Chapter 2
Divergence Operator

This chapter deals with solutions of the divergence in Sobolev spaces. We will say that a domain $\Omega \subset \mathbb{R}^n$ satisfies div_p if, for any $f \in L^p_0(\Omega)$, there exists $\mathbf{u} \in W^{1,p}_0(\Omega)^n$ such that

$$\text{div}\, \mathbf{u} = f \qquad \text{in } \ \Omega$$

and

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}$$

where the constant C depends only on Ω and p. First we consider the case of domains which are star-shaped with respect to a ball and give the construction introduced by Bogovskiǐ [14]. In his original paper Bogovskiǐ extended the existence of solutions to the case of Lipschitz domains using that this kind of domains can be written as a finite union of star-shaped domains. In the second section, we extend the construction to the class of John domains, this kind of domains includes the Lipschitz ones as well as many domains with fractal boundaries. The construction analyzed here was given in [3]. The proof that we present is a modification of the original one.

Of course, star-shaped domains are a particular case of John domains. The reason why we present first Bogovskiǐ's construction is because it is simpler and allows to present the main ideas with less technical difficulties. On the other hand, the analysis was extended in [47] to generalize the results for right-hand sides in negative order Sobolev spaces. We do not know whether the results in [47] can be extended to John domains.

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2.1 Solutions of the Divergence on Star-Shaped Domains

Let us begin by recalling the class of star-shaped domains.

Definition 2.1.1 *A bounded open $\Omega \subset \mathbb{R}^n$ is star-shaped with respect to a ball $B \subset \Omega$ if for every $y \in \Omega$ and every $z \in B$ the segment joining y and z is contained in Ω.*

Actually, given an arbitrary domain Ω and $f \in L^1(\Omega)$, Bogovskiĭ's construction gives a solution of $\operatorname{div} \mathbf{u} = f$, but in general, $\mathbf{u}$ will not vanish on the boundary of Ω. However, as we will show, if Ω is star-shaped with respect to a ball, then $\mathbf{u} = 0$ on $\partial\Omega$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter δ. Take $\omega \in C_0^\infty(\Omega)$ such that $\int_\Omega \omega = 1$ and define $G = (G_1, \cdots, G_n)$ as

$$G(x,y) = \int_0^1 \frac{(x-y)}{s}\, \omega\left(y + \frac{x-y}{s}\right) \frac{ds}{s^n} \tag{2.1.1}$$

The following lemma gives a bound for $G(x,y)$ that will be fundamental in our subsequent arguments.

Lemma 2.1. *For $y \in \Omega$ we have*

$$|G(x,y)| \le \|\omega\|_{L^\infty(\Omega)} \frac{\delta^n}{(n-1)|x-y|^{n-1}} \tag{2.1.2}$$

Proof. Since $\omega \in C_0^\infty(\Omega)$, it follows that the integrand in (2.1.1) vanishes for $z = y + (x-y)/s \notin \Omega$. Therefore, since $y \in \Omega$, we can restrict the integral defining $G(x,y)$ to those values of s such that $|z-y| \le \delta$, that is, $|x-y|/\delta \le s$, and so,

$$|G(x,y)| \le \delta \|\omega\|_{L^\infty(\Omega)} \int_{|x-y|/\delta}^1 \frac{ds}{s^n}$$

which immediately gives (2.1.2). □

In the next lemmas and theorem we introduce the explicit right inverse of the divergence.

Lemma 2.2. *For any $\varphi \in C_0^\infty(\Omega)$ we define $\varphi_\omega = \int_\Omega \varphi\, \omega$. Then, for $y \in \Omega$ we have*

$$(\varphi - \varphi_\omega)(y) = -\int_\Omega G(x,y) \cdot \nabla\varphi(x)\, dx$$

Proof. Extending by zero we can think $\varphi \in C_0^\infty(\mathbb{R}^n)$. Repeating the arguments given in the introduction, see (1.0.11), we have, for $y \in \Omega$,

$$(\varphi - \varphi_\omega)(y) = \int_\Omega \int_0^1 (y-z) \cdot \nabla\varphi(y + s(z-y))\, \omega(z)\, ds\, dz$$

and interchanging the order of integration and making the change of variable $x = y + s(z - y)$ we obtain

$$(\varphi - \varphi_\omega)(y) = \int_0^1 \int_\Omega \frac{(y-x)}{s} \cdot \nabla\varphi(x)\,\omega\Big(y + \frac{x-y}{s}\Big)\,dx\,\frac{ds}{s^n}$$

and the proof concludes by observing that we can interchange again the order of integration. Indeed, using the bound given in (2.1.2) for G, it is easy to see that the integral of the absolute value of the integrand is finite. □

By duality, we obtain the following fundamental result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain. Given $f \in L^1(\Omega)$ such that $\int_\Omega f = 0$, define*

$$\mathbf{u}(x) = \int_\Omega G(x,y)\,f(y)\,dy \tag{2.1.3}$$

then,

$$\operatorname{div}\mathbf{u} = f \qquad in \;\; \Omega$$

Proof. First, observe that, in view of (2.1.2), $\mathbf{u}$ is well defined and all its components belong to L^1_{loc}. In particular, $\operatorname{div}\mathbf{u}$ is well defined in the sense of distributions.

Now, using Lemma 2.2, for $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega f(y)\varphi(y)\,dy = \int_\Omega f(y)(\varphi - \varphi_\omega)(y)\,dy = -\int_\Omega \int_\Omega f(y)G(x,y)\cdot\nabla\varphi(x)\,dx\,dy$$

and interchanging the order of integration, which can be done using again (2.1.2), we obtain

$$\int_\Omega f(y)\varphi(y)\,dy = -\int_\Omega \mathbf{u}(x)\cdot\nabla\varphi(x)\,dx$$

which concludes the proof. □

Up to this point, we have not imposed any condition on the domain Ω other than boundedness. Assume now that $\Omega \subset \mathbb{R}^n$ is star-shaped with respect to a ball $B \subset \Omega$. The following lemma shows that, if we choose ω supported in B, then the function $\mathbf{u}$ defined in (2.1.3) vanishes on $\partial\Omega$. In principle, this will be true when $f \in L^p(\Omega)$ for some $p > n$ since, in this case, one can see that $\mathbf{u}$ defined in (2.1.3) is continuous. This will be proved in the next proposition. For other values of p, we can proceed by density to show that $\mathbf{u} \in W_0^{1,p}$ once we have proved that $\mathbf{u} \in W^{1,p}$.

In all what follows we extend f by zero outside of Ω, and therefore, we can think that $f \in L^p(\mathbb{R}^n)$ whenever $f \in L^p(\Omega)$, but we will write $f \in L^p(\Omega)$ to emphasize that f vanishes outside Ω. Analogously functions in $C_0^\infty(\Omega)$ will be thought as being in $C_0^\infty(\mathbb{R}^n)$. Moreover, we can make the following important observation.

Remark 2.1. The definition of $\mathbf{u}$ given in (2.1.3) can be extended to every $x \in \mathbb{R}^n$.

Proposition 2.1. *Let $f \in L^p(\Omega)$ for some $p > n$. If Ω is star-shaped with respect to a ball B and $\omega \in C_0^\infty(B)$, then $\mathbf{u}$ defined in (2.1.3) is continuous in $\mathbb{R}^n$ and vanishes outside Ω, in particular, $\mathbf{u}(x) = 0$ for all $x \in \partial\Omega$.*

Proof. First we observe that

$$G(x,y) = 0 \ \text{ whenever } \ x \notin \Omega, y \in \Omega \tag{2.1.4}$$

Indeed, in that case we have that, $z = y + (x-y)/s \notin B$, for any $s \in [0,1]$. Otherwise, since Ω is star-shaped with respect to B, $x = (1-s)y + sz$ would be in Ω. Therefore, recalling that $\omega \in C_0^\infty(B)$ and the definition of $G(x,y)$ we obtain (2.1.4. Consequently, $\mathbf{u} = 0$ for all $x \notin \Omega$.

Therefore, it is enough to prove continuity of $\mathbf{u}$ in an open bounded set containing $\overline{\Omega}$. Take x and $\bar{x}$ in a neighborhood of $\overline{\Omega}$. We have

$$G(x,y) - G(\bar{x},y) = \int_0^1 \left\{ \frac{(x-y)}{s}\, \omega\Big(y + \frac{x-y}{s}\Big) - \frac{(\bar{x}-y)}{s}\, \omega\Big(y + \frac{\bar{x}-y}{s}\Big) \right\} \frac{ds}{s^n}$$

Now, for y and z varying in a bounded domain, the function $z\omega(y+z)$ is Hölder α, for any $0 < \alpha < 1$, as a function of z, uniformly in y. Therefore, assuming, for example, $|x-y| \le |\bar{x}-y|$, and using that the integrand in the definition of $G(x,y)$ vanishes if $s < |x-y|/\delta$ we obtain

$$|G(x,y) - G(\bar{x},y)| \le C|x-\bar{x}|^\alpha \int_{|x-y|/\delta}^1 \frac{ds}{s^{n+\alpha}} \le C \frac{|x-\bar{x}|^\alpha}{|x-y|^{n-1+\alpha}}$$

with C depending only on δ, n, ω, and α. Then,

$$|\mathbf{u}(x) - \mathbf{u}(\bar{x})| \le C|x-\bar{x}|^\alpha \int_\Omega \frac{|f(y)|}{|x-y|^{n-1+\alpha}}\, dy$$

and the proof concludes by observing that, since $p > n$, we can choose $\alpha > 0$ such that $(n-1+\alpha)p' < n$, and using the Hölder inequality. □

We want to prove that $\mathbf{u} \in W^{1,p}(\Omega)^n$. It is not difficult to prove that $\mathbf{u} \in L^p(\Omega)^n$. Indeed, using the bound (2.1.2) we have

$$|\mathbf{u}(x)| \le C \int_\Omega \frac{|f(y)|}{|x-y|^{n-1}}\, dy,$$

and therefore, the Young inequality implies that $\mathbf{u} \in L^p(\Omega)^n$ and that

$$\|\mathbf{u}\|_{L^p(\Omega)^n} \le C\|f\|_{L^p(\Omega)} \tag{2.1.5}$$

with C depending only on n, δ, and ω.

The difficult part is to show that, for $1 < p < \infty$, $\frac{\partial u_i}{\partial x_j} \in L^p(\Omega)$ whenever $f \in L^p(\Omega)$, and this is our next goal.

A fundamental tool for our arguments is the Calderón-Zygmund singular integral operators theory [19, 20]. Also we will make use of the boundedness of the Hardy-Littlewood maximal operator. For the sake of completeness, we state in the next theorems the results on these subjects that we will use in this section and in the next one. With Σ we denote the unit sphere and with $d\sigma$ the corresponding surface measure.

