this definition is equivalent to the earlier one.

This notion is justified by the following important theorem:

Theorem 2.24 (Tychonoff) *An arbitrary product of compact topological spaces is compact.*

For the proof we need two lemmas on a generalization of the method of induction to the uncountable case and on the verification of compactness by considering only special open covers.

¹¹See Exercise 2.8 (p. 63) for an example, and Császár [118] or Kelley [273] for a general treatment.

¹²More precisely, but less intuitively, we could consider the functions $x : I \rightarrow X$ defined by the formula $x(i) := x_i$.

Definitions Let \mathcal{A} be a family of sets and $\mathcal{B} \subset \mathcal{A}$ a subfamily. We say that

- $A \in \mathcal{A}$ is a *maximal* element if it is not a proper subset of another element of \mathcal{A} ;
- \mathcal{B} is *monotone* if for any two sets $B_1, B_2 \in \mathcal{B}$ we have either $B_1 \subset B_2$ or $B_2 \subset B_1$;
- $A \in \mathcal{A}$ is an *upper bound* of \mathcal{B} if $B \subset A$ for all $B \in \mathcal{B}$.
- We similarly define the *lower bounds* and the *minimal* elements.

The following lemma is equivalent to the *axiom of choice* in set theory:

Lemma 2.25 (Zorn) *Let \mathcal{A} be a family of sets. If every monotone subfamily has an upper bound, then \mathcal{A} has at least one maximal element.*

Similarly, if every monotone subfamily has a lower bound, then \mathcal{A} has at least one minimal element.

Remark Applying the hypotheses of the lemma to the empty subfamily we see that \mathcal{A} has at least one element.

Proof See, e.g., Hajnal and Hamburger [218] or Kelley [273]. □

Definition A family \mathcal{S} of open sets in a topological space X is a *subbase* if for each open set V and for each $a \in V$ there exist finitely many sets $S_1, \dots, S_n \in \mathcal{S}$ such that

$$a \in S_1 \cap \dots \cap S_n \subset V.$$

Example The unbounded open intervals form a subbase in \mathbb{R} .

***Proposition 2.26 (Alexander)** *Let \mathcal{S} be a subbase in a topological space X . If every cover $\mathcal{U} \subset \mathcal{S}$ of X has a finite subcover, then X is compact.*

Proof Assume on the contrary that every cover $\mathcal{U} \subset \mathcal{S}$ of X has a finite subcover, but X is non-compact. Then it has an open cover without any finite subcover.

The family of such open covers satisfies the hypothesis of Zorn's lemma. Indeed, the union of any monotone subfamily is still an open cover without any finite subcover. For otherwise the finite subcover would already belong to some open cover of the subfamily. Applying Zorn's lemma we obtain the existence of a *maximal* open cover \mathcal{V} without any finite subcover.

In particular, $\mathcal{V} \cap \mathcal{S}$ has no finite subcover, and therefore it cannot cover X by our assumption on the subbase. We may thus fix a point $a \in X$ not covered by these sets, and then an open set $V \in \mathcal{V}$ containing a . Finally, since \mathcal{S} is a subbase, we may choose finitely many sets $S_1, \dots, S_n \in \mathcal{S}$ such that

$$a \in S_1 \cap \dots \cap S_n \subset V.$$

Since $\mathcal{V} \cap \mathcal{S}$ does not cover a , none of the sets S_j belong to \mathcal{V} . Therefore, using the maximality of \mathcal{V} , $\{S_j\} \cup \mathcal{V}$ already has a finite subcover for each j . There exists

therefore finitely many sets in \mathcal{V} such that their union A_j satisfies the equality $S_j \cup A_j = X$. Then

$$X = (S_1 \cap \cdots \cap S_n) \cup (A_1 \cup \cdots \cup A_n) \subset V \cup A_1 \cup \cdots \cup A_n,$$

contradicting our hypothesis that \mathcal{V} has no finite subcover. \square

**Proof of Theorem 2.24* Introducing the *projections* $\pi_i : X \rightarrow X_i$, $\pi_i(x) := x_i$, by the definition of the product topology the family

$$\mathcal{S} := \{\pi_i^{-1}(U_i) : U_i \text{ is open in } X_i, i \in I\}$$

is a subbase of X . Therefore it suffices to prove that every cover $\mathcal{U} \subset \mathcal{S}$ has a finite subcover.

