2 More on kinematics

In this chapter, we continue the study of kinematics by considering in more detail the motion of fluid parcels, by deriving expressions for the areal, volumetric, and mass flow rates across lines and surfaces drawn in a fluid, and by developing numerical methods for evaluating kinematic variables of interest in terms of derivatives and integrals of the velocity field. Mass conservation and physical conditions imposed at boundaries introduce mathematical constraints that motivate the description of a flow in terms of ancillary functions that expedite the mathematical analysis and considerably simplify the numerical computation.

2.1 Fundamental modes of fluid parcel motion

In Chapter 1, we pointed out that the nature of the motion of a small fluid parcel is determined by the relative motion of point particles residing inside the parcel. If variations in the point particle velocity are negligible compared to the average point particle velocity, the parcel exhibits rigid-body translation. Significant variations in the point particle velocity are responsible further general types of motion, including local rotation, deformation, and isotropic expansion.

To study the relative motion of point particles in the vicinity of a certain point, $\mathbf{x}_0 = (x_0, y_0, z_0)$, we consider differences in the corresponding velocity components evaluated at a point $\mathbf{x} = (x, y, z)$ that lies close to $\mathbf{x}_0$, and at the chosen point, $\mathbf{x}_0$, as shown in Figure 2.1.1. If the differences are small compared to the distance between the points $\mathbf{x}$ and $\mathbf{x}_0$, both measured in proper units, then the relative motion is negligible. If the differences are substantial, the relative motion is significant and needs to be properly analyzed.

C. Pozrikidis, *Fluid Dynamics*, DOI 10.1007/978-1-4899-7991-9_2

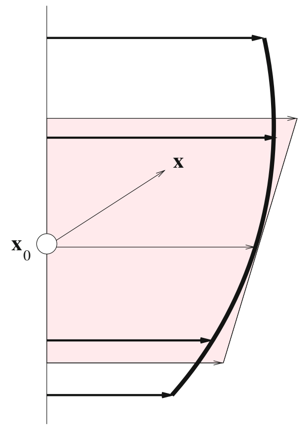


Figure 2.1.1 Illustration of relative motion of a fluid in the neighborhood of a point, $\mathbf{x}_0$. The bold line represents an actual velocity profile and the straight line represents the linearized velocity profile.

2.1.1 Function linearization

To prepare the ground for our analysis, we consider a scalar function of three independent variables that receives a triplet of numbers, (x, y, z), and generates a number, $f(x, y, z)$. If the function f is locally well behaved, and if the point $\mathbf{x}$ lies sufficiently near the point $\mathbf{x}_0$, then we expect that the value $f(x, y, z)$ will be close to the value $f(x_0, y_0, z_0)$. Stated differently, in the limit as $\mathbf{x}$ tends to $\mathbf{x}_0$, that is, all three scalar differences $x - x_0$, $y - y_0$, and $z - z_0$ tend to zero, the difference in the function values,

$$f(x, y, z) - f(x_0, y_0, z_0), \tag{2.1.1}$$

will vanish.

The variable point, $\mathbf{x}$, may approach the fixed point, $\mathbf{x}_0$, from different directions. Selecting the direction that is parallel to the x axis, we set $\mathbf{x} = (x, y_0, z_0)$, and consider the limit of the difference $f(x, y_0, z_0) - f(x_0, y_0, z_0)$ as $x - x_0$ tends to zero. Because the function f has been assumed well behaved, the ratio of the differences,

$$\frac{f(x, y_0, z_0) - f(x_0, y_0, z_0)}{x - x_0}, \tag{2.1.2}$$

tends to a finite number, which is defined as the first partial derivative of the function f with respect to the variable x evaluated at the point $\mathbf{x}_0$, and is denoted by $(\partial f/\partial x)(\mathbf{x}_0)$. Elementary calculus ensures that the partial derivative can be computed using the usual rules of differentiation of a function of one variable with respect to x, regarding all other independent variables as constant. For example, if $f = xyz$, then $\partial f/\partial x = yz$, and thus $(\partial f/\partial x)(\mathbf{x}_0) = y_0 z_0$, Setting the fraction shown in (2.1.2) equal to $(\partial f/\partial x)(\mathbf{x}_0)$, and solving the resulting equation for $f(x, y_0, z_0)$, we obtain

$$f(x, y_0, z_0) \simeq f(x_0, y_0, z_0) + (x - x_0)\left(\frac{\partial f}{\partial x}\right)_{\mathbf{x}_0}. \tag{2.1.3}$$

It is important to bear in mind that this equation is exact only in the limit as $\Delta x \equiv x - x_0$ tends to zero. For small but non-infinitesimal values of Δx, the difference between the left- and right-hand sides is on the order of Δx^2, which is small compared to Δx. For example, if Δx is equal to 0.01 is some units, then Δx^2 is equal to 0.0001 in corresponding units.

The point $\mathbf{x}$ may also approach the point $\mathbf{x}_0$ along the y or z axis, yielding the following counterparts of equation (2.1.3),

$$f(x_0, y, z_0) \simeq f(x_0, y_0, z_0) + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_{\mathbf{x}_0}, \tag{2.1.4}$$

and

$$f(x_0, y_0, z) \simeq f(x_0, y_0, z_0) + (z - z_0)\left(\frac{\partial f}{\partial z}\right)_{\mathbf{x}_0}. \tag{2.1.5}$$

Combining the arguments that led us to equations (2.1.3)–(2.1.5), we let the point $\mathbf{x}$ approach the point $\mathbf{x}_0$ from an arbitrary direction and derive the approximation

$$f(x, y, z) \simeq f(x_0, y_0, z_0) + (x - x_0)\left(\frac{\partial f}{\partial x}\right)_{\mathbf{x}_0} + (y - y_0)\left(\frac{\partial f}{\partial y}\right)_{\mathbf{x}_0} + (z - z_0)\left(\frac{\partial f}{\partial z}\right)_{\mathbf{x}_0}. \tag{2.1.6}$$

We pause to emphasize that relation (2.1.6) is exact only in the limit as all three spatial differences, $\Delta x = x - x_0$, $\Delta y = y - y_0$, and $\Delta z = z - z_0$, tend to zero. For small but non-infinitesimal values of any of these differences, the left-hand side of (2.1.6) differs from the right-hand side by an amount that is generally on the order of the maximum of Δx^2, Δy^2, or Δz^2.

Taylor series

Equation (2.1.6) can be rendered exact for any value of Δx, Δy, or Δz, by adding to the right-hand side a term called the remainder. As all three differences Δx, Δy, and Δz, tend to zero, the remainder vanishes faster than these differences. Elementary calculus shows that, if Δx, Δy, and Δz are sufficiently small, the remainder can be expressed as an infinite series involving products of powers of Δx, Δy, and Δz, called the Taylor series of the function f about the point $\mathbf{x}_0$.

The process of deriving (2.1.6) is called *linearization* of the function $f(\mathbf{x})$ about the point $\mathbf{x}_0$. The linearized form (2.1.6) states that, in the immediate vicinity of a point, $\mathbf{x}_0$, any regular function resembles a linear function of the shifted monomials Δx, Δy, and Δz. If all three first partial derivatives happen to vanish at the point $\mathbf{x}_0$, the function $f(\mathbf{x})$ behaves like a quadratic function; however, this is a rare exception.

Gradient of a scalar function

To economize our notation, we introduce the gradient of a function, f, denoted by $\boldsymbol{\nabla} f$, defined as the vector of the three partial derivatives,

$$\boldsymbol{\nabla} f \equiv \mathbf{e}_x \frac{\partial f}{\partial x} + \mathbf{e}_y \frac{\partial f}{\partial y} + \mathbf{e}_z \frac{\partial f}{\partial z}, \tag{2.1.7}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit vectors along the x, y, or z axes. The symbol $\boldsymbol{\nabla}$ is a vector operator called the del or gradient operator, defined as

$$\boldsymbol{\nabla} = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}. \tag{2.1.8}$$

Unlike a regular vector, ∇ may not stand alone, but must operate on a scalar function of position from the left to acquire a meaningful interpretation.

Inner vector product

As a second preliminary, we define the inner product of a pair of three-dimensional vectors,

$$\mathbf{f} = (f_x, f_y, f_z), \qquad \mathbf{g} = (g_x, g_y, g_z), \tag{2.1.9}$$

as the scalar

$$\mathbf{f} \cdot \mathbf{g} = f_x\, g_x + f_y\, g_y + f_z\, g_z. \tag{2.1.10}$$

In index notation,

$$\mathbf{f} \cdot \mathbf{g} \equiv f_i\, g_i, \tag{2.1.11}$$

where summation of the repeated index i is implied over x, y, and z, according to Einstein's repeated-index summation convention: *if an index appears twice in a product, then summation of that index is implied over its range.* In two dimensions, i is summed over x and y. An index may not appear more than twice in a product. An index that appears once is a *free index*

Interpretation of the inner vector product

It can be shown using the rule of cosines that the inner product defined in (2.1.10) is equal to the product of (a) the length of the first vector, $\mathbf{f}$, (b) the length of the second vector, $\mathbf{g}$, and (c) the cosine of the angle subtended between the two vectors, β,

$$\mathbf{f} \cdot \mathbf{g} = |\mathbf{f}|\, |\mathbf{g}| \cos\beta. \tag{2.1.12}$$

If the angle β is equal to $\frac{1}{2}\pi$, which means that the two vectors are orthogonal, the cosine of the angle is zero and the inner product vanishes. If the angle is zero, which means the two vectors are parallel, the inner product is equal to the product of the two vector lengths.

If the angle is equal to π, which means the two vectors are anti-parallel, the inner product is equal to the negative of the product of the two vector lengths.

If both $\mathbf{f}$ and $\mathbf{g}$ are unit vectors, that is, their lengths are equal to one unit of length, then the inner product is equal to the cosine of the angle subtended between the corresponding directions.

Linearized expansion in compact form

Using the preceding definitions, we state equation (2.1.6) in a compact vector form

$$f(\mathbf{x}) \simeq f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot (\boldsymbol{\nabla} f)_{\mathbf{x}_0}, \tag{2.1.13}$$

where the subscript $\mathbf{x}_0$ signifies that the gradient, $\boldsymbol{\nabla} f$, is evaluated at the point $\mathbf{x}_0$. The second term on the right-hand side of (2.1.13) is the inner product of the distance vector, $\mathbf{x} - \mathbf{x}_0$, and the gradient vector, ∇f, evaluated at a point of interest, $\mathbf{x}_0$. The magnitude of this term attains an extreme value when the two vectors are collinear.

2.1.2 Velocity gradient tensor

To derive the linearized form of the velocity field in the vicinity of a point, $\mathbf{x}_0$, we identify the function $f(\mathbf{x})$ with the x, y, or z velocity component, u_x, u_y, or u_z, and obtain the approximations

$$\begin{aligned}
u_x(\mathbf{x}) &\simeq u_x(\mathbf{x}_0) + (x - x_0)\left(\frac{\partial u_x}{\partial x}\right)_{\mathbf{x}_0} + (y - y_0)\left(\frac{\partial u_x}{\partial y}\right)_{\mathbf{x}_0} + (z - z_0)\left(\frac{\partial u_x}{\partial z}\right)_{\mathbf{x}_0}, \\
u_y(\mathbf{x}) &\simeq u_x(\mathbf{x}_0) + (x - x_0)\left(\frac{\partial u_y}{\partial x}\right)_{\mathbf{x}_0} + (y - y_0)\left(\frac{\partial u_y}{\partial y}\right)_{\mathbf{x}_0} + (z - z_0)\left(\frac{\partial u_y}{\partial z}\right)_{\mathbf{x}_0}, \\
u_z(\mathbf{x}) &\simeq u_z(\mathbf{x}_0) + (x - x_0)\left(\frac{\partial u_z}{\partial x}\right)_{\mathbf{x}_0} + (y - y_0)\left(\frac{\partial u_z}{\partial y}\right)_{\mathbf{x}_0} + (z - z_0)\left(\frac{\partial u_z}{\partial z}\right)_{\mathbf{x}_0}.
\end{aligned} \tag{2.1.14}$$

Collecting these equations into a unified vector form, we obtain the vector equation

$$\mathbf{u}(\mathbf{x}) \simeq \mathbf{u}(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{L}(\mathbf{x}_0), \tag{2.1.15}$$

where $\mathbf{L}$ is a 3×3 matrix called the velocity-gradient tensor, defined as

$$\mathbf{L} \equiv \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_z}{\partial x} \\ \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_z}{\partial y} \\ \dfrac{\partial u_x}{\partial z} & \dfrac{\partial u_y}{\partial z} & \dfrac{\partial u_z}{\partial z} \end{bmatrix}. \tag{2.1.16}$$

The notation $\mathbf{L}(\mathbf{x}_0)$ in (2.1.15) emphasizes that the nine components of the velocity-gradient tensor are evaluated at the chosen point $\mathbf{x}_0$ around which linearization has taken place. An actual and a linearized velocity profile is shown in Figure 2.1.1.

Denoting $x_1 = x$, $y_1 = y$, and $z_1 = x$, and also $u_1 = u_x$, $u_2 = u_y$, and $u_3 = u_z$. we compute the components of the velocity-gradient tensor

$$L_{ij} = \frac{\partial u_j}{\partial x_i} \tag{2.1.17}$$

for $i, j = 1, 2, 3$ or $i, j = x, y, z$.

An application

As an example, we consider the velocity field expressed by equations (1.4.8), repeated below for convenience,

$$\begin{aligned} u_x(x, y, z, t) &= a\,(y^2 + z^2) + x^3 yz\,(b + c\,t) + c\,e^{dxt}, \\ u_y(x, y, z, t) &= a\,(z^2 + x^2) + xy^3 z\,(b + c\,t) + c\,e^{dyt}, \\ u_z(x, y, z, t) &= a\,(x^2 + y^2) + xyz^3\,(b + c\,t) + c\,e^{dzt}, \end{aligned} \tag{2.1.18}$$

where a, b, c, and d are four constants. Applying the rules of partial differentiation, we obtain the associated velocity-gradient tensor

$$\mathbf{L} = \begin{bmatrix} 3x^2 yz\,(b + c\,t) + cdt\,e^{dxt} & 2ax + y^3 z\,(b + c\,t) & 2ax + yz^3 (b + c\,t) \\ 2ay + x^3 z\,(b + c\,t) & 3y^2 xz\,(b + c\,t) + cdt\,e^{dyt} & 2ay + xz^3 (b + c\,t) \\ 2az + x^3 y\,(b + c\,t) & 2az + xy^3\,(b + c\,t) & 3z^2 xy\,(b + c\,t) + cdt\,e^{dzt} \end{bmatrix}. \tag{2.1.19}$$

Placing the point $\mathbf{x}_0$ along the x axis, that is, setting $y_0 = 0$ and $z_0 = 0$, we find that

$$\mathbf{L}(x_0, 0, 0) = \begin{bmatrix} cdt\,e^{dx_0 t} & 2ax_0 & 2ax_0 \\ 0 & cdt & 0 \\ 0 & 0 & cdt \end{bmatrix}. \tag{2.1.20}$$

Thus, in the vicinity of the point $\mathbf{x}_0 = (x_0, 0, 0)$, the flow expressed by equations (2.1.18) can be approximated with a linear flow described by

$$\begin{aligned} u_x(x, y, z) &\simeq u_x(\mathbf{x}_0) + cdt\,e^{dx_0 t}\,(x - 1), \\ u_y(x, y, z) &\simeq u_y(\mathbf{x}_0) + 2a\,(x - 1) + cdt\,y, \\ u_z(x, y, z) &\simeq u_z(\mathbf{x}_0) + 2a\,(x - 1) + cdt\,z. \end{aligned} \tag{2.1.21}$$

The right-hand sides of equations (2.1.21) are linear functions of the spatial coordinates x, y, and z, but not necessarily linear functions of time, t.

What is a tensor?

The velocity-gradient tensor is a matrix containing the three first partial derivatives of the three components of the velocity with respect to x, y, or z, a total of nine scalar elements. Why have we called this matrix a tensor?

A tensor is a matrix whose elements are physical entities evaluated with reference to a chosen system of Cartesian coordinates. If the coordinate system is changed, for example, by translation or rotation, the elements of the matrix will also change to reflect the new Cartesian base. This change is analogous to that undergone by the components of the position or velocity vector when a new system of coordinates is introduced, as discussed in Section 1.5.

If the elements of the matrix corresponding to the new system are related to the elements corresponding to the old system by certain rules discussed in texts of matrix calculus and continuum mechanics mechanics, then the matrix is called a tensor.[1] Establishing whether or not a matrix is a tensor is important in deriving physical laws that relate matrices with different physical interpretations.

2.1.3 Relative motion of point particles

According to equation (2.1.15), the motion of a point particle near a point, $\mathbf{x}_0$, is governed by the equation

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{u}(\mathbf{X}) \simeq \mathbf{u}(\mathbf{x}_0) + (\mathbf{X} - \mathbf{x}_0) \cdot \mathbf{L}(\mathbf{x}_0), \tag{2.1.22}$$

where $\mathbf{X}$ is the position of the point particle and $\mathbf{u}(\mathbf{X})$ is the point-particle velocity, which is equal to the local and instantaneous fluid velocity.

The first term on the right-hand side of (2.1.22) states that a point particle located at the point $\mathbf{X}$ translates with the velocity of the point particle located at the point $\mathbf{x}_0$. The second term expresses the relative motion with respect to the point particle located at $\mathbf{x}_0$. Different velocity-gradient tensors, $\mathbf{L}(\mathbf{x}_0)$, represent different types of relative motion. Our next goal is to delineate the nature of the relative motion in terms of the components of $\mathbf{L}(\mathbf{x}_0)$.

2.1.4 Fundamental motions in two-dimensional flow

We begin by considering a two-dimensional flow in the xy plane and introduce the 2×2 velocity-gradient tensor

$$\mathbf{L} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_y}{\partial x} \\ \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_y}{\partial y} \end{bmatrix}. \tag{2.1.23}$$

In Section 1.6, we studied the velocity field associated with the linear flow expressed by equation (1.6.46), repeated below for convenience,

$$\begin{bmatrix} u_x & u_y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.1.24}$$

[1] Pozrikidis, C. (2011) *Introduction to Theoretical and Computational Fluid Dynamics.* Second Edition, Oxford University Press.

where a, b, c, and d are four constants with units of inverse time. Comparing equations (2.1.15) and (2.1.23) with equation (2.1.24), we set

$$a = \frac{\partial u_x}{\partial x}, \qquad b = \frac{\partial u_y}{\partial x}, \qquad c = \frac{\partial u_x}{\partial y}, \qquad d = \frac{\partial u_y}{\partial y}, \tag{2.1.25}$$

where all partial derivatives are evaluated at the point $\mathbf{x}_0$.

To study the nature of the linearized flow, we carry out the decomposition shown in equation (1.6.48), setting

$$\mathbf{L} = \mathbf{\Xi} + \mathbf{E} + \tfrac{1}{2}\,\alpha\,\mathbf{I}, \tag{2.1.26}$$

where

$$\mathbf{\Xi} \equiv \tfrac{1}{2} \begin{bmatrix} 0 & \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} \\ \dfrac{\partial u_x}{\partial y} - \dfrac{\partial u_y}{\partial x} & 0 \end{bmatrix} \tag{2.1.27}$$

is a skew-symmetric matrix with zero trace called the *vorticity tensor*,

$$\mathbf{E} \equiv \tfrac{1}{2} \begin{bmatrix} \dfrac{\partial u_x}{\partial x} - \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} - \dfrac{\partial u_x}{\partial x} \end{bmatrix} \tag{2.1.28}$$

is a symmetric matrix with zero trace called the *rate-of-deformation tensor*,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.1.29}$$

is the 2×2 identity matrix, and the scalar

$$\alpha = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \tag{2.1.30}$$

is the *rate of areal expansion.*

Areal expansion

The results of Section 1.6 suggest that a fluid parcel centered at the point $\mathbf{x}_0$ expands isotropically with an areal rate of expansion that is equal to the right-hand side of (2.1.30) evaluated at $\mathbf{x}_0$, as illustrated in Figure 2.1.2.

Rotation

Referring to equation (1.6.49), we find that a fluid parcel centered at the point $\mathbf{x}_0$ rotates in the xy plane around the point $\mathbf{x}_0$ with angular velocity

$$\Omega = \tfrac{1}{2}\,\Big(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\Big), \tag{2.1.31}$$

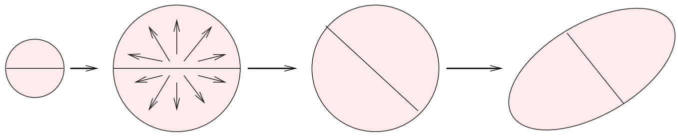


Figure 2.1.2 Expansion, rotation, and deformation of a small discoidal fluid parcel occurring during an infinitesimal period of time in a two-dimensional flow.

where the right-hand side is evaluated at $\mathbf{x}_0$, as shown in Figure 2.1.2. When Ω is positive, the parcel rotates in the counterclockwise direction; whereas, when Ω is negative, the parcel rotates in the clockwise direction.

Deformation

Our discussion in Section 1.6 suggests that the flow associated with the rate-of-deformation tensor, $\mathbf{E}$, expresses pure deformation in the absence of rotation or expansion, as illustrated in Figure 2.1.2.