Theorem 2.2. *Let $K(y,z)$ a function defined for $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$, $z \neq 0$ satisfying*

1. $K(y,\lambda z) = \lambda^{-n} K(y,z) \quad \forall \lambda > 0,\ y \in \mathbb{R}^n,\ 0 \neq z \in \mathbb{R}^n$
2. $\int_\Sigma K(y,\sigma) d\sigma = 0 \quad \forall y \in \mathbb{R}^n$.
3. $|K(y,z)| \leq \frac{C_1}{|z|^n} \quad \forall y \in \mathbb{R}^n$ *with* C_1 *independent of* y

Then, for any $1 < p < \infty$,

$$Tg(y) = \lim_{\varepsilon \to 0} T_\varepsilon g(y)$$

with

$$T_\varepsilon g(y) = \int_{|x-y|>\varepsilon} K(y, x-y) g(x) dx$$

defines a bounded operator in L^p and the convergence holds in the L^p norm. Moreover, there exists a constant C_2, depending on p, n, and C_1 such that, if

$$\widetilde{T}g(y) = \sup_{\varepsilon > 0} |T_\varepsilon g(y)|,$$

then

$$\left\| \widetilde{T}g \right\|_{L^p} \leq C \|g\|_{L^p}$$

Proof. See [20, Theorem 2]. □

For $g \in L^1_{loc}(\mathbb{R}^n)$ the Hardy-Littlewood maximal operator is defined by

$$Mg(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y)| dy.$$

Theorem 2.3. *For any $1 < p < \infty$ there exists a constant C depending only on p and n such that*

$$\|Mg\|_{L^p} \leq C \|g\|_{L^p}$$

Proof. See, for example, [34]. □

In the next lemma we give an expression for $\frac{\partial u_i}{\partial x_j}$ in terms of f. In order to do that we introduce a singular integral operator. It is convenient to introduce $\chi_\Omega(y)$, the characteristic function of Ω, in order to have a kernel which vanishes for y outside Ω. Of course, for f vanishing outside Ω this will not make any change, but to prove the bounds that we will need for the kernel it is important to have $\chi_\Omega(y)$ in its definition. We define

$$T_{ij} g(y) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \chi_\Omega(y) \frac{\partial G_i}{\partial x_j}(x,y) g(x)\, dx.$$

and its adjoint operator T^*_{ij}. The existence of this limit in L^p as well as the continuity of T_{ij}, for $1 < p < \infty$, will be proved below using Theorem 2.2. In particular, it will follow that

$$T^*_{ij} f(x) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \chi_\Omega(y) \frac{\partial G_i}{\partial x_j}(x,y) f(y)\, dy$$

Lemma 2.3. *For $1 \le i,j \le n$ we have*

$$\frac{\partial u_i}{\partial x_j} = T^*_{ij} f + \omega_{ij} f \qquad in \;\; \Omega \tag{2.1.6}$$

where

$$\omega_{ij}(y) = \int_{\mathbb{R}^n} \frac{z_i z_j}{|z|^2} \omega(y+z)\, dz \tag{2.1.7}$$

Proof. From the definition of G_i and using again (2.1.2) to interchange the order of integration we have, for any $\varphi \in C_0^\infty(\Omega)$,

$$-\int_\Omega u_i(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx = -\int_\Omega \int_\Omega G_i(x,y) f(y) \frac{\partial \varphi}{\partial x_j}(x)\, dx\, dy. \tag{2.1.8}$$

For any $y \in \Omega$,

$$\begin{aligned} -\int_\Omega G_i(x,y) \frac{\partial \varphi}{\partial x_j}(x)\, dx &= -\lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} G_i(x,y) \frac{\partial \varphi}{\partial x_j}(x)\, dx \\ &= \lim_{\varepsilon \to 0} \Big\{ \int_{|y-x|>\varepsilon} \frac{\partial G_i}{\partial x_j}(x,y) \varphi(x)\, dx \\ &\qquad - \int_{|y-\zeta|=\varepsilon} G_i(\zeta,y) \varphi(\zeta) \frac{(y_j - \zeta_j)}{|y-\zeta|}\, d\zeta \Big\} \end{aligned} \tag{2.1.9}$$

Now, we can decompose the second term on the right-hand side in the following way

$$\begin{aligned} \int_{|y-\zeta|=\varepsilon} G_i(\zeta,y) \phi(\zeta) \frac{(y_j-\zeta_j)}{|y-\zeta|}\, d\zeta &= \varphi(y) \int_{|y-\zeta|=\varepsilon} G_i(\zeta,y) \frac{(y_j-\zeta_j)}{|y-\zeta|}\, d\zeta \\ &+ \int_{|y-\zeta|=\varepsilon} G_i(\zeta,y)(\varphi(\zeta)-\varphi(y)) \frac{(y_j-\zeta_j)}{|y-\zeta|}\, d\zeta := \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon \end{aligned}$$

and it is easy to see that $\mathrm{II}_\varepsilon \to 0$. Indeed, using the bound given in (2.1.2) for G_i and the fact that φ has bounded derivatives we obtain that there exists a constant C depending only on δ, n and $\|\varphi\|_{W^{1,\infty}(\Omega)}$ such that

$$|\mathrm{II}_\varepsilon| \le C\varepsilon$$

On the other hand, we have

$$-\lim_{\varepsilon \to 0} \mathrm{I}_\varepsilon = -\lim_{\varepsilon \to 0} \varphi(y) \int_{|y-\zeta|=\varepsilon} \int_0^1 \frac{(\zeta_i - y_i)}{s} \omega\Big(y + \frac{\zeta - y}{s}\Big) \frac{(y_j-\zeta_j)}{|y-\zeta|} \frac{ds}{s^n}\, d\zeta$$

Then, making the change of variables $r = \varepsilon/s$ and $\sigma = (\zeta - y)/\varepsilon$ and denoting with Σ the unit sphere we obtain

$$
\begin{aligned}
-\lim_{\varepsilon\to 0} \mathrm{I}_\varepsilon &= \varphi(y) \lim_{\varepsilon\to 0} \int_{|y-\zeta|=\varepsilon} \int_\varepsilon^\infty (\zeta_i - y_i) \frac{(\zeta_j - y_j)}{|\zeta - y|} \omega\Big(y + r\frac{\zeta - y}{\varepsilon}\Big) \frac{r^{n-1}}{\varepsilon^n}\, dr\, d\zeta \\
&= \varphi(y) \lim_{\varepsilon\to 0} \int_\Sigma \int_\varepsilon^\infty \sigma_i \sigma_j \omega(y + r\sigma) r^{n-1}\, dr\, d\sigma \\
&= \varphi(y) \lim_{\varepsilon\to 0} \int_\Sigma \int_\varepsilon^\infty \frac{\sigma_i \sigma_j}{|\sigma|^2} \omega(y + r\sigma) r^{n-1}\, dr\, d\sigma \\
&= \varphi(y) \lim_{\varepsilon\to 0} \int_{|z|>\varepsilon} \frac{z_i z_j}{|z|^2} \omega(y+z)\, dz = \varphi(y)\omega_{ij}(y)
\end{aligned}
$$

and therefore, replacing in (2.1.9) we obtain that, for $y \in \Omega$,

$$
-\int_\Omega G_i(x,y) \frac{\partial \varphi}{\partial x_j}(x)\, dx = T_{ij}\varphi(y) + \omega_{ij}(y)\varphi(y), \tag{2.1.10}
$$

Then, (2.1.6) follows from (2.1.10) and (2.1.8). □

Remark 2.2. The previous lemma provides a different way of proving that $\mathbf{u}$ is a solution of the divergence. Moreover, we can consider $f \in L^1(\Omega)$, not necessarily with vanishing integral, and we have

$$
\operatorname{div} \mathbf{u} = f - \Big(\int_\Omega f\Big) \omega \qquad \text{in } \ \Omega \tag{2.1.11}
$$

Indeed, using the expressions for the derivatives given in Lemma 2.3 and observing that $\sum_{i=1}^n \omega_{ii} = 1$ we have that

$$
\operatorname{div} \mathbf{u} = f + \sum_{i=1}^n T_{ii}^* f \qquad \text{in } \ \Omega
$$

and so, we have to check that

$$
\sum_{i=1}^n T_{ii}^* f = -\Big(\int_\Omega f\Big) \omega \qquad \text{in } \ \Omega
$$

But, we have

$$
\sum_{i=1}^n T_{ii}^* f(x) = \lim_{\varepsilon\to 0} \sum_{i=1}^n \int_{|y-x|>\varepsilon} \chi_\Omega(y) \frac{\partial G_i}{\partial x_i}(x,y) f(y)\, dy \tag{2.1.12}
$$

and introducing $\eta_i(y,z) := z_i \omega(y+z)$ we obtain from (2.1.1) that

$$
\frac{\partial G_i}{\partial x_i}(x,y) = \int_0^1 \frac{\partial \eta_i}{\partial z_i}\Big(y, \frac{x-y}{s}\Big) \frac{ds}{s^{n+1}} \tag{2.1.13}
$$

but,

$$
\frac{\partial \eta_i}{\partial z_i}(y,z) = \omega(y+z) + z_i \frac{\partial \omega}{\partial z_i}(y+z)
$$

and so, making the change of variable $r = 1/s$ in (2.1.13) we obtain

$$\sum_{i=1}^{n} \frac{\partial G_i}{\partial x_i}(x,y) = \sum_{i=1}^{n} \int_1^\infty \frac{\partial \eta_i}{\partial z_i}(y, r(x-y))\,dr = \int_1^\infty \frac{d}{dr}[r^n \omega(y + r(x-y)]\,dr = -\omega(x)$$

which together with (2.1.12) gives (2.1.11).

Next, we will use the expression given in Lemma 2.3 to prove that $\frac{\partial u_i}{\partial x_j} \in L^p(\Omega)$. The kernel $\chi_\Omega(y)\frac{\partial G_i}{\partial x_j}(x,y)$, and so the operator T_{ij}, can be decomposed in two parts as follows:

$$\chi_\Omega(y)\frac{\partial G_i}{\partial x_j}(x,y) = \int_0^\infty \chi_\Omega(y)\frac{\partial \eta_i}{\partial z_j}\left(y, \frac{x-y}{s}\right)\frac{ds}{s^{n+1}} - \int_1^\infty \chi_\Omega(y)\frac{\partial \eta_i}{\partial z_j}\left(y, \frac{x-y}{s}\right)\frac{ds}{s^{n+1}}$$

$$:= K_1(y, x-y) + K_2(y, x-y)$$

and

$$T_{ij} = T_1 + T_2 \tag{2.1.14}$$

with

$$T_\ell g(y) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} K_\ell(y, x-y) g(x)\,dx \qquad \text{for} \qquad \ell = 1,2$$

First, we will show that the second part T_2 defines a bounded operator in $L^p(\Omega)$ for $1 \le p \le \infty$.

Lemma 2.4. *We have*

$$\|T_2 g\|_{L^p(\Omega)} \le \frac{(1+\delta)}{n}\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)}|\Omega|\,\|g\|_{L^p(\Omega)} \tag{2.1.15}$$

Proof. From the definition of η_i we can see that

$$\left|\frac{\partial \eta_i}{\partial z_j}\left(y, \frac{z}{s}\right)\right| \le \left(1 + \frac{|z|}{s}\right)\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)} \tag{2.1.16}$$

Now, since $\operatorname{supp}\omega \subset B \subset \Omega$ it follows that $\chi_\Omega(y)\frac{\partial \eta_i}{\partial z_j}(y, z/s)$ vanishes for $|z|/s > \delta$. In particular, the integral defining K_2 can be restricted to those values of s such that $s \ge |z|/\delta$, and so, from (2.1.16) we obtain

$$\left|\chi_\Omega(y)\frac{\partial \eta_i}{\partial z_j}\left(y, \frac{z}{s}\right)\right| \le (1+\delta)\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)}$$

Therefore,

$$|K_2(y,z)| \le (1+\delta)\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)}\int_1^\infty \frac{ds}{s^{n+1}} = \frac{1+\delta}{n}\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)}$$

and then,

$$|T_2 g(y)| \le \frac{(1+\delta)}{n} \|\omega\|_{W^{1,\infty}(\mathbb{R}^n)} \int_\Omega |g(x)|\, dx$$

and by the Hölder inequality we obtain (2.1.15). □

In view of the decomposition (2.1.14) it remains to analyze the continuity of T_1. With this goal, we will show, in the next two lemmas, that the kernel $K_1(y,z)$ satisfies the hypotheses of Theorem 2.2.