For each $i \in I$ we denote by \mathcal{U}_i the family of open sets U_i in X_i satisfying $\pi_i^{-1}(U_i) \in \mathcal{U}$. There exists an index $k \in I$ such that \mathcal{U}_k covers X_k . For otherwise there would exist a point $x \in X$ such that $\pi_i(x)$ is not covered by \mathcal{U}_i for any i . Then \mathcal{U} would not cover x , although it covers the whole space X by assumption.

If \mathcal{U}_k covers X_k , then the compactness of the latter implies the existence of a finite subcover

$$X_k = U_1 \cup \cdots \cup U_n.$$

This implies the equality

$$X = \pi_k^{-1}(U_1) \cup \cdots \cup \pi_k^{-1}(U_n);$$

we have thus found a finite subcover of \mathcal{U} . \square

2.5 * Convergence of Nets

Due to their countable character, sequences are less useful in topological spaces than in metric spaces.

Example Let X be an uncountable set (for example $X = \mathbb{R}$). The empty set and the complements of the countable sets form a topology on X . Although it is different from the discrete topology, the convergent sequences are the same in both of them: the eventually constant sequences.

The following notion may replace sequences in all topological spaces.

Definitions

- By a *net* in a set X ¹³ we mean a function $x : I \rightarrow X$ defined on a *directed set*, i.e., a non-empty set I endowed with a partial order having the following properties:
 - $i \geq j$ for all $i \in I$;
 - if $i \geq j$ and $j \geq k$, then $i \geq k$;
 - for all $i, j \in I$ there exists a $k \in I$ such that $k \geq i$ and $k \geq j$.

Usually we write x_i instead of $x(i)$, and (x_i) or $(x_i)_{i \in I}$ instead of x .

- We say that a net $(x_i)_{i \in I}$ *eventually* satisfies some condition (or satisfies this condition *for all sufficiently large i*) if there exists a $j \in I$ such that x_i satisfies this condition for all $i \geq j$.
- A net $(x_i)_{i \in I}$ in X *converges* to $a \in X$ if for each neighborhood U of a , $x_i \in U$ for all sufficiently large i . We also say in this case that a is a *limit* of (x_i) , and we write $x_i \rightarrow a$, $\lim x_i = a$ or $\lim (x_i)_{i \in I} = a$.
- We say that a net $(x_i)_{i \in I}$ *often* satisfies some condition if for each $j \in I$ there exists an $i \geq j$ such that x_i satisfies this condition.
- A net $(x_i)_{i \in I}$ in a *metric space* (X, d) is a *Cauchy net* if

$$\text{diam } \{x_j : j \geq i\} \rightarrow 0$$

in \mathbb{R} . Equivalently, this means that the net $(d(x_i, x_j))_{(i,j) \in I \times I}$ converges to 0, where $I \times I$ is endowed with the following order relation:

$$(i, j) \geq (k, \ell) \quad \text{if} \quad i \geq k \text{ and } j \geq \ell.$$

Examples

- Sequences correspond to the case $I = \mathbb{N}$ with the usual ordering.
- Consider the following partial ordering of the set \mathcal{U} of neighborhoods of a point a of a topological space:

$$U \geq V \iff U \subset V.$$

If we choose a point $x_U \in U$ for each $U \in \mathcal{U}$, then we obtain a net $(x_U)_{U \in \mathcal{U}}$ converging to a .

- Every Cauchy sequence is also a Cauchy net.