To compute the rate of deformation, G, we consider the eigenvalues of $\mathbf{E}$. Denoting $E_{xx} \equiv E_{11}$, introducing a similar notation for the other components, and taking into account that

$$E_{xx} + E_{yy} = 0, \qquad E_{xy} = E_{yx} \tag{2.1.32}$$

by construction, we find the eigenvalues

$$G = \pm\sqrt{E_{xx}^2 + E_{xy}^2}. \tag{2.1.33}$$

The corresponding eigenvectors define the principal directions of the rate of deformation, also called the rate of strain. It can be shown that, because $\mathbf{E}$ is symmetric, the two eigenvectors are mutually orthogonal. An eigenvalue of the rate-of-strain tensor expresses the rate of deformation of a circular fluid parcel centered at a point, $\mathbf{x}_0$, in the direction of the associated eigenvector.

A theorem of matrix calculus ensures that the sum of the eigenvalues of a matrix is equal to the sum of the diagonal elements; in the case of the rate-of-deformation tensor, $\mathbf{E}$, this is equal to zero by construction. Because of this property, the deformation conserves the area of a fluid parcel during the motion.

2.1.5 Fundamental motions in three-dimensional flow

To generalized the analysis of Section 2.1.4 to three-dimensional flow, we resolve the three-dimensional velocity-gradient tensor into three parts, as

$$\mathbf{L} = \mathbf{\Xi} + \mathbf{E} + \tfrac{1}{3}\,\alpha\,\mathbf{I}, \tag{2.1.34}$$

$$\mathbf{\Xi} \equiv \frac{1}{2} \begin{bmatrix} 0 & \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_z}{\partial x} - \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_x}{\partial y} - \dfrac{\partial u_y}{\partial x} & 0 & \dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_y}{\partial z} - \dfrac{\partial u_z}{\partial y} & 0 \end{bmatrix}$$

$$\mathbf{E} \equiv \begin{bmatrix} \dfrac{\partial u_x}{\partial x} - \frac{1}{3}\,\alpha & \frac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y}) & \frac{1}{2}(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z}) \\ \frac{1}{2}(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}) & \dfrac{\partial u_y}{\partial y} - \frac{1}{3}\,\alpha & \frac{1}{2}(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}) \\ \frac{1}{2}(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}) & \dfrac{1}{2}(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}) & \dfrac{\partial u_z}{\partial z} - \frac{1}{3}\,\alpha \end{bmatrix}$$

Table 2.1.1 Definition of the vorticity tensor, $\mathbf{\Xi}$, and rate-of-deformation tensor, $\mathbf{E}$, in a three-dimensional flow; the scalar $\alpha \equiv \boldsymbol{\nabla} \cdot \mathbf{u}$ is the volumetric rate of expansion.

where $\mathbf{\Xi}$ is the skew-symmetric vorticity tensor, $\mathbf{E}$ is the symmetric and traceless rate-of-deformation tensor,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.1.35}$$

is the 3×3 identity matrix, and the scalar coefficient

$$\alpha \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{2.1.36}$$

is the rate of volumetric expansion. Explicit expressions for the vorticity tensor, $\mathbf{\Xi}$, and rate-of-deformation tensor, $\mathbf{E}$, are given in Table 2.1.1.

The three terms on the right-hand side of (2.1.34) express, respectively, isotropic expansion, rotation, and pure deformation, as illustrated in Figure 2.1.3. Because of the fundamental significance of these motions, these terms merit individual attention in Sections 2.2–2.4.

2.1.6 Gradient in polar coordinates

We have defined the velocity-gradient tensor as the gradient of the velocity vector field. Expressions for the gradient operator in polar coordinates can be obtained by using geometrical transformation rules combined with the chain rule of differentiation.

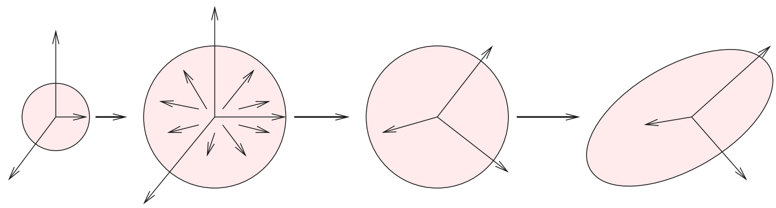


Figure 2.1.3 Expansion, rotation, and deformation of a small spherical fluid parcel occurring during an infinitesimal period of time in a three-dimensional flow.

Cylindrical polar coordinates

In the cylindrical polar coordinates depicted in Figure 1.3.2, the gradient of a scalar function, $f(\mathbf{x})$, is determined by its cylindrical polar components F_x, F_σ, and F_φ, as

$$\mathbf{F} \equiv \boldsymbol{\nabla} f = F_x\, \mathbf{e}_x + F_\sigma\, \mathbf{e}_\sigma + F_\varphi\, \mathbf{e}_\varphi. \tag{2.1.37}$$

Using the transformation rules shown in equations (1.3.20), we find that

$$F_\sigma = \cos\varphi\, \frac{\partial f}{\partial y} + \sin\varphi\, \frac{\partial f}{\partial z}, \qquad F_\varphi = -\sin\varphi\, \frac{\partial f}{\partial y} + \cos\varphi\, \frac{\partial f}{\partial z}. \tag{2.1.38}$$

To express the derivatives with respect to y and z in terms of derivatives with respect to cylindrical polar coordinates, we use the chain rule of differentiation along with the coordinate transformation rules (1.3.14) and (1.3.15), and find that

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \left(\frac{\partial f}{\partial x}\right)_{\sigma,\varphi} \left(\frac{\partial x}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial \sigma}\right)_{x,\varphi} \left(\frac{\partial \sigma}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial \varphi}\right)_{x,\sigma} \left(\frac{\partial \varphi}{\partial y}\right)_{x,z} \tag{2.1.39}$$

or

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \cos\varphi \left(\frac{\partial f}{\partial \sigma}\right)_{x,\varphi} - \frac{\sin\varphi}{\sigma} \left(\frac{\partial f}{\partial \varphi}\right)_{x,\sigma}, \tag{2.1.40}$$

and

$$\left(\frac{\partial f}{\partial z}\right)_{x,y} = \left(\frac{\partial f}{\partial x}\right)_{\sigma,\varphi} \left(\frac{\partial x}{\partial z}\right)_{x,y} + \left(\frac{\partial f}{\partial \sigma}\right)_{x,\varphi} \left(\frac{\partial \sigma}{\partial z}\right)_{x,y} + \left(\frac{\partial f}{\partial \varphi}\right)_{x,\sigma} \left(\frac{\partial \varphi}{\partial z}\right)_{x,y} \tag{2.1.41}$$

or

$$\left(\frac{\partial f}{\partial z}\right)_{x,y} = \sin\varphi \left(\frac{\partial f}{\partial \sigma}\right)_{x,\varphi} + \frac{\cos\varphi}{\sigma} \left(\frac{\partial f}{\partial \varphi}\right)_{x,\sigma}. \tag{2.1.42}$$

Substituting relations (2.1.40) and (2.1.42) into the right-hand sides of relations (2.1.38), we find that

$$F_x = \frac{\partial f}{\partial x}, \qquad F_\sigma = \frac{\partial f}{\partial \sigma}, \qquad F_\varphi = \frac{1}{\sigma}\, \frac{\partial f}{\partial \varphi}. \tag{2.1.43}$$

Equations (2.1.43) illustrate that the polar components of the gradient are equal to the partial derivatives with respect to the corresponding coordinates multiplied by an appropriate scaling factor.

Spherical polar coordinates

In the spherical polar coordinates depicted in Figure 1.3.3, the gradient of a scalar function, f, is defined by its spherical polar components F_r, F_θ, and F_φ, as

$$\mathbf{F} \equiv \boldsymbol{\nabla} f = F_r\,\mathbf{e}_r + F_\theta\,\mathbf{e}_\theta + F_\varphi\,\mathbf{e}_\varphi. \tag{2.1.44}$$

Working as in the case of cylindrical polar coordinates, we obtain

$$F_r = \frac{\partial f}{\partial r}, \qquad F_\theta = \frac{1}{r}\,\frac{\partial f}{\partial \theta}, \qquad F_\varphi = \frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \varphi}. \tag{2.1.45}$$

Note that the expression for F_φ is consistent with that given in the third relation of (2.1.43), subject to the substitution $\sigma = r\sin\theta$.

Plane polar coordinates

In the plane polar coordinates depicted in Figure 1.3.4, the gradient of a scalar function, f, is defined by its plane polar components, F_r and F_θ, as

$$\mathbf{F} \equiv \boldsymbol{\nabla} f = F_r\,\mathbf{e}_r + F_\theta\,\mathbf{e}_\theta. \tag{2.1.46}$$

Working as in the case of cylindrical coordinates, we obtain

$$F_r = \frac{\partial f}{\partial r}, \qquad F_\theta = \frac{1}{r}\,\frac{\partial f}{\partial \theta}. \tag{2.1.47}$$

Problems

2.1.1 *Inner vector product*

Prove the interpretation of the inner vector product discussed after equation (2.1.11). *Hint:* Use the law of cosines.

2.1.2 *Decomposition of a linearized flow*

(a) Linearize the velocity described by equations (1.5.2) around the origin of the y axis, and then decompose the velocity-gradient tensor of the linearized flow into the three constituents shown on the right-hand side of (2.1.34).

(b) Decompose the velocity gradient-tensor of the linearized flow expressed by equations (2.1.21) into the three constituents shown on the right-hand side of (2.1.34).

2.2 Fluid parcel expansion

The velocity field associated with the third term on the right-hand side of (2.1.34) is described by

$$\mathbf{u}^{\text{expansion}}(\mathbf{x}) = \tfrac{1}{3}\,\alpha(\mathbf{x}_0)\,(\mathbf{x}-\mathbf{x}_0)\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \tfrac{1}{3}\,\alpha(\mathbf{x}_0)\,(\mathbf{x}-\mathbf{x}_0). \tag{2.2.1}$$

Under the influence of this field, a spherical fluid parcel centered at the point $\mathbf{x}_0$ expands when the coefficient $\alpha(\mathbf{x}_0)$ is positive, or contracts when the coefficient $\alpha(\mathbf{x}_0)$ is negative, all the while retaining the spherical shape.

To see this behavior more clearly, we consider the motion of a point particle that lies at the surface of the spherical parcel. Using (2.2.1), we find that the radius of the parcel, $a(t)$, is given by

$$\frac{a(t)}{a(t=0)} = \mathrm{e}^{\frac{1}{3}\alpha t}. \tag{2.2.2}$$

Raising both sides to the third power and multiplying the result by the factor $\frac{4\pi}{3}$, we find that the ratio of the instantaneous parcel volume to the initial parcel volume is

$$\frac{\frac{4\pi}{3}a^3(t)}{\frac{4\pi}{3}a^3(t=0)} = \mathrm{e}^{\alpha t}. \tag{2.2.3}$$

This result explains why the constant α is called the rate of volumetric expansion.

Divergence of the velocity field

The rate of expansion defined in equation (2.1.36) can be expressed in a compact form that simplifies the notation. Taking the inner product of the del operator defined in (2.1.8) and the velocity, we find that

$$\boldsymbol{\nabla}\cdot\mathbf{u} \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}. \tag{2.2.4}$$

In index notation,

$$\boldsymbol{\nabla}\cdot\mathbf{u} \equiv \frac{\partial u_i}{\partial x_i}, \tag{2.2.5}$$

where summation over the repeated index i is implied for x, y, and z. In the case of two-dimensional flow in the xy plane, the derivative of u_z with respect to z does not appear. Accordingly, we write

$$\alpha = \boldsymbol{\nabla}\cdot\mathbf{u}. \tag{2.2.6}$$

The right-hand side of (2.2.6) is the *divergence of the velocity field.*

Solenoidal velocity fields

We have found that the rate of volumetric expansion at an arbitrary point in a three-dimensional flow and the rate of areal expansion at a point in a two-dimensional flow are equal to the divergence of the velocity evaluated at that point. If the divergence of the velocity vanishes everywhere in a flow, with the physical consequence that no parcel undergoes expansion but only exhibits translation, rotation, and deformation, then the velocity field is called solenoidal.

Problem

2.2.1 *Rate of expansion*

Derive the rate of expansion of the flow described by equations (2.1.18), and then evaluate the rate of expansion at the point $\mathbf{x}_0 = (1, 0, 1)$.

2.3 Fluid parcel rotation and vorticity

The velocity field associated with the first term on the right-hand side of (2.1.34) is given by

$$\mathbf{u}^{\text{rotation}}(x, y, z) = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{\Xi}(\mathbf{x}_0), \tag{2.3.1}$$

where $\mathbf{\Xi}$ is the vorticity tensor defined in Table 2.1.1.

A planar fluid parcel in a two-dimensional flow in the xy plane may only rotate around the z axis. In contrast, a three-dimensional fluid parcel in a three-dimensional flow may rotate around any arbitrary axis that passes through the designated center of rotation, $\mathbf{x}_0$, and points in any arbitrary direction.

The orientation, magnitude, and direction of rotation define an angular velocity vector, $\mathbf{\Omega}$, whose components can be deduced from the three upper triangular or three lower triangular entries of the vorticity tensor shown in Table 2.1.1, and are given by

$$\Omega_x = \tfrac{1}{2}\,(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}), \qquad \Omega_y = \tfrac{1}{2}\,(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}), \qquad \Omega_z = \tfrac{1}{2}\,(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}), \tag{2.3.2}$$

where the right-hand sides are evaluated at the designated parcel center, $\mathbf{x}_0$. As we look down into the vector $\mathbf{\Omega}$ from the tip of its arrow, the fluid rotates in the clockwise direction.

Equation (2.3.1) can be recast into a compact form in terms of the angular velocity vector as

$$\mathbf{u}^{\text{rotation}}(x, y, z) = (\mathbf{x} - \mathbf{x}_0) \cdot \begin{bmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{bmatrix}, \tag{2.3.3}$$

where $\mathbf{\Omega}$ derives from the velocity by way of (2.3.2).

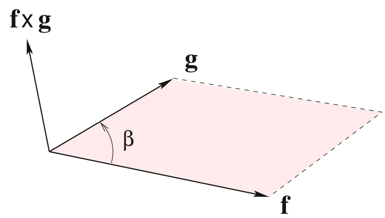


Figure 2.3.1 The outer product of two vectors, **f** and **g**, is a new vector that is perpendicular to the plane of **f** and **g**.

We note that the three components of the angular velocity vector arise by combining selected partial derivatives of the components of the velocity field in a particular fashion. Stated differently, the angular velocity vector field arises from the velocity field by operating with a differential operator, just as the rate of expansion arises from the velocity field by operating with the divergence operator ($\boldsymbol{\nabla}\cdot$), as discussed in Section 2.2.

Outer vector product

To identify the differential operator that generates the point particle angular velocity field, $\boldsymbol{\Omega}$, from the velocity field, $\mathbf{u}$, according to equations (2.3.2), we introduce the outer vector product. Consider a pair of vectors,

$$\mathbf{f} = (f_x, f_y, f_z), \qquad \mathbf{g} = (g_x, g_y, g_z). \tag{2.3.4}$$

The outer product of the first vector with the second vector, taken in this particular order, is a new vector, denoted as $\mathbf{f} \times \mathbf{g}$, defined as

$$\mathbf{f} \times \mathbf{g} = (f_y\, g_z - f_z\, g_y)\, \mathbf{e}_x + (f_z\, g_x - f_x\, g_z)\, \mathbf{e}_y + (f_x\, g_y - f_y\, g_x)\, \mathbf{e}_z, \tag{2.3.5}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit vectors along the x, y, or z axis. We find that

$$\mathbf{f} \times \mathbf{g} = -\mathbf{g} \times \mathbf{f}. \tag{2.3.6}$$

If the order of the two vectors is switched, a minus sign must be included.

Interpretation of the outer vector product

It can be shown that the outer-product vector $\mathbf{f} \times \mathbf{g}$ is normal to the plane containing the vectors $\mathbf{f}$ and $\mathbf{g}$, as illustrated in Figure 2.3.1. The magnitude of $\mathbf{f} \times \mathbf{g}$ is equal to the product of (a) the length of the vector $\mathbf{f}$, (b) the length of the vector $\mathbf{g}$, and (c) the absolute value of the sine of the angle, β, subtended between the two vectors.

The orientation of the outer-product vector $\mathbf{f} \times \mathbf{g}$ is such that, as we look down at the plane defined by $\mathbf{f}$ and $\mathbf{g}$ toward the negative direction of $\mathbf{f} \times \mathbf{g}$, the angle β measured in the counterclockwise direction from $\mathbf{f}$ is less than π. If β is equal to 0 or π, the two vectors are parallel or anti-parallel, the sine of the angle is zero, and the outer product vanishes.

The three directions defined by the the triplet of vectors $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{f} \times \mathbf{g}$, arranged in this particular order, form a right-handed system of coordinates. This is another way of saying that $\mathbf{f} \times \mathbf{g}$ arises from $\mathbf{f}$ and $\mathbf{g}$ according to the right-hand rule.

Now invoking the definition of the cross product, we recast equation (2.3.3) into the form

$$\mathbf{u}^{\text{rotation}}(x, y, z) = \boldsymbol{\Omega} \times (\mathbf{x} - \mathbf{x}_0), \tag{2.3.7}$$

which describes rigid-body rotation with angular velocity $\boldsymbol{\Omega}$ around the point $\mathbf{x}_0$, in agreement with the previously stated physical interpretation.

2.3.1 Curl and vorticity

Taking the outer product of the del operator and the velocity field, we obtain the curl of the velocity defined as the vorticity,

$$\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \mathbf{u} = (\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z})\,\mathbf{e}_x + (\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x})\,\mathbf{e}_y + (\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y})\,\mathbf{e}_z. \tag{2.3.8}$$

Comparing equation (2.3.8) with equations (2.3.2), we find that

$$\boldsymbol{\Omega} = \tfrac{1}{2}\,\boldsymbol{\omega}, \tag{2.3.9}$$

which shows that the angular-velocity vector is equal to half the vorticity vector, or half the curl of the velocity.

Irrotational flow

If the curl of a velocity field vanishes at every point in a flow, with the consequence that no spherical fluid parcel undergoes rotation, then the velocity field is called irrotational. The properties and computation of irrotational flow will be discussed in Chapter 3, and then again in Chapter 12 in the context of aerodynamics.

The alternating tensor

The long expression on the right-hand side of equation (2.3.5) defining the outer vector product is cumbersome. To simplify the notation, we introduce the three-index alternating tensor, ϵ_{ijk}, defined as follows:

1. If $i = j$, or $j = k$, or $k = i$, then $\epsilon_{ijk} = 0$. For example, $\epsilon_{xxy} = \epsilon_{zyz} = \epsilon_{zyy} = 0$.
2. If i, j, and k are all different, then $\epsilon_{ijk} = \pm 1$. The plus sign applies when the triplet ijk is a cyclic permutation of xyz, and the minus sign applies otherwise. For example, $\epsilon_{xyz} = \epsilon_{zxy} = \epsilon_{yzx} = 1$, but $\epsilon_{xzy} = -1$.

Two important properties of the alternating tensor stemming from its definition are

$$\epsilon_{ijk}\,\epsilon_{mjk} = 2\,\delta_{im}, \tag{2.3.10}$$

where double summation of the repeated indices j and k is implied on the left-hand side, and

$$\epsilon_{ijk}\,\epsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}, \tag{2.3.11}$$

where summation of the repeated index k is implied on the left-hand side. Kronecker's delta, δ_{ij}, represents the identity matrix: $\delta_{ij} = 1$ if $i = j$, or 0 if $i \neq j$. Additional properties of the alternating tensor are listed in Problem 2.3.2.

In terms of the alternating tensor, the ith component of the outer vector product $\mathbf{f} \times \mathbf{g}$ defined in equation (2.3.5) is given by

$$(\mathbf{f} \times \mathbf{g})_i = \epsilon_{ijk}\, f_j\, g_k, \tag{2.3.12}$$

where double summation of the two repeated indices j and k is implied on the right-hand side.

Using the definition (2.3.8), we find that the ith component of the vorticity is given by

$$\omega_i = \epsilon_{ijk}\,\frac{\partial u_k}{\partial x_j}. \tag{2.3.13}$$

Straightforward manipulation of (2.3.13) provides us with an expression for the vorticity vector in terms of the vorticity tensor,

$$\omega_i = \tfrac{1}{2}\,(\epsilon_{ijk}\frac{\partial u_k}{\partial x_j} + \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}) = \tfrac{1}{2}\,(\epsilon_{ijk}\frac{\partial u_k}{\partial x_j} - \epsilon_{ikj}\frac{\partial u_k}{\partial x_j}), \tag{2.3.14}$$

and then

$$\omega_i = \tfrac{1}{2}\,(\epsilon_{ijk}\frac{\partial u_k}{\partial x_j} - \epsilon_{ijk}\frac{\partial u_j}{\partial x_k}) = \epsilon_{ijk}\,\tfrac{1}{2}\,(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k}) \tag{2.3.15}$$

or

$$\omega_i = \epsilon_{ijk}\,\Xi_{jk}. \tag{2.3.16}$$

The inverse relationship is

$$\Xi_{ij} = \tfrac{1}{2}\,\epsilon_{ijk}\,\omega_k \tag{2.3.17}$$

(Problem 2.3.3).