Lemma 2.5. *We have*

$$|K_1(y,z)| \le \frac{(1+\delta)}{n} \|\omega\|_{W^{1,\infty}(\mathbb{R}^n)} \frac{\delta^n}{|z|^n} \tag{2.1.17}$$

Proof. By the same arguments used in the proof of Lemma 2.4 we obtain

$$|K_1(y,z)| \le (1+\delta)\|\omega\|_{W^{1,\infty}(\mathbb{R}^n)} \int_{|z|/\delta}^{\infty} \frac{ds}{s^{n+1}}$$

which immediately gives (2.1.17). □

Lemma 2.6. $K_1(y,z)$ *is homogeneous of degree* $-n$ *and with vanishing mean value on the unit sphere* Σ*, in the second variable.*

Proof. Given $\lambda > 0$, from the definition of K_1 and making the change of variable $t = s/\lambda$, we have

$$K_1(y,\lambda z) = \int_0^\infty \chi_\Omega(y) \frac{\partial \eta_i}{\partial z_j}\left(y, \frac{\lambda z}{s}\right) \frac{ds}{s^{n+1}} = \lambda^{-n} \int_0^\infty \chi_\Omega(y) \frac{\partial \eta_i}{\partial z_j}\left(y, \frac{z}{t}\right) \frac{dt}{t^{n+1}} = \lambda^{-n} K_1(y,z)$$

On the other hand, making the change of variable $r = 1/s$ in the integral defining K_1 we have

$$K_1(y,z) = \int_0^\infty \chi_\Omega(y) \frac{\partial \eta_i}{\partial z_j}(y, rz) r^{n-1}\, dr$$

and therefore,

$$\int_\Sigma K_1(y,\sigma)\, d\sigma = \int_\Sigma \int_0^\infty \chi_\Omega(y) \frac{\partial \eta_i}{\partial z_j}(y, r\sigma) r^{n-1}\, dr\, d\sigma = \int_{\mathbb{R}^n} \chi_\Omega(y) \frac{\partial \eta_i}{\partial z_j}(y,z)\, dz = 0$$

because $\eta_i(y,z)$ is a smooth function with compact support in the z variable. □

We can now state and prove the main result of this section.

Theorem 2.4. *Let* Ω *be bounded and star-shaped with respect to a ball* $B \subset \Omega$*. If* $f \in L^p(\Omega)$, $1 < p < \infty$, *and* $\int_\Omega f = 0$, *then, the function* $\mathbf{u}$ *defined in (2.1.3) is in* $W_0^{1,p}(\Omega)^n$ *and satisfies*

$$\operatorname{div} \mathbf{u} = f \qquad in \ \ \Omega. \tag{2.1.18}$$

Moreover, there exists a constant C depending only on Ω and p, such that

$$\|\mathbf{u}\|_{W_0^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)} \tag{2.1.19}$$

Proof. That $\mathbf{u} \in L^p(\Omega)^n$ and

$$\|\mathbf{u}\|_{L^p(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}$$

follows from the definition of $\mathbf{u}$ using (2.1.2) and the Young inequality.

Now we show that, for $1 \leq i, j \leq n$, there exists a constant C depending only on p, δ, n, and ω such that

$$\left\|\frac{\partial u_i}{\partial x_j}\right\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

To do that we use the expression for the derivatives given in Lemma 2.3. From (2.1.7) it follows immediately that ω_{ij} is a bounded function. Indeed, $\|\omega_{ij}\|_{L^\infty} \leq \|\omega\|_{L^1}$.

Then, it remains to prove that the operator T_{ij}^* is bounded in L^p, for $1 < p < \infty$. In view of Lemmas 2.5 and 2.6, it follows from Theorem 2.2 that the limit defining T_1 exists in the L^p norm and defines an operator which is continuous in L^p for $1 < p < \infty$. Then, the boundedness of T_{ij} in L^p, for $1 < p < \infty$, follows from the decomposition $T_{ij} = T_1 + T_2$ recalling that, as we proved in Lemma 2.1.15, T_2 is continuous in L^p. By a standard duality argument it follows that T_{ij}^* is also bounded for $1 < p < \infty$. Therefore, we have proved that $\mathbf{u} \in W^{1,p}(\Omega)$ and satisfies

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)} \tag{2.1.20}$$

Now, for $p > n$, it follows from Proposition 2.1 that $\mathbf{u}$ is continuous and vanishes on $\partial\Omega$. But, in [85], it is proved that for an arbitrary open set Ω, if a function is continuous and vanishes on $\partial\Omega$ and belongs to $W^{1,p}(\Omega)$, then it belongs to $W_0^{1,p}(\Omega)$.

On the other hand, for $1 < p \leq n$, take a sequence $f_m \in L^\infty(\Omega)$ such that $f_m \to f$ in $L^p(\Omega)$ and let

$$\mathbf{u}_m(x) = \int_\Omega G(x,y) f_m(y)dy.$$

Then, from (2.1.20) applied to $f - f_m$ it follows that $\mathbf{u}_m \to \mathbf{u}$ in $W^{1,p}(\Omega)^n$. But we already know that $\mathbf{u}_m \in W_0^{1,p}(\Omega)^n$, and therefore, $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ and the theorem is proved. $\square$

Remark 2.3. For a Lipschitz domain Ω, the existence of $\mathbf{u}$ satisfying (2.1.18) and (2.1.19) can be proved using the previous theorem and the fact that Ω can be written as a finite union of domains which are star-shaped with respect to a ball. We omit details (which can be found in [14], see also the decomposition technique described in Section 4.5) because in the next section we will generalize Bogovskiĭ's construction to a class of domains which contains the Lipschitz ones.

2.2 Solutions of the Divergence on John Domains

In view of the results of the previous section, an interesting problem is to find weaker sufficient conditions on a bounded domain Ω for the existence of $\mathbf{u}$ satisfying (2.1.18) and (2.1.19).

It is known that the domain cannot be arbitrary, indeed, several counterexamples of domains which do not satisfy this result have been published (see Section 4.4). From these counterexamples it follows that we have to consider a class of domains which excludes domains with external cusps. On the other hand, the Lipschitz condition is not necessary. In fact, it is known that if the result holds for two domains then it also holds for the union of them (see, for example, the argument given in [14]), and consequently, domains having internal cusps are allowed although they are not Lipschitz.

Consequently, it seems that a natural class of domains to be considered for our problem is that of the John domains (see definition below). For instance, it is known that a two dimensional domain with a piecewise smooth boundary is a John domain if and only if it does not have external cusps. These domains were first considered by F. John in his work on elasticity [61] and were named after him by Martio and Sarvas [79]. Further, John domains were used in the study of several problems in Analysis. For example, they were used by G. David and S. Semmes [30] in the analysis of quasiminimal surfaces of codimension one and by S. Buckley and P. Koskela [16] for the study of several inequalities. On the other hand, John domains are closely related with the extension domains of P. Jones [62]. Indeed the (ε,∞) domains, also called uniform domains, are John domains (but the converse is not true: a John domain can have an internal cusp while a uniform domain cannot).

As we will show in this section, the approach used to construct solutions of the divergence on star-shaped domains can be generalized to John domains. This generalization has been done in [3]. The key idea is to replace the segments used for the integration in (1.0.10) by appropriate curves.

There are several equivalent definitions of John domains. A usual one is the following. We will denote with $d(x)$ the distance from x to $\partial\Omega$.

Definition 2.2.1 *A bounded open $\Omega \subset \mathbb{R}^n$ is a John domain if there exist a positive constant c_1 and $x_0 \in \Omega$ such that, for every $y \in \Omega$ there exists a rectifiable curve $\mathscr{C}_{x_0,y} \subset \Omega$ joining y and x_0 with the following property:*

If $\ell(y)$ denotes the length of $\mathscr{C}_{x_0,y}$ and $\rho : [0,\ell(y)] \to \Omega$ is its parametrization by arc-length such that $\rho(0) = y$, $\rho(\ell(y)) = x_0$, then,

$$d(\rho(t)) \geq c_1 t \quad \forall t \in [0,\ell(y)] \tag{2.2.1}$$

The property given in this definition means that one can reach each point $y \in \Omega$ by a curve $\mathscr{C}_{x_0,y}$ such that any point $x \in \mathscr{C}_{x_0,y}$ is at a distance from the boundary of Ω greater than a fixed proportion of the length of the curve between y and x. This is why this property is sometimes called the “Twisted cone condition.”

In some papers condition (2.2.1) is replaced by the following two conditions. There exist two positive constants c_2 and c_3 such that

$$\ell(y) \leq c_2 \tag{2.2.2}$$

and

$$d(\rho(t)) \geq \frac{c_3}{\ell(y)} t \quad \forall t \in [0, \ell(y)] \tag{2.2.3}$$

where c_2 and c_3 are positive constants.

It is not difficult to see that both definitions are equivalent. Indeed, it is obvious that (2.2.2) and (2.2.3) imply (2.2.1) with $c_1 = c_3/c_2$. Conversely, taking $t = \ell(y)$ in (2.2.1) we obtain (2.2.2) with $c_2 = d(x_0)/c_1$. To prove (2.2.3), consider first $\ell(y) < d(x_0)/2$. In this case we have $d(\rho(t)) > d(x_0)/2 \geq d(x_0)t/2\ell(y)$ for all $t \in [0, \ell(y)]$. On the other hand, if $\ell(y) \geq d(x_0)/2$, it follows from (2.2.1) that $d(\rho(t)) \geq c_1 d(x_0)t/2\ell(y)$.

If Ω is a John domain, there is an infinite number of possible choices for the curves satisfying the properties required in Definition 2.2.1. To construct our solution of the divergence we will choose a family of curves verifying some extra conditions, in particular, close to y, the curve joining y and x_0 will be a segment. Moreover, we need to have some control on the variability of the curves as functions of y, indeed, measurability will be enough for our purposes. Also, for convenience we re-scale the curves in order to have the parameter in $[0, 1]$.

In the next lemma we state the properties that we will need and prove the existence of a family of curves satisfying them. We will make use of a Whitney decomposition of an open set. We refer the reader to [84] for a proof of the existence of such a decomposition for any open bounded set.

Definition 2.2.2 *Given $\Omega \subset \mathbb{R}^n$ an open bounded set, a* **Whitney decomposition** *of Ω is a family W of closed dyadic cubes with pairwise disjoint interiors and satisfying the following properties:*

1) $\Omega = \cup_{Q \in W} Q$

2) $diam(Q) \leq d(Q, \partial\Omega) \leq 4\, diam(Q) \quad \forall Q \in W$

3) $\frac{1}{4} diam(Q) \leq diam(\tilde{Q}) \leq 4\, diam(Q) \quad \forall Q, \tilde{Q} \in W \quad$ *such that* $Q \cap \tilde{Q} \neq \emptyset$

Given $Q \in W$, let x_Q be its center and Q^* the cube with the same center but expanded by a factor $9/8$, namely, $Q^* = \frac{9}{8}(Q - x_Q) + x_Q$. We will make use of the following facts which follow easily from the properties given in Definition 2.2.2.

$$d(Q^*, \partial\Omega) \sim diam(Q^*) \sim d(y) \qquad \forall y \in Q^*, \tag{2.2.4}$$

where $A \sim B$ means that there are constants c and C, which may depend on the dimension n but on nothing else, such that $cA \leq B \leq CA$. We will use the notation $\dot{\gamma}(s, y) := \frac{\partial \gamma}{\partial s}(s, y)$.