Now we generalize some results on sequences:

Proposition 2.27 *Let X and Y be topological spaces, $a \in X$ and $D, A \subset X$.*

- X is separated if and only if every net in X has at most one limit.*
- $a \in \overline{D} \iff D$ contains a net converging to a .*

¹³This notion is equivalent to the more elegant but less transparent notion of a *filter*. See Császár [118].

(c) *A is closed in X if and only if*

$$(x_i) \subset A \quad \text{and} \quad x_i \rightarrow a \implies a \in A.$$

(d) *A function $f : X \rightarrow Y$ is continuous at $a \in X$ if and only if*

$$x_i \rightarrow a \quad \text{in } X \implies f(x_i) \rightarrow f(a) \quad \text{in } Y.$$

(e) *A metric space is complete \iff every Cauchy net is convergent.*

Proof

- (a) Let X be separated and $x_i \rightarrow a$ in X . We show that $x_i \not\rightarrow b$ if $b \neq a$. For this we separate a and b by two disjoint neighborhoods U and V . Since $x_i \rightarrow a$, we have $x_i \in U$ for all sufficiently large i , say $i \geq j$. Then $x_i \notin V$ for all $i \geq j$, so that $x_i \not\rightarrow b$.

On the other hand, if X is not separated, then there exist two points a and b such that each neighborhood U of a meets each neighborhood V of b . Let us consider the set I of all such pairs (U, V) of neighborhoods with the following partial ordering:

$$(U, V) \geq (U_0, V_0) \iff U \subset U_0 \quad \text{and} \quad V \subset V_0.$$

Choosing a point $x_i \in U \cap V$ for each $i := (U, V) \in I$ we have $x_i \rightarrow a$ and $x_i \rightarrow b$.

- (b) If a net $(x_i) \subset D$ converges to a , then each neighborhood U of a contains x_i if i is sufficiently large, so that U meets D . Hence $a \in \overline{D}$ by definition. Conversely, if $a \in \overline{D}$, then choosing for each neighborhood U of a a point $x_U \in U \cap D$, we get a net $x_U \rightarrow a$.
- (c) If A is closed, $(x_i) \subset A$ and $x_i \rightarrow a$, then $a \in \overline{A} = A$ by (b). If A is not closed, then there exists a point $a \in \overline{A} \setminus A$, and then by (b) there exists a net $(x_i) \subset A$ satisfying $x_i \rightarrow a \notin A$.
- (d) Let $f : X \rightarrow Y$ be continuous at a , and $x_i \rightarrow a$ in X . We have to show that $f(x_i) \rightarrow f(a)$ in Y . If V is a neighborhood of $f(a)$, then $U := f^{-1}(V)$ is a neighborhood of a by continuity, hence $x_i \in U$ for all sufficiently large i . Then $f(x_i) \in V$ for all sufficiently large i . This proves that $f(x_i)$ converges to $f(a)$.
- If f is not continuous at a , then there exists a neighborhood V of $f(a)$ for which $f^{-1}(V)$ is not a neighborhood of a . Choose for each neighborhood U of a a point $x_U \in U \setminus f^{-1}(V)$. Then $x_U \rightarrow a$, but $f(x_U) \not\rightarrow f(a)$, because $f(x_U) \notin V$ for all U .
- (e) If $x_i \rightarrow a$ is a convergent net in a metric space (X, d) , then for each fixed $\varepsilon > 0$ there exists an index k such that $d(x_i, a) < \varepsilon/2$ for all $i \geq k$. Applying the triangle inequality we obtain that $d(x_i, x_j) < \varepsilon$ for all $i, j \geq k$, so that (x_i) is a Cauchy net.

If every Cauchy net is convergent in X , then in particular every Cauchy sequence is convergent, so that X is complete.

It remains to show that if X is complete, and (x_i) is a Cauchy net in X , then (x_i) is convergent. For each positive integer n we may choose an index $i(n)$ such that

$$d(x_i, x_j) < 1/n \quad \text{for all } i, j \geq i(n).$$

We may assume that $i(1) \leq i(2) \leq \dots$.¹⁴ Then $x_{i(1)}, x_{i(2)}, \dots$ is a Cauchy sequence, hence it converges to some point a .