2.3.2 Two-dimensional flow

Consider a two-dimensional flow in the xy plane. Inspecting the right-hand side of (2.3.8), we find that the x and y components of the vorticity vanish. The vorticity vector is then parallel to the z axis, and thus perpendicular to the plane of the flow,

$$\boldsymbol{\omega} = \omega_z\,\mathbf{e}_z, \tag{2.3.18}$$

where $\mathbf{e}_z$ is the unit vector along the z axis. The scalar ω_z is the strength of the vorticity, defined as

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}. \tag{2.3.19}$$

For example, in the case of simple shear flow, $u_x = \xi y$, $u_y = 0$, and $\omega_z = -\xi$, where the coefficient ξ is the shear rate.

Using the transformation rules discussed in Section 1.1, we find that the strength of the vorticity in the plane polar coordinates depicted in Figure 1.1.4 is given by

$$\omega_z = \frac{1}{r}\Big(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\Big). \tag{2.3.20}$$

In the case of rigid-body rotation with angular velocity Ω, $u_\theta = \Omega\, r$ $u_r = 0$, and $\omega_z = \frac{1}{2}\,\Omega$.

2.3.3 Axisymmetric flow

Consider an axisymmetric flow in the absence of swirling motion and refer to the polar cylindrical coordinates (x, σ, φ) depicted in Figure 1.1.2 and to the spherical polar coordinates (r, θ, φ) depicted in Figure 1.1.3.

A fluid patch that lies in an azimuthal plane, defined as plane of constant azimuthal angle φ, is able to rotate only around an axis that is perpendicular to this plane. Consequently, the vorticity vector points in the direction of increasing or decreasing azimuthal angle, φ. This observation suggests that the vorticity vector takes the form

$$\boldsymbol{\omega} = \omega_\varphi\, \mathbf{e}_\varphi, \tag{2.3.21}$$

where $\mathbf{e}_\varphi$ is the unit vector in the azimuthal direction and ω_φ is the corresponding vorticity component given by

$$\omega_\varphi = \frac{\partial u_\sigma}{\partial x} - \frac{\partial u_x}{\partial \sigma} = \frac{1}{r}\Big(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\Big). \tag{2.3.22}$$

Note that the expression in spherical polar coordinates, (r, θ), given on the right-hand side of (2.3.22) is identical to that in plane polar coordinates given in (2.3.20).

Problems

2.3.1 *Properties of the outer vector product*

(a) Show that $\mathbf{f} \times \mathbf{g} = -\mathbf{g} \times \mathbf{f}$, where the outer vector product, denoted by $\times$, is defined in equation (2.3.5).

(b) The outer vector product of two vectors, $\mathbf{f}$ and $\mathbf{g}$, can be identified with the determinant of a matrix,

$$\mathbf{f} \times \mathbf{g} = \det\left(\begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{bmatrix} \right). \tag{2.3.23}$$

Confirm that this rule is consistent with the definition of the curl of the velocity in (2.3.8).

2.3.2 *Properties of Kronecker's delta and alternating tensor*

Prove the properties

$$\delta_{ii} = 3, \qquad \epsilon_{ljk}\,\delta_{jk} = 0, \qquad a_j\,\delta_{jk} = a_k, \qquad A_{lj}\,\delta_{jk} = A_{lk}, \tag{2.3.24}$$

where δ_{ij} is Kronecker's delta representing the 3×3 identity matrix, $\mathbf{a}$ is an arbitrary vector, $\mathbf{A}$ is an arbitrary matrix, and summation is implied over a repeated index.

2.3.3 *Relation between the vorticity tensor and vector*

Prove relation (2.3.17). *Hint:* Express the vorticity in terms of the velocity as shown in (2.3.13), and then use property (2.3.11).

2.3.4 *The vorticity field is solenoidal*

Show that the divergence of the vorticity is identically zero, $\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0$, that is, the vorticity field is solenoidal.

2.4 Fluid parcel deformation

The velocity field associated with the second term on the right-hand side of (2.1.34) is

$$\mathbf{u}^{\text{deformation}}(x, y, z) = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{E}(\mathbf{x}_0), \tag{2.4.1}$$

where $\mathbf{E}$ is the symmetric and traceless rate-of-deformation tensor defined in Table 2.1.1.

To develop insights into the nature of the motion described by (2.4.1), we consider a special case where $\mathbf{E}(\mathbf{x}_0)$ is a diagonal matrix,

$$\mathbf{E}(\mathbf{x}_0) = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} - \frac{1}{3}\,\alpha & 0 & 0 \\ 0 & \dfrac{\partial u_y}{\partial y} - \frac{1}{3}\,\alpha & 0 \\ 0 & 0 & \dfrac{\partial u_z}{\partial z} - \frac{1}{3}\,\alpha \end{bmatrix}, \tag{2.4.2}$$

with the understanding that the derivatives on the right-hand side are evaluated at the point $\mathbf{x}_0$. The trace of the matrix on the right-hand side is zero, as required. The eigenvalues of a diagonal matrix are equal to the diagonal elements. The corresponding eigenvectors point along the x, y, or z axes.

Cursory inspection reveals that, under the action of the flow described by (2.4.1), subject to (2.4.2), a spherical fluid parcel centered at a point, $\mathbf{x}_0$, deforms to obtain an ellipsoidal shape while preserving its volume, as illustrated in Figure 2.1.2. The three eigenvalues of the rate-of-deformation tensor express the rate of deformation in three principal directions corresponding to the eigenvectors. If an eigenvalue is negative, the parcel is compressed in the corresponding direction to obtain an oblate shape.

More generally, the rate-of-deformation tensor has three real eigenvalues, λ_1, λ_2, and λ_3, that are found by setting the determinant of the following matrix to zero,

$$\mathbf{E} - \lambda\,\mathbf{I} = \begin{bmatrix} E_{xx} - \lambda & E_{xy} & E_{xz} \\ E_{yx} & E_{yy} - \lambda & E_{yz} \\ E_{zx} & E_{zy} & E_{zz} - \lambda \end{bmatrix}, \tag{2.4.3}$$

and then computing the roots of the emerging cubic equation for λ, where $\mathbf{I}$ is the 3×3 identity matrix. It can be shown that, because $\mathbf{E}$ is symmetric, all three eigenvalues are real and each eigenvalue has a distinct corresponding eigenvector. Moreover, the three eigenvectors are mutually orthogonal, pointing in the principal directions of the rate of strain.

Under the action of the flow stated in (2.4.1), a spherical fluid parcel centered at the point $\mathbf{x}_0$ deforms to obtain an ellipsoidal shape whose axes are generally inclined with respect to the x, y, and z axis. The three axes of the ellipsoid are parallel to the eigenvectors of $\mathbf{E}$, and the respective rates of deformation of the ellipsoid are equal to the corresponding eigenvalues. A theorem of matrix calculus states that the sum of the eigenvalues is equal to the sum of the diagonal elements of $\mathbf{E}$, which is zero. Because of this property, the deformation preserves the parcel volume.

Computation of the rates of strain

Setting the determinant of the matrix (2.4.3) to zero, we obtain a cubic algebraic equation for λ,

$$\lambda^3 + a\,\lambda^2 + b\,\lambda + c = 0, \tag{2.4.4}$$

where

$$\begin{aligned} a &= -\mathrm{trace}(\mathbf{E}) = -(E_{xx} + E_{yy} + E_{zz}), \\ b &= (E_{yy}E_{zz} - E_{yz}E_{zy}) + (E_{xx}E_{zz} - E_{xz}E_{zx}) + (E_{xx}E_{yy} - E_{xy}E_{yx}), \\ c &= \det(\mathbf{E}), \end{aligned} \tag{2.4.5}$$

and det stands for the determinant. Using Cardano's formulas, we find that the three roots of (2.4.4) are given by

$$\lambda_1 = -\frac{a}{3} + d\,\cos\frac{\chi}{3}, \qquad \lambda_{2,3} = -\frac{a}{3} - d\,\cos\frac{\chi \pm \pi}{3}, \tag{2.4.6}$$

where

$$d = 2\left(\tfrac{1}{3}\,|p|\right)^{1/2}, \qquad \chi = \arccos\Big(-\tfrac{1}{2}\,\frac{q}{\left(\tfrac{1}{3}\,|p|\right)^{3/2}}\Big) \tag{2.4.7}$$

and

$$p = b - \tfrac{1}{3}\,a^2, \qquad q = c + \tfrac{2}{27}\,a^3 - \tfrac{1}{3}\,ab. \tag{2.4.8}$$

In general, we may find three real eigenvalues or one real eigenvalue accompanied by a pair of complex conjugate eigenvalues.

In the case of the rate-of-deformation tensor, because the trace is zero, $a = 0$, we obtain the simplified expressions

$$\lambda_1 = d\,\cos\frac{\chi}{3}, \qquad \lambda_{2,3} = -d\,\cos\frac{\chi \pm \pi}{3}, \tag{2.4.9}$$

where

$$d = 2\left(\tfrac{1}{3}\,|b|\right)^{1/3}, \qquad \chi = \arccos\left(-\,\tfrac{1}{2}\,\frac{c}{\left(\tfrac{1}{3}\,|b|\right)^{3/2}}\right) \tag{2.4.10}$$

and b, c can be arbitrary.

Once the eigenvalues have been found, the eigenvectors are computed by solving a homogeneous system of three equations for three unknowns. For example, the eigenvector $\mathbf{e}^{(1)} = (e_x^{(1)}, e_y^{(1)}, e_z^{(1)})$ corresponding to the eigenvalue λ_1 is found by solving the homogeneous linear system

$$(\mathbf{E} - \lambda_1 \mathbf{I}) \cdot \mathbf{e}^{(1)} = \mathbf{0}, \tag{2.4.11}$$

which can be restated as

$$\begin{aligned}
(E_{xx} - \lambda_1)\,e_x^{(1)} + E_{xy}\,e_y^{(1)} &= -E_{xz}\,e_z^{(1)}, \\
E_{yx}\,e_x^{(1)} + (E_{yy} - \lambda_1)\,e_y^{(1)} &= -E_{yz}\,e_z^{(1)}, \\
E_{zx}\,e_x^{(1)} + E_{zy}\,e_y^{(1)} &= -(E_{zz} - \lambda_1)\,e_z^{(1)}.
\end{aligned} \tag{2.4.12}$$

To solve system (2.4.12), we may assign an arbitrary value to the first component, $e_z^{(1)}$, evaluate the first two right-hand sides, and solve the first two equations for $e_x^{(1)}$ and $e_y^{(1)}$ using, for example, Cramer's rule. The solution is guaranteed to also satisfy the third equation. A solution cannot be found if the eigenvector is perpendicular to the z axis, in which case $e_z^{(1)}$ is zero. If this occurs, we simply transfer the term involving $e_x^{(1)}$ or $e_y^{(1)}$ to the right-hand side instead, and solve for the other two components.

Problems

2.4.1 *Properties of eigenvalues*

(a) Confirm that the sum of the three eigenvalues given in (2.4.6) is equal to the trace of $\mathbf{E}$.

(b) Confirm that the product of the three eigenvalues given in (2.4.6) is equal to the determinant of the rate-of-deformation tensor, $\mathbf{E}$.

(c) Confirm that, when $\mathbf{E}$ is diagonal, formulas (2.4.6) identify the eigenvalues with the diagonal elements.

2.4.2 *Eigenvalues and eigenvectors*

Directory *05_eigen,* located inside directory *01_num_meth* of FDLIB, contains a program entitled *eigen33* that computes the eigenvalues of a 3×3 matrix. Use the program to compute the eigenvalues and eigenvectors of the rate of deformation tensor corresponding to the linearized flow (2.1.21) for $a = 1\ \mathrm{s}^{-1}$ and $cdt = 2\ \mathrm{s}^{-1}$.

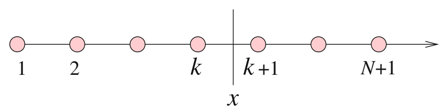
2.5 Numerical differentiation

We have mentioned that, in practice, the components of a velocity field are hardly ever given in analytical form by way of mathematical expressions. Instead, their values are either measured in the laboratory with probes, or computed by numerical methods at data points represented by grid nodes located in the domain of flow. The partial derivatives of the velocity are then recovered by a numerical procedure called numerical differentiation.

2.5.1 Numerical differentiation in one dimension

As a prelude to computing the partial derivatives of the components of the velocity from specified grid values, we consider computing the first derivative of a function, $f(x)$, of one independent variable, x, defined on a grid.

To be more specific, we assume that values of $f(x)$ are given at $N+1$ nodes of a one-dimensional uniform grid with nodes located at x_i for $i = 1, \ldots, N+1$, as shown below:



Our goal is to compute the derivative, $\mathrm{d}f/\mathrm{d}x$, at a point, x, that lies in the kth interval subtended between the nodes x_k and x_{k+1}.

First-order differentiation

In the simplest approach, the graph of the function $f(x)$ in the interval (x_k, x_{k+1}) is approximated with a straight line, as shown in Figure 1.7.2, and the derivative $\mathrm{d}f/\mathrm{d}x$ is approximated with the slope. Using equations (1.7.3) and (1.7.5), we derive the finite-difference approximation

$$f'(x) \simeq \frac{f_{k+1} - f_k}{x_{k+1} - x_k}, \tag{2.5.1}$$

where a prime denotes a derivative with respect to x.

Now identifying the evaluation point, x, with the grid point, x_k, we obtain the forward-difference approximation

$$f'(x_k) \simeq \frac{f_{k+1} - f_k}{x_{k+1} - x_k}. \tag{2.5.2}$$

The error associated with this approximation is proportional to the interval size, $h_k = x_{k+1} - x_k$.

Using instead the straight-line approximation for the $k-1$ interval, we obtain the backward-difference approximation

$$f'(x_k) \simeq \frac{f_k - f_{k-1}}{x_k - x_{k-1}}. \tag{2.5.3}$$

Formulas (2.5.2) and (2.5.3) carry a comparable amount of error due to the straight-line approximation.

To evaluate the derivative at the first point, $f'(x_1)$, we use a forward difference; to evaluate the derivative at the last point, $f'(x_{N+1})$, we use a backward difference; to evaluate $f'(x_i)$ at an interior grid point, where $i = 2, \ldots, N$, we use either a forward or a backward difference, whichever is deemed more convenient or appropriate.

Second-order differentiation

Numerical differentiation based on linear interpolation neglects the curvature of the graph of the function $f(x)$. To improve the accuracy of the interpolation, we approximate $f(x)$ with a parabolic function defined in the interval (x_k, x_{k+1}), as depicted in Figure 1.7.3, and then approximate the slope of the function, f', with the slope of the parabola. Differentiating (1.7.6), we derive the second-order finite-difference approximation

$$f'(x) \simeq 2\,a^{(k)}\,(x - x_k) + b^{(k)}, \tag{2.5.4}$$

where the coefficients $a^{(k)}$ and $b^{(k)}$ are given in (1.7.9).

Now identifying the evaluation point, x, with the grid point, x_k, we obtain the centered-difference approximation

$$f'(x_k) \simeq b^{(k)}. \tag{2.5.5}$$

When the grid points are spaced evenly, $x_k - x_{k-1} = x_{k+1} - x_k = h$, we obtain the simple form

$$f'(x_k) \simeq \frac{f_{k+1} - f_{k-1}}{2h}, \tag{2.5.6}$$

where h is the grid spacing. The error associated with this approximation is proportional to the square of the interval size, h^2.

The parabolic approximation allows us to also obtain an estimate for the second derivative, $\mathrm{d}^2 f/\mathrm{d}x^2$. Differentiating (1.7.6) twice with respect to x, we derive the finite-difference approximation

$$f''(x) \simeq 2a^{(k)}, \tag{2.5.7}$$

where the coefficient $a^{(k)}$ is given in (1.7.9).

When the grid points are spaced evenly along the x axis with separation h, we obtain the simpler formula

$$f''(x) \simeq \frac{f_{k+1} - 2f_k + f_{k-1}}{h^2}. \tag{2.5.8}$$

Identifying the evaluation point, x, with the grid point, x_k, we obtain the centered-difference approximation

$$f''(x_k) \simeq \frac{f_{k+1} - 2f_k + f_{k-1}}{h^2}. \tag{2.5.9}$$

The error associated with this approximation is proportional to the square of the interval size, h^2.

2.5.2 Numerical differentiation in two dimensions

Consider the computation of the first partial derivatives of a function of two independent variables, $f(x, y)$, $\partial f/\partial x$ and $\partial f/\partial y$, from given values of the function at the nodes of a two-dimensional grid defined by the intersection of the x-level lines, x_i for $i = 1, \ldots, N_x + 1$, and y-level lines, y_j for $j = 1, \ldots, N_y + 1$, as illustrated in Figure 1.7.2(b). The value of x lies in the k_xth x-interval confined between the x_{k_x} and x_{k_x+1} x-level lines, and the value of y lies in the k_yth y-interval confined between the y_{k_y} and y_{k_y+1} y-level lines.

First-order differentiation

Using the method of bilinear interpolation discussed in Section 1.7, we approximate the first partial derivatives of the function $f(x, y)$ with the partial derivatives of the bilinear function defined in equation (1.7.15). Considering the derivative with respect to x, we obtain the forward-difference approximation

$$\left(\frac{\partial f}{\partial x}\right)_{x,y} \simeq \left(\frac{\partial \Pi^{k_x,k_y}}{\partial x}\right)_{x,y}, \tag{2.5.10}$$

where Π^{k_x,k_y} is the bilinear function given in (1.7.15). Performing the differentiation, we obtain

$$\begin{aligned}\left(\frac{\partial f}{\partial x}\right)_{x,y} \simeq & \left(\frac{\partial w_{00}^{k_x,k_y}}{\partial x}\right)_{x,y} f(x_{k_x}, y_{k_y}) + \left(\frac{\partial w_{10}^{k_x,k_y}}{\partial x}\right)_{x,y} f(x_{k_x+1}, y_{k_y}) \\ & + \left(\frac{\partial w_{01}^{k_x,k_y}}{\partial x}\right)_{x,y} f(x_{k_x}, y_{k_y+1}) + \left(\frac{\partial w_{11}^{k_x,k_y}}{\partial x}\right)_{x,y} f(x_{k_x+1}, y_{k_y+1}).\end{aligned} \tag{2.5.11}$$

Using expressions (1.7.16) and (1.7.17), we obtain

$$\begin{aligned}\left(\frac{\partial f}{\partial x}\right)_{x,y} = & -\frac{y_{k_y+1} - y}{A} f(x_{k_x}, y_{k_y}) + \frac{y_{k_y+1} - y}{A} f(x_{k_x+1}, y_{k_y}) \\ & -\frac{y - y_{k_y}}{A} f(x_{k_x}, y_{k_y+1}) + \frac{y - y_{k_y}}{A} f(x_{k_x+1}, y_{k_y+1}).\end{aligned} \tag{2.5.12}$$

where

$$A = (x_{k_x+1} - x_{k_x})(y_{k_y+1} - y_{k_y}), \tag{2.5.13}$$

as given in (1.7.18). Using this formula, we derive the first-order, forward-difference approximation at the southwestern grid node,

$$\left(\frac{\partial f}{\partial x}\right)_{x_{k_x},\, y_{k_y}} \simeq \frac{f(x_{k_x+1}, y_{k_y}) - f(x_{k_x}, y_{k_y})}{x_{k_x+1} - x_{k_x}}. \tag{2.5.14}$$

A similar approximation for the y derivative yields

$$\left(\frac{\partial f}{\partial y}\right)_{x_{k_x},\, y_{k_y}} \simeq \frac{f(x_{k_x}, y_{k_y+1}) - f(x_{k_x}, y_{k_y})}{y_{k_y+1} - y_{k_y}}. \tag{2.5.15}$$

Both formulas express forward-difference approximations with respect to the respective variable, x or y.

Second-order differentiation

Second-order centered-difference formulas for evaluating the first partial derivative of a function at a grid point can be derived based on the one-dimensional formula (2.5.5). Using the expression for the coefficient $b^{(k)}$ given in (1.7.9), we obtain

$$\left(\frac{\partial f}{\partial x}\right)_{x_{k_x},\, y_{k_y}} \simeq \frac{(x_{k_x} - x_{k_x-1})\dfrac{f_{k_x+1,k_y} - f_{k_x,k_y}}{x_{k_x+1} - x_{k_x}} + (x_{k_x+1} - x_{k_x})\dfrac{f_{k_x,k_y} - f_{k_x-1,k_y}}{x_k - x_{k_x-1}}}{x_{k_x+1} - x_{k_x-1}}. \tag{2.5.16}$$

The corresponding expression for the derivative with respect to y is

$$\left(\frac{\partial f}{\partial y}\right)_{x_{k_x},\, y_{k_y}} \simeq \frac{(y_{k_y} - y_{k_y-1})\dfrac{f_{k_x,k_y+1} - f_{k_x,k_y}}{y_{k_y+1} - y_{k_y}} + (y_{k_y+1} - y_{k_y})\dfrac{f_{k_x,k_y} - f_{k_x,k_y-1}}{y_{k_y} - y_{k_y-1}}}{y_{k_y+1} - y_{k_y-1}}. \tag{2.5.17}$$

When the grid lines are spaced evenly,

$$x_{k_x} - x_{k_x-1} = x_{k_x+1} - x_{k_x} \equiv h_x, \tag{2.5.18}$$

and

$$y_{k_y} - y_{k_y-1} = y_{k_y+1} - y_{k_y} \equiv h_y, \tag{2.5.19}$$

we obtain the simpler formulas

$$\left(\frac{\partial f}{\partial x}\right)_{x_{k_x}, y_{k_y}} = \frac{f_{k_x+1,k_y} - f_{k_x-1,k_y}}{2h_x} \tag{2.5.20}$$

and

$$\left(\frac{\partial f}{\partial y}\right)_{x_{k_x}, y_{k_y}} = \frac{f_{k_x,k_y+1} - f_{k_x,k_y-1}}{2h_y}, \tag{2.5.21}$$

which express centered-difference approximations in x or y.

