Chapter 2
Some Recent Results on Distributed Control of Nonlinear Systems

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Abstract The spatially distributed structure of complex systems motivates the idea of distributed control. In a distributed control system, the subsystems are controlled by local controllers through information exchange with neighboring agents for coordination purposes. One of the major difficulties of distributed control is due to the complex characteristics such as nonlinearity, dimensionality, uncertainty, and information constraints. This chapter introduces small-gain methods for distributed control of nonlinear systems. In particular, a cyclic-small-gain result in digraphs is presented as an extension of the standard nonlinear small-gain theorem. It is shown that the new result is extremely useful for distributed control of nonlinear systems. Specifically, this chapter first gives a cyclic-small-gain design for distributed output-feedback control of nonlinear systems. Then, an application to formation control problem of nonholonomic mobile robots with a fixed information exchange topology is presented.

2.1 Introduction

Distributed control of multiagent systems under communication constraints has attracted tremendous attention from the control community over the past 10 years; see, for example, [45] using an adaptive gradient climbing strategy, [3, 5, 12, 21, 48] based on linear algebra and graph theory, [2, 53] using passivity and dissipativity theory, [4, 16, 19, 34, 42, 43, 50, 51] with Lyapunov methods, [17] using the nonlinear small-gain theorem, and [54, 57, 58] based on output regulation. The recent

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hot topics such as formation control, consensus, flocking, swarm, rendezvous and synchronization are all closely related to distributed coordinated control.

The distributed control problem for agents with second-order dynamics has been mainly studied from the perspective of second-order consensus and flocking. Considerable efforts have been devoted to solving the problems under switching information exchange topologies. Related results include [4, 12, 46, 47, 55]. Specifically, [12, 46] used potential functions to define Lyapunov functions and the topologies are allowed to be switching but undirected. Reference [47] presented a consensus result for double integrator systems based on a refined graph theoretical method. Reference [4] proposed a variable structure approach-based consensus design method for systems with switching but always connected information exchange topology. Several recent results on distributed control can also be found in [15, 32, 41, 49, 59]. It should be pointed out that most of the papers mentioned above do not consider systems under physical constraints (e.g., saturation of velocities), for which specific distributed nonlinear designs are expected.

As a practical application of distributed control, the formation control of autonomous mobile agents aims at forcing the agents to converge toward, and to maintain, specific relative positions, by using available information, e.g., relative position measurements. Recent formation control results can be found in [1, 7, 9, 10, 20, 30, 31, 44, 56], to name a few. The earlier results, e.g., [7, 56], assume a tree sensing structure to avoid the technical difficulties caused by the loop interconnections. In [9, 10, 31, 44], the assumption of tree sensing structures is relaxed at the price of using global position measurements. An exception is the wiggling controller developed by [33] to drive the robots to stationary points, which does not use global position measurements of the robots. In the results of coordinated path-following as presented in [1, 20, 30], the global position measurement issue can be easily addressed as each robot has access to its desired path. In our recent paper [38], thanks to the use of nonlinear small-gain techniques [25, 35], the requirement on global position measurements has been removed for formation control of unicycles with fixed sensing topologies.

The discussion in this chapter starts with an example of a multivehicle formation control system in which each vehicle is modeled by an integrator. In the case of leader-following with fixed topology, it is shown that the problem can be transformed into the stability problem of a specific dynamic network composed of ISS subsystems. This motivates a cyclic-small-gain result in digraphs, which is given in Sect. 2.2. It is shown that the new result is extremely useful for distributed control of nonlinear systems. Specifically, Sect. 2.3 presents a cyclic-small-gain design for distributed output-feedback control of nonlinear systems. In Sect. 2.4, we study the distributed formation control problem of nonholonomic mobile robots with a fixed information exchange topology.

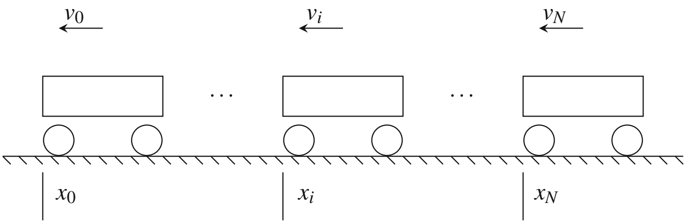

Fig. 2.1 A multivehicle system

2.2 A Cyclic-Small-Gain Result in Digraphs

Example 2.1 Consider a group of $N + 1$ vehicles (multivehicle system) as shown in Fig. 2.1, with each vehicle modeled by an integrator:

$$\dot{x}_i = v_i, \quad i = 0, \ldots, N, \tag{2.1}$$

where $x_i \in \mathbb{R}$ is the position and $v_i \in \mathbb{R}$ is the velocity of the ith vehicle. The vehicle with index 0 is the leader while the other vehicles are the followers. The objective is to control the follower vehicles to specific positions relative to the leader by adjusting the velocities v_i for $i = 1, \ldots, N$. More specifically, it is required that

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = d_{ij}, \quad i, j = 0, \ldots, N, \tag{2.2}$$

where constants d_{ij} represent the desired relative positions. Clearly, to define the problem well, $d_{ij} = d_{ik} + d_{kj}$ for any $i, j, k = 0, \ldots, N$ and $d_{ij} = -d_{ji}$ for any $i, j = 0, \ldots, N$. Also, by default, $d_{ii} = 0$ for any $i = 0, \ldots, N$. In the literature of distributed control, the vehicles are usually considered as agents and the multivehicle system is studied as a multiagent system.

