Chapter 2
Functions from $\mathbb{R}^p$ to $\mathbb{R}^q$

Consider a function $f\colon H \to \mathbb{R}^q$, where H is an arbitrary set, and let the coordinates of the vector $f(x)$ be denoted by $f_1(x), \ldots, f_q(x)$ for every $x \in H$. In this way we define the functions $f_1, \ldots, f_q$, where $f_i\colon H \to \mathbb{R}$ for every $i = 1, \ldots, q$. We call f_i the ith **coordinate function** or **component** of f.

The above defined concept is a generalization of the coordinate functions introduced by Definition 1.47. Indeed, let f be the identity function on $\mathbb{R}^p$, i.e., let $f(x) = x$ for every $x \in \mathbb{R}^p$. Then the coordinate functions of $f\colon \mathbb{R}^p \to \mathbb{R}^p$ are nothing but the functions $x = (x_1, \ldots, x_p) \mapsto x_i$ (with $i = 1, \ldots, p$).

Now let $f\colon H \to \mathbb{R}^q$ with $H \subset \mathbb{R}^p$. The coordinate functions of f are real-valued functions defined on the set H; therefore, they are p-variable real-valued functions. The limits, continuity, and differentiability of the function f could be defined using the corresponding properties of f's coordinate functions. However, it is easier, shorter, and more to the point to define these concepts directly for f, just copying the corresponding definitions for real-valued functions. Fortunately, as we shall see, the two approaches are equivalent to each other.

2.1 Limits and Continuity

Definition 2.1. Let $H \subset E \subset \mathbb{R}^p$, and let a be a limit point of H. The *limit of the function* $f\colon E \to \mathbb{R}^q$ *at* a *restricted to* H is $b \in \mathbb{R}^q$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - b| < \varepsilon$ whenever $x \in H$ and $0 < |x - a| < \delta$. *Notation:* $\lim_{x \to a,\ x \in H} f(x) = b$.

If the domain of the function f is equal to H (i.e., it is not greater than H), then we omit the part "restricted to the set H" of the definition above, and we simply say that the limit of f at a is b, with notation $\lim_{x \to a} f(x) = b$ or $f(x) \to b$, if $x \to a$.

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Obviously, $\lim_{x\to a,\ x\in H} f(x) = b$ if and only if for every neighborhood V of b there exists a punctured neighborhood $\dot{U}$ of a such that $f(x) \in V$ if $x \in H \cap \dot{U}$.

Theorem 2.2. *Let $H \subset E \subset \mathbb{R}^p$, let a be a limit point of H, and let $b = (b_1, \ldots, b_q) \in \mathbb{R}^q$. For every function $f\colon E \to \mathbb{R}^q$, we have $\lim_{x\to a,\ x\in H} f(x) = b$ if and only if $\lim_{x\to a,\, x\in H} f_i(x) = b_i \ (i = 1, \ldots, q)$ holds for every coordinate function f_i of f.*

Proof. The statement follows from the definitions, using the fact that for every point $y = (y_1, \ldots, y_q) \in \mathbb{R}^q$, we have $|y - b| \le |y_1 - b_1| + \ldots + |y_q - b_q|$ and $|y_i - b_i| \le |y - b|$, for each $i = 1, \ldots, q$. □

The **transference principle** follows from the theorem above: $\lim_{x\to a,\ x\in H} f(x) = b$ *if and only if for every sequence $x_n \in H \setminus \{a\}$*, we have that $x_n \to a$ *implies* $f(x_n) \to b$. (This statement is a generalization of the corresponding one dimensional theorem [7, Theorem 10.19].)

It is clear from Theorems 1.40 and 2.2 that *if* $\lim_{x\to a,\ x\in H} f(x) = b$ *and* $\lim_{x\to a,\ x\in H} g(x) = c$, *where* $b, c \in \mathbb{R}^q$, *then* $\lim_{x\to a,\ x\in H} \lambda f(x) = \lambda b$ *for every* $\lambda \in \mathbb{R}$. *Furthermore*, $\lim_{x\to a,\ x\in H}(f(x) + g(x)) = b + c$ *and* $\lim_{x\to a,\ x\in H}\langle f(x), g(x)\rangle = \langle b, c\rangle$.

Definition 2.3. Let $a \in H \subset E \subset \mathbb{R}^p$. We say that the function $f\colon E \to \mathbb{R}^q$ is *continuous at a restricted to the set H* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in H$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

If the domain of f is equal to H, then we can omit the part "restricted to the set H" from the definition.

If f is continuous at every point $a \in H$, then we say that f is continuous on the set H.

The following theorem follows from Theorem 2.2.

Theorem 2.4. *The function f is continuous at a point a restricted to the set H if and only if this is true for every coordinate function of f.* □

Clearly, *f is continuous at a restricted to H if and only if one of the following two conditions holds:*

(i) *a is an isolated point of H;*
(ii) *$a \in H \cap H'$ and* $\lim_{x\to a,\ x\in H} f(x) = f(a)$.

The **transference principle for continuity** can be easily verified: *the function $f\colon H \to \mathbb{R}^p$ is continuous at the point $a \in H$ restricted to the set H if and only if $f(x_n) \to f(a)$ holds for every sequence $x_n \in H$ with $x_n \to a$.*

This implies the following statement: *if the functions f and g are continuous at the point a restricted to the set H, then so are the functions $f + g$, $\langle f, g\rangle$ and λf for every $\lambda \in \mathbb{R}$.*

A theorem about the limit of composite functions follows.

Theorem 2.5. *Suppose that*

(i) *$H \subset \mathbb{R}^p$, $g\colon H \to \mathbb{R}^q$ and $\lim_{x\to a} g(x) = c$, where a is a limit point of H;*

(ii) *$g(H) \subset E \subset \mathbb{R}^q$, $f\colon E \to \mathbb{R}^s$ and $\lim_{x\to c} f(x) = b$;*

(iii) *$g(x) \neq c$ in a punctured neighborhood of a, or $c \in E$ and f is continuous at c restricted to the set $g(H)$.*

Then

$$\lim_{x\to a} f(g(x)) = b. \square \tag{2.1}$$

Corollary 2.6. *If g is continuous at a restricted to H, and f is continuous at the point $g(a)$ restricted to the set $g(H)$, then $f \circ g$ is also continuous at a restricted to H.* □

If we wish to generalize Weierstrass's theorem (Theorem 1.51) to functions mapping to $\mathbb{R}^q$, we have to keep in mind that for $q > 1$ there is no natural ordering of the points of $\mathbb{R}^q$. Therefore, we cannot speak about the largest or smallest value of a function. However, the statement on the boundedness still holds; moreover, we can state more.

Theorem 2.7. *Let $H \subset \mathbb{R}^p$ be bounded and closed, and let $f\colon H \to \mathbb{R}^q$ be continuous. Then the set $f(H)$ is bounded and closed in $\mathbb{R}^q$.*

Proof. Applying Weierstrass's theorem (Theorem 1.51) to the coordinate functions of f yields that the set $f(H)$ is bounded.

In order to prove that $f(H)$ is also closed we will use part (iii) of Theorem 1.17. Suppose that $y_n \in f(H)$ and $y_n \to b$. For every n we can choose a point $x_n \in H$ such that $f(x_n) = y_n$. The sequence (x_n) is bounded (since H is bounded). Thus, by the Bolzano–Weierstrass theorem, (x_n) has a convergent subsequence (x_{n_k}). If $x_{n_k} \to a$, then $a \in H$, because the set H is closed. Since the function f is continuous, it follows that

$$b = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(a),$$

and thus $b \in f(H)$. Then, by Theorem 1.17, the set $f(H)$ is closed. □

Recall the definition of injective functions. A mapping is **injective** (or **one-to-one**) if it takes on different values at different points of its domain. The following theorem states another important property of continuous functions with bounded and closed domains.

Theorem 2.8. *Let $H \subset \mathbb{R}^p$ be bounded and closed, and let $f\colon H \to \mathbb{R}^q$ be continuous. If f is injective on the set H, then f^{-1} is continuous on the set $f(H)$.*

Proof. Let $y_n \in f(H)$ and $y_n \to b \in f(H)$. Then we have $b = f(a)$ for a suitable $a \in H$. Let $x_n = f^{-1}(y_n)$ for every n; we need to prove that $x_n \to f^{-1}(b) = a$.

We prove by contradiction. Let us assume that the statement is not true. Then there exists $\varepsilon > 0$ such that $x_n \notin B(a, \varepsilon)$, i.e., $|x_n - a| \geq \varepsilon$ for infinitely many n. We may assume that this holds for every n, for otherwise, we could delete the terms of the sequence for which it does not hold. The sequence (x_n) is bounded (since H is bounded), and then, by the Bolzano–Weierstrass theorem, it has a convergent subsequence (x_{n_k}). If $x_{n_k} \to c$. And then $c \in H$, since H is closed. Also, $c \neq a$, since

$$|c - a| = \lim_{k\to\infty} |x_{n_k} - a| \geq \varepsilon.$$

Since the function f is continuous, it follows that

$$f(c) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = b = f(a),$$

which contradicts the assumption that f is injective. □

Remark 2.9. The condition of the boundedness of the set H cannot be omitted from the previous theorem, i.e., the inverse of a continuous and injective function on a closed domain is not necessarily continuous. Consider the following example. Let $p = q = 1,\ H = \mathbb{N}$ and let $f\colon \mathbb{N} \to \mathbb{R}$ be the function with $f(0) = 0$ and $f(n) = 1/n$ for every $n = 1, 2, \ldots$. The set H is closed (since every convergent sequence of H is constant begining from some index), the function f is continuous on H (since every point of H is an isolated point), and f is injective. On the other hand, f^{-1} is not continuous, since

$$0 = f^{-1}(0) \neq \lim_{n\to\infty} f^{-1}(1/n) = \lim_{n\to\infty} n = \infty.$$

(The condition of closedness of H cannot be omitted from the theorem either; see Exercise 2.2.)

Uniform continuity can be defined in the same way as in the case of real-valued functions.

Definition 2.10. We say that the function f is *uniformly continuous* on the set $H \subset \mathbb{R}^p$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such $|f(x) - f(y)| < \varepsilon$ holds whenever $x, y \in H$ and $|x - y| < \delta$ (where δ is independent of x and y).

Heine's theorem remains valid: *if $H \subset \mathbb{R}^p$ is a bounded and closed set and the function $f\colon H \to \mathbb{R}^q$ is continuous, then f is uniformly continuous on H.*

2.2 Differentiability

To define differentiability for an $\mathbb{R}^q$-valued function, we proceed as in the cases of limits and continuity; that is, we simply copy Definition 1.63. However, since we are dealing with functions that map from $\mathbb{R}^p$ to $\mathbb{R}^q$, we need to define linear maps from $\mathbb{R}^p$ to $\mathbb{R}^q$. For this reason we recall some basic notions of linear algebra.

We say that a function $A\colon \mathbb{R}^p \to \mathbb{R}^q$ is a **linear mapping** or a **linear transformation** if $A(x + y) = A(x) + A(y)$ and $A(\lambda x) = \lambda A(x)$ hold for every $x, y \in \mathbb{R}^p$

and $\lambda \in \mathbb{R}$. Clearly, the mapping $A\colon \mathbb{R}^p \to \mathbb{R}^q$ is linear if and only if each of its coordinate functions is linear.