2.5.3 Velocity gradient and related functions

The formulas derived in Section 2.5.2 can be applied to obtain approximations to the elements of the velocity-gradient tensor, rate-of-deformation tensor, vorticity vector, and rate of expansion, from specified values of the velocity at grid points. To illustrate the methodology, we consider a two-dimensional flow and employ a uniform grid with constant x and y grid spacings equal to h_x and h_y.

Using the second-order, centered-difference approximations (2.5.20) and (2.5.21), we find that the rate of expansion can be approximated with the finite-difference formula

$$(\nabla \cdot \mathbf{u})_{x_{k_x}, y_{k_y}} \simeq \frac{(u_x)_{k_x+1,k_y} - (u_x)_{k_x-1,k_y}}{2h_x} + \frac{(u_y)_{k_x,k_y+1} - (u_y)_{k_x,k_y-1}}{2h_y}. \tag{2.5.22}$$

The corresponding finite-difference approximation for the z component of the vorticity takes the form

$$\omega_z(x_{k_x}, y_{k_y}) \simeq \frac{(u_y)_{k_x+1,k_y} - (u_y)_{k_x-1,k_y}}{2h_x} - \frac{(u_x)_{k_x,k_y+1} - (u_x)_{k_x,k_y-1}}{2h_y}. \tag{2.5.23}$$

Similar finite-difference approximations can be written for the elements of the rate-of-deformation tensor, and subsequently used to obtain approximations to its eigenvalues and eigenvectors.

The following MATLAB function entitled *rec_2d_vgt*, located in directory *rec_2d* inside directory *02_grids* of FDLIB, computes the velocity gradient tensor at the nodes of a two-dimensional Cartesian grid:

```
function [Axx,Axy,Ayx,Ayy] = rec_2d_vgt ...
...
    (glx,gly,Nx,Ny,gux,guy)

%------------------------------
% compute the velocity gradient
% tensor L_ij at the grid points
%-------------------------------

%----------------
% interior points
% compute derivatives by central differences
%----------------

for i=2:Nx
  for j=2:Ny
    Lxx(i,j) = (gux(i+1,j)-gux(i-1,j))/(glx(i+1)-glx(i-1));
    Lxy(i,j) = (guy(i+1,j)-guy(i-1,j))/(glx(i+1)-glx(i-1));
    Lyx(i,j) = (gux(i,j+1)-gux(i,j-1))/(gly(j+1)-gly(j-1));
    Lyy(i,j) = (guy(i,j+1)-guy(i,j-1))/(gly(j+1)-gly(j-1));
```

```
  end
end

%----------
% left wall
% compute derivatives by central or forward differences
%----------

i=1;
for j=2:Ny
  Lxx(i,j) = (gux(i+1,j)-gux(i,j))/(glx(i+1)-glx(i));
  Lxy(i,j) = (guy(i+1,j)-guy(i,j))/(glx(i+1)-glx(i));
  Lyx(i,j) = (gux(i,j+1)-gux(i,j-1))/(gly(j+1)-gly(j-1));
  Lyy(i,j) = (guy(i,j+1)-guy(i,j-1))/(gly(j+1)-gly(j-1));
end

%----------
% bottom wall
% compute derivatives by central or forward differences
%----------

j=1;
for i=2:Nx
  Lxx(i,j) = (gux(i+1,j)-gux(i-1,j))/(glx(i+1)-glx(i-1));
  Lxy(i,j) = (guy(i+1,j)-guy(i-1,j))/(glx(i+1)-glx(i-1));
  Lyx(i,j) = (gux(i,j+1)-gux(i,j))/(gly(j+1)-gly(j));
  Lyy(i,j) = (guy(i,j+1)-guy(i,j))/(gly(j+1)-gly(j));
end

%----------
% right wall
% compute derivatives by central or backward differences
%----------

i=Nx+1;
for j=2:Ny
  Lxx(i,j) = (gux(i,j)-gux(i-1,j))/(glx(i)-glx(i-1));
  Lxy(i,j) = (guy(i,j)-guy(i-1,j))/(glx(i)-glx(i-1));
  Lyx(i,j) = (gux(i,j+1)-gux(i,j-1))/(gly(j+1)-gly(j-1));
  Lyy(i,j) = (guy(i,j+1)-guy(i,j-1))/(gly(j+1)-gly(j-1));
end

%----------
% top wall
% compute derivatives by central or backward differences
%----------

j=Ny+1;
for i=2:Nx
```

```
  Lxx(i,j) = (gux(i+1,j)-gux(i-1,j))/(glx(i+1)-glx(i-1));
  Lxy(i,j) = (guy(i+1,j)-guy(i-1,j))/(glx(i+1)-glx(i-1));
  Lyx(i,j) = (gux(i,j)-gux(i,j-1))/(gly(j)-gly(j-1));
  Lyy(i,j) = (guy(i,j)-guy(i,j-1))/(gly(j)-gly(j-1));
end

%--------------------
% four corner points
%--------------------

i=1; j=1;
Lxx(i,j) = (gux(i+1,j)-gux(i,j))/(glx(i+1)-glx(i));
Lxy(i,j) = (guy(i+1,j)-guy(i,j))/(glx(i+1)-glx(i));
Lyx(i,j) = (gux(i,j+1)-gux(i,j))/(gly(j+1)-gly(j));
Lyy(i,j) = (guy(i,j+1)-guy(i,j))/(gly(j+1)-gly(j));

i=Nx+1; j=1;
Lxx(i,j) = (gux(i,j)-gux(i-1,j))/(glx(i)-glx(i-1));
Lxy(i,j) = (guy(i,j)-guy(i-1,j))/(glx(i)-glx(i-1));
Lyx(i,j) = (gux(i,j+1)-gux(i,j))/(gly(j+1)-gly(j));
Lyy(i,j) = (guy(i,j+1)-guy(i,j))/(gly(j+1)-gly(j));

i=Nx+1; j=Ny+1;
Lxx(i,j) = (gux(i,j)-gux(i-1,j))/(glx(i)-glx(i-1));
Lxy(i,j) = (guy(i,j)-guy(i-1,j))/(glx(i)-glx(i-1));
Lyx(i,j) = (gux(i,j)-gux(i,j-1))/(gly(j)-gly(j-1));
Lyy(i,j) = (guy(i,j)-guy(i,j-1))/(gly(j)-gly(j-1));

i=1; j=Ny+1;
Lxx(i,j) = (gux(i+1,j)-gux(i,j))/(glx(i+1)-glx(i));
Lxy(i,j) = (guy(i+1,j)-guy(i,j))/(glx(i+1)-glx(i));
Lyx(i,j) = (gux(i,j)-gux(i,j-1))/(gly(j)-gly(j-1));
Lyy(i,j) = (guy(i,j)-guy(i,j-1))/(gly(j)-gly(j-1));

%-----
% done
%-----

return
```

The following MATLAB code appended to the code *rec_2d* discussed in Section 1.10, residing in directory *rec_2d* inside directory *02_grids* of FDLIB, computes various flow variables:

```
%---
% specify the grid velocities
%---

for i=1:Nx+1
```

```
  for j=1:Ny+1
    px = gx(i,j); py = gy(i,j);
    wnx = 2*pi/(bx-ax); wny = 2*pi/(by-ay);
    gux(i,j) = cos(wnx*px)*cos(wny*py);
    guy(i,j) = sin(wnx*px)*sin(wny*py);
  end
end

%---
% velocity gradient tensor
%----

Lxx,Lxy,Lyx,Lyy] = rec_2d_vgt (glx,gly,Nx,Ny,gux,guy);

%---
% compute the rate of expansion
%     the rate of strain tensor
%     the strains
%     the vorticity
%---

for i=1:Nx+1
  for j=1:Ny+1
    roe(i,j) = Lxx(i,j)+Lyy(i,j);   % rate of expansion
    omega(i,j) = Lxy(i,j)-Lyx(i,j);   % vorticity
    Exx(i,j) = Lxx(i,j)-0.5*roe(i,j);   % rate of deformation
    Exy(i,j) = 0.5*(Lxy(i,j)+Lyx(i,j));   % rate of deformation
    Eyx(i,j) = Exy(i,j);   % rate of deformation
    Eyy(i,j) = Lyy(i,j)-0.5*roe(i,j);   % rate of deformation
    det = 4.0*(Exx(i,j)^2+Exy(i,j)^2);   % eigenvalues
    srd = sqrt(det);
    strain1(i,j) = 0.5*srd;
    strain2(i,j) = -0.5*srd;

%---
% compute the eigenvectors of the rate of strain
%---

  if(abs(Exy(i,j))<0.0001)   % E is diagonal

  if(abs(Exx(i,j)-strain1(i,j))>0.0001)
    egv1x(i,j) = 0.0; egv1y(i,j) = 1.0;
  else
    egv1x(i,j) = 1.0; egv1y(i,j) = 0.0;
  end
  if(abs(Exx(i,j)-strain2(i,j))>0.0001)
    egv2x(i,j) = 0.0; egv2y(i,j) = 1.0;
  else
    egv2x(i,j) = 1.0; egv2y(i,j) = 0.0;
```

```
  end

  else   % E is not diagonal

  egv1x(i,j) = 1.0;
  egv1y(i,j) = -(Exx(i,j)-strain1(i,j))/Exy(i,j);
  egv2x(i,j) = 1.0;
  egv2y(i,j) = -(Exx(i,j)-strain2(i,j))/Exy(i,j);

  end

%---
% normalize the eigenvectors
%---

  fc1 = 1.0/sqrt(egv1x(i,j)^2+egv1y(i,j)^2);
  egv1_x(i,j) = fc1*egv1x(i,j); egv1_y(i,j) = fc1*egv1y(i,j);
  fc2 = 1.0/sqrt(egv2x(i,j)^2+egv2y(i,j)^2);
  egv2_x(i,j) = fc2*egv2x(i,j); egv2_y(i,j) = fc2*egv2y(i,j);

  end
end

%---
% plotting
%---

figure
mesh(glx,gly,omega')
xlabel('x','fontsize',15)
ylabel('y','fontsize',15)
zlabel('\omega','fontsize',15)

figure
mesh(glx,gly,roe')
xlabel('x','fontsize',15);
ylabel('y','fontsize',15)
zlabel('\alpha','fontsize',15)

figure
mesh(glx,gly,strain1')
xlabel('x','fontsize',15)
ylabel('y','fontsize',15)
zlabel('s\_1','fontsize',15)

figure
mesh(glx,gly,strain2')
xlabel('x','fontsize',15);
ylabel('y','fontsize',15)
```

```
zlabel('s\_2','fontsize',15)

figure
hold on
[ glx,gly,gx,gy ] = grid_2d (ax,bx,ay,by,Nx,Ny);
for i=1:Nx+1
  for j=1:Ny+1
    vector = draw_arrow_2d ...
      (gx(i,j),gy(i,j),egv1x(i,j)/Ny,egv1y(i,j)/Ny);
    plot(vector(:,1),vector(:,2));
  end
end
xlabel('x','fontsize',15)
ylabel('y','fontsize',15)
box on

figure
hold on
[glx,gly,gx,gy] = grid_2d (ax,bx,ay,by,Nx,Ny);
for i=1:Nx+1
  for j=1:Ny+1
    vector = draw_arrow_2d ...
      (gx(i,j),gy(i,j),egv2x(i,j)/Ny,egv2y(i,j)/Ny);
    plot(vector(:,1),vector(:,2));
  end
end
xlabel('x','fontsize',15)
ylabel('y','fontsize',15)
```

The graphics display generated by the code for the velocity field implemented in the code is shown in Figure 2.5.1.

Problems

2.5.1 *Numerical differentiation*

Use formula (2.5.9) to evaluate the second derivative of the exponential function $f(x) = e^x$ at $x = 0$ in terms of the values of $f(x)$ at $x = -h, 0, h$, for $h = 0.16$, 0.08, 0.04, 0.02, and 0.01. Compute and plot the difference between the numerical value and the exact value against h on a log-log scale. Assess and discuss the slope of the graph.

2.5.2 *Numerical differentiation of a two-dimensional flow*

Run the code *rec_2d* for a velocity field of your choice. Prepare and discuss plots of the vorticity, eigenvalues, and eigenvectors of the rate-of-strain tensor.

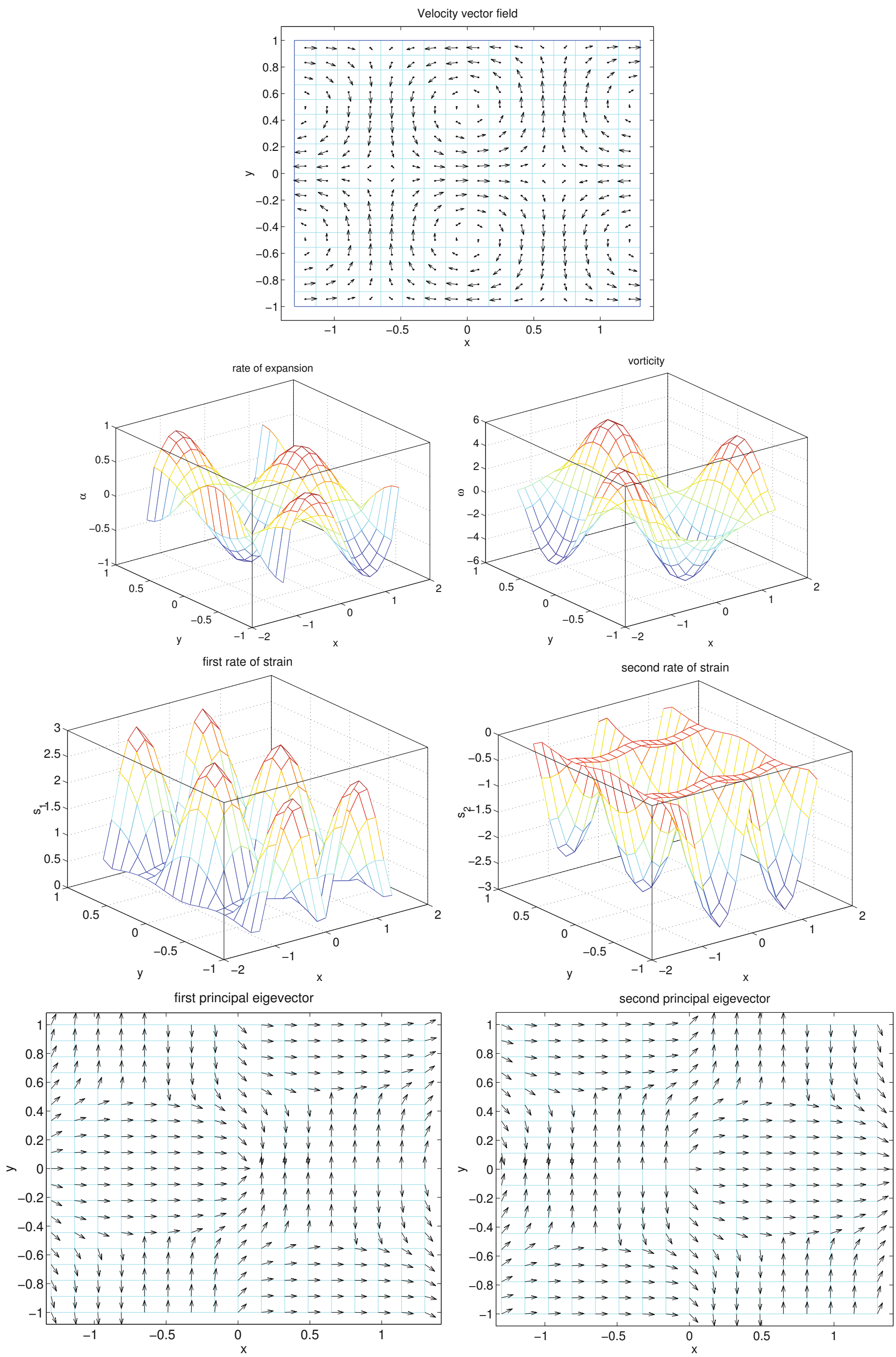


Figure 2.5.1 Velocity vector field, rate of expansion, vorticity, principal strains, and principal eigenvectors computed by numerical differentiation on a uniform Cartesian grid.

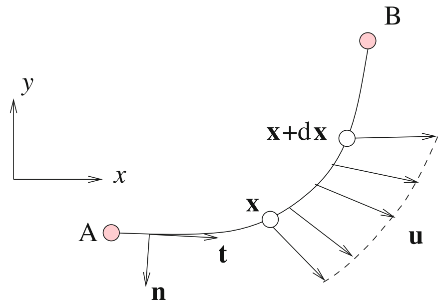


Figure 2.6.1 Illustration of a stationary open line that starts at a point, A, and ends at another point, B, used to define the areal flow rate and flux in a two-dimensional flow. When the end points A and B coincide, we obtain a closed loop.

2.6 Flow rates

Consider a two-dimensional flow in the xy plane and draw a *stationary* line that resides in its entirety inside the fluid. At any instant, point particles cross the line generating a net, positive or negative, areal flow rate in a designated direction. Our goal is to quantify this flow rate in terms of the shape of the line and the fluid velocity.

Unit tangent and unit normal vectors

First, we consider an open line that starts at a point, A, and ends at another point, B, as shown in Figure 2.6.1. As a preliminary, we introduce the unit tangent vector, $\mathbf{t} = (t_x, t_y)$, defined as the vector that is tangential to the line at a point, subject to the normalization condition

$$t_x^2 + t_y^2 = 1. \tag{2.6.1}$$

The direction of $\mathbf{t}$ is chosen such that, if we start moving along the line from point A in the direction of $\mathbf{t}$, we will finally end up at point B.

Next, we introduce the unit normal vector, $\mathbf{n} = (n_x, n_y)$, defined as the unit vector that is perpendicular to the line at a point. The magnitude of $\mathbf{n}$ is equal to unity,

$$n_x^2 + n_y^2 = 1. \tag{2.6.2}$$

The orientation of $\mathbf{n}$ is such that the tangent vector $\mathbf{t}$ arises by rotating $\mathbf{n}$ around the z axis in the counterclockwise direction by an angle equal to $\frac{1}{2}\pi$.

Now we consider an infinitesimal section of the line that starts at a point, $\mathbf{x}$, and ends at the point $\mathbf{x} + \mathrm{d}\mathbf{x}$, where the differential distance, $\mathrm{d}\mathbf{x} = (\mathrm{d}x, \mathrm{d}y)$ is parallel to, and points in the direction of the unit tangent vector, $\mathbf{t}$. The components of the unit tangent vector and unit normal vector are given by

$$t_x = \frac{\mathrm{d}x}{\mathrm{d}\ell}, \qquad t_y = \frac{\mathrm{d}y}{\mathrm{d}\ell} \tag{2.6.3}$$

and

$$n_x = \frac{\mathrm{d}y}{\mathrm{d}\ell}, \qquad n_y = -\frac{\mathrm{d}x}{\mathrm{d}\ell}, \tag{2.6.4}$$

where $\mathrm{d}\ell$ is the differential arc length of the infinitesimal section of the line, given by

$$\mathrm{d}\ell = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2}\,|\mathrm{d}x| \tag{2.6.5}$$

Because $\mathbf{t}$ and $\mathbf{n}$ are mutually orthogonal, their inner product is zero,

$$\mathbf{t}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{t} = 0. \tag{2.6.6}$$

To confirm this, we merely substitute (2.6.3) and (2.6.4) into the right-hand side of (2.1.10).

Normal and tangential velocities

Next, we consider a group of adjacent point particles distributed along the infinitesimal arc length, $\mathrm{d}\ell$, at a particular time instant, t. During an infinitesimal period of time, $\mathrm{d}t$, the point particles move to a new position, thus allowing other point particles located behind or in front of them to cross the line into the other side.