Lemma 2.7. *Let $\Omega \subset R^n$ be a bounded John domain and x_0, c_1 and $\rho(t)$ be as in Definition 2.2.1. Then, there exist a function $\gamma : [0,1] \times \Omega \to \Omega$ and positive constants c_J and k depending only on c_1, $diam(\Omega)$, $d(x_0)$ and n, such that*

1) $\gamma(0,y) = y$, $\gamma(1,y) = x_0$

2) $d(\gamma(s,y)) \geq c_J s$

3) $|\dot{\gamma}(s,y)| \leq c_J^{-1}$

4) $\{x \in \Omega : x = \gamma(s,y), 0 \leq s \leq kd(y)\}$ *is a segment and* $kd(y) \leq 1$

5) $\gamma(s,y)$ *and* $\dot{\gamma}(s,y)$ *are measurable functions.*

Proof. Let W be a Whitney decomposition of Ω and $Q_0 \in W$ be a cube containing x_0. Given $y \in \Omega$, let $Q \in W$ be such that $y \in Q$. We remark that if y belongs to the boundary of some $Q \in W$ then it belongs to more than one cube. We choose any of them arbitrarily (in any case this is not important because the set of those points is of measure zero).

Suppose first that $x_0 \in Q^*$. In this case we can take the curve as a segment, namely, $\gamma(s,y) = sx_0 + (1-s)y$. In fact, in view of (2.2.4), it is easy to see that $\gamma(s,y)$ satisfies 2) and 3) with c_J depending on $d(x_0)$. Also 4) is trivially satisfied for any k such that $kd(y) \leq 1$, we can take, for example, $k = 1/diam(\Omega)$.

Now, if $x_0 \notin Q^*$, let x_Q be the center of Q and take $\rho(t)$ as a parametrization of a curve joining x_Q and x_0 satisfying the conditions given in the definition of John domains. First we reparametrize ρ and define $\mu(s) = \rho(s\ell(x_Q))$. Then, $d(\mu(s)) \geq c_J s$ for $c_J \sim c_1 \ell(x_Q)$. But, since $x_0 \notin Q^*$, we obtain from properties 2) and 3) of Definition 2.2.2 that $\ell(x_Q) \geq |x_0 - x_Q| \geq cd(x_0)$ with c depending only on n. Therefore, 2) holds for μ with $c_J \sim c_1 d(x_0)$. Moreover, $|\dot{\mu}(s)| \leq d(x_0)/c_1$ and so we can choose c_J small enough such that μ also satisfies 3).

To define $\gamma(s,y)$ we modify $\mu(s)$ in the following way. Let s_1 be the first $s \in [0,1]$ such that $\mu(s) \in \partial Q^*$. Then we define

$$\gamma(s,y) = \begin{cases} (s/s_1)\mu(s_1) + (1-(s/s_1))y & \text{if } s \in [0,s_1] \\ \mu(s) & \text{if } s \in [s_1,1] \end{cases}$$

see Figure 2.1.

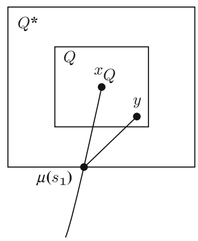

Fig. 2.1 Detail of the ending part of γ

Now, for $s \in [0, s_1]$, $\dot{\gamma}(s,y) = (\mu(s_1) - y)/s_1$. But, since $|\dot{\mu}(s)| \le d(x_0)/c_1$, $\mu(s_1) \in \partial Q^*$ and $\mu(0) = x_Q$, it is easy to check that $s_1 \ge c\,c_1 diam(Q^*)/d(x_0)$ with c depending only on n. Therefore, $|\dot{\gamma}(s,y)|$ is bounded by a constant which depends only on n, c_1, and $d(x_0)$. So, we can choose c_J small enough such that $\gamma(s,y)$ satisfies 3) on the interval $[0, s_1]$. On the other hand, for $s \in [0, s_1]$, both $\mu(s)$ and $\gamma(s,y)$ belong to Q^* and so $d(\gamma(s,y)) \sim d(\mu(s))$ and therefore 2) holds on this interval. Since $\gamma(s,y) = \mu(s)$ on $s \in [s_1, 1]$, 2) and 3) hold on all the interval $[0,1]$.

Using again that $s_1 \ge c\,c_1 diam(Q^*)/d(x_0)$, 4) follows from (2.2.4).

Finally, observe that 5) holds because $\gamma(s,y)$ and $\dot{\gamma}(s,y)$ are continuous for y in the interior of each $Q \in W$ and so they are continuous up to a set of measure zero. □

Our next goal is to introduce the solution of the divergence which generalizes to John domains that given in (2.1.3). To simplify notation we will assume, without loss of generality, that $x_0 = 0$.

Let c_J be the constant appearing in 2) and 3) of Lemma 2.7 and $\omega \in C_0^\infty(B(0, c_J/2))$ be such that $\int_\Omega \omega = 1$. Given a function $\varphi \in C_0^\infty(\Omega)$ we define $\varphi_\omega = \int_\Omega \varphi\omega$. The key point in our construction is to recover $\varphi - \varphi_\omega$ from its gradient. To do this we replace the segments used in the case of star-shaped domains by appropriate curves based on the function γ defined in Lemma 2.7. Observe that, taking $s = 1$ in 2) of Lemma 2.7, we obtain $c_J \le d(0)$ and so $B(0, c_J/2) \subset \Omega$.

Now, for any $y \in \Omega$ and any $z \in B(0, c_J/2)$ we define, for $s \in [0,1]$,

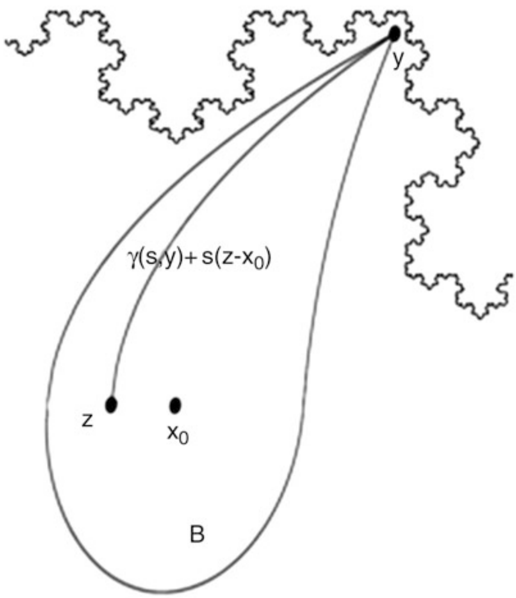

Fig. 2.2 A twisted cone inside of an irregular domain

$$\tilde{\gamma}(s,y,z) = \gamma(s,y) + sz \tag{2.2.5}$$

Then, it follows immediately from 1) of Lemma 2.7 that

$$\tilde{\gamma}(0,y,z) = y \quad \text{and} \quad \tilde{\gamma}(1,y,z) = z \tag{2.2.6}$$

Moreover,

$$\tilde{\gamma}(s,y,z) \in \Omega \qquad \forall s \in [0,1] \tag{2.2.7}$$

Indeed, using 2) of Lemma 2.7, we obtain

$$|\tilde{\gamma}(s,y,z) - \gamma(s,y)| = s|z| < \frac{sc_J}{2} < d(\gamma(s,y))$$

and therefore, (2.2.7) holds (Figure 2.2).

We can now introduce the function $G = (G_1, \cdots, G_n)$ which will be the kernel of the right inverse of the divergence.

For $x \in \mathbb{R}^n$ and $y \in \Omega$ we define

$$G(x,y) = \int_0^1 \left\{ \dot{\gamma}(s,y) + \frac{x-\gamma(s,y)}{s} \right\} \omega\left(\frac{x-\gamma(s,y)}{s}\right) \frac{ds}{s^n} \tag{2.2.8}$$

Observe that, from 5) of Lemma 2.7, we know that $G(x,y)$ is a measurable function.

Remark 2.4. The integral defining $G(x,y)$ can be restricted to $s \geq c|x-y|$ for c depending only on the constant c_J given in Lemma 2.7. Indeed, if $(x-\gamma(s,y))/s$ is in the support of ω, then

$$|x-y| \leq |x-\gamma(s,y)| + |\gamma(s,y)-\gamma(0,y)| \leq c_J s + \sqrt{n} c_J^{-1} s.$$

An important consequence of this remark is the bound for $G(x,y)$ given in the following lemma.

Lemma 2.8. *There exists a constant $C = C(n, c_J, \omega)$ such that*

$$|G(x,y)| \leq \frac{C}{|x-y|^{n-1}} \tag{2.2.9}$$

Proof. In view of Remark 2.4 we have

$$G(x,y) = \int_{c|x-y|}^1 \left\{ \dot{\gamma}(s,y) + \frac{x-\gamma(s,y)}{s} \right\} \omega\left(\frac{x-\gamma(s,y)}{s}\right) \frac{ds}{s^n}$$

But, for $(x-\gamma(s,y))/s$ in the support of ω, we have

$$\left| \dot{\gamma}(s,y) + \frac{x-\gamma(s,y)}{s} \right| \leq c_J^{-1} + \frac{c_J}{2}$$

where we have used property 3) of Lemma 2.7. Then, the integrand is bounded by $(c_J^{-1} + c_J/2)\|\omega\|_\infty s^{-n}$ and the estimate (2.2.9) follows by an elementary integration.
$\square$

The next lemma shows how $\varphi - \varphi_\omega$ can be recovered from its gradient by means of the kernel G. As a consequence of this result we obtain our solution of the divergence.

Lemma 2.9. *For $\varphi \in C^1(\Omega) \cap W^{1,1}(\Omega)$ we have that, for any $y \in \Omega$,*

$$(\varphi - \varphi_\omega)(y) = -\int_\Omega G(x,y) \cdot \nabla\varphi(x)dx.$$

Proof. Since $\int_\Omega \omega = 1$ we have, in view of (2.2.6), that for any $y \in \Omega$,

$$(\varphi - \varphi_\omega)(y) = \int_\Omega (\varphi(y) - \varphi(z))\omega(z)dz = -\int_\Omega \int_0^1 \dot{\tilde{\gamma}}(s,y,z) \cdot \nabla\varphi(\tilde{\gamma}(s,y,z))\omega(z)dsdz.$$

From (2.2.5) we obtain $\dot{\tilde{\gamma}}(s,y,z) = \dot{\gamma}(s,y) + z$. Then, interchanging the order of integration and making the change of variables $x = \tilde{\gamma}(s,y,z)$, we have $z = (x - \gamma(s,y))/s$ and $dz = dx/s^n$, and hence,

$$(\varphi - \varphi_\omega)(y) = -\int_\Omega \int_0^1 \left\{ \dot{\gamma}(s,y) + \frac{x - \gamma(s,y)}{s} \right\} \omega\left(\frac{x - \gamma(s,y)}{s}\right) \frac{ds}{s^n} \cdot \nabla\varphi(x)dx$$

which in view of the definition (2.2.8) concludes the proof $\square$

Theorem 2.5. *For $f \in L^1(\Omega)$ such that $\int_\Omega f = 0$ define*

$$\mathbf{u}(x) = \int_\Omega G(x,y)f(y)dy \tag{2.2.10}$$

then

$$\operatorname{div}\mathbf{u} = f.$$

Proof. The proof is exactly as that of Theorem 2.1, using now (2.2.9) to see that $\mathbf{u}$ is well defined and all its components belong to L^1_{loc}, and therefore, $\operatorname{div}\mathbf{u}$ is well defined in the sense of distributions.

Remark 2.5. As in the case of star-shaped domains, the definition of $\mathbf{u}$ given in (2.2.10) can be extended to every $x \in \mathbb{R}^n$.