Compared with global positions, the relative positions between vehicles are often easily measurable in practice, and are used for feedback in this example. Considering the position information exchange, agent j is called a neighbor of agent i if $(x_i - x_j)$ is available to agent i, and $\mathcal{N}_i \subseteq \{0, \ldots, N\}$ is used to denote the set of agent i's neighbors. We consider the case where each vehicle only uses the position differences with the vehicles right before and after it, i.e., $\mathcal{N}_i = \{i - 1, i + 1\}$ for $i = 1, \ldots, N - 1$ and $\mathcal{N}_N = \{N - 1\}$.

Define $\tilde{x}_i = x_i - x_0 - d_{i0}$ and $\tilde{v}_i = v_i - v_0$. By taking the derivative of $\tilde{x}_i$, we have

$$\dot{\tilde{x}}_i = \tilde{v}_i, \quad i = 1, \ldots, N. \tag{2.3}$$

According to the definition of $\tilde{x}_i$, $\tilde{x}_i - \tilde{x}_j = x_i - x_j - d_{ij}$. Thus, the control objective is achieved if $\lim_{t\to\infty}(\tilde{x}_i - \tilde{x}_0) = 0$. Also, $(\tilde{x}_i - \tilde{x}_j)$ is available to the control of the $\tilde{x}_i$-subsystem if $(x_i - x_j)$ is available to agent i. This problem is normally known as the consensus problem. If the position information exchange topology has a spanning tree with agent 0 as the root, then the following distributed control law is effective:

$$\tilde{v}_i = k_i \sum_{j \in \mathcal{N}_i} (\tilde{x}_j - \tilde{x}_i), \tag{2.4}$$

where k_i is a positive constant. Moreover, if the velocities v_i are required to be bounded, one may modify (2.4) as

$$\tilde{v}_i = \varphi_i \left(\sum_{j \in \mathcal{N}_i} (\tilde{x}_j - \tilde{x}_i) \right), \tag{2.5}$$

where $\varphi_i : \mathbb{R} \to [\underline{v}_i, \overline{v}_i]$ with constants $\underline{v}_i < 0 < \overline{v}_i$ is a continuous, strictly increasing function satisfying $\varphi_i(0) = 0$. With control law (2.5), $v_i \in [v_0 + \underline{v}_i, v_0 + \overline{v}_i]$. The validity of the control laws defined by (2.4) and (2.5) can be directly verified using the state agreement result in [34].

With control law (2.5), each $\tilde{x}_i$-subsystem can be rewritten as

$$\dot{\tilde{x}}_i = \varphi_i \left(\sum_{j \in \mathcal{N}_i} \tilde{x}_j - N_i \tilde{x}_i \right) := f_i(\tilde{x}), \tag{2.6}$$

where N_i is the size of $\mathcal{N}_i$ and $\tilde{x} = [\tilde{x}_0, \ldots, \tilde{x}_N]^T$. Define $V_i(\tilde{x}_i) = |\tilde{x}_i|$ as an ISS-Lyapunov function candidate for the $\tilde{x}_i$-subsystem for $i = 1, \ldots, N$. It can be verified that for any $\delta > 0$, there exists a continuous, positive definite α such that

$$V_i(\tilde{x}_i) \geq \frac{1}{(1 - \delta_i) N_i} \sum_{j \in \mathcal{N}_i} V_j(\tilde{x}_j) \Rightarrow \nabla V_i(\tilde{x}_i) f_i(\tilde{x}) \leq -\alpha_i(V_i(\tilde{x}_i)) \quad \text{a.e.}, \tag{2.7}$$

where, for convenience of notation, $V_0(\tilde{x}_0) = 0$. This shows the ISS of each $\tilde{x}_i$-subsystem with $i = 1, \ldots, N$. If the network of ISS subsystems is asymptotically stable, then the control objective is achieved.

We employ a digraph $\mathcal{G}_f$ to represent the underlying interconnection structure of the dynamic network. The vertices of the digraph correspond to agents $1, \ldots, N$, and for $i, j = 1, \ldots, N$, directed edge (j, i) exists in the graph if and only if $\tilde{x}_j$ is an input of the x_i-subsystem. We use $\overline{\mathcal{N}}_i$ to represent the set of neighbors of agent i in $\mathcal{G}_f$. Then, it is directly verified that $\overline{\mathcal{N}}_i = \mathcal{N}_i \backslash \{0\}$. Recall that $V_0(\tilde{x}_0) = 0$. Then, the $\mathcal{N}_i$ in (2.7) can be directly replaced by $\overline{\mathcal{N}}_i$. Figure 2.2 shows the digraph $\mathcal{G}_f$ for the case in which each follower vehicle uses the position differences with the vehicles right before and after it.