We prove that $x_i \rightarrow a$. For any given $\varepsilon > 0$ there exists a positive integer n such that $1/n < \varepsilon$. Then

$$d(x_i, x_{i(m)}) < 1/n \quad \text{for all } i \geq i(n) \quad \text{and } m \geq n.$$

Since $x_{i(m)} \rightarrow a$, letting $m \rightarrow \infty$ and using the continuity of the metric we conclude that

$$d(x_i, a) \leq 1/n < \varepsilon \quad \text{for all } i \geq i(n),$$

i.e., $x_i \rightarrow a$. □

The following examples show that the existence of convergent *subsequences* does not characterize the compact *topological* spaces.

Examples

- We consider the set X of functions $f : \mathbb{R} \rightarrow [0, 1]$ as the product $X := \prod_{t \in \mathbb{R}} I_t$ of the compact intervals $I_t = [0, 1]$. Convergence in X is pointwise convergence on \mathbb{R} . By Tychonoff's theorem (p. 53) X is compact.

Let us denote by $f_n(x)$ the n th binary digit of x ; if possible we consider finite expansions. Then the sequence (f_n) has no convergent subsequence. Indeed, for any given subsequence (f_{n_k}) consider a number $x \in [0, 1]$ whose n_k th binary digit is 0 or 1 according to whether k is even or odd. Then (f_{n_k}) is divergent at x .

- The functions $f : \mathbb{R} \rightarrow [0, 1]$ vanishing outside a countable set (that may depend on f) form a proper dense subset Y of X , so that Y is not compact. Nevertheless, in Y every sequence (f_n) has a convergent subsequence.

Indeed, since the union of countably many countable sets is still countable, there exists a countable set A outside which all function f_n vanish. Hence they belong to the compact subspace $Z := \prod_{t \in \mathbb{R}} J_t$ of Y , where $J_t = [0, 1]$ if $t \in A$, and $J_t = \{0\}$ otherwise. One may readily check¹⁵ that the topology of Z is metrizable with the metric

$$d(f, g) := \sum 2^{-n} |f(t_n) - g(t_n)|$$

¹⁴Use the last property in the definition of directed sets.

¹⁵Adapt the solution of Exercise 1.7, pp. 32 and 301.

where (t_n) is an arbitrary enumeration of the elements of A . Hence (f_n) has a convergent subsequence in the compact metric space Z , and then the convergence holds in X as well.

We may eliminate the above pathological situations by generalizing subsequences and cluster points.

Definition By a *subnet* of a net (x_i) we mean a net of the form $(x_{f(j)})_{j \in J}$ where the function $f : J \rightarrow I$ satisfies the following condition: for each $i \in I$ there exists a $j \in J$ such that

$$k \geq j \implies f(k) \geq i.$$

Remark Subsequences correspond to the special case where $I = J = \mathbb{N}$ and f is an increasing function.

The following variant of Lemma 1.20 (p. 24) holds:

Lemma 2.28 *Given a net $(x_i)_{i \in I}$ and a point a in a topological space, the following conditions are equivalent:*

- (a) x_i often belongs to each neighborhood V of a ;
- (b) x_i often belongs to each open set U containing a ;
- (c) there exists a subnet $x_{f(j)} \rightarrow a$.

Proof

(a) \implies (b) because U is a neighborhood of a .