Proposition 2.2. *Let $f \in L^p(\Omega)$ for some $p > n$. Then, $\mathbf{u}$ defined in (2.2.10) is continuous in $\mathbb{R}^n$ and vanishes outside Ω, in particular, $\mathbf{u}(x) = 0$ for all $x \in \partial\Omega$.*

Proof. First we observe that

$$G(x,y) = 0 \;\text{ whenever }\; x \notin \Omega, y \in \Omega$$

Indeed, it is enough to see that

$$\omega\left(\frac{x - \gamma(s,y)}{s}\right) = 0 \qquad \text{for } x \notin \Omega \;, y \in \Omega \;\text{ and }\; s \in [0,1] \tag{2.2.11}$$

But, in this case we have from property 2) of Lemma 2.7,

$$c_J s \le d(\gamma(s,y)) \le |\gamma(s,y) - x|$$

hence,

$$\frac{|\gamma(s,y) - x|}{s} \ge c_J$$

and therefore, (2.2.11) follows immediately since $\operatorname{supp}\omega \subset B(0, c_J/2)$. Consequently, $\mathbf{u} = 0$ for all $x \notin \Omega$. Take now x and $\bar{x}$ in a neighborhood of $\overline{\Omega}$. We have

$$G(x,y) - G(\bar{x},y) = \int_0^1 \left\{ \left(\dot{\gamma}(s,y) + \frac{x-\gamma(s,y)}{s}\right) \omega\left(\frac{x-\gamma(s,y)}{s}\right) - \left(\dot{\gamma}(s,y) + \frac{\bar{x}-\gamma(s,y)}{s}\right) \omega\left(\frac{\bar{x}-\gamma(s,y)}{s}\right) \right\} \frac{ds}{s^n}$$

and then, since $|\dot{\gamma}(s,y)| \le c_J^{-1}$,

$$|G(x,y) - G(\bar{x},y)| \le c_J^{-1} \int_0^1 \left| \omega\left(\frac{x-\gamma(s,y)}{s}\right) - \omega\left(\frac{\bar{x}-\gamma(s,y)}{s}\right) \right| \frac{ds}{s^n}$$
$$+ \int_0^1 \left| \left(\frac{x-\gamma(s,y)}{s}\right) \omega\left(\frac{x-\gamma(s,y)}{s}\right) - \left(\frac{\bar{x}-\gamma(s,y)}{s}\right) \omega\left(\frac{\bar{x}-\gamma(s,y)}{s}\right) \right| \frac{ds}{s^n}$$

But $\omega(z)$ and $z\omega(z)$ are Hölder α on bounded domains for any $0 < \alpha < 1$. Therefore, assuming, for example, $|x-y| \le |\bar{x}-y|$, and using that the integrand in the definition of $G(x,y)$ vanishes if $s < c|x-y|$ we obtain

$$|G(x,y) - G(\bar{x},y)| \le C|x-\bar{x}|^\alpha \int_{c|x-y|}^1 \frac{ds}{s^{n+\alpha}} \le C \frac{|x-\bar{x}|^\alpha}{|x-y|^{n-1+\alpha}}$$

and then,

$$|\mathbf{u}(x) - \mathbf{u}(\bar{x})| \le C|x-\bar{x}|^\alpha \int_\Omega \frac{|f(y)|}{|x-y|^{n-1+\alpha}}\, dy$$

now, since $p > n$ we can choose $\alpha > 0$ such that $(n-1+\alpha)p' < n$ and the proof concludes using the Hölder inequality. □

In what follows we will prove that $\mathbf{u}$ belongs to $W_0^{1,p}(\Omega)$. The argument is analogous to that used in the case of star-shaped domains, i, e., we will write the derivatives of the components of $\mathbf{u}$ as a singular integral operator acting on f. With this goal we introduce

$$T_{ij}g(y) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \chi_\Omega(y) \frac{\partial G_i}{\partial x_j}(x,y) g(x)\, dx.$$

and its adjoint operator T_{ij}^*. The existence of this limit in L^p, for $1 < p < \infty$, will be proved below using the Calderón-Zygmund operator theory. In particular, it will follow that

$$T_{ij}^* f(x) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \chi_\Omega(y) \frac{\partial G_i}{\partial x_j}(x,y) f(y)\, dy$$

Lemma 2.10. *For* $1 \le i, j \le n$ *we have*

$$\frac{\partial u_i}{\partial x_j} = T_{ij}^* f + \omega_{ij} f \quad in \quad \Omega$$

where

$$\omega_{ij}(y) = \int_{\mathbb{R}^n} \frac{z_i z_j}{|z|^2} \omega\left(-\dot{\gamma}(0,y) + z\right) dz$$

Proof. Proceeding as in Lemma 2.3 we have, for $\varphi \in C_0^\infty(\Omega)$ and $y \in \Omega$,

$$\int_\Omega \frac{\partial u_i}{\partial x_j}(x) \varphi(x)\, dx = \int_\Omega I(y) f(y) dy \tag{2.2.12}$$

where

$$I(y) = \lim_{\varepsilon \to 0} \left\{ \int_{|x-y|>\varepsilon} \frac{\partial G_i}{\partial x_j}(x,y) \varphi(x) dx - \int_{|y-\zeta|=\varepsilon} G_i(\zeta, y) \varphi(\zeta) \frac{(y_j - \zeta_j)}{|y-\zeta|} d\zeta \right\} \tag{2.2.13}$$

Proceeding as in Lemma 2.3, using now (2.2.9), the surface integral can be written as

$$-\int_{|y-\zeta|=\varepsilon} G_i(\zeta, y) \varphi(\zeta) \frac{(y_j - \zeta_j)}{|y-\zeta|} d\zeta = -\varphi(y) A_\varepsilon(y) + O(\varepsilon) \tag{2.2.14}$$

with

$$A_\varepsilon(y) := \int_{|y-\zeta|=\varepsilon} G_i(\zeta, y) \frac{(y_j - \zeta_j)}{|y-\zeta|} d\zeta$$

But, from the definition of G we have

$$A_\varepsilon(y) = \int_{|\zeta - y|=\varepsilon} \int_0^1 \left(\dot{\gamma}_i(s,y) + \frac{\zeta_i - \gamma_i(s,y)}{s} \right) \omega\left(\frac{\zeta - \gamma(s,y)}{s} \right) \frac{(y_j - \zeta_j)}{|y-\zeta|} \frac{ds}{s^n} d\zeta$$

and making the change of variables $r = \varepsilon / s$ we obtain

$$A_\varepsilon(y) = \int_{|\zeta - y|=\varepsilon} \int_\varepsilon^\infty \left(\dot{\gamma}_i(\varepsilon/r, y) + \frac{\zeta_i - \gamma_i(\varepsilon/r, y)}{\varepsilon/r} \right) \omega\left(\frac{\zeta - \gamma(\varepsilon/r, y)}{\varepsilon/r} \right) \frac{(y_j - \zeta_j)}{|y-\zeta|} \frac{r^{n-2}}{\varepsilon^{n-1}} dr d\zeta$$

while a further change of variables $\sigma = (\zeta - y)/\varepsilon$ yields

$$A_\varepsilon(y) = -\int_{|\sigma|=1} \int_\varepsilon^\infty \left(\dot{\gamma}_i(\varepsilon/r, y) + \frac{y_i + \varepsilon\sigma_i - \gamma_i(\varepsilon/r, y)}{\varepsilon/r} \right) \omega\left(\frac{y + \varepsilon\sigma - \gamma(\varepsilon/r, y)}{\varepsilon/r} \right) \sigma_j r^{n-2} dr d\sigma$$

$$= -\int_{|\sigma|=1} \int_\varepsilon^\infty \left(\dot{\gamma}_i(\varepsilon/r, y) + \frac{\gamma_i(0,y) - \gamma_i(\varepsilon/r, y)}{\varepsilon/r} + r\sigma_i \right) \omega\left(\frac{\gamma(0,y) - \gamma(\varepsilon/r, y)}{\varepsilon/r} + r\sigma \right) \sigma_j r^{n-2} dr d\sigma$$

where we have used that $y = \gamma(0,y)$. But, from 4) of Lemma 2.7 we know that $\dot{\gamma}(s,y)$ is continuous at $s=0$, and therefore, the integrand tends to $\sigma_i\sigma_j\omega\big(-\dot{\gamma}(0,y)+r\sigma\big)r^{n-1}$ for $\varepsilon \to 0$. Moreover, recalling that ω has compact support, we can restrict the integral to bounded r, and since the integrand is bounded we can apply the dominated convergence theorem to obtain

$$\lim_{\varepsilon\to 0} A_\varepsilon(y) = -\int_{|\sigma|=1}\int_0^\infty \sigma_i\sigma_j\omega\left(-\dot{\gamma}(0,y)+r\sigma\right)r^{n-1}\,dr\,d\sigma = -\int_{\mathbb{R}^n}\frac{z_iz_j}{|z|^2}\omega\left(-\dot{\gamma}(0,y)+z\right)dz.$$

Therefore, from (2.2.13) and (2.2.14), we conclude that

$$I(y) = T_{ij}\varphi(y) + \omega_{ij}(y)\varphi(y),$$

replacing in (2.2.12) the lemma is proved. □

Now, our goal is to prove the estimate

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

for $1 < p < \infty$.

In view of Lemma 2.10, and observing that the function ω_{ij} is bounded, our problem reduces to show that T_{ij}^* is a bounded operator in L^p for $1 < p < \infty$.

By duality, it is enough to prove that T_{ij} is bounded. To simplify notation we drop the subscripts i,j from the operator and introduce the function

$$\psi(a,z) = \frac{\partial}{\partial z_j}\Big((a+z_i)\omega(z)\Big)$$

for $a \in \mathbb{R}$ and $z \in \mathbb{R}^n$. Then, we have to prove the continuity of an operator of the form

$$Tg(y) = \lim_{\varepsilon\to 0} T_\varepsilon g(y) \tag{2.2.15}$$

where, for $\varepsilon > 0$, T_ε is given by

$$T_\varepsilon g(y) = \int_{|x-y|>\varepsilon} K(x,y)g(x)dx \tag{2.2.16}$$

with

$$K(x,y) = \chi_\Omega(y)\int_0^1 \psi\left(\dot{\gamma}_i(s,y), \frac{x-\gamma(s,y)}{s}\right)\frac{ds}{s^{n+1}} \tag{2.2.17}$$

where ψ is a bounded function such that its support in z is contained in that of ω. Since ψ is a derivative of a function with compact support we have

$$\int \psi(a,z)dz = 0. \tag{2.2.18}$$

Moreover, proceeding exactly as in Lemma 2.8 we can prove that

$$|K(x,y)| \leq \frac{C}{|x-y|^n} \tag{2.2.19}$$

with $C = C(n, c_J, \omega)$.

Lemma 2.11. *There exists a constant $C_3 = C_3(n, c_J)$ such that, if $K(x,y) \neq 0$, then*

$$|x-y| \leq C_3 d(x)$$

Proof. Recalling that $\psi(a,z)$ vanishes whenever $z \notin supp\, \omega \subset B(0, c_J/2)$, and using 2) of Lemma 2.7 we know that

$$|x - \gamma(s,y)| \leq \frac{c_J s}{2} \leq \frac{d(\gamma(s,y))}{2} \tag{2.2.20}$$

and so, recalling that $\gamma(0,y) = y$ and that $\dot{\gamma}(s,y) \leq c_J^{-1}$, we obtain

$$|x-y| \leq |x - \gamma(s,y)| + |\gamma(s,y) - \gamma(0,y)| \leq \frac{c_J s}{2} + \sqrt{n} c_J^{-1} s$$

therefore, using again 2) of Lemma 2.7, it follows that

$$|x-y| \leq C d(\gamma(s,y)) \tag{2.2.21}$$

with $C = C(n, c_J)$. But, the function d is Lipschitz with constant 1 and then, it follows from (2.2.20) that

$$d(\gamma(s,y)) - d(x) \leq |\gamma(s,y)) - x| \leq \frac{1}{2} d(\gamma(s,y))$$

and therefore,

$$d(\gamma(s,y)) \leq 2d(x)$$

which together with (2.2.21) concludes the proof. □

In order to prove the continuity of the operator defined in (2.2.15) and (2.2.16), in the next lemma we decompose it in three parts. The first one will be bounded using Theorem 2.2 while the other two parts using Theorem 2.3.