Let $a_{i1}x_1 + \ldots + a_{ip}x_p$ be the ith coordinate function of the mapping A $(i = 1, \ldots, q)$. We call

$$\begin{pmatrix} a_{11} & a_{12} \ldots a_{1p} \\ a_{21} & a_{22} \ldots a_{2p} \\ \vdots & \vdots \quad\quad \vdots \\ a_{q1} & a_{q2} \ldots a_{qp} \end{pmatrix} \tag{2.2}$$

the matrix of the mapping A. The matrix has q rows and p columns, and the ith row contains the coefficients of the ith coordinate function of A.

It is easy to see that if $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, then the vector $y = A(x)$ is the product of the matrix of A and the column vector consisting of the coordinates of x. That is,

$$A(x) = \begin{pmatrix} a_{11} & a_{12} \ldots a_{1p} \\ a_{21} & a_{22} \ldots a_{2p} \\ \vdots & \vdots \quad\quad \vdots \\ a_{q1} & a_{q2} \ldots a_{qp} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix}. \tag{2.3}$$

In other words, the ith coordinate of $A(x)$ is the scalar product of the ith row of A and x.

Definition 2.11. Let $H \subset \mathbb{R}^p$ and $a \in \operatorname{int} H$. We say that *the function $f\colon H \to \mathbb{R}^q$ is differentiable at the point a* if there exists a linear mapping $A\colon \mathbb{R}^p \to \mathbb{R}^q$ such that

$$f(x) = f(a) + A(x-a) + \varepsilon(x) \cdot |x-a| \tag{2.4}$$

for every $x \in H$, where $\varepsilon(x) \to 0$ if $x \to a$. (Here $\varepsilon\colon H \to \mathbb{R}^q$.)

Remark 2.12. Since the function ε can be defined to be 0 at the point a, *the differentiability of the function f is equivalent to* (2.4) *for an appropriate linear mapping A, where $\varepsilon(a) = 0$ and ε is continuous at a.*

We can formulate another equivalent condition: *for an appropriate linear mapping A we have $(f(x) - f(a) - A(x-a))/|x-a| \to 0$ as $x \to a$.*

Theorem 2.13. *The function $f\colon H \to \mathbb{R}^q$ $(H \subset \mathbb{R}^p)$ is differentiable at the point $a \in \operatorname{int} H$ if and only if every coordinate function f_i of f is differentiable at a. The jth entry of the ith row of the matrix of A from (2.4) is equal to the partial derivative $D_j f_i(a)$ for every $i = 1, \ldots, q$ and $j = 1, \ldots, p$.*

Proof. Suppose that (2.4) holds for every $x \in H$, where $\varepsilon(x) \to 0$ as $x \to a$. Since the vectors of the two sides of (2.4) are equal, their corresponding coordinates are equal as well. Thus, $f_i(x) = f_i(a) + A_i(x-a) + \varepsilon_i(x) \cdot |x-a|$ for every $x \in H$ and $i = 1, \ldots, q$, where f_i, A_i, ε_i denote the ith coordinate functions of f, A, and ε, respectively. Since A_i is linear and $\varepsilon_i(x) \to 0$ as $x \to a$ (following from the fact that $|\varepsilon_i(x)| \le |\varepsilon(x)|$ for every x), we get that f_i is differentiable at a. By Theorem 1.67, the jth coefficient of the linear function A_i is the $D_j f_i(a)$ partial derivative, which also proves the statement about the matrix A.

Now suppose that every coordinate function f_i of f is differentiable at a. By Theorem 1.67, $f_i(x) = f_i(a) + A_i(x-a) + \varepsilon_i(x)$, where $A_i(x) = D_1 f_i(a)x_1 + \ldots + D_p f_i(a)x_p$ and $\varepsilon_i(x) \to 0$ as $x \to a$. Let $A(x) = (A_1(x), \ldots, A_q(x))$ for every $x \in \mathbb{R}^p$, and let $\varepsilon(x) = (\varepsilon_1(x), \ldots, \varepsilon_q(x))$ for every $x \in H$. The mapping $A\colon \mathbb{R}^p \to \mathbb{R}^q$ is linear, and $\varepsilon(x) \to 0$ as $x \to a$ by Theorem 2.2. In addition, (2.4) holds for every $x \in H$. This proves that f is differentiable at the point a. □

Corollary 2.14. *If f is differentiable at a, then the linear mapping A from (2.4) is unique.* □

Definition 2.15. Let $f\colon H \to \mathbb{R}^q$ with $H \subset \mathbb{R}^p$, and let f be differentiable at $a \in \operatorname{int} H$. We say that the linear mapping $A\colon \mathbb{R}^p \to \mathbb{R}^q$ from (2.4) is *the derivative of the function f at the point a*, and we use the notation $f'(a)$. We call the matrix of the linear mapping $f'(a)$, i.e., the matrix of the partial derivatives $D_j f_i(a)$ $(j = 1, \ldots, p,\ i = 1, \ldots, q)$

$$\begin{pmatrix} D_1 f_1(a) & D_2 f_1(a) \ldots & D_p f_1(a) \\ D_1 f_2(a) & D_2 f_2(a) \ldots & D_p f_2(a) \\ \vdots & \vdots & \vdots \\ D_1 f_q(a) & D_2 f_q(a) \ldots & D_p f_q(a), \end{pmatrix}$$

the *Jacobian matrix*[1] of the function f at the point a.

The following statements are clear from Theorems 1.66, 1.67, 1.71, and 2.13.

Theorem 2.16.

(i) *If the function f is differentiable at the point a, then f is continuous at a. Furthermore, every partial derivative of every coordinate function of f exists and is finite at a.*

(ii) *If every partial derivative of every coordinate function of f exists and is finite in a neighborhood of the point a and is continuous at a, then f is differentiable at a.* □

Example 2.17. Consider the mapping

$$f(x, y) = (e^x \cos y, e^x \sin y) \qquad ((x, y) \in \mathbb{R}^2).$$

[1] Carl Jacobi (1804–1851), German mathematician.

The partial derivatives of f's coordinate functions are

$$\begin{aligned} D_1 f_1(x,y) &= e^x \cos y, \quad & D_2 f_1(x,y) &= -e^x \sin y, \\ D_1 f_2(x,y) &= e^x \sin y, \quad & D_2 f_2(x,y) &= e^x \cos y \end{aligned}$$

for every $(x,y) \in \mathbb{R}^2$. Since these partial derivatives are continuous everywhere, it follows from Theorem 2.16 that f is differentiable at every point (a,b) in the plane, and f's Jacobian matrix is

$$\begin{pmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{pmatrix}.$$

Thus, the derivative of f at (a,b) is the linear mapping

$$A(x,y) = ((e^a \cos b)x - (e^a \sin b)y, (e^a \sin b)x + (e^a \cos b)y).$$

Remark 2.18. Let us summarize the different objects we obtain by differentiating different kinds of mappings.

The derivative of a single-variable real function at a fixed point is a real number, namely the limit of the differential quotients.

The derivative of a curve $g\colon [a,b] \to \mathbb{R}^q$ at a given point is a vector of $\mathbb{R}^q$ whose coordinates are the derivatives of g's coordinate functions (see [7, Remark 16.22]).

The derivative of a p-variable real function is a vector of $\mathbb{R}^p$ (the gradient vector) whose components are the partial derivatives of the function at a given point.

Definition 2.15 takes another step toward further abstraction: the derivative of a map $\mathbb{R}^p \to \mathbb{R}^q$ is neither a number nor a vector, but a mapping.

As a consequence of this diversity, the derivative of a function $f\colon \mathbb{R} \to \mathbb{R}$ is a real number (if we consider f a function) but also a vector of dimension one (if we consider f a curve mapping into $\mathbb{R}$).

What's worse, the derivative of a mapping $\mathbb{R}^p \to \mathbb{R}^q$ is a vector for $q = 1$, but it is also a linear mapping, and for $p = q = 1$ it is a real number as well.

We should realize, however, that the essence of the derivative is the linear mapping with which we approximate the function, and the way we represent this linear mapping is less important. For a single-variable function f, the approximating linear function is $f(a) + f'(a)(x-a)$ defining the tangent line. This function is uniquely characterized by the coefficient $f'(a)$ (since it has to take the value $f(a)$ at a). Similarly, a linear function approximating a p-variable real function is the function $f(a) + \sum_{i=1}^{p} D_i f(a)(x_i - a_i)$ defining the tangent hyperplane. This can be characterized by the vector of its coefficients.

We could have circumvented these inconsistencies by defining the derivative of a function $f\colon \mathbb{R}^p \to \mathbb{R}^q$ not by a linear mapping, but by its matrix (i.e., its Jacobian matrix).[2] In most cases it is much more convenient to think of the derivative as a mapping and not as a matrix, which we will see in the next section. When we talk about mappings between more general spaces (called normed linear spaces), the linear mappings do not always have a matrix. In these cases we have to define the derivative as the linear mapping itself.

We have to accept the fact that the object describing the derivative depends on the dimensions of the corresponding spaces. Fortunately enough, whether we consider the derivative to be a number, a vector, or a mapping will always be clear from the context.

2.3 Differentiation Rules

Theorem 2.19. *If the functions f and g mapping to $\mathbb{R}^q$ are differentiable at the point $a \in \mathbb{R}^p$, then the functions $f+g$ and λf are also differentiable at a. Furthermore, $(f+g)'(a) = f'(a) + g'(a)$ and $(\lambda f)'(a) = \lambda f'(a)$ for every $\lambda \in \mathbb{R}$.*

Proof. The statement is obvious from Theorem 2.13. □

The following theorem concerns the differentiability of a composite function and its derivative.

Theorem 2.20. *Suppose that*

(i) $H \subset \mathbb{R}^p,\ g\colon H \to \mathbb{R}^q$, *and g is differentiable at the point $a \in \operatorname{int} H$;*

(ii) $g(a) \in \operatorname{int} E \subset \mathbb{R}^q,\ f\colon E \to \mathbb{R}^s$, *and f is differentiable at the point $g(a)$.*

Then the composite function $f \circ g$ is differentiable at a, with

$$(f \circ g)'(a) = f'(g(a)) \circ g'(a).$$

To prove this theorem we first need to show that every linear mapping has the Lipschitz property.

Lemma 2.21. *For every linear mapping $A\colon \mathbb{R}^p \to \mathbb{R}^q$ there exists a $K \geq 0$ such that $|A(x) - A(y)| \leq K \cdot |x-y|$ for every $x, y \in \mathbb{R}^p$.*

Proof. Let $e_1, \ldots, e_p$ be a basis of $\mathbb{R}^p$, and let $M = \max_{1 \leq i \leq p} |A(e_i)|$. Then, for every $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$ we have

$$|A(x)| = \left| \sum_{i=1}^{p} x_i \cdot A(e_i) \right| \leq \sum_{i=1}^{p} |x_i| \cdot M \leq Mp \cdot |x|.$$

Thus $|A(x) - A(y)| = |A(x-y)| \leq Mp \cdot |x-y|$ for every $x, y \in \mathbb{R}^p$, and hence $K = M\,p$ satisfies the requirements of the lemma. □

[2] However, the inconsistencies would not have disappeared entirely. For $p = 1$ (i.e., for curves mapping to $\mathbb{R}^q$) the Jacobian matrix is a $1 \times q$ matrix, in other words, it is a column vector, while the derivative of the curve is a row vector.