(b) \implies (c) Let us denote by \mathcal{U} the family of open sets containing a , and consider the set E of all pairs (i, U) in $I \times \mathcal{U}$ for which $x_i \in U$. Then E is a directed set for the partial order

$$(i, U) \geq (j, V) \iff i \geq j \quad \text{and} \quad U \subset V.$$

Indeed, the reflexivity and transitivity relations are obvious. Furthermore, if $(i, U), (j, V) \in E$, then by (b) there exists a $k \in I$ satisfying $k \geq i$, $k \geq j$ and $x_k \in U \cap V$ because $U \cap V \in \mathcal{U}$. We conclude that

$$(k, U \cap V) \geq (i, U), \quad (k, U \cap V) \geq (j, V), \quad \text{and} \quad (k, U \cap V) \in E.$$

Setting $f(i, U) := i$ we obtain a subnet $(x_{f(i, U)})_{(i, U) \in E}$ of $(x_i)_{i \in I}$. Indeed, fixing $V \in \mathcal{U}$ arbitrarily, for each $i \in I$ we have the implication

$$(k, W) \geq (i, V) \implies k \geq i \implies f(k, W) \geq i$$

by the definition of f .

Finally, for any fixed $U \in \mathcal{U}$ we have to find $(i, V) \in E$ such that $x_{f(k, W)} \in U$ for all $(k, W) \geq (i, V)$. It suffices to choose $V := U$ and $i \in I$ satisfying $x_i \in U$. Indeed, then we have the implication

$$(k, W) \geq (i, V) \implies x_{f(k, W)} = x_k \in W \subset U.$$

(c) \implies (a) There exists a $k_1 \in I$ such that $x_{f(j)} \in V$ for all $j \geq k_1$. Furthermore, there exists a $k_2 \in I$ such $f(j) \geq i$ for all $j \geq k_2$. Choose $k \in I$ satisfying $k \geq k_1$ and $k \geq k_2$, then $f(k) \geq i$ and $x_{f(k)} \in V$. \square

In view of Lemmas 1.20 (p. 24) and 2.28 the following definition is consistent with the former one given for metric spaces:

Definition In a topological space a is a *cluster point* of a net if the equivalent conditions of Lemma 2.28 are satisfied.

Remark Similarly to the last remark on p. 24, the limits and cluster points are the same for Cauchy nets.

Proposition 2.29 *Let X be a topological space.*

- (a) *If $x_i \rightarrow a$, then every subnet of (x_i) converges to a .*
- (b) *If $x_i \not\rightarrow a$, then (x_i) has a subnet of which a is not even a cluster point.*
- (c) *A set K in X is compact \iff every net $(x_i) \subset K$ has a cluster point in K .*
- (d) *Let X be a compact Hausdorff space. A net in X is convergent \iff it has a unique cluster point.*

Proof

- (a) Let $(x_{f(j)})$ be a subnet of (x_i) and U a neighborhood of a . Since $x_i \rightarrow a$, there exists an index i_0 such that $x_i \in U$ for all $i \geq i_0$. By the definition of a subnet there exists an index $j_0 \in J$ such that $f(j) \geq i_0$ for all $j \geq j_0$. Then $x_{f(j)} \in U$ for all $j \geq j_0$. This proves that $x_{f(j)} \rightarrow a$.
- (b) If $x_i \not\rightarrow a$, then a has a neighborhood U such that for each index i_0 there exists an $i \geq i_0$ satisfying $x_i \notin U$. Then considering the identical map $f : J \rightarrow I$ on the subset $J := \{i \in I : x_i \notin U\}$ of I we obtain a subnet $(x_j)_{j \in J}$ of (x_i) . Since no element of this subnet belongs to U , a cannot be its cluster point.
- (c) First we assume that K is not compact, and we fix an open cover $\{U_\alpha : \alpha \in A\}$ of K without any finite subcover. Consider the family I of the finite subsets of A with the partial ordering $i \geq j \iff i \supset j$. By our assumption we may choose a point

$$x_i \in K \setminus (U_{\alpha_1} \cup \dots \cup U_{\alpha_n})$$

for each $i := \{\alpha_1, \dots, \alpha_n\} \in I$. No subnet of $(x_i) \subset K$ converges to any $a \in K$ because a belongs to some U_α , and then $x_i \notin U_\alpha$ for all $i \geq \{\alpha\}$.