Notice that for any positive constants $a_1, \ldots, a_n$ satisfying $\sum_{i=1}^n 1/a_i \leq n$, it holds that $\sum_{i=1}^n d_i = \sum_{i=1}^n (1/a_i) a_i d_i \leq n \max_{i=1,\ldots,n}\{a_i d_i\}$, for all $d_1, \ldots, d_n \geq 0$. Then, property (2.7) implies

$$1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows \quad \cdots \quad \rightleftarrows N$$

Fig. 2.2 An example of information exchange digraph $\mathscr{G}_f$, for which each vehicle uses the position differences with the vehicles right before and after it. In this figure, $\overline{\mathscr{N}}_i = \{i-1, i+1\}$ for $i = 2, \ldots, N-1$, $\overline{\mathscr{N}}_1 = \{2\}$ and $\overline{\mathscr{N}}_N = \{N-1\}$

$$V_i(\tilde{x}_i) \geq \frac{\overline{N}_i}{(1-\delta_i)N_i} \max_{j \in \overline{\mathscr{N}}_i} \{a_{ij} V_j(\tilde{x}_j)\} \Rightarrow \nabla V_i(\tilde{x}_i) f_i(\tilde{x}) \leq -\alpha_i(V_i(\tilde{x}_i)), \tag{2.8}$$

where $\overline{N}_i$ is the size of $\overline{\mathscr{N}}_i$ and a_{ij} are positive constants satisfying $\sum_{j \in \overline{\mathscr{N}}_i} 1/a_{ij} \leq \overline{N}_i$. It can be observed that $N_i = \overline{N}_i + 1$ if $0 \in \mathscr{N}_i$ and $N_i = \overline{N}_i$ if $0 \notin \mathscr{N}_i$.

Given specific $a_{ij} > 0$, one can test the stability property of the closed-loop system by directly checking whether the cyclic-small-gain condition is satisfied. But, for a specific $\mathscr{G}_f$, can we find appropriate coefficients a_{ij} to satisfy the cyclic-small-gain condition, and how?

It should be noted that the effectiveness of control law (2.5) can be proved using the result in [34]. Here, our objective is to transform the problem into a stability problem of dynamic networks, and develop a result which is hopefully useful for more general distributed control problems.

The main result in this section answers the question in Example 2.1.

Consider a digraph $\mathscr{G}_f$ which has N vertices. For $i = 1, \ldots, N$, define $\overline{\mathscr{N}}_i$ such that if there is a directed edge (j, i) from the jth vertex to the i-th vertex, then $j \in \overline{\mathscr{N}}_i$. Each edge (j, i) is assigned a positive variable a_{ij}. For a simple cycle $\mathscr{O}$ of $\mathscr{G}_f$, denote $A_{\mathscr{O}}$ as the product of the positive values assigned to the edges of the cycle. For $i = 1, \ldots, N$, denote $\mathscr{C}(i)$ as the set of simple cycles of $\mathscr{G}_f$ through the i-th vertex.

Lemma 2.1 *If the digraph $\mathscr{G}_f$ has a spanning tree $\mathscr{T}_f$ with vertices $i_1^*, \ldots, i_q^*$ as the roots, then for any $\varepsilon > 0$, there exist $a_{ij} > 0$ for $i = 1, \ldots, N$, $j \in \overline{\mathscr{N}}_i$, such that*

$$\sum_{j \in \overline{\mathscr{N}}_i} \frac{1}{a_{ij}} \leq \overline{N}_i, \quad i = 1, \ldots, N \tag{2.9}$$

$$A_{\mathscr{O}} < 1 + \varepsilon, \quad \mathscr{O} \in \mathscr{C}(i_1^*) \cup \cdots \cup \mathscr{C}(i_q^*) \tag{2.10}$$

$$A_{\mathscr{O}} < 1, \quad \mathscr{O} \in \left(\bigcup_{i=1,\ldots,N} \mathscr{C}(i) \right) \Big\backslash \left(\mathscr{C}(i_1^*) \cup \cdots \cup \mathscr{C}(i_q^*) \right), \tag{2.11}$$

where $\overline{N}_i$ is the size of $\overline{\mathscr{N}}_i$.

Proof We only consider the case of $q = 1$. The case of $q \geq 2$ can be proved similarly. Denote i^* as the root of the tree.

Define $a^0_{ij} = 1$ for $1 \leq i \leq N$, $j \in \overline{\mathscr{N}}_i$. If $a_{ij} = a^0_{ij}$ for $1 \leq i \leq N$, $j \in \overline{\mathscr{N}}_i$, then

$$\sum_{j \in \overline{\mathscr{N}}_i} \frac{1}{a^0_{ij}} \leq \overline{N}_i, \quad i = 1, \ldots, N \tag{2.12}$$

$$A_{\mathscr{O}} = 1, \quad \mathscr{O} \in \bigcup_{i=1,\ldots,N} \mathscr{C}(i). \tag{2.13}$$

Consider one of the paths leading from root i^* in the spanning tree $\mathscr{T}_f$. Denote the path as $(p_1, \ldots, p_m)$ with $p_1 = i^*$.