Let $\mathcal{K}_A$ denote the set of numbers $K \geq 0$ that satisfy the conditions of Lemma 2.21. Obviously, the set $\mathcal{K}_A$ has a smallest element. Indeed, if $K_0 = \inf \mathcal{K}_A$, then $|A(x) - A(y)| \leq K_0 \cdot |x - y|$ also holds for every $x, y \in \mathbb{R}^p$, and thus $K_0 \in \mathcal{K}_A$.

Definition 2.22. The smallest number, K, satisfying the conditions of Lemma 2.21 is called the *norm* of A, and is denoted by $\|A\|$.

Proof of Theorem 2.20. Let $g'(a) = A$ and $f'(g(a)) = B$. We know that if x is close to a, then $g(a) + A(x - a)$ approximates $g(x)$ well, and if y is close to $g(a)$, then $f(g(a)) + B(y - g(a))$ approximates $f(y)$ well. Therefore, intuitively, if x is close to a, then

$$f(g(a)) + B(g(a) + A(x - a) - g(a)) = f(g(a)) + (BA)(x - a)$$

approximates $f(g(x))$ well; i.e., $(f \circ g)'(a) = BA$. Below we make this argument precise.

Since $g'(a) = A$, it follows that

$$g(x) = g(a) + A(x - a) + \varepsilon(x) \cdot |x - a|, \tag{2.5}$$

where $\lim_{x \to a} \varepsilon(x) = 0$. Let us choose $\delta > 0$ such that $|x - a| < \delta$ implies $x \in H$ and $|\varepsilon(x)| < 1$. Then

$$\begin{aligned} |g(x) - g(a)| \leq |A(x - a)| + |\varepsilon(x)| \cdot |x - a| \leq \|A\| \cdot |x - a| + |x - a| = \\ = (\|A\| + 1) \cdot |x - a| \end{aligned} \tag{2.6}$$

for every $|x - a| < \delta$. On the other hand, $f'(g(a)) = B$ implies

$$f(y) = f(g(a)) + B(y - g(a)) + \eta(y) \cdot |y - g(a)|, \tag{2.7}$$

where $\lim_{y \to g(a)} \eta(y) = \eta(g(a)) = 0$. Now g is continuous at the point a by (2.6) (or by Theorem 2.16), whence $g(x) \in E$ if x is close enough to a. Applying (2.7) with $y = g(x)$ and using also (2.5), we get

$$\begin{aligned} f(g(x)) - f(g(a)) &= B(g(x) - g(a)) + \eta(g(x)) \cdot |g(x) - g(a)| = \\ &= B(A(x - a)) + B(\varepsilon(x)) \cdot |x - a| + \eta(g(x)) \cdot |g(x) - g(a)| = \\ &= (B \circ A)(x - a) + r(x), \end{aligned} \tag{2.8}$$

where $r(x) = B(\varepsilon(x)) \cdot |x - a| + \eta(g(x)) \cdot |g(x) - g(a)|$. Then, by (2.6),

$$|r(x)| \leq \|B\| \cdot |\varepsilon(x)| \cdot |x - a| + |\eta(g(x))| \cdot (\|A\| + 1) \cdot |x - a| = \theta(x) \cdot |x - a|,$$

where

$$\theta(x) = \|B\| \cdot |\varepsilon(x)| + (\|A\| + 1) \cdot |\eta(g(x))| \to 0$$

if $x \to a$, since $\eta(g(a)) = 0$ and η is continuous at $g(a)$. Therefore, (2.8) implies that the function $f \circ g$ is differentiable at a, and $(f \circ g)'(a) = B \circ A$. $\square$

Corollary 2.23. (Differentiation of composite functions)
Suppose that the real-valued function f is differentiable at the point $b=(b_1, \ldots, b_q) \in \mathbb{R}^q$, and the real-valued functions $g_1, \ldots, g_q$ are differentiable at the point $a \in \mathbb{R}^p$, where $g_i(a) = b_i$ for every $i = 1, \ldots, q$. Then the function $F(x) = f(g_1(x), \ldots, (g_q(x))$ is differentiable at the point a, and

$$D_j F(a) = \sum_{i=1}^{q} D_i f(b) \cdot D_j g_i(a) \tag{2.9}$$

holds for every $j = 1, \ldots, p$.

Proof. Let $g_1, \ldots, g_q$ be defined in $B(a, \delta)$, and let $G(x) = (g_1(x), \ldots, g_q(x))$ for every $x \in B(a, \delta)$. By Theorem 2.13, the mapping $G\colon B(a, \delta) \to \mathbb{R}^q$ is differentiable at a. Since $F = f \circ G$, Theorem 2.20 implies that F is differentiable at a and its Jacobian matrix (i.e., the vector $F'(a)$) is equal to the product of the Jacobian matrix of f at the point b (i.e., the vector $f'(b)$) and the Jacobian matrix of G at the point a. The jth coordinate of the vector $F'(a)$ is equal to $D_j F(a)$. On the other hand (by the rules of matrix multiplication), the jth coordinate of the vector $F'(a)$ is equal to the scalar product of the vector $f'(b)$ and the jth column of the Jacobian matrix of G. This is exactly equation (2.9). $\square$

Remark 2.24. The formula (2.9) is easy to memorize in the following form. Let $y_1, \ldots, y_q$ denote the variables of f, and let us write also y_i instead of g_i. We get

$$\frac{\partial F}{\partial x_j} = \frac{\partial f}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_j} + \frac{\partial f}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_j} + \ldots + \frac{\partial f}{\partial y_q} \cdot \frac{\partial y_q}{\partial x_j}.$$

The differentiability of products and fractions follows easily from Corollary 2.23.

Theorem 2.25. *Let f and g be real-valued functions differentiable at the point $a \in \mathbb{R}^p$. Then $f \cdot g$, and assuming $g(a) \neq 0$, f/g is also differentiable at a.*

Proof. The function $\varphi(x, y) = x \cdot y$ is differentiable everywhere on $\mathbb{R}^2$. Since $f(x) \cdot g(x) = \varphi(f(x), g(x))$, Corollary 2.23 gives the differentiability of $f \cdot g$ at a. The differentiability of f/g follows similarly, using the fact that the rational function x/y is differentiable on the set $\{(x, y) \in \mathbb{R}^2 \colon y \neq 0\}$. $\square$

Note that the partial derivatives of $f \cdot g$ and f/g can be obtained using (2.9) (or using the rules of differentiating single-variable functions). (See Exercise 1.92.)

The differentiation rule for the inverse of one-variable functions (see [7, Theorem 12.20]) can be generalized to multivariable functions as follows.

Theorem 2.26. *Suppose that $H \subset \mathbb{R}^p$, the function $f\colon H \to \mathbb{R}^p$ is differentiable at the point $a \in \operatorname{int} H$, and the mapping $f'(a)$ is invertible. Let $f(a) = b$, $\delta > 0$, and* *let* $g\colon B(b,\delta) \to \mathbb{R}^p$ *be a continuous function that satisfies* $g(b) = a$ *and* $f(g(x)) = x$ *for every* $x \in B(b,\delta)$.

Then the function g *is differentiable at* b, *and* $g'(b) = (f'(a))^{-1}$, *where* $(f'(a))^{-1}$ *is the inverse of the linear mapping* $f'(a)$.

Proof. Without loss of generality, we may assume that $a = b = 0$ (otherwise, we replace the functions f and g by $f(x+a) - b$ and $g(x+b) - a$, respectively).

First we also assume that $f'(0)$ is the identity mapping. Then $|f(x) - x|/|x| \to 0$ as $x \to 0$. Since $\lim_{x\to 0} g(x) = 0$ and $g \neq 0$ on the set $B(0,\delta) \setminus \{0\}$, it follows from Theorem 2.5 on the limit of composite functions that $|f(g(x)) - g(x)|/|g(x)| \to 0$ as $x \to 0$. Since $f(g(x)) = x$, we find that $|x - g(x)|/|g(x)| \to 0$ as $x \to 0$.

Now we prove that $g'(0)$ is also the identity mapping, i.e., $\lim_{x\to 0} |g(x) - x|/|x| = 0$. First note that $|x - g(x)| \le |g(x)|/2$ for every $x \in B(0,\delta')$ for a small enough δ'. Thus $x \in B(0,\delta')$ implies

$$|g(x)| \le |g(x) - x| + |x| \le (|g(x)|/2) + |x|,$$

whence $|g(x)| \le 2|x|$, and

$$\frac{|x - g(x)|}{|x|} = \frac{|x - g(x)|}{|g(x)|} \cdot \frac{|g(x)|}{|x|} \le 2 \cdot \frac{|x - g(x)|}{|g(x)|}.$$

Therefore, $\lim_{x\to 0} |g(x) - x|/|x| = 0$ holds. We have proved that g is differentiable at the origin, and its derivative is the identity mapping there.

Now we consider the general case (still assuming $a = b = 0$). Let $f'(0) = A$. By Theorem 2.20, $f_1 = A^{-1} \circ f$ is differentiable at the origin, and its derivative is the linear mapping $A^{-1} \circ A$, which is the identity. The function $g_1 = g \circ A$ is continuous in a neighborhood of the origin, with $f_1(g_1(x)) = x$ in this neighborhood. Thus, the special case proved above implies that $g_1'(0)$ is also the identity mapping. Since $g = g_1 \circ A^{-1}$, Theorem 2.20 on the differentiability of composite functions implies that g is differentiable at the origin, and its derivative is A^{-1} there. $\square$

Exercises

2.1. Let $H \subset \mathbb{R}^p$. Show that the mapping $f\colon H \to \mathbb{R}^q$ is continuous on H if and only if for every open set $V \subset \mathbb{R}^q$ there is an open set $U \subset \mathbb{R}^p$ such that $f^{-1}(V) = H \cap U$.

2.2. Give an example of a bounded set $H \subset \mathbb{R}^p$ and a continuous, injective function $f\colon H \to \mathbb{R}^q$ such that f^{-1} is not continuous on the set $f(H)$.

2.3. Show that if $A = (a_{ij})$ $(i = 1, \ldots, q,\ j = 1, \ldots, p)$, then

$$\|A\| \le \sqrt{\sum_{i=1}^{q}\sum_{j=1}^{p} a_{ij}^2}.$$

Give an example when strict inequality holds.

2.4. Show that if $A = (a_{ij})$ $(i = 1, \ldots, q, j = 1, \ldots, p)$, then

$$\max_{1\le i\le q,\ 1\le j\le p} |a_{ij}| \le \|A\|,$$

furthermore,

$$\max_{1\le i\le q} \sqrt{\sum_{j=1}^{p} a_{ij}^2} \le \|A\|.$$

Give an example when strict inequality holds.

2.5. Let the linear mapping $A\colon \mathbb{R}^p \to \mathbb{R}^p$ be invertible. Show the existence of some $\delta > 0$ and $K \ge 0$ such that $\|B^{-1} - A^{-1}\| \le K \cdot \|B - A\|$ for every B that satisfies $\|B - A\| < \delta$.