To compute the net area of fluid that has crossed the infinitesimal arc length $\mathrm{d}\ell$, we resolve the velocity of the point particles into a normal component and a tangential component, writing

$$\mathbf{u} = u_n\,\mathbf{n} + u_t\,\mathbf{t}. \tag{2.6.7}$$

The normal and tangential velocities, u_n and u_t, can be computed readily in terms of the inner vector product defined in equation (2.1.10). Taking the inner product of the unit normal vector with both sides of (2.6.7), and using (2.6.6) and (2.6.2), we obtain

$$u_n = \mathbf{u}\cdot\mathbf{n} = u_x\,n_x + u_y\,n_y. \tag{2.6.8}$$

Taking the inner product of the unit tangent vector with both sides of (2.6.7), and using (2.6.6) and (2.6.1), we obtain

$$u_t = \mathbf{u}\cdot\mathbf{t} = u_x\,t_x + u_y\,t_y. \tag{2.6.9}$$

2.6.1 Areal flow rate and flux

By definition, the local areal flow rate across an infinitesimal section, $\mathrm{d}\ell$, is the area of fluid, $\mathrm{d}A_f$, that crosses the infinitesimal section during an infinitesimal period of time, $\mathrm{d}t$, given by

$$\frac{\mathrm{d}A_f}{\mathrm{d}t} = u_n\,\mathrm{d}\ell, \tag{2.6.10}$$

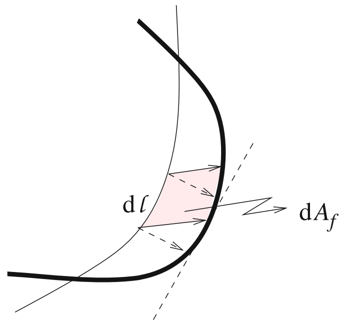


Figure 2.6.2 Two point particles move from the thin line to the bold line over small period of time, $\mathrm{d}t$, thereby allowing for an areal flow rate, $\mathrm{d}A_f$. The particles can be assumed to move first normal to the bold line (dashed vectors) and then tangential to the line to reach their final destination.

as shown in Figure 2.6.2. To see why we selected the normal component of the velocity on the right-hand side, we observe that, if the normal component vanishes, the particles move tangentially to the line and fluid does not cross the line. In general, although point particles move both in the tangential and normal directions, only the normal motion contributes to the local areal flow rate.

The corresponding local areal flux, q, is defined as the ratio of the local areal flow rate, $\mathrm{d}A_f/\mathrm{d}t$, to the infinitesimal length of the line across which transport takes place,

$$q \equiv \frac{\mathrm{d}A_f}{\mathrm{d}t\,\mathrm{d}\ell} = u_n. \tag{2.6.11}$$

We have found that the local areal flux is merely the normal component of the fluid velocity.

Substituting expression (2.6.8) for the normal velocity component into (2.6.10), and using (2.6.4), we obtain

$$\frac{\mathrm{d}A_f}{\mathrm{d}t} = u_n\,\mathrm{d}\ell = q\,\mathrm{d}\ell \tag{2.6.12}$$

and then

$$\frac{\mathrm{d}A_f}{\mathrm{d}t} = (u_x\,n_x + u_y\,n_y)\,\mathrm{d}\ell = u_x\,\mathrm{d}y - u_y\,\mathrm{d}x. \tag{2.6.13}$$

These expressions allow us to evaluate the local areal flow rate in terms of the components of the velocity.

2.6.2 Areal flow rate across a line

To compute the areal flow rate across the stationary open line depicted in Figure 2.6.1, denoted by Q_{areal}, we subdivide the line into an infinite collection of infinitesimal sections with differential lengths, $\mathrm{d}\ell$, and add all contributions. In mathematical terms, we integrate the local areal flux along the line with respect to arc length, finding that

$$Q^{\text{areal}} = \int_{\mathrm{A}}^{\mathrm{B}} q \, \mathrm{d}\ell = \int_{\mathrm{A}}^{\mathrm{B}} \frac{\mathrm{d}A_f}{\mathrm{d}t \, \mathrm{d}\ell} \, \mathrm{d}\ell = \int_{\mathrm{A}}^{\mathrm{B}} \frac{\mathrm{d}A_f}{\mathrm{d}t} = \int_{\mathrm{A}}^{\mathrm{B}} (u_x n_x + u_y n_y) \, \mathrm{d}\ell, \tag{2.6.14}$$

and then

$$Q^{\text{areal}} = \int_{A}^{B} u_n \, \mathrm{d}\ell = \int_{A}^{B} \big(u_x \, \mathrm{d}y - u_y \, \mathrm{d}x \big). \tag{2.6.15}$$

The integral on the right-hand side of (2.6.15) allows us to evaluate Q^{areal} in terms of the geometry of the line and the two velocity components.

Note that the areal flow rate, Q^{areal}, has units of area divided by time. The associated volumetric flow rate with units of volume divided by time, is given by

$$Q = w \, Q^{\text{areal}}, \tag{2.6.16}$$

where w is a chosen width along the z axis.

Parcel expansion

The integral representation for the areal flow rate is also applicable in the case of a closed line, $\mathcal{L}$, described as a loop, as shown in Figure 2.6.3. In that case, the last point, B, simply coincides with the first point, A, yielding a closed integral,

$$Q^{\text{areal}} = \oint_{\mathcal{L}} u_n \, \mathrm{d}\ell = \oint_{\mathcal{L}} \big(u_x \, \mathrm{d}y - u_y \, \mathrm{d}x \big). \tag{2.6.17}$$

In fact, the areal flow rate across a closed loop is equal to the rate of change of the area of the fluid parcel that is enclosed by the loop at a certain instant, A_p, that is,

$$\frac{\mathrm{d}A_p}{\mathrm{d}t} = Q^{\text{areal}}. \tag{2.6.18}$$

The area of the parcel can change only if the fluid occupying the parcel is compressible.

2.6.3 Analytical integration

If a line has a sufficiently simple shape and the components of the velocity are known functions of position with simple forms, the integrals in (2.6.15) can be computed by standard analytical methods.

As an example, we consider a line that has the shape of a section of a circle of radius a centered at a point, $\mathbf{x}_c$, with end points corresponding to polar angles θ_{A} and θ_{B}. Points along the circular arc are described by the equations

$$x = x_c + a \cos\theta, \qquad y = y_c + a \sin\theta. \tag{2.6.19}$$

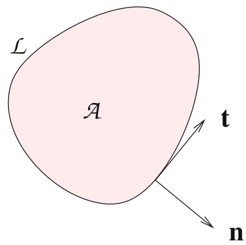


Figure 2.6.3 When the end points of a line coincide, we obtain a closed loop enclosing an area, $\mathcal{A}$.

Differentiating these equations with respect to θ, we obtain

$$\mathrm{d}x = -a\,\sin\theta\,\mathrm{d}\theta, \qquad \mathrm{d}y = a\,\cos\theta\,\mathrm{d}\theta. \tag{2.6.20}$$

Substituting these expressions into the last integral in (2.6.15), we obtain

$$Q^{\text{areal}} = a\int_{\theta_{\mathrm{A}}}^{\theta_{\mathrm{B}}} (\,u_x\cos\theta + u_y\sin\theta\,)\,\mathrm{d}\theta. \tag{2.6.21}$$

Substituting the expressions for the velocity components in terms of the angle θ, we obtain an integral representation in terms of θ.

As an application, we assume that

$$u_x = \frac{\alpha}{2\pi}\,\frac{1}{r}\,\cos\theta, \qquad u_y = \frac{\alpha}{2\pi}\,\frac{1}{r}\,\sin\theta, \tag{2.6.22}$$

where α is a constant and r is the distance from the origin. The flow rate is given by

$$Q^{\text{areal}} = a\,\frac{\alpha}{2\pi}\,\frac{1}{a}\int_{\theta_{\mathrm{A}}}^{\theta_{\mathrm{B}}}\mathrm{d}\theta = \frac{\alpha}{2\pi}\,(\theta_B - \theta_A). \tag{2.6.23}$$

If the circular segment forms a complete circle and the integration is performed in the counterclockwise direction from $\theta_{\mathrm{A}} = \theta_0$ to $\theta_{\mathrm{B}} = 2\pi - \theta_0$, then $Q_{\text{areal}} = \alpha$, independent of the radius, a, where θ_0 is an arbitrary angle.

2.6.4 Numerical integration

Under most conditions, we will not be able to compute the line integrals in (2.6.15) exactly by analytical methods and we must resort to numerical computation.

To perform numerical integration, we mark the location of a line with $N+1$ sequential nodes denoted by $\mathbf{x}_i$ for $i = 1, \ldots, N+1$, as depicted in Figure 2.6.4. The first node coincides with the first end point, A, and the last node coincides with the second end point, B. If the line is closed, the first node labeled 1 coincides with the last node labeled $N+1$.

Next, we approximate the shape of the line between two successive nodes labeled i and $i+1$ with a straight segment that passes through these nodes, denoted by E_i, where E

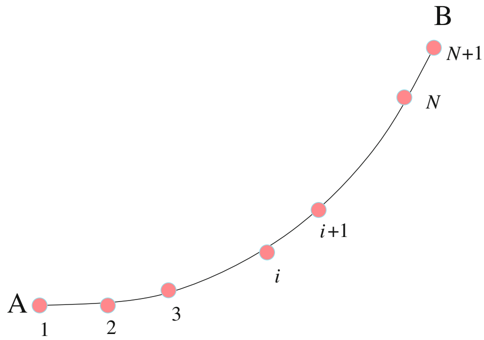


Figure 2.6.4 An array of points along a line in the xy plane is introduced to compute the areal flow rate across a line by numerical methods.

stands for *element.* The union of the N elements forms a polygonal line, called a polyline, starting at the first end point, A, and ending at the second end point, B.

Trapezoidal rule

A key step in developing a numerical approximation is the replacement of the line integrals in (2.6.15) with the sum of integrals over the elements, and the approximation of the velocity components over each element with the average of the values at the element end points. With these approximations, the last integral in (2.6.15) takes the form

$$Q^{\text{areal}} = \sum_{i=1}^{N} \Big[\frac{u_x(\mathbf{x}_i) + u_x(\mathbf{x}_{i+1})}{2} \, (y_{i+1} - y_i) - \frac{u_y(\mathbf{x}_i) + u_y(\mathbf{x}_{i+1})}{2} \, (x_{i+1} - x_i) \Big]. \qquad (2.6.24)$$

Writing out the sum and rearranging, we obtain

$$\begin{aligned} Q^{\text{areal}} = {} & \tfrac{1}{2} \big[u_x(\mathbf{x}_1)(y_2 - y_1) - u_y(\mathbf{x}_1)(x_2 - x_1) \big] \\ & + \tfrac{1}{2} \sum_{i=2}^{N} \big[u_x(\mathbf{x}_i)(y_{i+1} - y_{i-1}) - u_y(\mathbf{x}_i)(x_{i+1} - x_{i-1}) \big] \\ & + \tfrac{1}{2} \big[u_x(\mathbf{x}_{N+1})(y_{N+1} - y_N) - u_y(\mathbf{x}_{N+1})(x_{N+1} - x_N) \big]. \end{aligned} \qquad (2.6.25)$$

If the line is closed, nodes labeled 1 and $N+1$ coincide, and the first and last contributions on the right-hand side of (2.6.25) combine to yield the simpler form

$$Q^{\text{areal}} = \tfrac{1}{2} \sum_{i=1}^{N} \big[u_x(\mathbf{x}_i)(y_{i+1} - y_{i-1}) - u_y(\mathbf{x}_i)(x_{i+1} - x_{i-1}) \big], \qquad (2.6.26)$$

where the wrapped point labeled 0 coincides with the penultimate point labeled N.

The computation of the right-hand sides of (2.6.25) and (2.6.26) requires knowledge of the velocity components at the nodes. In practice, the nodal values are either given explicitly or computed by interpolation from grid values, as discussed in Section 1.7.

2.6.5 The Gauss divergence theorem in two dimensions

Consider a closed loop in the xy plane, denoted by $\mathcal{L}$, and a vector function of position $\mathbf{h} = (h_x, h_y)$, where $h_x(x, y)$ and $h_y(x, y)$ are two scalar functions. The normal component of $\mathbf{h}$ along $\mathcal{L}$ is given by the inner vector product

$$h_n \equiv \mathbf{h} \cdot \mathbf{n} = h_x \, n_x + h_y \, n_y. \tag{2.6.27}$$

The divergence of $\mathbf{h}$ is a scalar function of position given by

$$\boldsymbol{\nabla} \cdot \mathbf{h} \equiv \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y}. \tag{2.6.28}$$

The Gauss divergence theorem states that the line integral of the normal component, h_n, along the loop, $\mathcal{L}$, is equal to the integral of the divergence of $\mathbf{h}$ over the area $\mathcal{A}$ enclosed by $\mathcal{L}$,

$$\oint_{\mathcal{L}} \mathbf{h} \cdot \mathbf{n} \, \mathrm{d}\ell = \iint_{\mathcal{A}} \boldsymbol{\nabla} \cdot \mathbf{h} \, \mathrm{d}A, \tag{2.6.29}$$

where $\mathbf{n}$ is the unit vector normal to $\mathcal{L}$ pointing outward, $\mathrm{d}\ell$ is a differential arc length, and $\mathrm{d}A$ is a differential area.

Areal flow rate across a loop

Now we consider the areal flow rate across a closed loop, as shown in Figure 2.6.3. Applying (2.6.29) with $\mathbf{h} = \mathbf{u}$, we find that the areal flow rate across this loop is equal to the areal integral of the divergence of the velocity over the area enclosed by the loop,

$$Q^{\mathrm{areal}} = \oint_{\mathcal{L}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\ell = \iint_{\mathcal{A}} \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}A, \tag{2.6.30}$$

where the unit normal vector points outward, as shown in Figure 2.6.3. The expression on the right-hand side of (2.6.30) allows us to compute the instantaneous areal flow rate across a closed loop in terms of the integral of the rate of expansion over the enclosed area.

Incompressible fluids

It is clear from expression (2.6.30) that, if the velocity field is solenoidal, that is, the divergence of the velocity vanishes at every point,

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2.6.31}$$

then the areal flow rate across any closed loop is zero. In physical terms, fluid parcels deform and rotate but do not expand. As a consequence, the amount of fluid entering an area that is enclosed by a stationary closed loop is equal to the amount of fluid exiting the loop during any period of time.

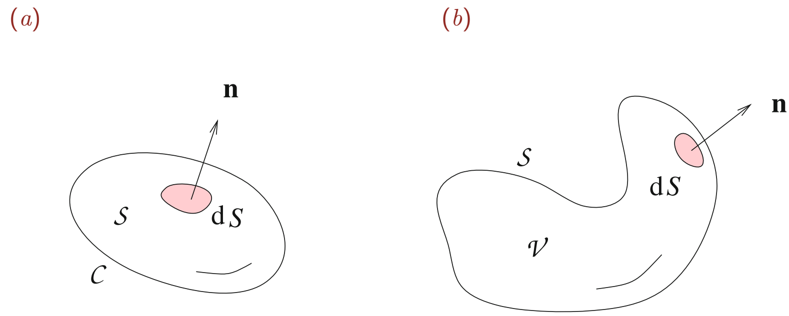


Figure 2.6.5 (a) Illustration of an open surface, $\mathcal{S}$, in a three-dimensional flow, bounded by a closed line, $\mathcal{C}$, (b) Illustration of a closed surface, $\mathcal{S}$, enclosing a volume, $\mathcal{V}$.

2.6.6 Flow rate in a three-dimensional flow

The preceding discussion for two-dimensional flow can be extended in a straightforward fashion to three-dimensional flow. To carry out this extension, we replace the line integrals along open or closed loops with surface integrals over open or closed surfaces residing inside the flow. The volumetric flow rate across an open or closed surface, $\mathcal{S}$, is given by the surface integral

$$Q = \iint_{\mathcal{S}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S. \tag{2.6.32}$$

The unit vector normal to $\mathcal{S}$, denoted by $\mathbf{n}$, and the differential area of a surface element, $\mathrm{d}S$, are defined in Figure 2.6.5. Note that Q has units of volume divided by time.

If V_p is the volume of a parcel confined by a closed surface, then the rate of change of the parcel volume is

$$\frac{\mathrm{d}V_p}{\mathrm{d}t} = Q. \tag{2.6.33}$$

Parcel expansion or shrinkage is possible only if the fluid is compressible.

2.6.7 Gauss divergence theorem in three dimensions

Consider a closed surface, $\mathcal{S}$, and a vector function of position, $\mathbf{h} = (h_x, h_y, h_z)$. The normal component of $\mathbf{h}$ over $\mathcal{S}$ is given by the inner product

$$h_n \equiv \mathbf{h} \cdot \mathbf{n} = h_x n_x + h_y n_y + h_z n_z. \tag{2.6.34}$$

The divergence of $\mathbf{h}$ is defined as

$$\boldsymbol{\nabla} \cdot \mathbf{h} \equiv \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z}. \tag{2.6.35}$$

The Gauss divergence theorem states that the surface integral of h_n over $\mathcal{S}$ is equal to the integral of the divergence of $\mathbf{h}$ over the volume $\mathcal{V}$ enclosed by $\mathcal{S}$,

$$\iint_{\mathcal{S}} \mathbf{h} \cdot \mathbf{n} \, \mathrm{d}S = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{h} \, \mathrm{d}V, \tag{2.6.36}$$

where $\mathbf{n}$ is the unit vector normal to the surface $\mathcal{S}$ pointing outward, $\mathrm{d}S$ is a differential surface area and $\mathrm{d}V$ is a differential volume.

Flow rate

Now we consider the flow rate across a closed surface $\mathcal{V}$, given in (2.6.32). Applying (2.6.36) for the fluid velocity, $\mathbf{h} = \mathbf{u}$, we obtain

$$Q = \iint_{\mathcal{S}} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}V. \tag{2.6.37}$$

This expression shows that, if the velocity field is solenoidal, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, the volumetric flow rate across any closed surface enclosing fluid alone must vanish.

2.6.8 Axisymmetric flow

Next, we consider an axisymmetric flow and draw a line that begins at a point, A, and ends at another point, B, in a azimuthal plane, as illustrated in Figure 2.6.6. The volumetric flow rate across the axisymmetric surface that arises by rotating the line around the x axis is given by

$$Q = 2\pi \int_{\mathrm{A}}^{\mathrm{B}} \sigma \, (u_x \, n_x + u_\sigma \, n_\sigma) \, \mathrm{d}\ell, \tag{2.6.38}$$

where $\mathrm{d}\ell$ is the differential arc length along the generating line, u_x it the velocity component along the x axis, and u_σ is the velocity component normal to the x axis.

Expression (2.6.38) arises by adding the fluxes across all elementary axisymmetric surfaces confined between two parallel planes that are perpendicular to the x axis and are separated by an infinitesimal distance, $\mathrm{d}x$, corresponding to the arc length, $\mathrm{d}\ell$, taking into consideration that the surface area of an elementary surface centered at a ring of radius σ is equal to $2\pi\sigma\mathrm{d}\ell$.

Substituting the components of the normal vector,

$$n_x = \frac{\mathrm{d}\sigma}{\mathrm{d}\ell}, \qquad n_\sigma = -\frac{\mathrm{d}x}{\mathrm{d}\ell}, \tag{2.6.39}$$

we obtain

$$Q = 2\pi \int_{\mathrm{A}}^{\mathrm{B}} \sigma \, (u_x \, \mathrm{d}\sigma - u_\sigma \, \mathrm{d}x). \tag{2.6.40}$$

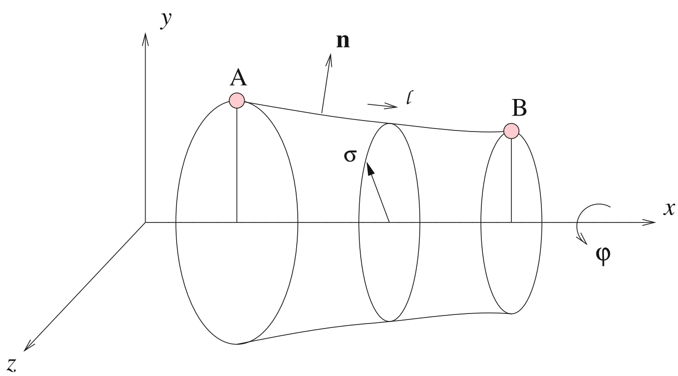


Figure 2.6.6 Illustration of an axisymmetric surface whose trace in an azimuthal plane is an open line that begins at a point, A, and ends at another point, B.

Problems

2.6.1 *Flow rate across an ellipse*

Consider a closed loop in the xy plane in the shape of a horizontal ellipse centered at a point, $\mathbf{x}_c = (x_c, y_c)$, with major and minor semi-axes equal to a and b. The elliptical shape is described in parametric form by the equations

$$x = x_c + a\cos\eta, \qquad y = y_c + b\sin\eta, \tag{2.6.41}$$

where η is the native parameter of the ellipse ranging in the interval $(0, 2\pi]$. We will assume that, in plane polar coordinates in the xy plane with origin at the center of the ellipse, (r, θ), the velocity components are given by

$$u_x = \frac{\alpha}{2\pi}\,\frac{1}{r}\,\cos\theta, \qquad u_y = \frac{\alpha}{2\pi}\,\frac{1}{r}\,\sin\theta, \tag{2.6.42}$$

where α is a constant. Show that

$$\tan\theta = \frac{b}{a}\,\tan\eta \tag{2.6.43}$$

and derive an expression for the flow rate across the ellipse as an integral with respect to η.

2.6.2 💻 *Flow rate across an ellipse*

With reference to Problem 2.6.1, write a code that computes the flow rate across the ellipse using a numerical method based on equation (2.6.26). Perform computations for ellipses with aspect ratios, $a/b = 1$, $2, 4$, and 8, in each case for $N = 8$, $16, 32$, and 64 numerical divisions. Discuss the results of your computations.