Now assume that no subnet of the net $(x_i) \subset K$ converges to any point of K . Then for each $a \in K$ there is a neighborhood U_a of a and an index i_a such that

$x_i \notin U_a$ for all $i \geq i_a$. We prove that the open cover

$$K \subset \bigcup_{a \in K} U_a$$

has no open subcover.

Indeed, choosing arbitrarily a finite number of points $a_1, \dots, a_n \in K$, there exists an index i such that $i \geq i_{a_1}, \dots, i \geq i_{a_n}$, and then

$$x_i \notin U_{a_1} \cup \dots \cup U_{a_n}.$$

Hence K is not compact.

- (d) We already know that every net (x_i) has at least one cluster point a . If (x_i) does not converge to a , then (x_i) has by (b) a subnet $(x_{f(j)})$ of which a is not even a cluster point. By (c) this subnet also has at least one cluster point b , necessarily different from a . But then b is also a cluster point of (x_i) by (a), so that (x_i) has at least two cluster points.

If $x_i \rightarrow a$, then (x_i) cannot have a cluster point different from a . Indeed, if $x_{f(j)} \rightarrow b$ for some subnet, then $x_{f(j)} \rightarrow a$ by (a), and therefore $a = b$ by Proposition 2.27 (a). \square

2.6 Exercises

Exercise 2.1 Let A be a set in a topological space X . Prove the following equalities:

$$\text{int}(X \setminus A) = \text{ext } A;$$

$$\text{ext}(X \setminus A) = \text{int } A;$$

$$\partial(X \setminus A) = \partial A;$$

$$\overline{A} = (\text{int } A) \cup \partial A = A \cup \partial A = X \setminus \text{ext } A.$$

Exercise 2.2 Let $A \subset X$ and $a \in X$. Prove the following:

$$a \in \text{int } A \iff A \text{ is a neighborhood of } a;$$

$$a \in \text{ext } A \iff X \setminus A \text{ is a neighborhood of } a;$$

$$a \in \overline{A} \iff \text{each neighborhood of } a \text{ meets } A;$$

$$a \in \partial A \iff \text{each neighborhood of } a \text{ meets both } A \text{ and } X \setminus A.$$

Exercise 2.3 Prove the following statements:

- (i) any two non-empty open intervals are homeomorphic;
- (ii) any two non-degenerate bounded closed intervals are homeomorphic;
- (iii) any two non-empty half-closed intervals are homeomorphic;
- (iv) $(0, 1)$, $[0, 1]$ and $[0, 1)$ are pairwise non-homeomorphic;
- (v) no interval is homeomorphic to a circle of the plane.

Exercise 2.4 (Accumulation Points) Given a point a and a set D in a topological space X , prove that the following properties are equivalent:

- (i) every neighborhood of a meets D ;
- (ii) every open set containing a meets D ;
- (iii) there exists a net in D that converges to a .

We say that a is an *accumulation point* of a set $A \subset X$ if the above conditions are satisfied with $D := A \setminus \{a\}$. Show that this definition is compatible with the former one in metric spaces.

Show that in a compact topological space every infinite set has an accumulation point, but the converse statement may fail.

Exercise 2.5 (New Characterization of Compact Sets) We say that a family of sets has the *finite intersection property* if any finite number of sets of the family has a non-empty intersection.

Given a set K in a topological space, prove that the following properties are equivalent:

- (i) K is compact;
- (ii) if a family of closed sets in K has the finite intersection property, then the whole family has a non-empty intersection.

Exercise 2.6 (Initial Topology)

- (i) Let Y be a non-empty set in a topological space X , and consider the embedding $i : Y \rightarrow X$, $i(x) := x$. Prove that the subspace topology on Y is the weakest topology on Y for which i is continuous.
- (ii) Let $(X_i)_{i \in I}$ be an arbitrary non-empty family of topological spaces, and X the direct product of the sets X_i . Prove that the product topology on X is the weakest topology on X for which all projections $\pi_i : X \rightarrow X_i$ are continuous.
- (iii) Given a non-empty set X , a family $(Y_i)_{i \in I}$ of topological spaces and a corresponding family of functions $f_i : X \rightarrow Y_i$, prove that there exists a weakest topology on X for which all functions f_i are continuous. It is called an *initial topology* on X .