2.6. Let $1 \le i \le q$ and $1 \le j \le p$ be fixed. Show that a_{ij} is a continuous (furthermore, Lipschitz) function of A, i.e., there exists K such that $|a_{ij} - b_{ij}| \le K \cdot \|A - B\|$. (Here a_{ij} and b_{ij} are the jth entries of the ith row of the matrices A and B, respectively.)

2.7. Find all differentiable functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ that satisfy $D_1 f \equiv D_2 f$. (S)

2.8. Let the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable on the plane, and let $D_1 f(x, x^2) = D_2 f(x, x^2) = 0$ for every x. Show that $f(x, x^2)$ is constant.

2.9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable on the plane. Let $f(0,0) = 0$, $D_1 f(x, x^3) = x$ and $D_2 f(x, x^3) = x^3$ for every x. Find $f(1,1)$.

2.10. Let $H \subset \mathbb{R}^p$, and let $f\colon H \to \mathbb{R}$ be differentiable at the point $a \in \operatorname{int} H$. We call the set $S = \{x \in \mathbb{R}^p\colon f(x) = f(a)\}$ the **contour line** corresponding to a. Show that the contour line is perpendicular to the gradient $f'(a)$ in the following sense: if $g\colon (c,d) \to \mathbb{R}^p$ is a differentiable curve whose graph lies in S and $g(t_0) = a$ for some $t_0 \in (c,d)$, then $g'(t_0)$ and $f'(a)$ are perpendicular to each other. (The zero vector is perpendicular to every vector.)

2.11. We say that the function $f\colon \mathbb{R}^p \setminus \{0\} \to \mathbb{R}$ is a **homogeneous function with degree** k (where k is a fixed real), if $f(tx) = t^k \cdot f(x)$ holds for every $x \in \mathbb{R}^p \setminus \{0\}$ and $t \in \mathbb{R}$, $t > 0$. **Euler's theorem**[3] states that if $f\colon \mathbb{R}^p \setminus \{0\} \to \mathbb{R}$ is

[3] Leonhard Euler (1707–1783), Swiss mathematician.

differentiable and homogeneous with degree k, then $x_1 \cdot D_1 f + \ldots + x_p \cdot D_p f = k \cdot f$ for every $x = (x_1, \ldots, x_p) \in \mathbb{R}^p \setminus \{0\}$.
Double-check the theorem for some particular functions (e.g., $xy/\sqrt{x^2+y^2}$, $xy/(x^2+y^2)$, $\sqrt{x^2+y^2}$, etc.).

2.12. Prove Euler's theorem.

2.13. Let the function $f\colon \mathbb{R}^p \to \mathbb{R}^q$ be differentiable at the points of the segment $[a, b]$, where $a, b \in \mathbb{R}^p$. True or false? There exists a point $c \in [a, b]$ such that $f(b) - f(a) = f'(c)(b - a)$. (I.e., can we generalize the mean value theorem (Theorem 1.79) for vector valued functions?) (H S)

2.4 Implicit and Inverse Functions

Solving an equation means that the unknown quantity, given only implicitly by the equation, is made explicit. For example, x is defined implicitly by the quadratic equation $ax^2 + bx + c = 0$, and as we solve this equation, we express x explicitly in terms of the parameters a, b, c. In order to make the nature of this problem more transparent, let's write x_1, x_2, x_3 in place of a, b, c and y in place of x. Then we are given the function $f(x_1, x_2, x_3, y) = x_1 y^2 + x_2 y + x_3$ of four variables, and we have to find a function $\varphi(x_1, x_2, x_3)$ satisfying

$$f(x_1, x_2, x_3, \varphi(x_1, x_2, x_3)) = 0. \tag{2.10}$$

In this case we say that the function $y = \varphi(x_1, x_2, x_3)$ is the solution of equation (2.10). As we know, there is no solution on the set $A = \{(x_1, x_2, x_3)\colon x_2^2 - 4x_1x_3 < 0\} \subset \mathbb{R}^3$, and there are continuous solutions on the set $B = \{(x_1, x_2, x_3) : x_1 \neq 0,\ x_2^2 - 4x_1x_3 \geq 0\} \subset \mathbb{R}^3$, namely each of the functions

$$\varphi_1 = (-x_2 + \sqrt{x_2^2 - 4x_1x_3})/(2x_1), \qquad \varphi_2 = (-x_2 - \sqrt{x_2^2 - 4x_1x_3})/(2x_1)$$

is a continuous solution on B.

Finding the inverse of a function means solving an equation as well. A function φ is the inverse of the function g exactly when the unique solution of the equation $x - g(y) = 0$ is $y = \varphi(x)$.

In general, we cannot expect that the solution y can be given by a (closed) formula of the parameters. Even $f(x, y)$ is not always defined by a closed formula. However, even assuming that $f(x, y)$ is given by a formula, we cannot ensure that y belongs to the same family of functions that we used to express f. For example, $f(x, y) = x - y^3$ is a polynomial, but the solution $y = \sqrt[3]{x}$ of the equation $f(x, y) = 0$ cannot. Based on this observation, it is not very surprising that there exists a function $f(x, y)$ such that f can be expressed by elementary functions, but the solution y of the equation $f(x, y) = 0$ is not. Consider the function $g(x) = x + \sin x$. Then g is strictly monotonically increasing and continuous on the real line, and furthermore it assumes every real value, and thus it has an inverse on $\mathbb{R}$. It can be shown that the inverse of g cannot be expressed by elementary functions only. That is, the equation $x - y - \sin y = 0$ has a unique solution, but the solution is not an elementary function.

The same phenomenon is illustrated by a famous theorem of algebra stating that the roots of a general quintic polynomial cannot be obtained from the coefficients by rational operations and by extractions of roots. That is, there does not exist a function $y = \varphi(x_1, \ldots, x_6)$ defined only by the basic algebraic operations and extraction of roots of the coordinate functions $x_1, \ldots, x_6$ such that y is the solution of the equation $x_1 y^5 + \ldots + x_5 y + x_6 = 0$ on a nonempty, open subset of $\mathbb{R}^6$.

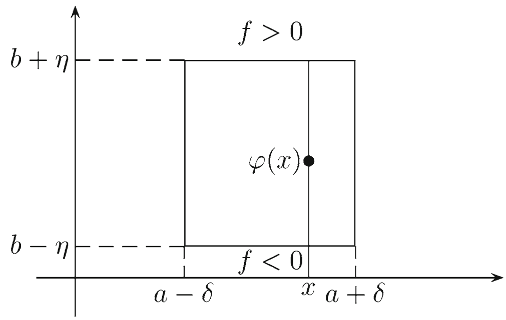
Therefore, solving y explicitly does not necessarily mean expressing y by a (closed) formula; it means only establishing the existence or nonexistence of the solution and describing its properties when it does exist. The simplest related theorem is the following.

Theorem 2.27. *Let f be a two-variable real function such that f is zero at the point $(a,b) \in \mathbb{R}^2$ and continuous on the square $[a-\eta, a+\eta] \times [b-\eta, b+\eta]$ for an appropriate $\eta > 0$. If the section f_x is strictly monotone at every $x \in [a-\eta, a+\eta]$, then there exists a positive real δ such that*

(i) *for every $x \in (a-\delta, a+\delta)$, there exists a unique $\varphi(x) \in (b-\eta, b+\eta)$ such that $f(x, \varphi(x)) = 0$, and furthermore,*

(ii) *the function φ is continuous on the interval $(a-\delta, a+\delta)$.*

Proof. We know that the section f_a is strictly monotonically. Without loss of generality, we may assume that f_a is strictly monotone increasing (the proof of the other case is exactly the same), and thus $f_a(b-\eta) < f_a(b) = f(a,b) = 0 < f_a(b+\eta)$.



2.1. Figure

Let $\varepsilon > 0$ be small enough to imply $f_a(b-\eta) < -\varepsilon$ and $\varepsilon < f_a(b+\eta)$.

Since f is continuous at the points $(a, b-\eta)$, $(a, b+\eta)$, there exists $0 < \delta < \eta$ such that $|f(x, b-\eta) - f(a, b-\eta)| < \varepsilon$ and $|f(x, b+\eta) - f(a, b+\eta)| < \varepsilon$ for every $x \in (a-\delta, a+\delta)$. That is, if $x \in (a-\delta, a+\delta)$, then

$$f(x, b-\eta) < 0 < f(x, b+\eta).$$

Since f_x is strictly monotone and continuous on the interval $[b-\eta, b+\eta]$, it follows from Bolzano's theorem that there is a unique $\varphi(x) \in (b-\eta, b+\eta)$ such that $f(x, \varphi(x)) = 0$. Thus, we have proved statement (i).

Let $x_0 \in (a-\delta, a+\delta)$ and $\varepsilon > 0$ be fixed. Choose positive numbers δ_1 and $\eta_1 < \varepsilon$ such that

$$(x_0 - \delta_1, x_0 + \delta_1) \subset (a-\delta, a+\delta)$$

and

$$(\varphi(x_0) - \eta_1, \varphi(x_0) + \eta_1) \subset (b-\eta, b+\eta)$$

hold. Following the steps of the first part, we end up with a number $0 < \delta' < \delta_1$ such that for every $x \in (x_0 - \delta', x_0 + \delta')$ there exists a unique

$$y \in (\varphi(x_0) - \eta_1, \varphi(x_0) + \eta_1) \subset (b-\eta, b+\eta)$$

with $f(x, y) = 0$. By (i), $\varphi(x)$ is the only such number, and hence $y = \varphi(x)$. Thus, for $|x - x_0| < \delta'$ we have $|\varphi(x) - \varphi(x_0)| < \eta_1 < \varepsilon$. Therefore, φ is continuous at x_0. □

Corollary 2.28. (Implicit function theorem for single-variable functions) *Suppose that the two-variable function f is zero at the point $(a, b) \in \mathbb{R}^2$ and continuous in a neighborhood of (a, b). Let the partial derivative $D_2 f$ exist and be finite and nonzero in a neighborhood of (a, b). Then there exist positive numbers δ and η such that*

(i) *for every $x \in (a-\delta, a+\delta)$ there exists a unique number $\varphi(x) \in (b-\eta, b+\eta)$ with $f(x, \varphi(x)) = 0$, furthermore,*
(ii) *the function φ is continuous in the interval $(a-\delta, a+\delta)$.*

Proof. It follows from the assumptions that there is a rectangle $(a_1, a_2) \times (b_1, b_2)$ containing (a, b) in its interior such that f is continuous, $D_2 f$ exists and is finite and nonzero in $(a_1, a_2) \times (b_1, b_2)$. The section f_x is strictly monotone in the interval (b_1, b_2) for every $x \in (a_1, a_2)$, since it is differentiable and, by Darboux's theorem[4] [7, Theorem 13.44], its derivative must be everywhere positive or everywhere negative in the interval (b_1, b_2). Then an application of Theorem 2.27 to the rectangle $(a_1, a_2) \times (b_1, b_2)$ finishes the proof. □

Remark 2.29. We will see later that if f is continuously differentiable at (a, b), then the function φ is continuously differentiable at the point a (see Theorem 2.40).