2.7 Mass conservation and the continuity equation

In Section 1.5, we defined a point particle as an idealized entity arising in the limit as the size of a small fluid parcel becomes decreasingly small and eventually infinitesimal. In this limit, the ratio between the mass of the parcel and the volume of the parcel tends to a finite, nonzero, and non-infinite value, which is defined as the fluid density, ρ. To indicate that the density is a function of position and time in a fluid, we write

$$\rho(\mathbf{x}, t), \tag{2.7.1}$$

with the understanding that the density at a particular point in a flow is equal to the density of the point particle that happens to be at that position at the designated time.

2.7.1 Mass flux and mass flow rate

Consider a two-dimensional flow in the xy plane and draw a *stationary line* that begins at a point, A, and ends at another point, B, as illustrated in Figure 2.6.1. At any instant, point particles cross this line, thereby generating a net *mass flow rate* in a specified direction. Our goal is to quantify this mass flow rate in terms of the shape of the line and the velocity and density distributions in the fluid.

Repeating the analysis of Section 2.6, we find that the mass flux across an infinitesimal section of the line is given by the following counterpart of equation (2.6.11),

$$q_{\text{mass}} = \rho\, u_n, \tag{2.7.2}$$

where $u_n = \mathbf{u} \cdot \mathbf{n}$ is the component of the fluid velocity normal to the line. The mass flow rate across a line that begins at a point, A, and ends at another point, B, is given by the following counterpart of equation (2.6.15),

$$Q^{\text{areal}}_{\text{mass}} = \int_{\text{A}}^{\text{B}} q_{\text{mass}}\, \mathrm{d}\ell = \int_{\text{A}}^{\text{B}} \rho\, \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}\ell = \int_{\text{A}}^{\text{B}} \rho\, \big(u_x\, \mathrm{d}y - u_y\, \mathrm{d}x \big). \tag{2.7.3}$$

Note that $Q^{\text{areal}}_{\text{mass}}$ has units of mass divided by length and time. The integrals in (2.7.3) can be computed by analytical or numerical methods, as discussed in Section 2.6.

2.7.2 Mass flow rate across a closed line

The net mass flow rate outward from a closed line in a two-dimensional flow can be expressed in terms of a closed line integral in the form

$$Q^{\text{areal}}_{\text{mass}} = \oint q_{\text{mass}}\, \mathrm{d}\ell = \oint \rho\, \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}\ell = \oint \rho\, \big(u_x\, \mathrm{d}y - u_y\, \mathrm{d}x \big), \tag{2.7.4}$$

where the unit normal vector, $\mathbf{n}$, points into the exterior of the area enclosed by the closed line, as depicted in Figure 2.6.3.

The Gauss divergence theorem expressed by equation (2.6.29) states that the line integral in (2.7.4) is equal to the integral of the divergence of the velocity multiplied by the fluid density over the area enclosed by the line, $\mathcal{A}$,

$$Q_{\text{mass}}^{\text{areal}} = \iint_{\mathcal{A}} \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u})\, \mathrm{d}x\, \mathrm{d}y, \tag{2.7.5}$$

where

$$\boldsymbol{\nabla} \cdot (\rho\, \mathbf{u}) = \frac{\partial(\rho\, u_x)}{\partial x} + \frac{\partial(\rho\, u_y)}{\partial y} \tag{2.7.6}$$

is the divergence of the mass velocity, $\rho\, \mathbf{u}$.

For future reference, we expand the derivatives of the products on the left-hand side of (2.7.6) using the rules of product differentiation, finding that

$$\boldsymbol{\nabla} \cdot (\rho\, \mathbf{u}) = \frac{\partial \rho}{\partial x}\, u_x + \frac{\partial u_x}{\partial x}\, \rho + \frac{\partial \rho}{\partial y}\, u_y + \frac{\partial u_y}{\partial y}\, \rho \tag{2.7.7}$$

or

$$\boldsymbol{\nabla} \cdot (\rho\, \mathbf{u}) = \mathbf{u} \cdot \nabla \rho + \rho\, \boldsymbol{\nabla} \cdot \mathbf{u}. \tag{2.7.8}$$

We have introduced the vector of the first partial derivatives of the density,

$$\boldsymbol{\nabla} \rho = (\frac{\partial \rho}{\partial x}, \quad \frac{\partial \rho}{\partial y}), \tag{2.7.9}$$

defined as the gradient of the density.

2.7.3 The continuity equation

The first principle of thermodynamics mandates that the rate of change of the mass residing inside an area, $\mathcal{A}$, that is enclosed by a stationary closed line, $\mathcal{L}$, given by

$$\iint_{\mathcal{A}} \rho\, \mathrm{d}A, \tag{2.7.10}$$

is equal to the mass flow rate inward across the line, which is equal to the negative of the mass flow rate outward across the line. If the outward mass flow rate is positive, the rate of the change of mass enclosed by the line is negative, reflecting a reduction in time.

In terms of the mass flow rate defined in equation (2.7.4) and expressed as an areal integral in equation (2.7.5), mass conservation requires that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathcal{A}} \rho\, \mathrm{d}A = -Q_{\text{mass}}^{\text{areal}} = -\oint_{\mathcal{L}} \rho\, \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}\ell = -\iint_{\mathcal{A}} \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u})\, \mathrm{d}A. \tag{2.7.11}$$

Since the area $\mathcal{A}$ is fixed in space, we can interchange the order of the time differentiation and space integration on the left-hand side of (2.7.11), and then combine the two integrals to obtain

$$\iint_{\mathcal{A}} \Big(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u}) \Big)\, \mathrm{d}A = 0. \tag{2.7.12}$$

Since the shape of the area $\mathcal{A}$ is arbitrary, the integrand on the right-hand side of (2.7.12) must be identically zero, yielding a partial differential equation in time-space expressing mass conservation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u}) = 0, \tag{2.7.13}$$

called the *continuity equation.* This terminology emphasizes that, in the absence of singularities in the form of point sources and sinks, mass neither appears nor disappears in the flow and the fluid must move in a continuous fashion in the available domain of flow.

Combining equations (2.7.8) and (2.7.13), we derive an alternative form of the continuity equation,

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho + \rho\, \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2.7.14}$$

involving the vectorial density gradient, $\boldsymbol{\nabla}\rho$, and the scalar rate of expansion, $\boldsymbol{\nabla} \cdot \mathbf{u}$.

Differential mass balance

It is instructive to derive the continuity equation based on a mass balance over a small stationary rectangular control area in the xy plane, as shown in Figure 2.7.1. Balancing the rate of mass accumulation inside the control area with the rates of mass crossing the four edges, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho\, \mathrm{d}x\, \mathrm{d}y\Big) = \big(\rho\, u_x \mathrm{d}y\big)_x - \big(\rho\, u_x \mathrm{d}y\big)_{x+\mathrm{d}x} + \big(\rho\, u_y \mathrm{d}x\big)_y - \big(\rho\, u_y \mathrm{d}x\big)_{y+\mathrm{d}y}. \tag{2.7.15}$$

Dividing both sides by $\mathrm{d}x\, \mathrm{d}y$ and noting that the variables, x, y, and t are independent, we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\big(\rho\, u_x\big)_x - \big(\rho\, u_x\big)_{x+\mathrm{d}x}}{\mathrm{d}x} + \frac{\big(\rho\, u_y \mathrm{d}x\big)_y - \big(\rho\, u_y \mathrm{d}x\big)_{y+\mathrm{d}y}}{\mathrm{d}y}. \tag{2.7.16}$$

To derive the continuity equation (2.7.13), we merely invoke the definition of the partial derivative.

2.7.4 Three-dimensional flow

Our discussion earlier in this section for two-dimensional flow can be generalized in a straightforward fashion to three-dimensional flow. To carry out this extension, we replace the line integrals with surface integrals over a closed or open surface. The mass flow rate

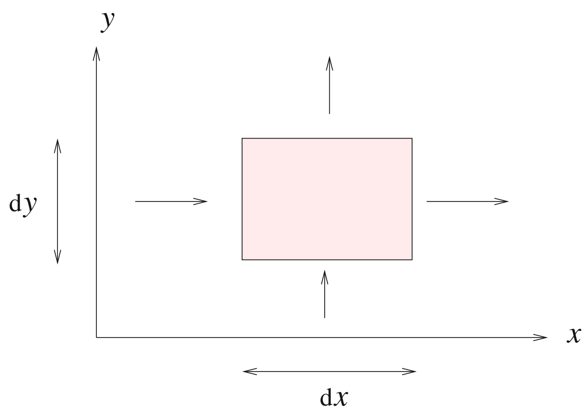


Figure 2.7.1 To derive the continuity equation for two-dimensional flow, we write a mass balance over an infinitesimal rectangular control area.

across a stationary, open or closed surface $\mathcal{S}$ depicted in Figure 2.6.5 is given by the surface integral

$$Q_{\text{mass}} = \iint_{\mathcal{S}} \rho\, \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S, \tag{2.7.17}$$

involving the normal velocity component, $u_n = \mathbf{u} \cdot \mathbf{n}$.

If the surface is closed and the unit normal vector points outward, as shown in Figure 2.6.5(b), we may use the divergence theorem to convert the surface integral on the right-hand side of (2.7.17) into an integral of the rate of expansion over the volume $\mathcal{V}$ enclosed by the surface, obtaining

$$Q_{\text{mass}} = \iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u})\, \mathrm{d}V. \tag{2.7.18}$$

The counterpart of the mass balance equation (2.7.11) is

$$\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, \mathrm{d}V = -Q_{\text{mass}} = -\iint_{\mathcal{S}} \rho\, \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S = -\iiint_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u})\, \mathrm{d}V. \tag{2.7.19}$$

Since the area D is fixed in space, we can interchange the order of

The continuity equation expressed by (2.7.13) or (2.7.14) stands true, with the understanding that $\boldsymbol{\nabla}\rho$ is the three-dimensional density gradient with components

$$\nabla \rho = \Big(\frac{\partial \rho}{\partial x}, \quad \frac{\partial \rho}{\partial y}, \quad \frac{\partial \rho}{\partial z} \Big) \tag{2.7.20}$$

defined over the domain of flow.

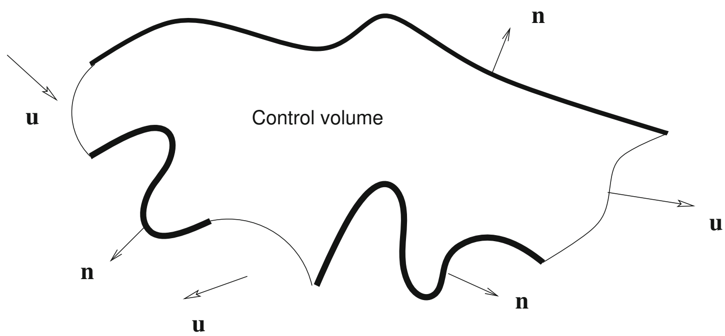


Figure 2.7.2 Illustration of a stationary control volume in a flow (cv) bounded by solid or fluid surfaces.

2.7.5 Control volume and integral mass balance

In the context of transport phenomena, a volume, $\mathcal{V}$, bounded by a closed surface, $\mathcal{S}$, is regarded as a control volume (cv), as shown in Figure 2.7.2. Equation (2.7.11) requires that

$$\iiint_{\text{cv}} \frac{\partial \rho}{\partial t}\,\mathrm{d}V + \iint_{\text{cv}} \rho\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = 0, \tag{2.7.21}$$

physically stating that mass accumulation in a stationary control volume is due to convective motion through the boundaries of the control volume. Equation (2.7.21) expresses an integral or macroscopic mass balance.

2.7.6 Rigid-body translation

When a fluid translates as a rigid body, the fluid velocity, $\mathbf{u}$, has a constant and possibly time-dependent value, $\mathbf{U}(t)$. In this case, the continuity equation (2.7.13) simplifies to a linear convection equation,

$$\frac{\partial \rho}{\partial t} + \mathbf{U}\cdot\boldsymbol{\nabla}\rho = 0. \tag{2.7.22}$$

Consider a steady flow where $\mathbf{U}$ is independent of time. Using equation (2.7.22), we find that, if $\rho_0(\mathbf{x})$ is the density field at $t = 0$, then

$$\rho(\mathbf{x}, t) = \rho_0(\mathbf{x} - \mathbf{U}\,t) \tag{2.7.23}$$

will be the density field at any other time, t. Physically, the density at the point $\mathbf{x} = \mathbf{x}_0 - \mathbf{U}\,t$ at time t is equal to the density at the point $\mathbf{x}_0$ at $t = 0$. We may say that the density field is *convected* by the uniform flow.

To confirm (2.7.23), we introduce an auxiliary vector variable, $\mathbf{w} = \mathbf{x} - \mathbf{U}\,t$, with components

$$w_x \equiv x - U_x\,t, \qquad w_y \equiv y - U_y\,t, \qquad w_z \equiv z - U_z\,t. \tag{2.7.24}$$

Using the chain rule of differentiation, we write

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho_0}{\partial w_x}\frac{\partial w_x}{\partial t} + \frac{\partial \rho_0}{\partial w_y}\frac{\partial w_y}{\partial t} + \frac{\partial \rho_0}{\partial w_z}\frac{\partial w_z}{\partial t}, \tag{2.7.25}$$

and then

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho_0}{\partial w_x}(-U_x) + \frac{\partial \rho_0}{\partial w_y}(-U_y) + \frac{\partial \rho_0}{\partial w_z}(-U_z). \tag{2.7.26}$$

The proof follows by observing that

$$\frac{\partial \rho_0}{\partial w_x} = \frac{\partial \rho_0}{\partial x}, \qquad \frac{\partial \rho_0}{\partial w_y} = \frac{\partial \rho_0}{\partial y}, \qquad \frac{\partial \rho_0}{\partial w_z} = \frac{\partial \rho_0}{\partial z}. \tag{2.7.27}$$

2.7.7 Evolution equation for the density

The continuity equation can be regarded as an evolution equation for the density, determined by the fluid velocity. To see this, we recast equation (2.7.13) into the form

$$\boxed{\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \mathbf{u}).} \tag{2.7.28}$$

Evaluating the right-hand side of (2.7.28) at a certain point, $\mathbf{x}$, in terms of the local and instantaneous velocity and density, we obtain an expression for the local and current rate of change of the density in time.

Temporal discretization

Consider the change in density occurring during a small time interval, Δt, following the current time t. Evaluating both sides of equation (2.7.28) at a point, $\mathbf{x}$, and approximating the right-hand side with a first-order forward difference, we obtain

$$\frac{\rho(\mathbf{x}, t + \Delta t) - \rho(\mathbf{x}, t)}{\Delta t} = -\boldsymbol{\nabla} \cdot (\rho \mathbf{u}), \tag{2.7.29}$$

where the right-hand side is evaluated at $(\mathbf{x}, t)$. Solving for $\rho(\mathbf{x}, t + \Delta t)$, we obtain

$$\boxed{\rho(\mathbf{x}, t + \Delta t) = \rho(\mathbf{x}, t) - \Delta t\, \boldsymbol{\nabla} \cdot (\rho\, \mathbf{u}),} \tag{2.7.30}$$

which provides us with an explicit expression for $\rho(\mathbf{x}, t + \Delta t)$ in terms of the density and velocity at the current time, t.

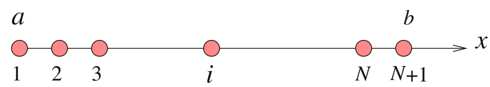
Finite-difference method

In practice, equation (2.7.28) is solved by numerical methods. Consider an idealized one-dimensional flow along the x axis representing, for example, the flow along a conduit with a known axial velocity, $u(x, t)$. The one-dimensional version of the continuity equation (2.7.28) is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho\, u)}{\partial x}. \tag{2.7.31}$$

The solution must be found inside a specified interval, $a \le x \le b$, subject to an initial condition that specifies the density distribution at the designated origin of time, $\rho(x, t = 0)$, and a boundary condition that specifies the density at the left end of the solution domain, $x = a$.

To develop the numerical method, we divide the solution domain into N intervals defined by $N + 1$ nodes, x_i for $i = 1, \ldots, N + 1$, as shown below:



The first node coincides with the left end point, $x = a$, and the last node coincides with the right end point, $x = b$. Our goal is to generate the values of ρ at the nodes at a sequence of time instants separated by the time interval Δt. To simplify the notation, we denote the density at the ith node at the kth time level, corresponding to time $t_k = k\,\Delta t$, by ρ_i^k.

Evaluating both sides of (2.7.31) at the ith node at the kth time level, and approximating the time derivative on the left-hand side with a first-order forward difference and the spatial derivative on the right-hand side with a first-order backward difference, we derive the finite-difference approximation

$$\frac{\rho_i^{k+1} - \rho_i^k}{\Delta t} = -\frac{(\rho u)_i^k - (\rho u)_{i-1}^k}{x_i - x_{i-1}}. \tag{2.7.32}$$

Solving for ρ_i^{k+1}, we obtain the updating formula

$$\rho_i^{k+1} = \rho_i^k - \frac{\Delta t}{x_i - x_{i-1}} \left[(\rho\, u)_i^k - (\rho u)_{i-1}^k \right]. \tag{2.7.33}$$

Algorithm

The numerical method involves the following steps:

1. Specify the initial values ρ_i^0 for $i = 1, \ldots, N + 1$.
2. Use equation (2.7.33) to compute ρ_i^1 for $i = 2, \ldots, N + 1$.
3. Use the left end boundary value to set the value of ρ_1^1.
4. Use equation (2.7.33) to compute ρ_i^2 for $i = 2, \ldots, N + 1$.
5. Use the left end boundary value to set the value of ρ_1^2.
6. Stop, or continue for further steps.

Note that a boundary condition at the right end of the solution domain is not required. Numerical analysis shows that the success of this method depends on the size of the time step, Δt, and sign of the convection velocity, u.

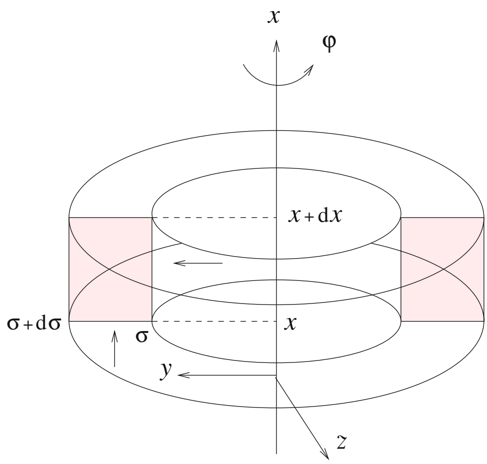


Figure 2.7.2 To derive the continuity equation for axisymmetric flow, we write a mass balance over an infinitesimal toroidal control volume in cylindrical polar coordinates.

2.7.8 *Continuity equation for axisymmetric flow*

Consider an axisymmetric flow and introduce cylindrical polar coordinates, (x, σ, φ), as shown in Figure 2.7.2. We will demonstrate that the continuity equation takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} + \frac{1}{\sigma}\frac{\partial\,(\sigma\rho\,u_\sigma)}{\partial \sigma} = 0. \tag{2.7.34}$$

To derive this equation, we perform a differential mass balance over a toroidal control volume with two sides parallel sides at x and $x+\mathrm{d}x$ and the other two coaxial sides at σ and $\sigma+\mathrm{d}\sigma$, as shown in Figure 2.7.2. The volume of the differential control volume is

$$\mathrm{d}V_{\mathrm{cv}} = 2\pi\sigma\,\mathrm{d}x\,\mathrm{d}\sigma \tag{2.7.35}$$

and the mass of the fluid residing inside the control volume at any instant is $\mathrm{d}m_{\mathrm{cv}} = \rho\,\mathrm{d}V_{\mathrm{cv}}$.