In view of the previous lemma we have

$$T_\varepsilon g(y) = \int_{\varepsilon < |x-y| \leq C_3 d(x)} K(x,y) g(x) dx.$$

Since for $|x-y| \leq d(x)/2$ we have $d(y)/3 \leq d(x)/2$, and assuming $C_3 > 1/2$, we can decompose the operator as

$$T_\varepsilon g(y) = T_{1,\varepsilon} g(y) + T_2 g(y) + T_3 g(y) \tag{2.2.22}$$

with

$$T_{1,\varepsilon}g(y) = \int_{\varepsilon<|x-y|\le d(y)/3} K(x,y)g(x)dx, \tag{2.2.23}$$

$$T_2 g(y) = \int_{d(y)/3<|x-y|\le d(x)/2} K(x,y)g(x)dx$$

and

$$T_3 g(y) = \int_{d(x)/2<|x-y|\le C_3 d(x)} K(x,y)g(x)dx$$

Lemma 2.12. *For $1 < p < \infty$, there exists a constant C depending on C_3, ω, n, and p such that*

$$\|T_2 g\|_{L^p} + \|T_3 g\|_{L^p} \le C\|g\|_{L^p} \tag{2.2.24}$$

Proof. To bound T_2 observe that, $|x-y| \le d(x)/2$ implies $d(x) \le 2d(y)$, and therefore, using (2.2.19), we have

$$|T_2 g(y)| \le CMg(y), \tag{2.2.25}$$

and therefore, it follows from Theorem 2.3 that T_2 is bounded in L^p.

On the other hand, for any $f \in L^{p'}$, we have

$$\int T_3 g(y) f(y)dy = \int\int_{d(x)/2<|x-y|\le C_3 d(x)} K(x,y)g(x)f(y)dxdy, \tag{2.2.26}$$

changing the order of integration and using again (2.2.19) we obtain

$$\left|\int T_3 g(y)f(y)dy\right| \le C\int \left\{\frac{1}{d(x)^n}\int_{|x-y|\le C_3 d(x)} |f(y)|dy\right\}|g(x)|dx \le C\int Mf(x)|g(x)|dx$$

which by duality and using again the boundedness of the maximal operator given in Theorem 2.3 gives the bound for T_3. □

Finally, we will show that the singular part $T_{1,\varepsilon}$ can be bounded using Theorems 2.2 and 2.3. With this goal we introduce

$$Sg(y) = \lim_{\varepsilon\to 0} S_\varepsilon g(y) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} H(y,x-y)g(x)dx \tag{2.2.27}$$

where

$$H(y,z) = \chi_\Omega(y)\int_0^\infty \psi\left(\dot{\gamma}_i(0,y), \frac{z}{s} - \dot{\gamma}(0,y)\right)\frac{ds}{s^{n+1}}$$

Lemma 2.13. *The kernel $H(y,z)$ satisfies the hypotheses of Theorem 2.2.*

Proof. To prove 1), given $\lambda > 0$ we make the change of variable $t = s/\lambda$. Then, we have

$$H(y,\lambda z) = \int_0^\infty \psi\left(\dot{\gamma}_i(0,y), \frac{\lambda z}{s} - \dot{\gamma}(0,y)\right) \frac{ds}{s^{n+1}}$$
$$= \lambda^{-n} \int_0^\infty \psi\left(\dot{\gamma}_i(0,y), \frac{z}{t} - \dot{\gamma}(0,y)\right) \frac{dt}{t^{n+1}} = \lambda^{-n} H(y,z)$$

To prove 2), we make now the change of variable $r = 1/s$ in the integral defining $H(y,z)$ to obtain

$$H(y,z) = \chi_\Omega(y) \int_0^\infty \psi\left(\dot{\gamma}_i(0,y), rz - \dot{\gamma}(0,y)\right) r^{n-1} dr$$

and therefore,

$$\int_\Sigma H(y,\sigma)d\sigma = \chi_\Omega(y) \int_\Sigma \int_0^\infty \psi\left(\dot{\gamma}_i(0,y), r\sigma - \dot{\gamma}(0,y)\right) r^{n-1} dr\, d\sigma$$
$$= \chi_\Omega(y) \int_{\mathbb{R}^n} \psi\left(\dot{\gamma}_i(0,y), z - \dot{\gamma}(0,y)\right) dz = 0$$

where we have used (2.2.18).

Finally, since the support of ψ in its second variable is contained in $B(0, c_J/2)$ and $|\dot{\gamma}(0,y)| \le c_J^{-1}$, there exists a constant C depending only on c_J such that

$$|H(y,z)| \le \int_{|z|/C}^\infty \left|\psi\left(\dot{\gamma}_i(0,y), \frac{z}{s} - \dot{\gamma}(0,y)\right)\right| \frac{ds}{s^{n+1}}$$

which, using that ψ is bounded, implies 3). $\square$

Corollary 2.1. *For $1 < p < \infty$, the operator*

$$Sg(y) = \lim_{\varepsilon \to 0} S_\varepsilon g(y)$$

defined in (2.2.27) is a bounded operator in L^p and the convergence holds in the L^p norm. Moreover, there exists a constant C, depending on p, n, c_J, and ω such that, if

$$\widetilde{S}g(y) = \sup_{\varepsilon > 0} |S_\varepsilon g(y)|,$$

then

$$\left\|\widetilde{S}g\right\|_{L^p} \le C\|g\|_{L^p}$$

Proof. It follows immediately from Lemma 2.13 and Theorem 2.2. $\square$

Corollary 2.2. *For $1 < p < \infty$, the operator*

$$T_1 g(y) = \lim_{\varepsilon \to 0} T_{1,\varepsilon} g(y),$$

with $T_{1,\varepsilon}$ as in (2.2.23), defines a bounded operator in L^p and the convergence holds in the L^p norm.

Proof. According to 4) in Lemma 2.7, for $0 \leq s \leq kd(y)$, we have $\gamma(s,y) = y + \dot{\gamma}(0,y)s$, and therefore, the kernel defined in (2.2.17) can be written as

$$K(x,y) = \chi_\Omega(y)\left\{\int_0^{kd(y)} \psi\left(\dot{\gamma}_i(0,y), \frac{x-y}{s} - \dot{\gamma}(0,y)\right)\frac{ds}{s^{n+1}} + \int_{kd(y)}^1 \psi\left(\dot{\gamma}_i(s,y), \frac{x-\gamma(s,y)}{s}\right)\frac{ds}{s^{n+1}}\right\}$$

and then, defining

$$J(x,y) = \chi_\Omega(y)\left\{-\int_{kd(y)}^{\infty} \psi\left(\dot{\gamma}_i(0,y), \frac{x-y}{s} - \dot{\gamma}(0,y)\right)\frac{ds}{s^{n+1}} + \int_{kd(y)}^1 \psi\left(\dot{\gamma}_i(s,y), \frac{x-\gamma(s,y)}{s}\right)\frac{ds}{s^{n+1}}\right\}$$

we obtain

$$K(x,y) = H(y,x-y) + J(x,y)$$

or, in other words,

$$T_{1,\varepsilon}g(y) = \int_{\varepsilon<|x-y|\leq d(y)/3} H(y,x-y)g(x)dx + \int_{\varepsilon<|x-y|\leq d(y)/3} J(x,y)g(x)dx$$

which can be rewritten as

$$T_{1,\varepsilon}g(y) = S_\varepsilon g(y) - \int_{|x-y|>d(y)/3} H(y,x-y)g(x)dx + \int_{\varepsilon<|x-y|\leq d(y)/3} J(x,y)g(x)dx.$$

Now, it is easy to see that there exists a constant, depending only on n, k and the L^∞-norm of ψ such that $|J(x,y)| \leq C/d(y)^n$, and therefore, the third term on the right-hand side is controlled by $Mg(y)$. Then, it follows from Corollary 2.1, that the limit defining $T_1 g(y)$ exists in L^p, and moreover,

$$|T_1 g(y)| \leq C\left\{|Sg(y)| + \widetilde{S}g(y) + Mg(y)\right\},$$

and we conclude the proof applying again Corollary 2.1 and Theorem 2.3. □

Summing up we obtain our main theorem.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain with constant c_1 with respect to $x_0 = 0$. If $f \in L^p(\Omega)$, $1 < p < \infty$, and $\int_\Omega f = 0$, then the function $\mathbf{u}$ defined in (2.2.10) is in $W_0^{1,p}(\Omega)^n$ and satisfies*

$$\operatorname{div} \mathbf{u} = f \qquad in \;\; \Omega.$$

Moreover, there exists a constant $C = C(c_1, d(x_0), diam(\Omega), n, p)$ such that

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)} \tag{2.2.28}$$

Proof. First, using the bound for G given in (2.2.9) we obtain, by an application of the Young inequality, that $\mathbf{u} \in L^p(\Omega)^n$ and

$$\|\mathbf{u}\|_{L^p(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}. \tag{2.2.29}$$

From Theorem 2.5 we know that $\operatorname{div} \mathbf{u} = f$. On the other hand, from Lemma 2.10 we know that

$$\frac{\partial u_i}{\partial x_j} = T_{ij}^* f + \omega_{ij} f \ \text{ in } \ \Omega.$$

with $\omega_{ij}(y)$ bounded, indeed, $\|\omega_{ij}\|_{L^\infty} \leq \|\omega\|_{L^1}$. Then, (2.2.28) is a consequence of (2.2.29) and the boundedness of T_{ij}^* or, by duality, of T_{ij}. But this follows from (2.2.22), (2.2.24) and Corollary 2.2. In all the estimates used to obtain (2.2.28) the constants depends only on $p, n, c_J, diam\,(\Omega)$. But, from Lemma 2.7, the constant c_J depend on c_1 and $d(x_0)$.

It only remains to show that $\mathbf{u} \in W_0^{1,p}$. For $p > n$ we have proved in Proposition 2.2 that $\mathbf{u}$ is continuous and vanishes on $\partial\Omega$, and then, as in the case of star-shaped domains, it follows from [85] that $\mathbf{u} \in W_0^{1,p}(\Omega)^n$. For $1 < p \leq n$ we proceed by density as we have done in the proof of Theorem 2.4. □

In some applications it is of interest to have a generalization of Theorem 2.6 to weighted Sobolev spaces. Below we will show that such a result can be obtained as a consequence of a general theorem for singular integral operators for weights in the Muckenhoupt class A_p.

For $1 < p < \infty$, a non-negative function w defined in $\mathbb{R}^n$ is in A_p if

$$\sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{p-1} < \infty, \tag{2.2.30}$$

where the supremum is taken over all cubes with edges parallel to the coordinate axes.

In the following theorems we will use the sharp maximal function defined as

$$M^\# f(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy$$

and the Fefferman-Stein inequality which says that

$$\|g\|_{L^p_w} \leq C\|M^\# g\|_{L^p_w} \tag{2.2.31}$$

for any $g \in L^p_w$ (see, for example, [34]).

Theorem 2.7. *Given a singular integral operator*

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} K(x,y)f(y)dy$$

which is continuous in L^p, for $1 < p < \infty$, and such that $K(x,y)$ satisfies

$$|K(x,y) - K(\bar{x},y)| \leq \frac{C|x-\bar{x}|}{|x-y|^{n+1}} \ , \qquad \textit{for} \quad |x-y| \geq 2|x-\bar{x}|$$

then, for any $s > 1$,

$$M^{\#}Tf(x) \leq C(M|f|^s(x))^{1/s}$$

Proof. This estimate is well known and its proof can be found in several books, although the hypotheses on the operator are not stated usually as we are doing here. However, it is easy to check that the proof given in [34, Lemma 7.9] only uses the hypotheses given above. □

Theorem 2.8. *Under the hypotheses of Theorem 2.6, if $w \in A_p$, there exists a constant $C = C(c_1, d(x_0), diam(\Omega), n, p, w)$ such that*

$$\|\mathbf{u}\|_{W^{1,p}(\Omega,w)^n} \leq C\|f\|_{L^p(\Omega,w)} \tag{2.2.32}$$

Proof. It is enough to bound $\partial u_i/\partial x_j$. The estimate for u will follow from the Poincaré inequality, which is known to hold for A_p weights (see, for example, [33]).