For the single-variable case, it is not difficult to show that the differentiability of f at (a, b) and $D_2 f(a, b) \neq 0$ implies the differentiability of φ at a (see Exercise 2.15). We can calculate $\varphi'(a)$ by applying the differentiation rule of composite functions. Since $f(x, \varphi(x)) = 0$ in a neighborhood of the point a, its derivative is also zero there. Thus,

[4] Jean Gaston Darboux (1842–1917), French mathematician. Darboux's theorem states that if $f\colon [a, b] \to \mathbb{R}$ is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

$$D_1 f(a,b) \cdot 1 + D_2 f(a,b) \cdot \varphi'(a) = 0$$

holds, from which we obtain $\varphi'(a) = -D_1 f(a,b)/D_2 f(a,b)$.

Example 2.30. The function $f(x,y) = x^2 + y^2 - 1$ is continuous and (infinitely) differentiable everywhere. If $a^2 + b^2 = 1$ and $-1 < a < 1$, then $D_2 f(a,b) = 2b \neq 0$, and the conditions of Corollary 2.28 are satisfied. Thus, there exists some function ϕ such that ϕ is continuous in a neighborhood of a, $\varphi(a) = b$, and $x^2 + \varphi(x)^2 - 1 = 0$. Namely, if $b > 0$, then the function $\varphi(x) = \sqrt{1-x^2}$ on the interval $(-1,1)$ is such a function. If, however, $b < 0$, then the function $\varphi(x) = -\sqrt{1-x^2}$ satisfies the conditions on the interval $(-1,1)$.

On the other hand, if $a = 1$, then there is no such function in any neighborhood of a, since $x > 1$ implies $x^2 + y^2 - 1 > 0$ for every y. The conditions of Corollary 2.28 are not satisfied here, since $a = 1$ implies $b = 0$ and $D_2 f(1,0) = 0$. The same happens in the $a = -1$ case.

Our next goal is to generalize Corollary 2.28 to multivariable functions.

Corollary 2.28 gives a sufficient condition for the existence of the inverse of a function—at least locally. The inverse of an arbitrary function g is given by the solution of the equation $f(x,y) = 0$, where $f(x,y) = x - g(y)$. Let $g(b) = a$; thus $f(a,b) = 0$. By Corollary 2.28, if g is differentiable in a neighborhood of b such that $g'(x) \neq 0$ in this neighborhood, then there exists a continuous function φ in a neighborhood $(a-\delta, a+\delta)$ of a such that $\varphi(a) = b$ and $g(\varphi(x)) = x$ on $(a-\delta, a+\delta)$.

We expect that a generalization of Corollary 2.28 to multivariable functions would also give a sufficient condition for the existence of the inverse locally. Therefore, we first consider the question of the existence of the inverse function.

Proving the existence of the inverse of a multivariable function is substantially more difficult than for one-variable functions; this is a case in which the analogy with the single-variable case exists but is far from being sufficient. The question is how to decide whether or not a given function is injective on a given set. For a continuous single-variable, real function defined on an interval, the answer is quite simple: the function is injective if it is strictly monotone. (This follows from the Bolzano–Darboux theorem,[5] see [7, Theorem 10.57].) It is not clear, however, how to generalize this condition to continuous multivariable, or vector-valued functions.

Yet another problem is related to the existence of a “global” inverse. Let $f\colon I \to \mathbb{R}$ be continuous, where $I \subset \mathbb{R}$ is an interval. Given that every point of I has a neighborhood on which the function f is injective, we can easily show that f is injective on the whole interval. Thus, global injectivity follows from local injectivity for single-variable continuous real functions. However, this does not hold for vector-valued or multivariable functions! Let $g\colon \mathbb{R} \to \mathbb{R}^2$ be a curve with $g(t) = (\cos t, \sin t)$ for every $t \in \mathbb{R}$. The mapping g is injective on every interval

[5] The Bolzano–Darboux’s theorem states that if $f\colon [a,b] \to \mathbb{R}$ is continuous, then f takes on every value between $f(a)$ and $f(b)$.

shorter than 2π (and maps to the unit circle), but g is not injective globally, since it is periodic with period 2π. Similarly, let $f(x,y) = (e^x \cos y, e^x \sin y)$ for every $(x,y) \in \mathbb{R}^2$. The mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is injective on every disk of the plane with radius less than π, but f is not injective globally, since $f(x, y+2\pi) = f(x,y)$ for every $(x,y) \in \mathbb{R}^2$.

Unfortunately, we cannot help this; it seems that there are no natural, simple sufficient conditions for the global injectivity of a vector-valued or multivariable function. Thus, we have to restrict our investigations to the question of local injectivity.

Let the mapping f be differentiable in a neighborhood of the point a. Since the mapping $f(a) + f'(a)(x-a)$ approximates f well locally, we might think that given the injectivity of the linear mapping $f'(a)$, f will also be injective on a neighborhood of a. However, this is not always so, not even in the simple special case of $p = q = 1$. There are everywhere differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(0) \neq 0$, but f is nonmonotone on every neighborhood of 0. (See [7, Remark 12.45.4].) Let $f'(0) = b$. By the general definition of the derivative, $f'(0)$ is the linear mapping $x \mapsto b \cdot x$, which is injective. Nonetheless, f is not injective on any neighborhood of 0.

Thus, we need to have stricter assumptions if we wish to prove the local injectivity of f. One can show that if *the linear mapping $f'(x)$ is injective for every x in a neighborhood of a, then f is injective in a neighborhood of a.* The proof involves more advanced topological tools, and hence it is omitted here. We will prove only the special case in which the partial derivatives of f are continuous at a.

Definition 2.31. Let $H \subset \mathbb{R}^p$ and $f\colon H \to \mathbb{R}^q$. We say that the mapping f is *continuously differentiable* at the point $a \in \operatorname{int} H$ if f is differentiable in a neighborhood of a, and the partial derivatives of the coordinate functions of f are continuous at a.

Theorem 2.32. (Local injectivity theorem) *Let $H \subset \mathbb{R}^p$ and $f\colon H \to \mathbb{R}^q$, with $p \le q$. If f is continuously differentiable at the point $a \in \operatorname{int} H$ and the linear mapping $f'(a)\colon \mathbb{R}^p \to \mathbb{R}^q$ is injective, then f is injective in a neighborhood of a.*

Lemma 2.33. *Let $H \subset \mathbb{R}^p$, and let the function $f\colon H \to \mathbb{R}^q$ be differentiable at the points of the segment $[a,b] \subset H$. If $|D_j f_i(x)| \le K$ for every $i = 1,\dots,q$, $j = 1,\dots,p$ and $x \in [a,b]$, then $|f(b) - f(a)| \le Kpq \cdot |b-a|$.*

Proof. Applying the mean value theorem (Theorem 1.79) to the coordinate function f_i yields

$$f_i(b) - f_i(a) = \sum_{j=1}^{p} D_j f_i(c_i)(b_i - a_i)$$

for an appropriate point $c_i \in [a,b]$. Thus,

$$|f_i(b) - f_i(a)| \le \sum_{j=1}^{p} K \cdot |b_i - a_i| \le Kp \cdot |b-a|$$

for every i, and

$$|f(b) - f(a)| \le \sum_{i=1}^{q} |f_i(b) - f_i(a)| \le Kpq \cdot |b - a|. \qquad \square$$

Proof of Theorem 2.32. First, we assume that $p = q$ and $f'(a)$ is the identity map, i.e., $f'(a)(x) = x$ for every $x \in \mathbb{R}^p$. By the definition of the derivative this means that $\lim_{x\to a} |g(x)|/|x - a| = 0$, where $g(x) = f(x) - f(a) - (x - a)$ for every $x \in H$. Obviously, g is continuously differentiable at the point a, and $g'(a)$ is the constant zero mapping. It follows that $D_j g_i(a) = 0$ for every $i, j = 1, \dots, p$. Since g is continuously differentiable at a, we can choose some $\delta > 0$ such that $|D_j g_i(x)| \le 1/(2p^2)$ holds for every $x \in B(a, \delta)$ and every $i, j = 1, \dots, p$. By Lemma 2.33, we have $|g(x) - g(y)| \le |x - y|/2$ for every $x, y \in B(a, \delta)$. If $x, y \in B(a, \delta)$ and $x \ne y$, then $f(x) \ne f(y)$; otherwise, $f(y) = f(x)$ would imply $g(y) - g(x) = x - y$, which is impossible. We have proved that f is injective on the ball $B(a, \delta)$.

Consider the general case. Let A denote the injective linear mapping $f'(a)$. Let the range of A be V; it is a linear subspace of $\mathbb{R}^q$ (including the case $V = \mathbb{R}^q$). Let $B(y) = A^{-1}(y)$ for every $y \in V$. Obviously, B is a well-defined linear mapping from V to $\mathbb{R}^p$. Extend B linearly to $\mathbb{R}^q$, and let us denote this extension by B as well. (The existence of such an extension is easy to show.) Then the mapping $B \circ A$ is the identity map on $\mathbb{R}^p$.

Clearly, the derivative of the linear mapping B is itself B everywhere. Then, it follows from Theorem 2.20 on the differentiation rules of composite functions that $B \circ f$ is differentiable in a neighborhood of a with $(B \circ f)'(x) = B \circ f'(x)$ there. Then the Jacobian matrix of $B \circ f$ at the point x is equal to the (matrix) product of the matrices of B and $f'(x)$. Thus, every partial derivative of every coordinate function of $B \circ f$ is a linear combination of the partial derivatives $D_j f_i$. This implies that $B \circ f$ is continuously differentiable at the point a. Since $(B \circ f)'(a) = B \circ f'(a) = B \circ A$ is the identity, the already proven special case implies the injectivity of $B \circ f$ in a neighborhood of a. Then f itself has to be injective in this neighborhood. $\square$

Remarks 2.34. **1.** Let $A\colon \mathbb{R}^p \to \mathbb{R}^q$ be a linear mapping. It is well known that A cannot be injective if $p > q$. Indeed, in this case the dimension of the null space of A, i.e, the linear subspace $\{x \in \mathbb{R}^p : A(x) = 0\}$, is $p - q > 0$, and thus there exists a point $x \ne 0$ such that $A(x) = 0$. This implies that A can be injective only when $p \le q$.
2. The local injectivity theorem turns the question of a function's local injectivity into a question of the injectivity of a linear mapping. The latter is easy to answer. A linear mapping $A\colon \mathbb{R}^p \to \mathbb{R}^q$ is injective if and only if $A(x) \ne 0$ for every vector $x \in \mathbb{R}^p$, $x \ne 0$. Furthermore, it is well known that A is injective if and only if the rank of its matrix is p. This means that the matrix of A has p linearly independent rows, or equivalently, the matrix has a nonzero $p \times p$ subdeterminant.

A linear mapping $A\colon \mathbb{R}^p \to \mathbb{R}^q$ is called **surjective** , if its range is $\mathbb{R}^q$. Since the range of A can be at most p-dimensional, A can be surjective only if $p \ge q$. The following statement is the dual of Theorem 2.32.