Balancing the rate of accumulation of fluid inside the control volume with the rates of convection of mass across the four sides, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho\,2\pi\sigma\mathrm{d}x\,\mathrm{d}\sigma\Big) &= \big(\rho\,u_x 2\pi\sigma\mathrm{d}\sigma\big)_x - \big(\rho\,u_x 2\pi\sigma\mathrm{d}\sigma\big)_{x+\mathrm{d}x} \\ &\quad + \big(\rho\,u_\sigma 2\pi\sigma\mathrm{d}x\big)_\sigma - \big(\rho\,u_\sigma 2\pi\sigma\mathrm{d}x\big)_{\sigma+\mathrm{d}\sigma}.\end{aligned} \tag{2.7.36}$$

Simplifying, we obtain

$$\sigma\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho\,\mathrm{d}x\,\mathrm{d}\sigma\Big) = \sigma\,\big(\rho\,u_x\mathrm{d}\sigma\big)_x - \sigma\,\big(\rho\,u_x\mathrm{d}\sigma\big)_{x+\mathrm{d}x} + \big(\rho\,u_\sigma\sigma\mathrm{d}x\big)_\sigma - \big(\rho\,u_\sigma 2\sigma\mathrm{d}x\big)_{\sigma+\mathrm{d}\sigma}. \tag{2.7.37}$$

Now dividing both sides by $\sigma\mathrm{d}x\mathrm{d}\sigma$ and noting that x, σ, and t are independent variables, we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\big(\rho\,u_x\big)_x - \big(\rho\,u_x\big)_{x+\mathrm{d}x}}{\mathrm{d}x} + \frac{1}{\sigma}\,\frac{\big(\sigma\rho\,u_\sigma\mathrm{d}x\big)_\sigma - \big(\sigma\rho\,u_\sigma\mathrm{d}x\big)_{\sigma+\mathrm{d}\sigma}}{\mathrm{d}\sigma}. \tag{2.7.38}$$

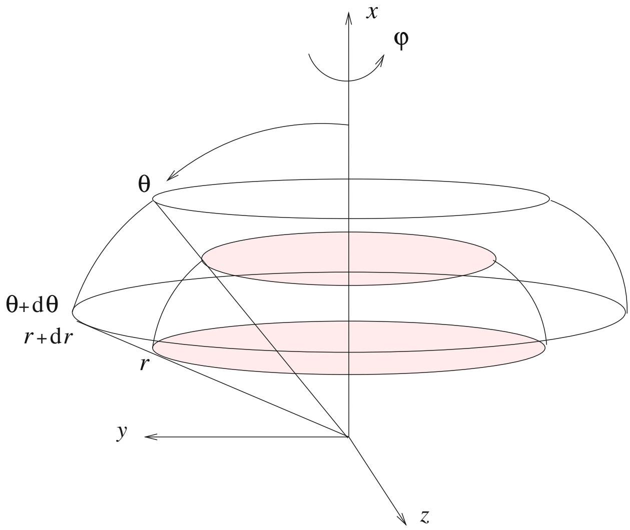


Figure 2.7.3 To derive the continuity equation for axisymmetric flow, we write a mass balance over an infinitesimal control volume in spherical polar coordinates.

To derive (2.7.34), we invoke the definition of the partial derivative and transfer all terms to the left-hand side. Note that the density, ρ, remains inside the derivatives.

In spherical polar coordinates, (r, θ, φ), equation (2.7.34) takes the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 \rho\, u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta \rho\, u_\theta)}{\partial \theta} = 0. \tag{2.7.39}$$

To derive this equation, we perform a differential mass balance over a toroidal control volume with two faces at r and $r + \mathrm{d}r$ and the other two faces at θ and $\theta + \mathrm{d}\theta$, as shown in Figure 2.7.3.

Problems

2.7.1 *Convection under constant velocity*

Consider the one-dimensional flow discussed in the text where the density field is governed by (2.7.31) with the velocity u being a constant. Sketch a profile of the density distribution along the x axis at the initial time, $t = 0$, and at a subsequent time.

2.7.2 *Steady state*

Consider a steady one-dimensional flow with a specified velocity distribution, $u(x)$. Derive an expression for the density distribution at steady state based on (2.7.31). Discuss the behavior of the density at a point where the velocity is zero.

2.7.3 *Finite-difference method*

Consider a steady, one-dimensional, periodic flow along the x axis with sinusoidal velocity distribution,

$$u(x) = U\left[1 + \epsilon \cos(2\pi x/L)\right], \tag{2.7.40}$$

where U is a constant velocity, ϵ is a specified dimensionless constant, and L is the period. Write a computer program that uses the numerical method discussed in the text to compute the evolution of the density over one spatial period, L, subject to a uniform initial distribution. Run the program for $\epsilon = 0$, 0.2, 0.4, and 0.8, prepare graphs of the density distribution at different times, and discuss the behavior of the solution at long times.

2.8 Properties of point particles

The physical properties of a homogeneous fluid parcel consisting of a single chemical species are determined by the number of molecules, the kinetic energy, the potential energy, and the thermal energy of the molecules that comprise the parcel. Each one of these physical properties is *extensive*, in that, the larger the parcel volume, the higher the magnitude of the physical property.

As the size of a parcel tends to zero, the ratio between the value of an extensive property and the parcel volume tends to a limit that is regarded as an *intensive* physical property of the point particle that emerges from the parcel immediately before the molecular nature of the fluid becomes apparent.

For example, we have already seen that, as the volume of a parcel tends to zero, the ratio between the mass of the parcel and the volume of the parcel tends to a finite limit that is defined as the fluid density, ρ. Similarly, the ratio of the number of molecules residing within the parcel and the volume of the parcel tends to the molecular number density, and the ratio of the potential energy of the molecules and the volume of the parcel tends to the specific potential energy.

2.8.1 The material derivative

To prepare the ground for establishing evolution laws governing the motion and physical state of a fluid, we seek corresponding laws determining the rate of change of physical and kinematic properties of point particles moving with the local fluid velocity. Kinematic properties include the point particle velocity and its first time derivative defined as the point particle acceleration, the vorticity, and the rate of strain.

A key concept is the material derivative, defined as the *rate of change of a physical or kinematic property following a point particle.* Our first objective is to derive an expression for the material derivative in terms of Eulerian derivatives; that is, partial derivatives with respect to spatial coordinates and time.

Taylor series expansion

Consider the material derivative of the density of a point particle which, at a certain time $t_{,0}$, is located at the point $\mathbf{x}_0$. In three-dimensional flow, the density is a function of four independent variables, including the three Cartesian coordinates, (x, y, z), determining position in space, and time, t.

We begin by linearizing the density field, $\rho(x, y, z, t)$, around (x_0, y_0, z_0, t_0), as discussed in Section 2.1. Adding time dependence to equation (2.1.6) and identifying the generic function $f(\mathbf{x}, t)$ with the density, we obtain the linearized form

$$\begin{aligned}\rho(\mathbf{x}, t) \simeq \rho(\mathbf{x}_0, t_0) + (t - t_0) \left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{x}_0, t_0} \\ +(x - x_0) \left(\frac{\partial \rho}{\partial x}\right)_{\mathbf{x}_0, t_0} + (y - y_0) \left(\frac{\partial \rho}{\partial y}\right)_{\mathbf{x}_0, t_0} + (z - z_0) \left(\frac{\partial \rho}{\partial z}\right)_{\mathbf{x}_0, t_0},\end{aligned} \tag{2.8.1}$$

where $\mathbf{x}_0 = (x_0, y_0, z_0)$. Next, we bring the first term on the right-hand side, $\rho(\mathbf{x}_0)$, to the left-hand side, and divide both sides of the resulting equation by the time elapsed, $t - t_0$, to derive the expression

$$\frac{\rho(\mathbf{x}, t) - \rho(\mathbf{x}_0, t_0)}{t - t_0} = \left(\frac{\partial \rho}{\partial t}\right)_{\mathbf{x}_0, t_0} + \frac{x - x_0}{t - t_0} \left(\frac{\partial \rho}{\partial x}\right)_{\mathbf{x}_0, t_0} + \frac{y - y_0}{t - t_0} \left(\frac{\partial \rho}{\partial y}\right)_{\mathbf{x}_0, t_0} + \frac{z - z_0}{t - t_0} \left(\frac{\partial \rho}{\partial z}\right)_{\mathbf{x}_0, t_0}, \tag{2.8.2}$$

which is applicable at any point, $\mathbf{x}$, in the neighborhood of a chosen point of interest, $\mathbf{x}_0$, and for time t near t_0.

Moving with the fluid

The second key step involves the judicious choice of the field point, $\mathbf{x}$. This point is selected so that, if a point particle is located at the position $\mathbf{x}_0$ at time t_0, then the same point particle is located at the position $\mathbf{x}$ at a later time, t. By definition then, the left-hand side of (2.8.2) reduces to the material derivative.

Since the point particle moves with the fluid velocity, the three fractions on the right-hand side of (2.8.2) are equal the three components of the fluid velocity, u_x, u_y, and u_z. Denoting the material derivative by $\mathrm{D}/\mathrm{D}t$, we find that

$$\left(\frac{\mathrm{D}\rho}{\mathrm{D}t}\right)_{\mathbf{x}_0, t_0} = \left(\frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z}\right)_{\mathbf{x}_0, t_0}. \tag{2.8.3}$$

In terms of the density gradient defined in (2.7.20), equation (2.8.3) takes the simpler form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho, \tag{2.8.4}$$

where both sides are evaluated at the arbitrary point, $\mathbf{x}_0$, at an arbitrary time instant, t_0.

Lagrangian and Eulerian derivatives

Equation (2.8.3) allows us to compute the material derivative of the density, sometimes also called the Lagrangian derivative, in terms of Eulerian derivatives, that is, in terms of partial derivatives of the density with respect to time and spatial coordinates, x, y, and z. In numerical practice, the partial derivatives are computed by finite-difference approximations, as discussed in Section 2.5.

2.8.2 The continuity equation

Comparing equations (2.8.4) and (2.7.14), we find that, in terms of the material derivative of the density, the continuity equation takes the form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \tag{2.8.5}$$

which reveals that the rate of change of the density of a point particle is determined exclusively by the local rate of expansion, $\boldsymbol{\nabla}\cdot\mathbf{u}$. However, the inverse interpretation is physically more appropriate: *the structure of the velocity field is determined, in part, by the rate of change of the density of all point particles.*

Consider a small fluid parcel with volume δV_p, density ρ, and mass $\delta m_p = \rho\,\delta V_p$. Mass conservation requires that δm_p remains constant in time, $\mathrm{D}\,\delta m_p/\mathrm{D}t = 0$. Expanding the material derivative, we obtain

$$\frac{\mathrm{D}(\rho\,\delta V_p)}{\mathrm{D}t} = \delta V_p\,\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\frac{\mathrm{D}\,\delta V_p}{\mathrm{D}t} = 0 \tag{2.8.6}$$

(Problem 2.8.1). Using the continuity equation (2.8.5) and rearranging, we find that

$$\frac{1}{\delta V_p}\,\frac{\mathrm{D}\,\delta V_p}{\mathrm{D}t} = \boldsymbol{\nabla}\cdot\mathbf{u}, \tag{2.8.7}$$

which reinforces our interpretation of the divergence of the velocity as the rate of volumetric expansion.

2.8.3 Point particle acceleration

The acceleration of a point particle, $\mathbf{a}$, is defined as the rate of change of the point particle velocity. Invoking the definition of the material derivative, we find that the x component of the acceleration is equal to the material derivative of the x component of the point particle velocity, which is equal to the local fluid velocity, $a_x = \mathrm{D}u_x/\mathrm{D}t$. Similar arguments reveal that

$$a_x = \frac{\mathrm{D}u_x}{\mathrm{D}t}, \qquad a_y = \frac{\mathrm{D}u_y}{\mathrm{D}t}, \qquad a_z = \frac{\mathrm{D}u_z}{\mathrm{D}t}. \tag{2.8.8}$$

In vector form,

$$\mathbf{a} = \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}. \tag{2.8.9}$$

Not surprisingly, the acceleration vector is the material derivative of the velocity vector.

Replacing ρ in equation (2.8.4) with u_x, u_y, or u_z, we find that the three Cartesian components of the point particle acceleration are given by

$$\begin{aligned}
a_x &\equiv \frac{\mathrm{D}u_x}{\mathrm{D}t} = \frac{\partial u_x}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} u_x = \frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z},\\
a_y &\equiv \frac{\mathrm{D}u_y}{\mathrm{D}t} = \frac{\partial u_y}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} u_y = \frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z},\\
a_z &\equiv \frac{\mathrm{D}u_z}{\mathrm{D}t} = \frac{\partial u_z}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} u_z = \frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}.
\end{aligned} \tag{2.8.10}$$

The three scalar equations (2.8.10) can be collected conveniently into a vector form,

$$\mathbf{a} \equiv \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\mathbf{L} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}, \tag{2.8.11}$$

where $\mathbf{L} = \boldsymbol{\nabla}\mathbf{u}$ is the velocity-gradient tensor defined in equation (2.1.16), with components $L_{ij} = \partial u_j/\partial x_i$. In index notation, the jth component of (2.8.11) takes the form

$$\frac{\mathrm{D}u_j}{\mathrm{D}t} = \frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i}, \tag{2.8.12}$$

where summation is implied over the repeated index i.

Linear momentum

The linear momentum of a small fluid parcel is the product of the mass of the parcel, $\delta m_p = \rho\,\delta V_p$, and the parcel velocity, $\mathbf{u}$. Requiring mass conservation, that is, demanding that δm_p remains constant in time, we find that the rate of change of the linear momentum can be expressed in terms of the point particle acceleration in the form

$$\frac{\mathrm{D}\,(\delta m_p\mathbf{u})}{\mathrm{D}t} = \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\rho\,\delta V_p = \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\delta m_p. \tag{2.8.13}$$

Thus, the mass of an infinitesimal parcel can be extracted from the material derivative, just like a constant can be extracted from an ordinary derivative.

Cylindrical polar coordinates

In the cylindrical polar coordinates defined in Figure 1.3.2, the point particle acceleration is expressed in terms of its cylindrical polar components, a_x, a_σ, and a_φ, as

$$\mathbf{a} = a_x\,\mathbf{e}_x + a_\sigma\,\mathbf{e}_\sigma + a_\varphi\,\mathbf{e}_\varphi. \tag{2.8.14}$$

Using the transformation rules shown in (1.3.20), we find that

$$a_\sigma = \cos\varphi\,a_y + \sin\varphi\,a_z, \qquad a_\varphi = -\sin\varphi\,a_y + \cos\varphi\,a_z. \tag{2.8.15}$$

Substituting the right-hand sides of the second and third relations in (2.8.10) into the right-hand sides of the equations in (2.8.15), and then using the chain rule of differentiation to convert derivatives with respect to x, y, and z to derivatives with respect to x, σ, and φ in the resulting equations as well as in the first equation in (2.8.10), we obtain

$$a_x = \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_\sigma \frac{\partial u_x}{\partial \sigma} + \frac{u_\varphi}{\sigma} \frac{\partial u_x}{\partial \varphi},$$

$$a_\sigma = \frac{\partial u_\sigma}{\partial t} + u_x \frac{\partial u_\sigma}{\partial x} + u_\sigma \frac{\partial u_\sigma}{\partial \sigma} + \frac{u_\varphi}{\sigma} \frac{\partial u_\sigma}{\partial \varphi} - \frac{u_\varphi^2}{\sigma}, \tag{2.8.16}$$

$$a_\varphi = \frac{\partial u_\varphi}{\partial t} + u_x \frac{\partial u_\varphi}{\partial x} + u_\sigma \frac{\partial u_\varphi}{\partial \sigma} + \frac{u_\varphi}{\sigma} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\sigma u_\varphi}{\sigma}.$$

Using the expression for the gradient of a function in cylindrical polar coordinates defined in equations (2.1.37) and (2.1.43), we recast expressions (2.8.16) into compact form involving the material derivative,

$$a_x = \frac{\partial u_x}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} u_x = \frac{\mathrm{D} u_x}{\mathrm{D}t},$$

$$a_\sigma = \frac{\partial u_\sigma}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} u_\sigma - \frac{u_\varphi^2}{\sigma} = \frac{\mathrm{D} u_\sigma}{\mathrm{D}t} - \frac{u_\varphi^2}{\sigma},$$

$$a_\varphi = \frac{\partial u_\varphi}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} u_\varphi + \frac{u_\sigma u_\varphi}{\sigma} = \frac{\mathrm{D} u_\sigma}{\mathrm{D}t} + \frac{u_\sigma u_\varphi}{\sigma}. \tag{2.8.17}$$

These expressions illustrate that the cylindrical polar components of the acceleration are *not* simply equal to the material derivative of the corresponding polar components of the velocity.

Spherical polar coordinates

In the spherical polar coordinates depicted in Figure 1.3.3, the point particle acceleration is expressed in terms of its spherical polar components, a_r, a_θ, and a_φ, as

$$\mathbf{a} = a_r\, \mathbf{e}_r + a_\theta\, \mathbf{e}_\theta + a_\varphi\, \mathbf{e}_\varphi. \tag{2.8.18}$$

Working as previously for the cylindrical polar coordinates, we find the somewhat more involved expressions

$$a_r = \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_\varphi}{r \sin\theta} \frac{\partial u_r}{\partial \varphi} - \frac{u_\theta^2 + u_\varphi^2}{r},$$

$$a_\theta = \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\varphi}{r \sin\theta} \frac{\partial u_\theta}{\partial \varphi} + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r}, \tag{2.8.19}$$

$$a_\varphi = \frac{\partial u_\varphi}{\partial t} + u_r \frac{\partial u_\varphi}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\varphi}{\partial \theta} + \frac{u_\varphi}{r \sin\theta} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r u_\varphi}{r} + \frac{u_\theta u_\varphi}{r} \cot\theta,$$

which can be expressed in a more compact form involving the material derivative,

$$a_r = \frac{\mathrm{D} u_r}{\mathrm{D}t} - \frac{u_\theta^2 + u_\varphi^2}{r}, \qquad a_\theta = \frac{\mathrm{D} u_\theta}{\mathrm{D}t} + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r}, \tag{2.8.20}$$

$$a_\varphi = \frac{\mathrm{D} u_\theta}{\mathrm{D}t} + \frac{u_r u_\varphi}{r} + \frac{u_\theta u_\varphi}{r} \cot\theta.$$

These expressions illustrate that the spherical polar components of the acceleration are *not* simply equal to the material derivative of the corresponding polar components of the velocity.

Plane polar coordinates

In the system of plane polar coordinates depicted in Figure 1.3.4, the point particle acceleration is expressed in terms of its plane polar components, a_r and a_θ, as

$$\mathbf{a} = a_r\,\mathbf{e}_r + a_\theta\,\mathbf{e}_\theta. \tag{2.8.21}$$

Working in the familiar way, we obtain

$$\begin{aligned} a_r &= \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial \theta} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = \frac{\mathrm{D}u_r}{\mathrm{D}t} - \frac{u_\theta^2}{r}, \\ a_\theta &= \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = \frac{\mathrm{D}u_\theta}{\mathrm{D}t} + \frac{u_r u_\theta}{r}. \end{aligned} \tag{2.8.22}$$

Note that these components are related to the σ and φ components in polar cylindrical coordinates.

Acceleration at a point with zero vorticity

If all components of the vorticity vector are zero at a certain point in a flow, the velocity gradient tensor is symmetric at that point. Consequently, selected partial derivatives of the velocity must be such that the three terms enclosed by the parentheses on the right-hand side of (2.3.8) are zero,

$$\frac{\partial u_z}{\partial y} = \frac{\partial u_y}{\partial z}, \qquad \frac{\partial u_x}{\partial z} = \frac{\partial u_z}{\partial x}, \qquad \frac{\partial u_y}{\partial x} = \frac{\partial u_x}{\partial y}. \tag{2.8.23}$$

The sum of the last three terms on the right-hand side of the first equation in (2.8.10) may then be written as

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_y}{\partial x} + u_z \frac{\partial u_z}{\partial x} = \tfrac{1}{2}\frac{\partial u_x^2}{\partial x} + \tfrac{1}{2}\frac{\partial u_y^2}{\partial x} + \tfrac{1}{2}\frac{\partial u_z^2}{\partial x} = \tfrac{1}{2}\frac{\partial (u_x^2 + u_y^2 + u_z^2)}{\partial x}. \tag{2.8.24}$$

Working in a similar fashion with the y and z components, and collecting the derived expressions into a vector form, we obtain

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = \tfrac{1}{2}\,\boldsymbol{\nabla} u^2 = \Big(\frac{\partial u^2}{\partial x}, \quad \frac{\partial u^2}{\partial y}, \quad \frac{\partial u^2}{\partial z}\Big), \tag{2.8.25}$$

where

$$u^2 \equiv u_x^2 + u_y^2 + u_z^2 \tag{2.8.26}$$

is the square of the magnitude of the velocity, and $\boldsymbol{\nabla} u^2$ is its gradient. The point particle acceleration may thus be expressed in the alternative form

$$\mathbf{a} \equiv \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + \tfrac{1}{2}\,\boldsymbol{\nabla} u^2. \tag{2.8.27}$$

The first term on the right-hand side of (2.8.27) is zero in a steady flow. The point particle acceleration is then equal to half the gradient of the square of the magnitude of the local velocity, which is a measure of the local kinetic energy of the fluid. We conclude that the acceleration is oriented in the direction of maximum change of kinetic energy indicated by the gradient.

In Chapter 6, we will see that the simplified expression (2.8.27) serves as a point of departure for the theoretical analysis and numerical computation of irrotational flows.

Problems

2.8.1 *Properties of the material derivative*

Consider two scalar physical or kinematic fluid properties, such as the density or a component of the velocity, denoted, respectively, by f and g. Prove that the following usual rule of product differentiation applies,

$$\frac{\mathrm{D}(fg)}{\mathrm{D}t} = g\,\frac{\mathrm{D}f}{\mathrm{D}t} + f\,\frac{\mathrm{D}g}{\mathrm{D}t}, \tag{2.8.28}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative.

2.8.2 *Point particle acceleration in rotational flow*

Show that the counterpart of equation (2.8.25) at a point where the vorticity $\boldsymbol{\omega}$ is not necessarily zero is the inclusive equation

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = \tfrac{1}{2}\,\boldsymbol{\nabla}u^2 - \mathbf{u}\times\boldsymbol{\omega}, \tag{2.8.29}$$

where u the magnitude of the velocity. How does this expression simplify at a point where the velocity vector is parallel to the vorticity vector?