One can find $a^1_{p_2 p_1} = a^0_{p_2 p_1} + \varepsilon^0_{p_2 p_1} > 0$ with $\varepsilon^0_{p_2 p_1} > 0$ and $a^1_{p_2 j} = a^0_{p_2 j} - \varepsilon_{p_2 j} > 0$ with $\varepsilon_{p_2 j} > 0$ for $j \in \overline{\mathscr{N}}_{p_2} \backslash \{p_1\}$ such that if $a_{ij} = a^1_{ij}$ for $i = p_2$ and $a_{ij} = a^0_{ij}$ for $i \neq p_2$, then (2.12) is satisfied, and also

$$A_{\mathscr{O}} < 1 + \varepsilon' \text{ for } \mathscr{O} \in \mathscr{C}(p_1), \tag{2.14}$$

$$A_{\mathscr{O}} < 1 \text{ for } \mathscr{O} \in \mathscr{C}(p_2) \backslash \mathscr{C}(p_1) \tag{2.15}$$

with $0 < \varepsilon' < \varepsilon$.

Then, one can find $a^1_{p_3 p_2} = a^0_{p_3 p_2} + \varepsilon^0_{p_3 p_2} > 0$ with $\varepsilon^0_{p_3 p_2} > 0$ and $a^1_{p_3 j} = a^0_{p_3 j} - \varepsilon^0_{p_3 j} > 0$ with $\varepsilon^0_{p_3 j} > 0$ for $j \in \overline{\mathscr{N}}_{p_3} \backslash \{p_2\}$ such that if $a_{ij} = a^1_{ij}$ for $i \in \{p_2, p_3\}$, and $a_{ij} = a^0_{ij}$ for $i \notin \{p_2, p_3\}$, then (2.12) is satisfied, and also

$$A_{\mathscr{O}} < 1 + \varepsilon'' \text{ for } \mathscr{O} \in \mathscr{C}(p_1), \tag{2.16}$$

$$A_{\mathscr{O}} < 1 \text{ for } \mathscr{O} \in (\mathscr{C}(p_2) \cup \mathscr{C}(p_3)) \backslash \mathscr{C}(p_1) \tag{2.17}$$

with $0 < \varepsilon' \leq \varepsilon'' < \varepsilon$.

By doing this for $i = p_2, \ldots, p_m$, we can find $a^1_{ij} > 0$ for $i \in \{p_2, \ldots, p_m\}$, $j \in \overline{\mathscr{N}}_i$, such that

$$A_{\mathscr{O}} < 1 + \varepsilon_1 \text{ for } \mathscr{O} \in \mathscr{C}(p_1), \tag{2.18}$$

$$A_{\mathscr{O}} < 1 \text{ for } \mathscr{O} \in (\mathscr{C}(p_2) \cup \cdots \cup \mathscr{C}(p_m)) \backslash \mathscr{C}(p_1) \tag{2.19}$$

with $0 < \varepsilon_0 < \varepsilon$.

By considering each path leading from the root i^* in the spanning tree one by one, we can find $a^1_{ij} > 0$ for $i \in \{1, \ldots, N\}$, $j \in \overline{\mathscr{N}}_i$, such that if $a_{ij} = a^1_{ij}$ for $i \in \{1, \ldots, N\}$, $j \in \overline{\mathscr{N}}_i$, then (2.12) and (2.11) are satisfied and

$$A_{\mathscr{O}} < 1 + \varepsilon^1 \text{ for } \mathscr{O} \in \mathscr{C}(i^*_1) \cup \cdots \cup \mathscr{C}(i^*_q), \tag{2.20}$$

where $0 < \varepsilon^1 < \varepsilon$.

Note that the left-hand sides of inequalities (2.9), (2.10), and (2.11) continuously depend on a_{ij} for $i \in \{1, \ldots, N\}$, $j \in \overline{\mathscr{N}}_i$. One can find $a^2_{ij} > 0$ for $i \in \{1, \ldots, N\}$,

$j \in \overline{\mathcal{N}}_i$, such that if $a_{ij} = a_{ij}^2$ for $i \in \{1, \ldots, N\}$, $j \in \overline{\mathcal{N}}_i$, then conditions (2.9), (2.10), and (2.11) are satisfied. □

Example 2.2 Continue Example 2.1. Define $\mathcal{L} = \{i \in \{1, \ldots, N\} : 0 \in \mathcal{N}_i\}$. Considering the relation between N_i and $\overline{N}_i$, and $\overline{N}_i \leq N$, the cyclic-small-gain condition can be satisfied by the network of ISS subsystems with property (2.8) if

$$A_{\mathcal{O}} < \frac{(1-\bar{\delta})^N (N+1)}{N}, \quad \mathcal{O} \in \bigcup_{i \in \mathcal{L}} \mathcal{C}(i), \tag{2.21}$$

$$A_{\mathcal{O}} < (1-\bar{\delta})^N, \quad \mathcal{O} \in \left(\bigcup_{i \in \{1,\ldots,N\}} \mathcal{C}(i) \right) \Big\backslash \left(\bigcup_{i \in \mathcal{L}} \mathcal{C}(i) \right), \tag{2.22}$$

where $\bar{\delta} = \max_{i=1,\ldots,N}\{\delta_i\}$.

