

# Preface

Given a domain  $\Omega$  and a function  $f$  with vanishing mean in  $\Omega$ , is it possible to find a zero trace field  $\mathbf{u}$  such that

$$\operatorname{div} \mathbf{u} = f$$

and

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}?$$

If the answer is positive, how irregular can  $\Omega$  be? And if it is negative, is it possible to enlarge the space of involved functions in which  $\mathbf{u}$  is sought in order to obtain solutions?

This problem, that is, the existence of a right inverse of the *divergence operator*, also called  $\operatorname{div}_p$  in this book, is strongly related to the Korn inequality and to the Stokes equations among many other relevant problems and inequalities. The purpose of this book is to present in a unified and coherent way an overview of many of these central topics.

Our approach follows somehow our own line of research in these matters collecting most of our production related to the field. Notwithstanding that, many proofs have been improved or shortened and the material organized in a coherent fashion that hopefully helps the reader to get smoothly into the subject. Moreover, at each stage, we tried to survey as many contributions as we could covering both classical papers and recent developments. Naturally, length and time restrictions necessarily imposed a limit. We tried to focus on specific aspects and therefore the reader may - and surely will - find omissions that genuinely should deserve to have a place in this book. Our apologies, in advance, for them.

The book is organized as follows: several equivalences of the problem of existence of a right inverse of the divergence are explored in Chapter 1. Chapter 2 is devoted to present the construction of solutions for  $\operatorname{div}_p$  elaborated by Bogovskii for star-shaped domains and to show how that result can be generalized to more intricate domains, specifically to John domains. Besides geometrical technicalities, the main ideas are the same and the key tool is ultimately, in both cases, the Calderón-Zygmund singular integrals theory. The last sections of that chapter treat the relation of that problem with improved Poincaré inequalities. Chapter 3, in turn, focuses on

Korn's inequalities beginning with several equivalences, for the so-called *second case* of the inequality, in arbitrary domains. Then, Korn's inequalities on John domains are derived by different alternative approaches using, for instance, the results previously obtained for the divergence operator as well as improved Poincaré inequalities. The latter approach turns out to be useful even for obtaining weighted versions of these inequalities on Hölder- $\alpha$  domains and for domains with external cusps. This is extensively treated along Chapter 4 where also families of counterexamples are exhibited to prove that the presented results are sharp. Moreover, in Chapter 4 more irregular cases are treated. In the last part, cusps are allowed to have rough boundaries, besides the singular point at the tip. Weighted Korn's inequalities are also derived for them using elementary decomposition techniques.

Finally, a brief and even informal derivation of some basic equations of *continuum mechanics* and a short proof delimiting what powers of the distance to a set function belong to the Muckenhoupt's  $A_p$  class together with some supplementary material is given in three appendices.

We would like to end this preface with just a few more words. We devoted, in a nonsystematic way, more than twelve years to the issues treated in this book. We did it, as it usually happens, without a master plan in mind and following most of the time personal tastes and interests. Looking back we noticed that a very rich net of connections was simply lost in the vast literature dealing with these topics. A net, maybe not ignored by the expert, we wished to expose clearly to a larger audience along the following pages.

Last but not least, we want to thank our coauthors throughout all these years. Without them, we would have never reached this point. When we were finishing this book we received the sad news that our friend Maria Amelia Muschietti passed away. An important part of the contents presented here corresponds to work done in collaboration with her. This book is dedicated to her memory.

Buenos Aires, Argentina  
December, 2016

Gabriel Acosta  
Ricardo G. Durán

Divergence Operator and Related Inequalities

Acosta, G.; Durán, R.G.

2017, XIII, 124 p. 7 illus., Softcover

ISBN: 978-1-4939-6983-8