2.8.3 *Point particle motion in one-dimensional flow*

Consider an idealized one-dimensional flow along the x axis with velocity $u(x,t)$ satisfying the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0. \tag{2.8.30}$$

Explain why point particles in this flow travel with a time-independent velocity that is equal to the velocity assigned to them at the initial instant; different point particles may travel with different velocities.

2.9 Incompressible fluids and stream functions

If the volume of a fluid parcel is preserved as the parcel is convected in a flow, the fluid residing inside the parcel is incompressible. In contrast, if the volume of the parcel is allowed to change in time, the fluid residing inside the parcel is compressible.

Mass conservation requires that the mass of any fluid parcel is conserved irrespective of whether the fluid is compressible or incompressible.

Since both the mass and the volume of an arbitrary incompressible fluid parcel are conserved during the motion, the density of the point particles that comprise the parcel remain constant in time. Using the physical interpretation of the material derivative, $\mathrm{D}/\mathrm{D}t$, we derive the mathematical statement of incompressibility,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \tag{2.9.1}$$

It is important to bear in mind that the density of an incompressible fluid is not necessarily uniform throughout the domain of flow. Different point particles may have different densities, but the density of each individual point particle is conserved during the motion.

2.9.1 Kinematic consequence of incompressibility

Using the incompressibility condition expressed by equation (2.9.1), we find that the continuity equation (2.8.5) for an incompressible fluid simplifies to

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2.9.2}$$

which states that the velocity field should be *solenoidal*. By definition, the divergence of any solenoidal vector field is identically zero. Consequently, the rate of expansion α defined in equation (2.2.6) is identically zero. An incompressible fluid parcel may undergo translation, rotation, and isochoric (volume-preserving) deformation, but not expansion. The word *isochoric* is composed from the Greek words $\iota\sigma o\varsigma$ which means equal, and the word $\chi\omega\rho o\varsigma$ which means volume or space.

It is important to bear in mind that the stipulation (2.9.1) is the defining property of an incompressible fluid, while the simplified form of the continuity equation (2.9.2) is a consequence of mass conservation.

2.9.2 Mathematical consequence of incompressibility

Equation (2.9.2) states that the x, y, and z components of the velocity of an incompressible fluid may not be prescribed arbitrarily, but must be such that the differential constraint imposed on them by the requirement that the velocity field be solenoidal is satisfied throughout the domain of flow at any time. In contrast, the three components of the velocity of a compressible fluid may be arbitrary; the density of the point particles will then adjust to ensure mass conservation, as dictated by the continuity equation.

A second important consequence of incompressibility is that, because the evolution of the density is governed by the kinematic constraint (2.9.1), an equation of state relating the pressure to the density to the temperature is not needed. The important significance of this consequence will be discussed further in Chapters 4 and 8 in the context of hydrodynamics.

2.9.3 Stream function for two-dimensional flow

The continuity equation for a two-dimensional flow in the xy plane stated in (2.9.2) takes the form

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0. \tag{2.9.3}$$

In computing the velocity field of an incompressible fluid by analytical or numerical methods, it is convenient to satisfy this constraint at the outset and concentrate on satisfying boundary conditions and other constraints that arise by balancing forces and torques, as will be discussed in later chapters.

To achieve this, we may express the two velocity components in terms of a scalar function, ψ, called the stream function, as

$$u_x = \frac{\partial \psi}{\partial y}, \qquad u_y = -\frac{\partial \psi}{\partial x}. \tag{2.9.4}$$

If the two velocity components, u_x and u_y derive from ψ by equations (2.9.4), then the satisfaction of the incompressibility constraint (2.9.3) is guaranteed. To confirm this, we substitute (2.9.4) into (2.9.3) and find that

$$\frac{\partial^2 \psi}{\partial x\,\partial y} - \frac{\partial^2 \psi}{\partial y\,\partial x} = 0. \tag{2.9.5}$$

Since the order of partial differentiation with respect to the two independent spatial variables x and y is immaterial, the equality is satisfied.

Extensional flow

As an example, we consider a two-dimensional flow with velocity components

$$u_x = \xi x, \qquad u_y = -\xi y \tag{2.9.6}$$

describing an extensional flow, where ξ is a constant with units of inverse time. It can be verified readily that the continuity equation is fulfilled, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. Substituting these expressions into (2.9.3), we confirm that the fluid is incompressible. The stream function corresponding to this flow is given by

$$\psi = \xi x y + c, \tag{2.9.7}$$

where c is an unspecified and inconsequential constant.

Non-uniqueness of the stream function

The example discussed in the last section illustrates that the stream function of a specified two-dimensional flow is not unique. Cursory inspection of equation (2.9.4) shows that an arbitrary constant may be added to a particular stream function to yield another perfectly acceptable stream function describing the same flow. However, this ambiguity is neither essential nor alarming. In performing analytical or numerical computation, the arbitrary constant simply provides us with one degree of freedom that can be used to simplify numerical and algebraic manipulations.

Physical interpretation

Consider the areal flow rate, Q^{areal}, across a line that begins at a point, A, and ends at another point, B, as illustrated in Figure 2.6.1. Substituting expressions (2.9.4) into the right-hand side of the last integral in (2.6.15) for the areal flow rate, we obtain

$$Q^{\text{areal}} = \int_{\mathrm{A}}^{\mathrm{B}} \left(\frac{\partial \psi}{\partial y}\,\mathrm{d}y + \frac{\partial \psi}{\partial x}\,\mathrm{d}x \right). \tag{2.9.8}$$

We may then write

$$Q^{\text{areal}} = \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}\psi = \psi_{\mathrm{B}} - \psi_{\mathrm{A}}, \tag{2.9.9}$$

where ψ_{A} and ψ_{B} are the values of the stream function at the end points, A and B.

Equation (2.9.9) shows that the difference in the values of the stream function between two points is equal to the areal flow rate across any arbitrary line that begins at the first point and ends at the second point. Because the fluid is incompressible, the flow rate is independent of the actual shape of the line, provided that the line begins and ends at two specified points.

Vorticity

The z component of the vorticity of a two-dimensional flow in the xy plane was given in equation (2.3.19) in terms of selected derivatives of the velocity,

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}. \tag{2.9.10}$$

Substituting expressions (2.9.4), we find that

$$\omega_z = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \equiv -\nabla^2 \psi, \tag{2.9.11}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.9.12}$$

is the Laplacian operator in the xy plane, as discussed in Section 3.2. Thus, the z component of the vorticity is equal to the negative of the Laplacian of the stream function.

If the stream function satisfies Laplace's equation, $\nabla^2\psi = 0$, the velocity field is solenoidal and the flow is irrotational. A function that satisfies Laplace's equation is called harmonic.

Plane polar coordinates

Departing from equations (2.9.4) and (2.3.19), and using the rules of coordinate transformation, we derive the velocity components of a two-dimensional flow in plane polar coordinates, (r, θ), in terms of the stream function,

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}. \tag{2.9.13}$$

The vorticity is

$$\omega_z = -\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial \psi}{\partial r}\Big) - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \equiv -\nabla^2 \psi, \tag{2.9.14}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2.9.15}$$

is the Laplacian operator in plane polar coordinates.

Expressions (2.9.13) satisfy the continuity equation in plane polar coordinates,

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} = 0 \tag{2.9.16}$$

for any differentiable and single valued stream function, ψ.

2.9.4 Stream function for axisymmetric flow

In the case of axisymmetric flow without swirling motion, we express all dependent and independent variables in the continuity equation, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, in cylindrical polar coordinates, (x, σ, φ). After carrying out a fair amount of algebra using the chain rule, we find that the continuity equation takes the form of a constraint on the axial and radial velocity components, u_x and u_σ,

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{1}{\sigma}\frac{\partial\,(\sigma u_\sigma)}{\partial \sigma} = 0. \tag{2.9.17}$$

To ensure the satisfaction of this equation, we express the axial and radial components of the velocity in terms of an axisymmetric stream function, ψ, also called the Stokes stream function, defined by the equations

$$u_x = \frac{1}{\sigma}\frac{\partial \psi}{\partial \sigma}, \qquad u_\sigma = -\frac{1}{\sigma}\frac{\partial \psi}{\partial x}. \tag{2.9.18}$$

Notice the minus sign in the second expression. Straightforward substitutions confirm that the velocity components given in (2.9.18) satisfy the continuity equation (2.9.17) for any regular stream function, ψ.

Extensional flow

As an example, we consider an axisymmetric flow with velocity components

$$u_x = \xi x, \qquad u_\sigma = -\tfrac{1}{2}\,\xi\sigma, \tag{2.9.19}$$

representing an extensional flow, where ξ is a constant with units of inverse time. Substituting these expressions into (2.9.17), we confirm that the left-hand side vanishes and the fluid is incompressible. The corresponding stream function is given by

$$\psi = \tfrac{1}{2}\,\xi x\sigma^2 + c, \tag{2.9.20}$$

where c is an unspecified constant.

Physical interpretation

Working as in Section 2.9.3 for two-dimensional flow, we find that the volumetric flow rate across an axisymmetric surface whose trace in an azimuthal plane of constant angle φ starts at a point, A, and ends at another point, B, as illustrated in Figure 2.6.6, is

$$Q = \psi_\mathrm{B} - \psi_\mathrm{A} \tag{2.9.21}$$

(Problem 2.9.2). This result is consistent with the units of the axisymmetric stream function, velocity multiplied by length squared, evident from equations (2.9.18). In contrast, the stream function for two-dimensional has units of velocity multiplied by length.

Vorticity

The azimuthal component of the vorticity in an axisymmetric flow was given in equation (2.3.22) in terms of derivatives of the cylindrical polar components of the velocity,

$$\omega_\varphi = \frac{\partial u_\sigma}{\partial x} - \frac{\partial u_x}{\partial \sigma}. \tag{2.9.22}$$

Substituting expressions (2.9.18), we obtain

$$\omega_\varphi = -\frac{1}{\sigma}\,\mathcal{E}^2\psi = -\frac{1}{\sigma}\,\Big(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial \sigma^2} - \frac{1}{\sigma}\frac{\partial\psi}{\partial\sigma}\Big), \tag{2.9.23}$$

where $\mathcal{E}^2$ is a second-order linear differential operator defined as

$$\mathcal{E}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \sigma^2} - \frac{1}{\sigma}\frac{\partial}{\partial\sigma}. \tag{2.9.24}$$

If the stream function is such that the right-hand side of (2.9.23) is zero throughout the domain of flow, the flow is irrotational.

Spherical polar coordinates

Departing from equations (2.9.18) and (2.3.22), and using the rules of coordinate transformation, we derive the velocity components in spherical polar coordinates, (r, θ, φ),

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}. \tag{2.9.25}$$

The azimuthal component of the vorticity is given by

$$\omega_\varphi = -\frac{1}{r \sin\theta} \mathcal{E}^2 \psi, \tag{2.9.26}$$

where $\mathcal{E}^2$ is the second-order differential operator defined in (2.9.24). In spherical polar coordinates,

$$\mathcal{E}^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\Big(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\Big) = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2} \frac{\partial}{\partial \theta}. \tag{2.9.27}$$

If the stream function is such that the right-hand side of (2.9.26) is zero throughout the domain of flow, the flow is irrotational.

Problems

2.9.1 *Stream function for two-dimensional flow*

Derive the Cartesian components of the velocity and the z vorticity component of a two-dimensional flow whose stream function is (a) $\psi = \frac{1}{2}\xi\, y^2$ or (b) $\psi = \frac{1}{2}\xi\,(x^2 - y^2)$, where ξ is a constant. Deduce the units of ξ and discuss the nature of each flow.

2.9.2 *Stream function of axisymmetric flow*

Substitute expressions (2.9.18) into the right-hand side of (2.6.38) and perform the integration to confirm (2.9.21).

2.10 Kinematic conditions at boundaries

In real life, a flow occurs in a domain that is bounded by stationary or moving surfaces with different constitutions and physical properties. Examples include the flow in an internal combustion engine generated by the motion of an engine piston, the flow induced by the motion of an aircraft or ground vehicle, the flow induced by the sedimentation of an aerosol particle in the atmosphere, the flow induced by a small bubble rising in a carbonated beverage, and the flow induced by the motion of an elephant running through the Savannah to escape a mouse.

Types of boundary conditions

In the context of kinematics, boundaries are classified into the following four main categories:

1. *Impermeable solid boundaries:* examples include the surface of a rigid or flexible solid body, such as a vibrating radio antenna or a swimming microorganism.

2. *Permeable solid boundaries:* examples include the surface of a porous medium, such as a rock bed or a biological tissue composed of cells separated by gaps in the intervening spaces.

3. *Sharp interfaces between immiscible fluids:* examples include the free surface of the ocean and the interface between oil and vinegar in an Italian salad dressing.

4. *Diffuse interfaces between miscible fluids:* examples include the fuzzy edge of a river discharging into the ocean and the ambiguous edge of a smoke ring rising in still air.

Different boundary conditions are imposed on each of these surfaces according to the prevailing physical context.

2.10.1 The no-penetration boundary condition

By definition, a point particle moving with the fluid velocity may not cross an impermeable solid boundary or a sharp interface between two immiscible fluids, but is required to lie on one side of the boundary or interface at all times. As a consequence, the velocity of a point particle that lies at a stationary or moving impermeable boundary or sharp interface must be consistent with, but not necessarily equal to, the velocity of the boundary or interface. To ensure compatibility, the no-penetration boundary condition is required.

Impermeable solid boundaries

Consider a flow that is bounded by an impermeable solid, but not necessarily rigid, boundary (rubber is a non-rigid, elastic yet solid boundary.) The no-penetration boundary condition requires that the component of the fluid velocity normal to the boundary is equal to the component of the boundary velocity normal to its instantaneous shape. The tangential component of the velocity is left unspecified. If the boundary is stationary, the normal component of the fluid velocity must vanish.

To derive the mathematical statement of the no-penetration condition, we introduce the unit vector normal to the boundary at a point, $\mathbf{n}$, and the velocity of the boundary, $\mathbf{v}^{\mathrm{B}}$, where the orientation of $\mathbf{n}$ is left unspecified. If the boundary is stationary, the boundary velocity is zero, $\mathbf{v}^{\mathrm{B}} = \mathbf{0}$; if the boundary translates as a rigid body, $\mathbf{v}^{\mathrm{B}}$ is constant; if the boundary rotates as a rigid body or exhibits some type of deformation, $\mathbf{v}^{\mathrm{B}}$ is a function of position, as will be discussed later in this section.

In all cases, the no-penetration boundary condition requires that

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{v}^{\mathrm{B}} \cdot \mathbf{n}, \tag{2.10.1}$$

where both sides are evaluated at a point on the boundary.

Rigid-body motion

Consider an impermeable rigid boundary that translates with velocity $\mathbf{U}^{\rm B}$ while rotating with angular velocity $\mathbf{\Omega}^{\rm B}$ around a specified center of rotation, $\mathbf{x}_{\rm R}$. The angular velocity vector, $\mathbf{\Omega}^{\rm B}$, passes through the center of rotation, $\mathbf{x}_{\rm R}$. The magnitude and orientation of $\mathbf{\Omega}^{\rm B}$ express the rate of direction and direction of rotation. As we look down at the angular velocity vector from above, the body rotates in the counterclockwise direction.

In terms of the velocity of translation and angular velocity of rotation, the velocity at a point $\mathbf{x}$ that lies at the boundary is given by the expression

$$\mathbf{v}^{\rm B} = \mathbf{U}^{\rm B} + \mathbf{\Omega}^{\rm B} \times (\mathbf{x} - \mathbf{x}_{\rm R}), \tag{2.10.2}$$

where $\times$ denotes the outer vector product defined in equation (2.3.5). In component form,

$$\begin{aligned} \mathbf{v}^{\rm B} = {} & [\, U_x^{\rm B} + \Omega_y^{\rm B}\,(z - z_{\rm R}) - \Omega_z^{\rm B}\,(y - y_{\rm R})\,]\,\mathbf{e}_x \\ & + [\, U_y^{\rm B} + \Omega_z^{\rm B}\,(x - x_R) - \Omega_x^{\rm B}\,(z - z_{\rm R})\,]\,\mathbf{e}_y \\ & + [\, U_z^{\rm B} + \Omega_x^{\rm B}\,(y - y_{\rm R}) - \Omega_y^{\rm B}\,(x - x_{\rm R})\,]\,\mathbf{e}_z, \end{aligned} \tag{2.10.3}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit vectors along the x, y, or z axes.

In the case of two-dimensional flow in the xy plane, the z velocity component is zero, $U_z^{\rm B} = 0$, and the angular velocity vector is parallel to the z axis, $\Omega_x^{\rm B} = 0$ and $\Omega_y^{\rm B} = 0$, yielding the simplified form

$$\mathbf{v}^{\rm B} = \big[\, U_x^{\rm B} - \Omega_z^{\rm B}\,(y - y_{\rm R})\,\big]\,\mathbf{e}_x + \big[\, U_y^{\rm B} + \Omega_z^{\rm B}\,(x - x_{\rm R})\,\big]\,\mathbf{e}_y, \tag{2.10.4}$$

which is linear in x and y.

The no-penetration boundary condition arises by substituting expression (2.10.3) or (2.10.4) into the right-hand side of (2.10.1), respectively, for three-dimensional or two-dimensional flow. If the boundary is stationary, $\mathbf{v}^{\rm B} = \mathbf{0}$, we obtain the simple form

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0}, \tag{2.10.5}$$

where the direction of the unit normal vector, $\mathbf{n}$, is unspecified.

The no-penetration condition in terms of the stream function

Next, we consider an incompressible fluid in a two-dimensional flow and express the velocity in terms of the stream function, ψ, defined in equations (2.9.4). The no-penetration boundary condition (2.10.1) requires that

$$\mathbf{u} \cdot \mathbf{n} = u_x\, n_x + u_y\, n_y = \frac{\partial \psi}{\partial y}\, n_x - \frac{\partial \psi}{\partial x}\, n_y = \mathbf{v}^{\rm B} \cdot \mathbf{n}. \tag{2.10.6}$$

Substituting expressions (2.6.4) for the components of the normal vector in terms of differential displacements along the boundary, we obtain

$$\frac{\partial \psi}{\partial y}\,\frac{{\rm d}y}{{\rm d}\ell} + \frac{\partial \psi}{\partial x}\,\frac{{\rm d}x}{{\rm d}\ell} = \frac{{\rm d}\psi}{{\rm d}\ell} = \mathbf{v}^{\rm B} \cdot \mathbf{n}, \tag{2.10.7}$$

where $\mathrm{d}\ell$ is an infinitesimal arc length measured along the boundary from an arbitrary origin.

If the boundary is stationary, the right-hand side of (2.1.8) is zero, $\mathrm{d}\psi/\mathrm{d}\ell = 0$, and the stream function is constant over the boundary. The no-penetration boundary condition takes the simple form

$$\psi = \psi_0, \tag{2.10.8}$$

where the constant ψ_0 is either assigned arbitrarily or computed as part of the solution.

Similar arguments can be made to show that the stream function is constant over an impermeable stationary boundary in axisymmetric flow (Problem 2.10.2(*b*)).

Sharp interfaces

Next, we consider the no-penetration condition over a stationary or moving sharp interface separating two immiscible fluids. Physical arguments suggest that the normal component of the fluid velocity on one side of the interface must be equal to the normal component of the velocity on the other side of the interface. However, the tangential velocities may be different.

To derive the mathematical statement of the no-slip condition, we introduce the velocity on one side of the interface, denoted by $\mathbf{u}^{(1)}$, and the velocity on the other side of the interface, denoted by $\mathbf{u}^{(2)}$, and require that

$$\mathbf{u}^{(1)} \cdot \mathbf{n} = \mathbf{u}^{(2)} \cdot \mathbf{n}, \tag{2.10.9}$$

where $\mathbf{n}$ is the unit vector normal to the interface. Both sides are evaluated at a point at the interface with an unspecified direction of the unit normal vector, $\mathbf{n}$.

Problems

2.10.1 *Changing the center of rotation*

The center of rotation of a rigid body can be placed at any arbitrary position. Suppose that we choose a point, $\mathbf{x}'_{\mathrm{R}}$, instead of the point $\mathbf{x}_{\mathrm{R}}$ discussed in the text. The counterpart of equation (2.10.2) is

$$\mathbf{v}^{\mathrm{B}} = \mathbf{U}^{\mathrm{B}'} + \boldsymbol{\Omega}^{\mathrm{B}'} \times (\mathbf{x} - \mathbf{x}'_{\mathrm{R}}). \tag{2.10.10}$$

Set the right-hand side of (2.10.10) equal to the right-hand side of (2.10.2) to derive expressions for $\mathbf{U}^{\mathrm{B}'}$ and $\boldsymbol{\Omega}^{\mathrm{B}'}$ in terms of $\mathbf{U}^{B}$ and $\boldsymbol{\Omega}^{B}$, and *vice versa*.

2.10.2 *Stream functions*

(*a*) Use the no-penetration boundary condition to derive an expression for the stream function over a translating but non-rotating impermeable boundary in two-dimensional flow.

(*b*) Show that the no-penetration condition over a stationary boundary in axisymmetric flow takes the form expressed by (2.10.8).

http://www.springer.com/978-1-4899-7990-2

Fluid Dynamics
Theory, Computation, and Numerical Simulation
Pozrikidis, C.
2017, XIX, 901 p., Hardcover
ISBN: 978-1-4899-7990-2