Using Lemma 2.1, if graph $\mathcal{G}_f$ has a spanning tree with the agents belonging to $\mathcal{L}$ as the roots, one can find a constant $\bar{\delta} > 0$ and constants $a_{ij} > 0$ satisfying $\sum_{j \in \overline{\mathcal{N}}_i} 1/a_{ij} \leq \overline{N}_i$ such that conditions (2.21) and (2.22) are satisfied. The graph shown in Fig. 2.2 satisfies this condition.

Lemma 2.1 proves very useful in constructing distributed controllers for nonlinear agents to achieve convergence of their outputs to an agreement value. It provides for a form of gain assignment in the network coupling.

2.3 Distributed Output-Feedback Control

In this section, the basic idea of cyclic-small-gain design for distributed control is generalized to high-order nonlinear systems. Consider a group of N nonlinear agents, of which each agent i ($1 \leq i \leq N$) is in the output-feedback form:

$$\dot{x}_{ij} = x_{i(j+1)} + \Delta_{ij}(y_i, w_i), \quad 1 \leq j \leq n_i - 1 \tag{2.23}$$

$$\dot{x}_{in_i} = u_i + \Delta_{in_i}(y_i, w_i) \tag{2.24}$$

$$y_i = x_{i1}, \tag{2.25}$$

where $[x_{i1}, \ldots, x_{in_i}]^T := x_i \in \mathbb{R}^{n_i}$ with $x_{ij} \in \mathbb{R}$ ($1 \leq j \leq n_i$) is the state, $u_i \in \mathbb{R}$ is the control input, $y_i \in \mathbb{R}$ is the output, $[x_{i2}, \ldots, x_{in_i}]^T$ is the unmeasured portion of the state, $w_i \in \mathbb{R}^{n_{w_i}}$ represents external disturbances, and Δ_{ij}'s ($1 \leq j \leq n_i$) are unknown locally Lipschitz functions.

The objective of this section is to develop a new class of distributed controllers for the multiagent system based on available information such that the outputs y_i for $1 \leq i \leq N$ converge to the same desired agreement value y_0. This problem is called the output agreement problem in this chapter.

Different from decentralized control, the major objective of distributed control is to control the agents in a coordinated way for some desired group behavior. For the output agreement problem, the objective is to control the agents so that the outputs converge to a desired common value. Information exchange between the agents is required for coordination purposes. In practice, the information exchange is subject to constraints. As considered in Example 2.1, the position x_0 of the leader vehicle is only available to some of the follower vehicles, and the formation control objective is achieved through information exchange between the neighboring vehicles.

For distributed control of the multiagent nonlinear system (2.23)–(2.25), we employ a digraph $\mathscr{G}^c$ to represent the information exchange topology between the agents. Digraph $\mathscr{G}^c$ contains N vertices corresponding to the N agents and M directed edges corresponding to the information exchange links. Specifically, if $y_i - y_k$ is available to the local controller design of agent i, then there is a directed link from agent k to agent i and agent k is called a neighbor of agent i; otherwise, there is no link from agent k to agent i. Set $\mathscr{N}_i \subseteq \{1, \ldots, N\}$ is used to represent agent i's neighbors. In this section, an agent is not considered as a neighbor of itself and thus $i \notin \mathscr{N}_i$ for $1 \le i \le N$. Agent i is called an informed agent if it has access to the knowledge of the agreement value y_0 for its local controller design. Let $\mathscr{L} \subseteq \{1, \ldots, N\}$ represent the set of all the informed agents.

The following assumption is made on the agreement value and system (2.23)–(2.25).

Assumption 2.1 There exists a nonempty set $\Omega \subseteq \mathbb{R}$ such that

1. $y_0 \in \Omega$;
2. for each $1 \le i \le N$, $1 \le j \le n_i$,

$$|\Delta_{ij}(y_i, w_i) - \Delta_{ij}(z_i, 0)| \le \psi_{\Delta_{ij}}(|[y_i - z_i, w_i^T]^T|) \tag{2.26}$$

   for all $[y_i, w_i^T]^T \in \mathbb{R}^{1+n_{w_i}}$ and all $z_i \in \Omega$, where $\psi_{\Delta_{ij}} \in \mathscr{K}_\infty$ is Lipschitz on compact sets and known.