Now, using Lemma 2.10 and that $\omega_{i,j}$ is bounded, we have to prove that

$$\|T^* f\|_{W^{1,p}_w(\Omega)^n} \leq C\|f\|_{L^p_w(\Omega)}$$

where

$$T^* f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x,y) f(y) dy$$

with $K(x,y)$ given in (2.2.17). Proceeding again as in Lemma 2.8 we obtain

$$|\nabla_x K(x,y)| \leq \frac{C}{|x-y|^{n+1}},$$

and consequently,

$$|K(x,y) - K(\bar{x},y)| \leq \frac{C|x-\bar{x}|}{|x-y|^{n+1}}, \qquad \text{for} \quad |x-y| \geq 2|x-\bar{x}|,$$

therefore, since we already know that T^* is a continuous operator in L^p, for $1 < p < \infty$, T^* satisfies the hypotheses of Theorem 2.7. Consequently, we have

$$M^{\#}T^* f(x) \leq C(M|f|^s(x))^{1/s}$$

for any $s > 1$.

Now the result follows from this estimate combined with (2.2.31) for $g = T^* f$. We omit details and refer the reader to the proof of Theorem 7.11 in [34]. □

2.3 Improved Poincaré Inequality and Equivalences

This section deals with the so-called improved Poincaré inequality, namely, for $f \in L^p_0(\Omega)$,

$$\|f\|_{L^p(\Omega)} \leq C\|d\nabla f\|_{W^{1,p}(\Omega)^n} \tag{2.3.1}$$

where the constant C depends only on the domain Ω.

This inequality has many applications. In fact, we will show below that it provides a different way to prove the existence of solutions of the divergence in Sobolev spaces. Moreover, we will see in Chapter 3 that it can be used to prove Korn type inequalities.

The next theorem is a particular case of results given in [33]. In its proof we will use the following well-known result.

Lemma 2.14. *For $\beta > 0$ and $g \in L^1_{loc}$,*

$$\int_{|x-y|\leq\beta} \frac{|g(y)|}{|x-y|^{n-1}} dy \leq C\beta Mg(x)$$

with a constant C independent of β and g.

Proof.

$$\int_{|x-y|\leq\beta} \frac{|g(y)|}{|x-y|^{n-1}} dy \leq \sum_{k=0}^{\infty} \int_{2^{-(k+1)}\beta<|x-y|\leq 2^{-k}\beta} \frac{|g(y)|}{|x-y|^{n-1}} dy$$
$$\leq C\beta \sum_{k=0}^{\infty} \frac{2^{-k}}{|B(x,2^{-k}\beta)|} \int_{|x-y|\leq 2^{-k}\beta} |g(y)|dy \leq C\beta Mg(x)$$

□

Theorem 2.9. *If $\Omega \subset \mathbb{R}^n$ is a bounded John domain, then the improved Poincaré inequality (2.3.1) holds for $1 \leq p < \infty$.*

Proof. By Lemma 3.1 it is enough to prove

$$\|f - f_\omega\|_{L^p(\Omega)} \leq C\|d\nabla f\|_{L^p(\Omega)^n}$$

with ω is as in the previous section.

By density we can assume that $f \in C^1(\Omega)$. Indeed, the density of $C^1(\Omega)$ in $W^{1,p}(\Omega,1,d^p)$ can be proved by the same argument used in the unweighted case (see, for example, [40]).

As in Lemma 2.9 we can show that

$$f(y) - f_\omega = -\int_\Omega G(x,y) \cdot \nabla f(x)dx.$$

with $G(x,y)$ given in (2.2.8). Then, given $g \in L^{p'}(\Omega)$, we have

$$\int_\Omega (f(y) - f_\omega)g(y)dy = \int_\Omega \int_\Omega G(x,y) \cdot \nabla f(x)g(y)dxdy$$

Interchanging the order of integration and using Lemmas 2.8 and 2.11, we obtain

$$\int_\Omega (f(y) - f_\omega)g(y)dy \leq C \int_\Omega \left\{ \int_{|x-y| \leq C_3 d(x)} \frac{|g(y)|}{|x-y|^{n-1}} dy \right\} |\nabla f(x)|dx,$$

and therefore, using Lemma 2.14,

$$\int_\Omega (f(y) - f_\omega)g(y)dy \leq C \int_\Omega Mg(x)d(x)|\nabla f(x)|dx \leq C\|Mg\|_{L^{p'}(\Omega)}\|d\nabla f\|_{L^p(\Omega)}$$

and the proof concludes using the continuity of the maximal operator in $L^{p'}$ and duality. □

Remark 2.6. Note that the previous theorem includes the limit case $p = 1$. Clearly, the argument does not apply to $p = \infty$ because it would require the continuity of the maximal operator in L^1. Moreover, the improved Poincaré is not valid in L^∞, indeed, an easy counterexample can be given taking $\Omega = (0,1)$ and $f(x) = \log x$.

The improved Poincaré can be used to prove a decomposition of functions with vanishing integral as a sum of locally supported functions with the same property, actually, it is equivalent to the existence of this decomposition. This decomposition is useful to obtain global from local results, more precisely, to extend to very general domains results which are known for cubes.

In the next theorem we analyze the relation between the existence of solutions of the divergence, the improved Poincaré inequality and the decomposition of functions. The arguments are contained in [35, 36, 60].

We will make use of the Whitney decomposition introduced in (2.2.2) denoting now Q_j, $j \in \mathbb{N}$, the cubes. It is known (see, for example, [84]) that there exists a family of functions $\phi_j \in C_0^\infty(Q_j^*)$ associated with the decomposition such that $\sum_j \phi_j = \chi_\Omega$, $\|\phi_j\|_{L^\infty} \leq C$ and $\|\nabla\phi_j\|_{L^\infty} \leq C/d_j$, where d_j denotes the distance of Q_j to the boundary of Ω.

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain and $1 < p < \infty$. Consider the following statements,*

1. $\|f\|_{L^{p'}(\Omega)} \leq C\|d\nabla f\|_{L^{p'}(\Omega)^n} \qquad \forall f \in L_0^{p'}(\Omega)$

2. $\forall f \in L_0^p(\Omega)$, *there exists* $\mathbf{u} \in L^p(\Omega)^n$ *such that*

$$div\, \mathbf{u} = f \quad in\ \Omega\ , \ \mathbf{u} \cdot n = 0 \quad on\ \partial\Omega$$

and

$$\left\| \frac{\mathbf{u}}{d} \right\|_{L^p(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}$$

3. $\forall f \in L^p_0(\Omega)$*, there exists a decomposition*

$$f = \sum_j f_j$$

such that

$$f_j \in L^p_0(\Omega), \quad \operatorname{supp} f_j \subset Q_j^*, \quad \textit{and} \quad \|f\|^p_{L^p(\Omega)} \sim \sum_j \|f_j\|^p_{L^p(Q_j^*)}$$

4. $\forall f \in L^p_0(\Omega)$*, there exists* $\mathbf{u} \in W^{1,p}_0(\Omega)^n$ *such that*

$$\mathit{div}\, \mathbf{u} = f \quad \mathit{in}\ \Omega$$

and

$$\|D\mathbf{u}\|_{L^p(\Omega)^n} \le C\|f\|_{L^p(\Omega)}$$

Then,

$$(1) \iff (2) \iff (3) \Longrightarrow (4)$$

and the constants are equivalent, i.e., the ratio between two of them is bounded by above and below by positive constants depending only on n and p.

Proof. $(1) \Rightarrow (2)$: For $f \in L^p_0(\Omega)$

$$\mathscr{L}(\nabla g) = \int_\Omega fg$$

defines a linear form on the subspace of $L^{p'}(\Omega)^n$ formed by the gradient vector fields. Note that $\mathscr{L}$ is well defined because $\int_\Omega f = 0$. Moreover, it follows from (1) that

$$|\mathscr{L}(\nabla g)| = \left| \int_\Omega f(g - g_\Omega) \right| \le C\|f\|_{L^p(\Omega)} \|d\nabla g\|_{L^{p'}(\Omega)^n}.$$

By the Hahn-Banach theorem $\mathscr{L}$ can be extended as a linear continuous functional

$$\mathscr{L} : L^{p'}(\Omega, d^{p'})^n \longrightarrow \mathbb{R}$$

and therefore, by duality, there exists $\mathbf{u} \in L^p(\Omega, d^{-p})^n$ such that

$$\mathscr{L}(\mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} \quad \text{and} \quad \left\| \frac{\mathbf{u}}{d} \right\|_{L^p(\Omega)^n} \le C\|f\|_{L^p(\Omega)}$$

in particular,

$$\int_\Omega \mathbf{u} \cdot \nabla g = \int_\Omega fg \quad \forall g \in W^{1,p'}(\Omega)$$

which is equivalent to

$$\operatorname{div} \mathbf{u} = f \quad \text{in}\ \Omega\ ,\ \mathbf{u} \cdot n = 0 \quad \text{on}\ \partial\Omega$$

and so (2) holds.

$(2) \Rightarrow (1)$: Given $f \in L^p_0(\Omega)$ we have

$$\|f\|_{L^{p'}(\Omega)} = \sup_{g \in L^p_0(\Omega), \|g\|_p = 1} \int_\Omega fg \tag{2.3.2}$$

Now, for $g \in L^p_0(\Omega)$, let $\mathbf{u} \in L^p(\Omega)^n$ be the solution of $\operatorname{div} \mathbf{u} = g$ given by (2). Then,

$$\int_\Omega fg = \int_\Omega f \operatorname{div} \mathbf{u} = \int_\Omega \nabla f \cdot \mathbf{u}$$

$$\leq \|d\nabla f\|_{L^{p'}(\Omega)^n} \left\| \frac{\mathbf{u}}{d} \right\|_{L^p(\Omega)} \leq C \|d\nabla f\|_{L^{p'}(\Omega)^n} \|g\|_{L^p(\Omega)}$$

which together with (2.3.2) implies (1).

$(2) \Rightarrow (3)$: Given $f \in L^p_0(\Omega)$ let $\mathbf{u} \in L^p(\Omega)^n$ given by (2). We define

$$f_j = \operatorname{div}(\phi_j \mathbf{u})$$

then

$$f = \operatorname{div} \mathbf{u} = \operatorname{div}\left(\mathbf{u} \sum_j \phi_j\right) = \sum_j \operatorname{div}(\phi_j \mathbf{u}) = \sum_j f_j$$

Since $\operatorname{supp} \phi_j \subset Q_j^*$ we have $\operatorname{supp} f_j \subset Q_j^*$ and $\int_\Omega f_j = 0$.

Moreover, from the finite superposition of the Whitney decomposition we have

$$|f(x)|^p \leq C \sum_j |f_j(x)|^p$$

and then

$$\|f\|^p_{L^p(\Omega)} \leq C \sum_j \|f_j\|^p_{L^p(Q_j^*)}$$

where the constant C depends only on p and n.