Theorem 2.35. *Let $H \subset \mathbb{R}^p$ and let $f\colon H \to \mathbb{R}^q$, where $p \ge q$. If f is continuously differentiable at the point $a \in \operatorname{int} H$ and the linear mapping $f'(a)\colon \mathbb{R}^p \to \mathbb{R}^q$ is surjective, then the range of f contains a neighborhood of $f(a)$.*

We need to show that if b is close to $f(a)$, then the equation $f(x) = b$ has a solution. We prove this with the help of iterates, which are useful in several cases of solving equations[6].

The most widely used version of this method is given by the following theorem.

We say that the mapping $f\colon H \to H$ has a **fixed point** at $x \in H$ if $f(x) = x$. Let $f\colon H \to \mathbb{R}^q$, where $H \subset \mathbb{R}^p$. The mapping f is called a **contraction**, if there exists a number $\lambda < 1$ such that $|f(y) - f(x)| \le \lambda \cdot |y - x|$ for every $x, y \in H$. (That is, f is contraction if it is Lipschitz with a constant less than 1.)

Theorem 2.36. (Banach's[7] fixed-point theorem) *If $H \subset \mathbb{R}^p$ is a nonempty closed set, then every contraction $f\colon H \to H$ has a fixed point.*

Proof. Let $|f(y) - f(x)| \le \lambda \cdot |y - x|$ for every $x, y \in H$, with $0 < \lambda < 1$. Let $x_0 \in H$ be an arbitrarily chosen point, and consider the sequence of points x_n defined by the recurrence $x_n = f(x_{n-1})$ $(n = 1, 2, \ldots)$. (Since f maps H into itself, x_n is defined for every natural number n.) We prove that the sequence x_n is convergent and tends to a fixed point of f.

Let $|x_1 - x_0| = d$. By induction, we get $|x_{n+1} - x_n| \le \lambda^n d$ for every $n \ge 0$. Indeed, this is clear for $n = 0$, and if it holds for $(n-1)$, then

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \le \lambda \cdot |x_n - x_{n-1}| \le \lambda \cdot \lambda^{n-1} d = \lambda^n d.$$

Now we show that (x_n) satisfies the Cauchy criterion (Theorem 1.8). Indeed, for every $\varepsilon > 0$, the convergence of the infinite series $\sum \lambda^n$ implies the existence of some index N such that $|\lambda^n + \ldots + \lambda^m| < \varepsilon$ holds for every $N \le n < m$. For $N \le n < m$ we have

$$\begin{aligned} |x_m - x_n| \le |x_{n+1} - x_n| + |x_{n+2} - x_{n+1}| + \ldots + |x_m - x_{m-1}| \le \\ \le |\lambda^n + \ldots + \lambda^{m-1}| d < \varepsilon d. \end{aligned}$$

Thus, by Theorem 1.8, (x_n) is convergent. If $x_n \to c$, then $c \in H$ follows from the fact that H is closed. Since $|x_{n+1} - f(c)| = |f(x_n) - f(c)| \le \lambda \cdot |x_n - c|$, we have $x_{n+1} \to f(c)$, which implies $f(c) = c$, i.e., c is a fixed point of f. □

[6] Regarding the solution of equations using iterates, see Exercises 6.4 and 6.5 of [7]. In (a)–(d) of Exercise 6.4 the equations $x = \sqrt{a + x}$, $x = 1/(2 - x)$, $x = 1/(4 - x)$, $x = 1/(1 + x)$ are solved using iterates by defining sequences converging to the respective solutions. The solution of the equation $x^2 = a$ using the same method can be found in Exercise 6.5.

[7] Stefan Banach (1892–1945), Polish mathematician.

Proof of Theorem 2.35. We may assume that $a = 0$ and $f(a) = 0$ (otherwise, we replace f by the function $f(x+a) - f(a)$).

First, assume that $p = q$ and $f'(0)$ is the identity map. Let $g(x) = f(x) - x$ for every $x \in H$. As we saw in the proof of Theorem 2.32, there exists a $\delta > 0$ such that $|g(x) - g(y)| \le |x - y|/2$ for every $x, y \in B(0, \delta)$. We may assume that this inequality also holds for every $x, y \in \overline{B}(0, \delta)$, for otherwise, we could choose a smaller δ. We prove that the range of f contains the ball $B(0, \delta/2)$.

Let $b \in B(0, \delta/2)$ be fixed. The mapping $h(x) = b - g(x)$ maps the closed ball $\overline{B}(0, \delta)$ into itself, since $|x| \le \delta$ implies

$$|h(x)| \le |b| + |g(x)| \le (\delta/2) + |x|/2 \le \delta.$$

Furthermore, since $|h(x) - h(y)| = |g(x) - g(y)| \le |x - y|/2$ for every $x \in \overline{B}(0, \delta)$, it follows that h is a contraction, and then, by Banach's fixed point theorem, it has a fixed-point. If x is such a fixed point, then $x = h(x) = b - g(x) = b + x - f(x)$, i.e., $f(x) = b$.

Now consider the general case $p \le q$. (still assuming $a = 0$ and $f(a) = 0$). Let $e_1, \ldots, e_q$ be a basis of the linear space $\mathbb{R}^q$, and let the points $x_1, \ldots, x_q \in \mathbb{R}^p$ be such that $f'(0)(x_i) = e_i$ $(i = 1, \ldots, q)$. (Such points exist, since the linear mapping $f'(0)$ is surjective.) There exists a linear mapping $A\colon \mathbb{R}^q \to \mathbb{R}^p$ such that $A(e_i) = x_i$ $(i = 1, \ldots, q)$.

Since $0 \in \operatorname{int} H$, we must have $B(0, r) \subset H$ for an appropriate $r > 0$. The mapping A is linear, and thus it is continuous, even Lipschitz by Lemma 2.21. Thus, there exists an $\eta > 0$ such that $|A(x)| < r$ for every $|x| < \eta$. Applying Theorem 2.20 (the differentiation rule for composite functions), we obtain that $f \circ A\colon B(0, \eta) \to \mathbb{R}^q$ is differentiable in the ball $B(0, \eta) \subset \mathbb{R}^q$. We have $(f \circ A)'(0) = f'(0) \circ A$ (since the derivative of the linear mapping A is itself), which is the identity on $\mathbb{R}^q$, by the construction of A. It is easy to see that $f \circ A$ is continuously differentiable at the origin. Thus, by the already proven special case, the range of $f \circ A$ contains a neighborhood of the origin. Then the same is true for f. □

Corollary 2.37. (Open mapping theorem) *Let $H \subset \mathbb{R}^p$ be an open set, and let $f\colon H \to \mathbb{R}^q$ be continuously differentiable at the points of H. If the linear mapping $f'(x)$ is surjective for every $x \in H$, then $f(H)$ is an open set in $\mathbb{R}^q$.*

Proof. If $H \ne \emptyset$, then the assumptions imply $p \ge q$. Let $b \in f(H)$ be arbitrary. Then $b = f(a)$ for a suitable $a \in H$. By Theorem 2.35, $f(H)$ contains a neighborhood of b. Since this is true for every $b \in f(H)$, it follows that $f(H)$ is open. □

The name of Corollary 2.37 comes from the fact that a function $f\colon \mathbb{R}^p \to \mathbb{R}^q$ is called an **open mapping** if $f(G)$ is an open set for every open set $G \subset \mathbb{R}^p$.

Using Theorems 2.32 and 2.35 one obtains a sufficient condition for the existence of a local inverse.

Theorem 2.38. (Inverse function theorem) *Let $H \subset \mathbb{R}^p$ and $a \in \operatorname{int} H$. If $f : H \to \mathbb{R}^p$ is continuously differentiable at a and the linear mapping $f'(a) : \mathbb{R}^p \to \mathbb{R}^p$ is invertible, then there exist positive numbers δ and η such that*

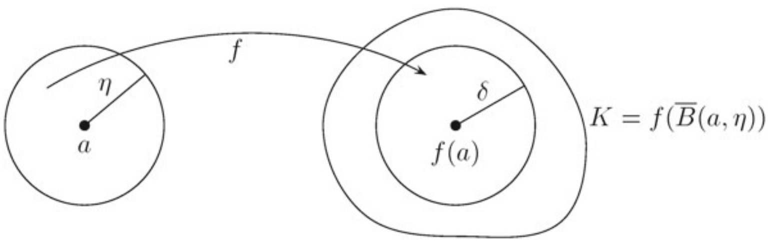
(i) *for every $x \in B(f(a), \delta)$ there exists a unique $\varphi(x) \in B(a, \eta)$ such that $f(\varphi(x)) = x$,*

(ii) *the function φ defined this way is differentiable on the ball $B(f(a), \delta)$ and is continuously differentiable at the point $f(a)$, and furthermore,*

(iii) *$f'(x)$ is invertible at every $x \in B(a, \eta)$, and $\varphi'(f(x)) = f'(x)^{-1}$ for every $x \in B(f(a), \delta)$.*

If f is continuously differentiable in a neighborhood of a, then we can choose δ and η such that φ is continuously differentiable in $B(f(a), \delta)$.

Proof. By Theorem 2.32, f is injective on some ball $B(a, \eta)$. We may assume that f is differentiable and injective on the closed ball $\overline{B}(a, \eta)$, since otherwise, we could choose a smaller η. Let $K = f(\overline{B}(a, \eta))$. For every $x \in K$ let $\varphi(x)$ denote the unique point in $\overline{B}(a, \eta)$ such that $f(\varphi(x)) = x$. It follows from Theorem 2.8 that the function φ is continuous on the set K.



2.2. Figure

Since an invertible linear mapping that maps $\mathbb{R}^p$ into itself is necessarily surjective as well, we find, by Theorem 2.35, that $f(B(a, \eta))$ contains a ball $B(f(a), \delta)$. Obviously, for every point $x \in B(f(a), \delta)$ there exists a unique point in $B(a, \eta)$ whose image by f equals x, namely, the point $\varphi(x)$. This proves (i).

A linear mapping that maps $\mathbb{R}^p$ to itself is injective if and only if the determinant of the mapping's matrix is nonzero. By assumption, the determinant of f's Jacobian matrix at the point a is nonzero. Since the Jacobian matrix is a polynomial in the partial derivatives $D_j f_i$, it follows that the determinant of the Jacobian matrix is continuous at a, implying that it is nonzero in a neighborhood of a. We have proved that the linear mapping $f'(x)$ is injective for every point x close enough to a. By taking a smaller η if necessary, we may assume that $f'(x)$ is injective for every $x \in B(a, \eta)$.

Then it follows from Theorem 2.26 (the differentiation rule for inverse functions) that φ is differentiable in $B(f(a), \delta)$ and (iii) holds on this ball.

We now prove that φ is continuously differentiable at the point $f(a)$. This follows from the equality $\varphi'(f(x)) = f'(x)^{-1}$. Indeed, this implies that the partial derivative $D_j\varphi_i(x)$ equals the jth entry of the ith row in the matrix of the inverse of $f'(\varphi(x))$. Now it is well known that the jth element of the ith row of the inverse of a matrix is equal to A_{ij}/D, where A_{ij} is an appropriate subdeterminant and D is the determinant of the matrix itself (which is nonzero). The point is that the entries of the inverse matrix can be written as rational functions of the entries of the original matrix. Since $D_j f_i(\varphi(x))$, i.e., the entries of the matrix of the mapping $f'(\varphi(x))$ are continuous at $f(a)$, their rational functions are also continuous at $f(a)$.