It should be noted that a priori information on the bounds of y_0 (and thus Ω) is usually known in practice. In this case, condition 2 in Assumption 2.1 can be guaranteed if for each z_i, there exists a $\psi_{\Delta_{ij}}^{z_i} \in \mathscr{K}_\infty$ that is Lipschitz on compact sets such that

$$\begin{aligned} |\Delta_{ij}(y_i, w_i) - \Delta_{ij}(z_i, 0)| &= |\Delta_{ij}((y_i - z_i) + z_i, w_i) - \Delta_{ij}(z_i, 0)| \\ &\le \psi_{\Delta_{ij}}^{z_i}(|[y_i - z_i, w_i^T]^T|). \end{aligned} \tag{2.27}$$

Then, $\psi_{\Delta_{ij}}$ can be defined as $\psi_{\Delta_{ij}}(s) = \sup_{z_i \in \Omega} \psi_{\Delta_{ij}}^{z_i}(s)$ for $s \in \mathbb{R}_+$. In fact, there always exists a $\psi_{\Delta_{ij}}^{z_i} \in \mathscr{K}_\infty$ that is Lipschitz on compact sets to fulfill condition (2.27) if Δ_{ij} is locally Lipschitz.

It is also assumed that the external disturbances are bounded.

Assumption 2.2 For each $i = 1, \ldots, N$, there exists a $\bar{w}_i \geq 0$ such that

$$|w_i(t)| \leq \bar{w}_i \tag{2.28}$$

for all $t \geq 0$.

The basic idea is to design observer-based local controllers for the agents such that each controlled agent i is IOS, and moreover, has the UO property. Then, the cyclic-small-gain theorem in digraphs can be used to guarantee the IOS of the closed-loop multiagent system and then the achievement of output agreement.

By introducing a dynamic compensator

$$\dot{u}_i = v_i \tag{2.29}$$

and defining $x'_{i1} = y_i - y_0$ and $x'_{i(j+1)} = x_{i(j+1)} + \Delta_{ij}(y_0, 0)$ for $1 \leq j \leq n_i$, we can transform each agent i defined by (2.23)–(2.25) into the form of

$$\dot{x}'_{ij} = x'_{i(j+1)} + \Delta_{ij}(y_i, w_i) - \Delta_{ij}(y_0, 0), \quad 1 \leq j \leq n_i + 1 \tag{2.30}$$

$$\dot{x}'_{in_i} = v_i + \Delta_{in_i}(y_i, w_i) - \Delta_{in_i}(y_0, 0) \tag{2.31}$$

$$y'_i = x'_{i1} \tag{2.32}$$

with the output tracking error $y'_i = y_i - y_0$ as the new output and v_i as the new control input.

Moreover, the dynamic compensator (2.29) guarantees that the origin is an equilibrium of the transformed agent system (2.30)–(2.32) if it is disturbance-free, and the distributed control objective can be achieved if the equilibrium at the origin of each transformed agent system is stabilized.

The local controller for each agent i is designed by directly using the available y_i^m, defined as follows:

$$y_i^m = \frac{1}{N_i + 1}\left(\sum_{k \in \mathcal{N}_i}(y_i - y_k) + (y_i - y_0)\right), \quad i \in \mathcal{L} \tag{2.33}$$

$$y_i^m = \frac{1}{N_i}\sum_{k \in \mathcal{N}_i}(y_i - y_k), \quad i \in \{1, \ldots, N\}\backslash\mathcal{L}, \tag{2.34}$$

where N_i is the size of $\mathcal{N}_i$. For the convenience of discussions, we represent y_i^m with the new outputs as

$$y_i^m = y'_i - \mu_i \tag{2.35}$$

with

$$\mu_i = \frac{1}{N_i + 1} \sum_{k \in \mathcal{N}_i} y'_k, \quad i \in \mathcal{L} \tag{2.36}$$

$$\mu_i = \frac{1}{N_i} \sum_{k \in \mathcal{N}_i} y'_k, \quad i \in \{1, \ldots, N\} \backslash \mathcal{L}. \tag{2.37}$$

2.3.1 Distributed Output-Feedback Controller

Owing to the output-feedback structure, we design a local observer for each transformed agent system (2.30)–(2.32):

$$\dot{\xi}_{i1} = \xi_{i2} + L_{i2}\xi_{i1} + \rho_{i1}(\xi_{i1} - y_i^m) \tag{2.38}$$

$$\dot{\xi}_{ij} = \xi_{i(j+1)} + L_{i(j+1)}\xi_{i1} - L_{ij}(\xi_{i2} + L_{i2}\xi_{i1}), \quad 2 \le j \le n_i \tag{2.39}$$

$$\dot{\xi}_{i(n_i+1)} = v_i - L_{i(n_i+1)}(\xi_{i2} + L_{i2}\xi_{i1}), \tag{2.40}$$

where $\rho_{i1} : \mathbb{R} \to \mathbb{R}$ is an odd and strictly decreasing function, and $L_{i2}, \ldots, L_{in_i}$ are positive constants. In the observer, ξ_{i1} is an estimate of y'_i, and ξ_{ij} is an estimate of $x'_{ij} - L_{ij} y'_i$ for $2 \le j \le n_i + 1$.