Exercise 2.7 (Quotient Topology and Final Topology)

- (i) Consider a function $f : X \rightarrow Y$ where X is a topological space. Prove that there exists a strongest topology on Y for which f is continuous.

Henceforth we consider this topology on Y . A map $f : X \rightarrow Y$ is called *open* if it sends open sets into open sets, and *closed* if it sends closed sets into closed sets.

- (ii) f is open $\iff f^{-1}(f(U)) \subset X$ is open for each open set $U \subset X$.
 (iii) f is closed $\iff f^{-1}(f(F)) \subset X$ is closed for each closed set $F \subset X$.

If there is an equivalence relation on X and $f(x)$ denotes the equivalence class of x for each $x \in X$, then it is called the *quotient topology* on $Y := f(X)$.

- (iv) Given a non-empty set Y , a family $(X_i)_{i \in I}$ of topological spaces and a corresponding family of functions $f_i : X_i \rightarrow Y$, prove that there exists a finest topology on Y for which all functions f_i are continuous. It is called a *final topology* on Y .

Exercise 2.8 (Alexandroff's One-Point Compactification) If X is a non-compact topological space, then add the new symbol ∞ to X , and consider the larger set $\bar{X} := X \cup \{\infty\}$. Prove the following properties¹⁶:

- (i) The open subsets of X and the sets $\bar{X} \setminus K$, where K runs over the closed compact subsets of X , form a topology \mathcal{T} on \bar{X} .
 (ii) This topology is compact.
 (iii) X is a dense subspace of \bar{X} .

Exercise 2.9 Prove that an arbitrary (finite or infinite) product of connected spaces is connected.

Exercise 2.10 (Peano Curve) We recall from Exercise 1.8 (p. 32) that Cantor's ternary set C consists of those points $t \in [0, 1]$ which can be written in the form

$$t = 2\left(\frac{t_1}{3} + \frac{t_2}{3^2} + \cdots + \frac{t_n}{3^n} + \cdots\right)$$

with suitable integers $t_n \in \{0, 1\}$.

- (i) Prove that the formula

$$t \mapsto \left(\frac{t_1}{2} + \frac{t_3}{2^2} + \cdots + \frac{t_{2n-1}}{2^n} + \cdots, \frac{t_2}{2} + \frac{t_4}{2^2} + \cdots + \frac{t_{2n}}{2^n} + \cdots\right)$$

defines a continuous function f of C onto $[0, 1] \times [0, 1]$.

- (ii) Extend f to a continuous function of $[0, 1]$ onto $[0, 1] \times [0, 1]$.
 (iii) For each positive integer N , construct a continuous function of $[0, 1]$ onto $[0, 1]^N$.

¹⁶This is a generalization of the definition of the complex sphere in complex analysis.

Exercise 2.11 (Semi-continuity) A function $f : X \rightarrow \mathbb{R}$ on a topological space X is called *upper semi-continuous* if $\{x \in X : f(x) < \alpha\}$ is an open set for every $\alpha \in \mathbb{R}$, and *lower semi-continuous* if $\{x \in X : f(x) > \alpha\}$ is an open set for every $\alpha \in \mathbb{R}$. Prove the following results:

- (i) f is upper semi-continuous $\iff -f$ is lower semi-continuous.
- (ii) f is continuous \iff it is both upper and lower semi-continuous.
- (iii) Let $\{f_i : i \in I\}$ be a non-empty family of upper semi-continuous functions on X . If $f := \inf_{i \in I} f_i$ is finite-valued everywhere, then f is upper semi-continuous.
- (iv) If X is compact and f is upper semi-continuous, then f has a maximal value.



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