Thus, if f is continuously differentiable on the ball $B(a,\eta)$, then φ is continuously differentiable on $B(f(a),\delta)$. $\square$

Now we turn to what is called the implicit function theorem, that is, to the generalization of Corollary 2.28 to multivariable functions. Intuitively, the statement of the theorem is the following. Let the equations

$$\begin{aligned} f_1(x_1,\ldots,x_p,y_1,\ldots,y_q) &= 0,\\ f_2(x_1,\ldots,x_p,y_1,\ldots,y_q) &= 0,\\ &\vdots\\ f_q(x_1,\ldots,x_p,y_1,\ldots,y_q) &= 0 \end{aligned} \tag{2.11}$$

be given, together with a solution $(a_1,\ldots,a_p,b_1,\ldots,b_q)$. Our goal is to express the unknowns $y_1,\ldots,y_q$ as functions of the variables $x_1,\ldots,x_p$ in a neighborhood of the point $a = (a_1,\ldots,a_p)$. In other words, we want to prove that there are functions $y_j = y_j(x_1,\ldots,x_p)$ $(j = 1,\ldots,q)$ with the following properties: they satisfy (2.11) in a neighborhood of a, and $y_j(a_1,\ldots,a_p) = b_j$ for every $j = 1,\ldots,q$.

Let us use the following notation. If $x = (x_1,\ldots,x_p) \in \mathbb{R}^p$ and $y = (y_1,\ldots,y_q) \in \mathbb{R}^q$, then (x,y) denotes the vector $(x_1,\ldots,x_p,y_1,\ldots,y_q) \in \mathbb{R}^{p+q}$.

If the function f is defined on a subset of $\mathbb{R}^{p+q}$ and $a = (a_1,\ldots,a_p) \in \mathbb{R}^p$, then f_a denotes the **section function,** obtained by putting $a_1,\ldots,a_p$ in place of $x_1,\ldots,x_p$. That is, f_a is defined at the points $y = (y_1,\ldots,y_q) \in \mathbb{R}^q$ that satisfy $(a,y) \in D(f)$, and $f_a(y) = f(a,y)$ for every such point y. The section f^b can be defined for $b = (b_1,\ldots,b_q) \in \mathbb{R}^q$ in a similar manner. The following lemma is the generalization of the fact that differentiability implies partial differentiability.

Lemma 2.39. *Let $H \subset \mathbb{R}^{p+q}$, and let the function $f\colon H \to \mathbb{R}^s$ be differentiable at the point $(a,b) \in \operatorname{int} H$, where $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$. Then the section function f_a is differentiable at the point b, and the section function f^b is differentiable at the point a. If $(f^b)'(a) = A$, $(f_a)'(b) = B$, and $f'(a,b) = C$, then $A(x) = C(x,0)$ and $B(y) = C(0,y)$ for every $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$.*

Proof. Let $r(x,y) = f(x,y) - f(a,b) - C(x-a,y-b)$. Since $f'(a,b) = C$, we have $r(x,y)/|(x,y)-(a,b)| \to 0$ if $(x,y) \to (a,b)$. Since $f(a,y) - f(a,b) - C(0,y-b) = r(a,y)$, it follows that $(f_a)'(b)$ equals the linear mapping $y \mapsto$

$C(0,y)$ ($y \in \mathbb{R}^q$). A similar argument shows that $(f^b)'(a)(x) = C(x,0)$ for every $x \in \mathbb{R}^p$. $\square$

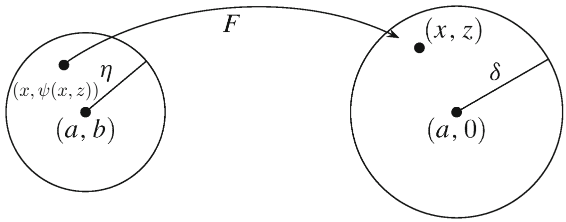
Theorem 2.40. (Implicit function theorem) *Let $H \subset \mathbb{R}^{p+q}$ and $(a,b) \in \operatorname{int} H$, where $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$. Suppose that the function $f\colon H \to \mathbb{R}^q$ vanishes at the point (a,b) (i.e., $f(a,b)$ is the null vector of $\mathbb{R}^q$). If f is continuously differentiable at (a,b) and the linear mapping $(f_a)'(b)$ is injective, then there are positive numbers δ and η such that*

(i) *for every $x \in B(a,\delta)$ there exists a unique point $\varphi(x) \in B(b,\eta)$ such that $f(x,\varphi(x)) = 0$,*

(ii) *the function φ defined this way is differentiable in the ball $B(a,\delta)$ and continuously differentiable at the point a.*

Proof. Let $F(x,y) = (x, f(x,y))$ for every $(x,y) \in H$, where $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. Then F maps the set H into $\mathbb{R}^{p+q}$. We will prove that F is continuously differentiable at the point (a,b), and the linear mapping $F'(a,b)$ is invertible.

Let us proceed with the proof of the theorem, assuming the statements above. Note that $F(a,b) = (a,0)$. Applying the inverse function theorem to F, we obtain positive numbers δ and η such that $F'(x,y)$ is injective for every $(x,y) \in B((a,b),\eta)$, for every point $(x,z) \in B((a,0),\delta)$ there exists a unique point $(x,\psi(x,z)) \in B((a,b),\eta)$ such that $F(x,\psi(x,z)) = (x,z)$, and furthermore, the function ψ defined this way is differentiable on the ball $B((a,0),\delta)$ and is continuously differentiable at the point $(a,0)$. From the definition of the mapping F it follows that $f(x,\psi(x,z)) = z$ for every point $(x,z) \in B((a,0),\delta)$.



2.3. Figure

Let $\varphi(x) = \psi(x,0)$ for every point $x \in \mathbb{R}^p$ with $|x-a| < \delta$. The definition makes sense, since $|x-a| < \delta$ implies $(x,0) \in B((a,0),\delta)$. It is clear that φ is differentiable on the ball $B(a,\delta)$ of $\mathbb{R}^p$ and continuously differentiable at the point a, and $f(x,\varphi(x)) = 0$ holds for every $x \in B(a,\delta)$.

We now prove the claims on F. First we prove that if f is differentiable at a point (x_0,y_0) and its derivative there is $f'(x_0,y_0) = C$, then F is also differentiable at the given point and $F'(x_0,y_0) = E$, with $E(x,y) = (x, C(x,y))$ for every $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. Indeed, by the definition of the derivative, $\lim_{(x,y)\to(x_0,y_0)} r(x,y)/|(x,y)-(x_0,y_0)| = 0$, where

$$r(x,y) = f(x,y) - f(x_0,y_0) - C(x-x_0, y-y_0)$$

for every $(x, y) \in H$. Thus,

$$\begin{aligned} F(x,y) - F(x_0,y_0) &= (x, f(x,y)) - (x_0, f(x_0,y_0)) = \\ &= (x - x_0, f(x,y) - f(x_0,y_0)) = \\ &= (x - x_0, C(x - x_0, y - y_0) + r(x,y)) = \\ &= E(x - x_0, y - y_0) + t(x,y) \end{aligned}$$

follows, where $t(x,y) = (0, r(x,y))$. Obviously,

$$\lim_{(x,y)\to(x_0,y_0)} t(x,y)/|(x,y) - (x_0,y_0)| = 0,$$

where E is indeed the derivative of the mapping F at the point (x_0, y_0). This proves that F is differentiable in a neighborhood of the point (a, b).

We now prove that if $(f_{x_0})'(y_0)$ is injective, then $F'(x_0, y_0) = E$ is also injective. Since the mapping E is linear, we need to prove that if the vector $(x, y) \in \mathbb{R}^{p+q}$ is nonzero, then $E(x, y) \neq 0$. By Lemma 2.39, $(f_{x_0})'(y_0)$ is equal to the linear mapping $(x, y) \mapsto C(0, y)$ ($x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$). By assumption, this mapping is injective on $\mathbb{R}^q$, thus $C(0, y) \neq 0$ holds if $y \neq 0$. We know that $E(x, y) = (x, C(x, y))$ for every $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$. If $x \neq 0$, then $E(x, y) \neq 0$ is clear. On the other hand, if $x = 0$ and $y \neq 0$, then $E(0, y) = (0, C(0, y)) \neq 0$, since $C(0, y) \neq 0$. We have proved that $(x, y) \neq 0$ implies $E(x, y) \neq 0$, i.e., E is injective. Since we assumed the injectivity of $(f_a)'(b)$, t follows that $F'(a, b)$ is also injective.

Let the coordinate functions of F and f be F_i and f_i, respectively. Obviously, $F_i(x, y) = x_i$ for every $i = 1, \ldots, p$, and $F_i(x, y) = f_{i-p}(x, y)$ for every $i = p + 1, \ldots, q$. Since the partial derivatives $D_j f_i$ are continuous at the point (a, b), it follows that the partial derivatives $D_j F_i(x, y)$ are also continuous at the point (a, b) for every $i, j = 1, \ldots, p + q$. Therefore, F is continuously differentiable at (a, b). □

Remarks 2.41. **1.** It is easy to compute the derivative of the function φ of Theorem 2.40. Let $c \in \mathbb{R}^p$, $|c - a| < \delta$, and let $\varphi(c) = d$. It is easy to see that the derivative of the mapping $x \mapsto (x, \varphi(x))$ at the point c is the linear mapping $x \mapsto (x, A(x))$ with $\varphi'(c) = A$.

Let $f'(c, d) = C$. It follows from the derivation rule for composite functions that the derivative of the function $f(x, \varphi(x))$ at the point c is the linear function $C(x, A(x))$. Since $f(x, \varphi(x)) = 0$ for every $|x - a| < \delta$, this derivative is zero, i.e.,

$$0 = C(x, A(x)) = C(x, 0) + C(0, A(x)).$$

By Lemma 2.39, $C(x, 0) = (f^d)'(c)(x)$ and $C(0, y) = (f_c)'(d)(y)$, i.e., the linear mapping $(f^d)'(c) + (f_c)'(d) \circ A$ is identically zero. This implies

$$\varphi'(c) = A = -\left((f_c)'(d)\right)^{-1} \circ (f^d)'(c).$$

We get

$$\varphi'(x) = -\left(f'_x(\varphi(x))\right)^{-1} \circ (f^{\varphi(x)})'(x)$$

for every $x \in B(a, \delta)$.

2. If f satisfies the conditions of Theorem 2.40 and f is continuously differentiable in a neighborhood of the point (a, b), then *we can choose* δ *and* η *such that* φ *is continuously differentiable on the ball* $B(a, \delta)$.

It suffices to choose δ and η such that in addition to parts (i) and (ii) of the theorem, we also require that f be continuously differentiable on the ball $B((a, b), \eta)$. In this case φ will be continuously differentiable at every point (c, d) of the ball $B(a, \delta)$. This follows from Theorem 2.40 applied to the point (c, d) instead of the point (a, b).

As an important application of the implicit function theorem we give a method for finding the conditional extremal points of a function.