Here, (2.38) is constructed to estimate y'_i using y_i^m which is influenced by the outputs y'_k ($k \in \mathcal{N}_i$) of the neighbor agents (see (2.35)). The nonlinear function ρ_{i1} in (2.38) is used to assign an appropriate *nonlinear* gain to the observation error system. As shown later, it is the key to make each controlled agent IOS with specific gains satisfying the cyclic-small-gain condition.

With the estimates, a nonlinear local control law is designed as

$$e_{i1} = \xi_{i1}, \tag{2.41}$$

$$e_{ij} = \xi_{ij} - \kappa_{i(j-1)}(e_{i(j-1)}), \quad 2 \le j \le n_i + 1 \tag{2.42}$$

$$v_i = \kappa_{i(n_i+1)}(e_{i(n_i+1)}), \tag{2.43}$$

where $\kappa_{i1}, \ldots, \kappa_{i(n_i+1)}$ are continuously differentiable, odd, strictly decreasing, and radially unbounded functions.

Consider $Z_i = [x'_{i1}, \ldots, x'_{i(n_i+1)}, \xi_{i1}, \ldots, \xi_{i(n_i+1)}]^T$ as the internal state of each controlled agent composed of the transformed agent system (2.30)–(2.32) and the local observer-based controller (2.38)–(2.43). The block diagram of controlled agent i with μ_i as the input and y'_i as the output is shown in Fig. 2.3.

The following proposition presents the UO and IOS properties of each controlled agent i.

Proposition 2.1 *Each controlled agent i composed of* (2.30)–(2.32) *and* (2.38)–(2.43) *has the following UO and IOS properties with μ_i as the input and y'_i as the*

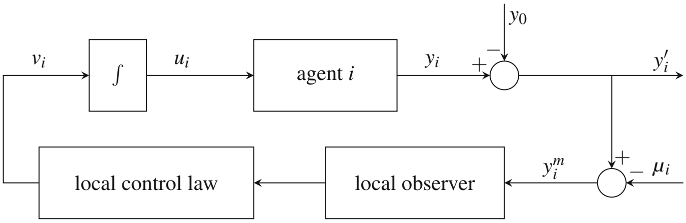

Fig. 2.3 The block diagram of each controlled agent i

output: for all $t \geq 0$,

$$|Z_i(t)| \leq \alpha_i^{UO}(|Z_{i0}| + \|\mu_i\|_{[0,t]}) \tag{2.44}$$

$$|y_i'(t)| \leq \max\left\{\beta_i(|Z_{i0}|, t), \chi_i(\|\mu_i\|_{[0,t]}), \gamma_i(\|w_i\|_{[0,t]})\right\}, \tag{2.45}$$

for any initial state $Z_i(0) = Z_{i0}$ *and any* μ_i, w_i, *where* $\beta_i \in \mathcal{KL}$ *and* $\chi_i, \gamma_i, \alpha_i \in \mathcal{K}_\infty$. *Moreover,* γ_i *can be designed to be arbitrarily small, and for any specified constant* $b_i > 1$, χ_i *can be designed such that* $\chi_i(s) \leq b_i s$, *for all* $s \geq 0$.

Due to space limitations, the proof of Proposition 2.1 is not provided here. The interested reader may consult [39] for reference.

2.3.2 Cyclic-Small-Gain Synthesis

With the proposed distributed output-feedback controller, the closed-loop multiagent system has been transformed into a network of IOS subsystems. This subsection presents the main result of output agreement and provides a proof based on the cyclic-small-gain result in digraphs.

Theorem 2.1 *Consider the multiagent system in the form of* (2.23)–(2.25) *satisfying Assumptions 2.1 and 2.2. If there is at least one informed agent, i.e.,* $\mathcal{L} \neq \emptyset$, *and the communication digraph* $\mathcal{G}^c$ *has a spanning tree with the informed agents as the roots, then we can design distributed observers* (2.38)–(2.40) *and distributed control laws* (2.29), (2.41)–(2.43) *such that all the signals in the closed-loop multiagent system are bounded, and the output* y_i *of each agent* i *can be steered to within an arbitrarily small neighborhood of the desired agreement value* y_0. *Moreover, if* $w_i = 0$ *for* $i = 1, \ldots, N$, *then each output* y_i *asymptotically converges to* y_0.

Proof Notice that for any constants $a_1, \ldots, a_n > 0$ satisfying $\sum_{i=1}^n (1/a_i) \leq n$, it holds that

$$\sum_{i=1}^n d_i = \sum_{i=1}^n \frac{1}{a_i} a_i d_i \leq n \max_{1 \leq i \leq n} \{a_i d_i\} \tag{2.46}$$

for all $d_1, \dots, d_n \geq 0$.