To prove the other inequality we use again the finite superposition and that $\|\phi_j\|_{L^\infty} \leq 1$ and $\|\nabla \phi_j\|_{L^\infty} \leq C/d_j$. Then, we have

$$\|f_j\|^p_{L^p(Q_j^*)} \leq C \left\{ \|f\|^p_{L^p(Q_j^*)} + \left\| \frac{\mathbf{u}}{d} \right\|^p_{L^p(Q_j^*)^n} \right\}$$

and therefore it follows from (2) that

$$\sum_j \|f_j\|^p_{L^p(Q_j^*)} \leq C \|f\|^p_{L^p(\Omega)}$$

$(3) \Rightarrow (2)$: Given $f \in L^p_0(\Omega)$ we write $f = \sum_j f_j$ according to (3). From the results in Section 2.1 we know that, for each j, there exists $\mathbf{u}_j \in W_0^{1,p}(Q_j^*)^n$ such that

$$\operatorname{div} \mathbf{u}_j = f_j \quad \text{and} \quad \|D\mathbf{u}_j\|_{L^p(Q_j^*)} \leq C \|f_j\|_{L^p(Q_j^*)}$$

where the constant is independent of the size of the cube. Then, $\mathbf{u} = \sum_j \mathbf{u}_j \in W_0^{1,p}(\Omega)^n$ is the required $\mathbf{u}$. Indeed, $\operatorname{div}\mathbf{u} = f$ and the estimate

$$\left\|\frac{\mathbf{u}}{d}\right\|_{L^p(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}$$

follows applying the Poincaré inequality on each Q_j^*. Indeed

$$\left\|\frac{\mathbf{u}_j}{d}\right\|_{L^p(Q_j^*)^n} \sim \frac{1}{d_j}\|\mathbf{u}_j\|_{L^p(Q_j^*)^n} \leq C\|D\mathbf{u}_j\|_{L^p(Q_j^*)^n} \leq C\|f_j\|_{L^p(Q_j^*)}$$

then

$$\left\|\frac{\mathbf{u}}{d}\right\|^p_{L^p(Q_j^*)^n} \leq C\sum_j\left\|\frac{\mathbf{u}_j}{d}\right\|^p_{L^p(Q_j^*)^n} \leq C\sum_j\|f_j\|^p_{L^p(Q_j^*)} \leq C\|f\|^p_{L^p(\Omega)}$$

as we wanted to show.

$(3) \Rightarrow (4)$: It is proved exactly as $(3) \Rightarrow (2)$.

The equivalence between the constants follows easily from the proofs. □

As we mentioned above, the results given in this section provide a different argument to prove the existence of solutions of the divergence. This is summarized in the following corollary. The same argument has been used in [32]

Corollary 2.3. *If $\Omega \subset \mathbb{R}^n$ is a bounded John domain and $1 \leq p < \infty$ then for any $f \in L_0^p(\Omega)$ there exists $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ such that*

$$\mathit{div}\,\mathbf{u} = f \quad \mathit{in}\ \Omega$$

and

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}$$

Proof. It is an immediate consequence of Theorems 2.9 and 2.10. □

Remark 2.7. We don't know whether (4) implies the other statements given in Theorem 2.10 for a general domain. However, it is easy to see that $(4) \Rightarrow (2)$ holds for domains satisfying the Hardy inequality, i.e., there exists a constant depending only on Ω and p such that

$$\left\|\frac{v}{d}\right\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^p(\Omega)^n} \qquad \forall v \in W_0^{1,p}(\Omega)$$

Moreover, it is known that this inequality is valid for any domain different from $\mathbb{R}^n$ when $p > n$ [73]. Therefore, for $p > n$ we have the stronger statement

$$(1) \iff (2) \iff (3) \iff (4)$$

for any domain $\Omega \not\subseteq \mathbb{R}^n$.

2.4 A Partial Converse Result

An interesting problem is to characterize the bounded domains for which the results considered in Theorem 2.10 are valid. According to the previous section we know that all of them hold for John domains. As we have mentioned at the beginning of Section 2.2, it is known that div_p is not valid for some domains with external cusps. Since the class of John domains is very general and excludes external cusps it seems a natural question whether

$$\Omega \quad \text{satisfies } \operatorname{div}_p \iff \Omega \quad \text{is a John domain.} \tag{2.4.1}$$

As far as we know the answer is not known. However, a partial answer can be given. Indeed, (2.4.1) is true if the bounded domain Ω satisfies the separation property. We omit the technical definition of this property and refer the reader to [16] where it was introduced. In that paper it is also proved that, in the two dimensional case, any simply connected domain satisfies the separation property.

For $1 \le p < n$ we say that Ω satisfies the Sobolev-Poincaré inequality for p if there exists a constant depending only on p and Ω such that

$$\|f\|_{L^{p^*}(\Omega)} \le C\|\nabla f\|_{L^p(\Omega)^n} \qquad \forall f \in W^{1,p}(\Omega) \cap L^p_0(\Omega),$$

where $p^* = pn/(n-p)$. In [16] the authors prove that, if Ω is a bounded domain that satisfies the separation property as well as the Sobolev-Poincaré inequality for some $1 < p < n$, then it is a John domain.

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the separation property. Then, Ω satisfies div_p, for some $1 < p < \infty$, if and only if Ω is a John domain.*

Proof. From the previous section we already know that if Ω is a John domain then div_p is valid for all $1 < p < \infty$. The converse was proved in [3] in the case $1 < p < n$ showing that div_p implies the Sobolev-Poincaré for $q = (p^*)'$ and applying the result in [16]. Indeed, given $f \in W^{1,q}(\Omega) \cap L^q_0(\Omega)$ and $g \in L^p_0(\Omega)$, take $\mathbf{u} \in W^{1,p}_0(\Omega)^n$ such that $\operatorname{div} \mathbf{u} = g$ and $\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \le C\|g\|_{L^p(\Omega)}$. Now, by the Sobolev-Poincaré for functions in $W^{1,p}_0(\Omega)^n$ and observing that $q' = p^*$ we have

$$\begin{aligned}\int_\Omega fg = \int_\Omega f \operatorname{div} \mathbf{u} = -\int_\Omega \nabla f \cdot \mathbf{u} &\le \|\nabla f\|_{L^q(\Omega)^n} \|\mathbf{u}\|_{L^{q'}(\Omega)^n} \\ &\le C\|\nabla f\|_{L^q(\Omega)^n} \|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \le C\|\nabla f\|_{L^q(\Omega)^n} \|g\|_{L^p(\Omega)}\end{aligned}$$

and the argument concludes observing that $p = (q^*)'$ and using duality.

For the case $n < p$ the result was proved in [60] generalizing the arguments of [16] to show that, under the separation property, the improved Poincaré inequality implies that Ω is a John domain, and then using that div_p implies the improved Poincaré for p' (see Theorem 2.10 and Remark 2.7). We can also use the following argument: in [35] it is proved that, if the improved Poincaré is valid for some $q \ge 1$ then it is valid for all r such that $q \le r < \infty$, actually, in that paper the proof is written for $q = 1$ but the argument can be easily extended to any q. Now, assuming div_p for some $p > n$ we have that Ω satisfies the improved Poincaré for p', and therefore, for p. Then, using again Theorem 2.10, we obtain that Ω satisfies $div_{p'}$. But $p' < n$, and so, Ω is a John domain. Finally, the case $p = n$ was proved in [59]. $\square$

2.5 Comments and References

An interesting problem that has been widely considered is that of the dependence on the domain of the constants involved in all the inequalities considered. Ideally, given a particular Ω and an inequality, one would like to know the best constant possible, but this is a too difficult problem that can be solved only in very particular cases. For example, the constant for div_2 is known for circles, ellipses, spheres, and spherical shells. We refer the reader to [56] and the references therein.

A less ambitious problem is to obtain estimates of the constants in terms of geometric properties of the domains. There are many works in this direction. Important tools in this problem are results like Theorem 2.10 which allow to translate information for some inequality to another one.

Consider, for example, the estimate

$$\|D\mathbf{u}\|_{L^p(\Omega)} \le C_{p,div,\Omega}\|f\|_{L^p(\Omega)}, \tag{2.5.1}$$

where $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ is some solution of $\operatorname{div}\mathbf{u} = f$.

One could try to obtain information tracing constants in the proofs given in the previous sections for the estimates for the solutions of the divergence defined there. However, the arguments are based on the general Calderón-Zygmund singular integral operators theory and the constant that one obtains from that theory seems to be nonoptimal for our particular case. As it was pointed out in [46], for the case of a domain of diameter d which is star-shaped with respect to a ball B of radius ρ, it follows from [21] that, for $1 < p < \infty$, $C_{p,div,\Omega} \le C_{n,p}(d/\rho)^{n+1}$.

However, at least in the case $p = 2$, this estimate can be improved. Indeed, this has been done in [36] where the result given in Section 2.1 is proved for the case $p = 2$ using a different argument. Instead of relying on the general theory, the proof in [36] is based on elementary properties of the Fourier transform. In this way it is proved that, for the solution $\mathbf{u}$ defined in Section 2.1,

$$C_{2,div,\Omega} \le C_n \frac{d}{\rho}\left(\frac{|\Omega|}{|B|}\right)^{\frac{n-2}{2(n-1)}}\left(\log\frac{|\Omega|}{|B|}\right)^{\frac{n}{2(n-1)}} \tag{2.5.2}$$

In particular, in the two dimensional case we have, for any $\varepsilon > 0$,

$$C_{2,div,\Omega} \le C_\varepsilon (d/\rho)^{1+\varepsilon}$$

This estimate has been improved in [28] removing the ε and obtaining

$$C_{2,div,\Omega} \le C(d/\rho) \tag{2.5.3}$$

The result in that paper is not for the solution analyzed in Section 2.1 but for a different one. The authors use the equivalence between $C_{2,div,\Omega}$ and the constant in the so-called Friedrichs inequality. Let h and g be real valued functions such that $\int_\Omega h = 0$ and $h + ig$ is an holomorphic function in Ω. Under appropriate assumptions on the domain Ω, Friedrichs proved that there exists a constant $C_{fr,\Omega}$ such that

$$\|h\|_{L^2(\Omega)} \le C_{fr,\Omega} \|g\|_{L^2(\Omega)}, \tag{2.5.4}$$

Assuming that Ω is a smooth domain it was proved in [57] that, if $C_{2,div,\Omega}$ and $C_{fr,\Omega}$ are the best possible constants in (2.5.1) (with $p = 2$) and (2.5.4), respectively, then

$$C^2_{2,div,\Omega} = C^2_{fr,\Omega} + 1$$

This result was extended for arbitrary bounded domains in [28], and using this equivalence and complex variable arguments the authors proved (2.5.3).

This result is optimal, indeed, consider the rectangular domain $\Omega_{a,\varepsilon} = (-a, +a) \times (-\varepsilon, \varepsilon)$, with a and ε positive constants, and take $h(x_1, x_2) = x_1$ and $g(x_1, x_2) = x_2$. Then, an elementary computation shows that the Friedrichs inequality applied to these functions gives

$$C_{fr,\Omega_{a,\varepsilon}} \ge (a/\varepsilon)$$

and consequently, for these domains,

$$C_{2,div,\Omega} \ge c_1(d/\rho) \tag{2.5.5}$$

where c_1 is a constant independent of Ω.

We do not know whether (2.5.2) can be improved for $n > 3$ nor whether similar estimates can be proved for $p \ne 2$.

For the particular case of convex domains it is possible to use (1) $\Rightarrow$ (4) in Theorem 2.10 to prove, for $1 < p < \infty$ and arbitrary dimension n, that

$$C_{p,div,\Omega} \le C(d/\rho)$$

with C depending only on n and p. This has been done in [36] for $p = 2$ but the arguments there can be easily extended to $1 < p < \infty$.

To end our comments on the constant in (2.5.1) let us mention the papers [27] and [11] where the behavior of the constant for domains with corners and continuity with respect to the domain were analyzed.

Finally, several papers have considered the existence of solutions of the divergence in higher order Sobolev spaces under appropriate assumptions on the right-hand side f (see [9, 83, 29]).

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