Definition 2.42. Let $a \in H \subset \mathbb{R}^p$, $F\colon H \to \mathbb{R}^q$, and let $F(a) = 0$. Let the p-variable real function f be defined in a neighborhood of a, and let $\delta > 0$ be such that $f(x) \le f(a)$ for every point $x \in B(a, \delta)$ that satisfies $F(x) = 0$. Then we say that *the function* f *has a conditional local maximum point at the point* a *with the condition* $F = 0$. Conditional local minima can be defined in a similar manner. If f has a conditional local maximum or minimum at the point a with the condition $F = 0$, then we say that f *has a conditional local extremum at the point* a *with the condition* $F = 0$.

Example 2.43. Suppose we want to find the maximum of the function $f(x, y, z) = x + 2y + 3z$ on the sphere $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$. By Weierstras's theorem, f has a maximal value on the bounded and closed set S. If f takes on this greatest value at the point a, then f has a conditional local maximum at a with the condition $x^2 + y^2 + z^2 - 1 = 0$.

Theorem 2.44. (Lagrange[8] multiplier method) *Let* $H \subset \mathbb{R}^p$, *and suppose that* $F\colon H \to \mathbb{R}^q$ *vanishes and is continuously differentiable at the point* $a \in \operatorname{int} H$. *Let us denote the coordinate functions of* F *by* $F_1, \ldots, F_q$.

If the p*-variable real function* f *is differentiable at* a *and* f *has a conditional local extremum at the point* a *with the condition* $F = 0$, *then there are real numbers* $\lambda, \lambda_1, \ldots, \lambda_q$ *such that at least one of these numbers is nonzero, and the partial derivatives of the function* $\lambda f + \lambda_1 F_1 + \ldots + \lambda_q F_q$ *are zero at* a.

The $p = 2$, $q = 1$ special case of the theorem above states that the gradients of f and F are parallel to each other at the conditional local extremum points. Intuitively, this can be proved as follows. Condition $F(x, y) = 0$ defines a curve in the plane. If we move along this curve, then we move perpendicularly to the gradient of F at each point of the curve (see Exercise 2.10). As we reach a conditional local

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[8] Joseph-Louis Lagrange (1736–1813), Italian-French mathematician.

extremum point of f, we go neither upward nor downward on the graph of f, and thus the gradient of f is also perpendicular to the curve. That is, the two gradients are parallel to each other.

Proof of Theorem 2.44. Consider the matrix

$$\begin{pmatrix} D_1F_1(a) & D_2F_1(a) & \dots & D_pF_1(a) \\ D_1F_2(a) & D_2F_2(a) & \dots & D_pF_2(a) \\ \vdots & \vdots & \dots & \vdots \\ D_1F_q(a) & D_2F_q(a) & \dots & D_pF_q(a) \\ D_1f(a) & D_2f(a) & \dots & D_pf(a) \end{pmatrix}. \tag{2.12}$$

We need to prove that the rows of this matrix are linearly dependent. Indeed, in this case there are real numbers $\lambda_1, \dots, \lambda_q, \lambda$ such that at least one of these numbers is nonzero, and the linear combination of the row vectors with coefficients $\lambda_1, \dots, \lambda_q, \lambda$ is zero. Then every partial derivative of $\lambda_1 F_1 + \dots + \lambda_q F_q + \lambda f$ is zero at a, and this is what we want to prove.

If $p \le q$, then the statement holds trivially. Indeed, the matrix has p columns, and its rank is at most p. Thus, $q + 1 > p$ row vectors must be linearly dependent.

Therefore, we may assume that $p > q$. We may also assume that the first q row vectors of the matrix (the gradient vectors $F_1'(a), \dots, F_q'(a)$) are linearly independent, since otherwise, there would be nothing to prove.

The vectors $F_1'(a), \dots, F_q'(a)$ are the row vectors of the Jacobian matrix of F at the point a. Since these are linearly independent, the rank of the Jacobian matrix is q, and the matrix has q linearly independent column vectors. Permuting the coordinates of $\mathbb{R}^q$ if necessary, we may assume that the last q columns of the Jacobian matrix are linearly independent.

Let $s = p - q$. Put $b = (a_1, \dots, a_s) \in \mathbb{R}^s$ and $c = (a_{s+1}, \dots, a_p)$; then $a = (b, c)$. The Jacobian matrix of the section $F_b \colon \mathbb{R}^q \to \mathbb{R}^q$ at the point c consists of the last q column vectors of the matrix of $F'(a)$. Since these are linearly independent, the linear mapping $(F_b')(c)$ is injective. Therefore, we may apply the implicit function theorem. We obtain $\delta > 0$ and a differentiable function $\varphi \colon B(b, \delta) \to \mathbb{R}^q$ such that $\varphi(b) = c$ and $F(x, \varphi(x)) = 0$ for every $x \in B(b, \delta)$. (Here, $B(b, \delta)$ denotes the ball with center b and radius δ in $\mathbb{R}^s$.)

We know that f has a conditional local extremum point at $a = (b, c)$ with the condition $F = 0$. Let us assume that this is a local maximum. This means that if $x \in \mathbb{R}^s$, $y \in \mathbb{R}^q$ and the point (x, y) is close enough to a, and $F(x, y) = 0$, then $f(x, y) \le f(a)$. Consequently, if x is close enough to b, then $f(x, \varphi(x)) \le f(b, \varphi(b))$. In other words, the function $f(x, \varphi(x))$ has a local maximum at the point b. By Theorem 1.60, the partial derivatives of $f(x, \varphi(x))$ are zero at the point b. If $\varphi_1, \dots, \varphi_q$ are the coordinate functions of φ, then applying Corollary 2.23, we find that for every $i = 1, \dots, s$ we have

$$D_i f(a) + \sum_{j=1}^{q} D_{s+j} f(a) \cdot D_i \varphi_j(b) = 0. \tag{2.13}$$

For every $k = 1, \ldots, q$ the function $F_k(x, \varphi(x))$ is constant and equal to zero in a neighborhood of the point b, thus its partial derivatives are zero at b. We get

$$D_i F_k(a) + \sum_{j=1}^{q} D_{s+j} F_k(a) \cdot D_i \varphi_j(b) = 0 \tag{2.14}$$

for every $k = 1, \ldots, q$ and $i = 1, \ldots, s$. Equations (2.13) and (2.14) imply that the first s column vectors of the matrix of (2.12) are linear combinations of the last q column vectors. In other words, the rank of the matrix is at most q. Since the matrix has $q + 1$ rows, they are linearly dependent. □

Remark 2.45. If we want to find the conditional local extremum points a of the function f with condition $F = 0$, then according to Theorem 2.44, we need to find λ, $\lambda_1, \ldots, \lambda_q$ such that $\lambda D_i f(a) + \lambda_1 D_i F_1(a) + \ldots + \lambda_q D_i F_q(a) = 0$ for every $i = 1, \ldots, p$. These equations, together with the conditions $F_k(a) = 0$ ($k = 1, \ldots, q$), form a set of $p + q$ equations in $p + q + 1$ unknowns $a_1, \ldots, a_p, \lambda$, $\lambda_1, \ldots, \lambda_q$. We can also add the equation

$$\lambda^2 + \lambda_1^2 + \ldots + \lambda_q^2 = 1$$

to our system of equations, since instead of λ, $\lambda_1, \ldots, \lambda_q$, we could also take $\nu \cdot \lambda$, $\nu \cdot \lambda_1, \ldots, \nu \cdot \lambda_q$, where $\nu = 1/(\lambda^2 + \lambda_1^2 + \ldots + \lambda_q^2)$. We now have exactly as many equations as unknowns. Should we be lucky enough, these equations are "independent" and they have only a finite number of solutions. Checking these solutions one by one, we can find, in principle, the set of actual conditional local extremum points.

Example 2.46. In Example 2.43 we have seen that the function $f(x, y, z) = x + 2y + 3z$ has a greatest value on the sphere $S = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$. If f takes on this greatest value at the point $a = (u, v, w)$, then f has a conditional local maximum at a with the condition $x^2 + y^2 + z^2 - 1 = 0$. Each of the functions mentioned above is continuously differentiable, and thus we can apply Theorem 2.44. We get that there are real numbers λ, μ such that they are not both zero and the partial derivatives of the function $\lambda(x + 2y + 3z) + \mu(x^2 + y^2 + z^2 - 1)$ are zero at the point (u, v, w). Thus, the equations

$$\lambda + 2\mu u = 0, \qquad 2\lambda + 2\mu v = 0, \qquad 3\lambda + 2\mu w = 0, \tag{2.15}$$

and $u^2 + v^2 + w^2 = 1$ hold. The Equations (2.15) imply $\mu \neq 0$, since $\mu = 0$ would imply $\lambda = 0$. Thus, applying (2.15) again gives us $v = 2u$ and $w = 3u$, implying $u^2 + (2u)^2 + (3u)^2 = 1$, $u = \pm 1/\sqrt{14}$, i.e., $(u, v, w) = (1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$ or $(u, v, w) = (-1/\sqrt{14}, -2/\sqrt{14}, -3/\sqrt{14})$.

The function $f(x, y, z) = x + 2y + 3z$ also has a least value on the sphere S. Since f is not constant on S, the points where f takes its maximum and its minimum must be different. This means that there are at least two conditional local extremal points. Our calculations above imply that there are exactly two such extremal points, and it is also clear that f assumes its greatest value at the point $(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$ while it takes its least value at the point $(-1/\sqrt{14}, -2/\sqrt{14}, -3/\sqrt{14})$ on S.

Exercises

2.14. Show that in Corollary 2.28 the condition on the finiteness of the partial derivative $D_2 f$ can be omitted. (H)

2.15. Let the function f of Corollary 2.28 be differentiable at the point (a, b). Show directly (i.e., without applying Theorem 2.40) that the function φ is differentiable at the point a and $\varphi'(a) = -D_1 f(a, b)/D_2 f(a, b)$.

2.16. Let $f : I \to \mathbb{R}$ be continuous, where $I \subset \mathbb{R}$ is an interval. Show that if every point of I has a neighborhood where f is injective, then f is injective on the whole interval.

2.17. Let $f(x, y) = (e^x \cos y, e^x \sin y)$ for every $(x, y) \in \mathbb{R}^2$.

(a) Show that $f'(a, b)$ is injective at every $(a, b) \in \mathbb{R}^2$.
(b) Show that f is injective in every open disk with radius π.
(c) Let $G = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Define a continuous map $\varphi : G \to \mathbb{R}^2$ such that $\varphi(1, 0) = (0, 0)$ and $f \circ \varphi$ is the identity on G. (S)

2.18. Show that a contraction can have at most one fixed point.

2.19. Let $B \subset \mathbb{R}^p$ be an open ball. Show that there exists a contraction $f : B \to B$ with no fixed points.

2.20. We call the mapping $f : \mathbb{R}^p \to \mathbb{R}^p$ a **similarity with ratio** λ if $|f(x) - f(y)| = \lambda \cdot |x - y|$ holds for every $x, y \in \mathbb{R}^p$. Show that if $0 < \lambda < 1$, then every similarity with ratio λ has exactly one fixed point.

2.21. Find the largest value of $x - y + 3z$ on the ellipsoid $x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1$.

2.22. Find the largest value of xy with the condition $x^2 + y^2 = 1$.

2.23. Find the largest value of xyz with the condition $x^2 + y^2 + z^2 = 3$.

2.24. Find the largest value of xyz with the condition $x + y + z = 5$, $xy + yz + xz = 8$.

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