Recall the definition of μ_i in (2.36) and (2.37). We have

$$|\mu_i| \leq \delta_i \max_{k \in \mathcal{N}_i} \{a_{ik} |y'_k|\}, \tag{2.47}$$

where $\delta_i = \frac{N_i}{N_i + 1}$ if $i \in \mathcal{L}$, $\delta_i = 1$ if $i \notin \mathcal{L}$, and a_{ik} are positive constants satisfying

$$\sum_{k \in \mathcal{N}_i} \frac{1}{a_{ik}} \leq N_i. \tag{2.48}$$

Then, using the fact that the $\mathcal{N}_i$ in (2.47) are time invariant, property (2.45) implies

$$|y'_i(t)| \leq \max \left\{ \beta_i(|Z_{i0}|, t), b_i \delta_i \max_{k \in \mathcal{N}_i} \{a_{ik} \|y'_k\|_{[0,t]}\}, \gamma_i(\|w_i\|_{[0,t]}) \right\} \tag{2.49}$$

for any initial state Z_{i0} and any w_i, for all $t \geq 0$.

It can be observed that the interconnection topology of the controlled agents is in accordance with the information exchange topology, represented by digraph $\mathcal{G}^c$. For $i \in \mathcal{N}$, $k \in \mathcal{N}_i$, we assign the positive value a_{ik} to the edge (k, i) in $\mathcal{G}^c$. Denote $\mathcal{C}$ as the set of all simple cycles in $\mathcal{G}^c$ and $\mathcal{C}_{\mathcal{L}}$ as the set of all simple cycles through the vertices belonging to $\mathcal{L}$. Denote $A_{\mathcal{O}}$ as the product of the positive values assigned to the edges of the cycle $\mathcal{O} \in \mathcal{C}$.

Note that b_i can be designed to be arbitrarily close to one. By using the cyclic-small-gain theorem for networks of IOS systems, the closed-loop multiagent system is IOS if

$$A_{\mathcal{O}} \frac{N}{N+1} < 1, \quad \mathcal{O} \in \mathcal{C}_{\mathcal{L}} \tag{2.50}$$

$$A_{\mathcal{O}} < 1, \quad \mathcal{O} \in \mathcal{C} \backslash \mathcal{C}_{\mathcal{L}}. \tag{2.51}$$

If $\mathcal{G}^c$ has a spanning tree with vertices belonging to $\mathcal{L}$ as the roots, then according to Lemma 2.1, there exist positive constants a_{ik} satisfying (2.48), (2.50), and (2.51). Then, the closed-loop distributed system is UO and IOS with w_i as the inputs and y'_i as the outputs. With Assumption 2.2, the external disturbances w_i are bounded. The boundedness of the signals of the closed-loop distributed system can be directly verified under Assumption 2.2.

By designing the IOS gains γ_i arbitrarily small (this can be done according to Proposition 2.1), the influence of the external disturbances w_i is made arbitrarily small, and y'_i can be driven to within an arbitrarily small neighborhood of the origin. Equivalently, y_i can be driven to within an arbitrarily small neighborhood of y_0. In the case of $w_i = 0$ for $i = 1, \dots, N$, each output y_i asymptotically converges to y_0. This ends the proof of Theorem 2.1. □

2.3.3 Robustness to Time Delays of Information Exchange

If there are communication delays, then y_i^m as defined in (2.33) and (2.34) should be modified as

$$y_i^m(t) = \frac{1}{N_i+1}\left(\sum_{k\in\mathcal{N}_i}(y_i(t) - y_k(t-\tau_{ik}(t))) + (y_i(t) - y_0)\right), \quad i \in \mathcal{L} \tag{2.52}$$

$$y_i^m(t) = \frac{1}{N_i}\sum_{k\in\mathcal{N}_i}(y_i(t) - y_k(t-\tau_{ik}(t))), \quad i \in \{1,\ldots,N\}\backslash\mathcal{L}, \tag{2.53}$$

where $\tau_{ik} : \mathbb{R}_+ \to \mathbb{R}_+$ represents nonconstant time delays of exchanged information.

In this case, $y_i^m(t)$ can still be written in the form of $y_i^m(t) = y_i'(t) - \mu_i(t)$ with

$$\mu_i(t) = \frac{1}{N_i+1}\sum_{k\in\mathcal{N}_i} y_k'(t-\tau_{ik}(t)), \quad i \in \mathcal{L} \tag{2.54}$$

$$\mu_i(t) = \frac{1}{N_i}\sum_{k\in\mathcal{N}_i} y_k'(t-\tau_{ik}(t)), \quad i \in \{1,\ldots,N\}\backslash\mathcal{L}. \tag{2.55}$$

We assume that there exists a $\bar{\tau} \geq 0$ such that, for $i = 1,\ldots,N$, $k \in \mathcal{N}_i$, $0 \leq \tau_{ik}(t) \leq \bar{\tau}$ holds for all $t \geq 0$. By considering μ_i and w_i as the external inputs, each controlled agent i composed of (2.30)–(2.32) and (2.38)–(2.43) is still UO and property (2.49) should be modified as

$$|y_i'(t)| \leq \max\left\{\beta_i(|Z_{i0}|, t), b_i\delta_i \max_{k\in\mathcal{N}_i}\{a_{ik}\|y_k'\|_{[-\bar{\tau},\infty)}\}, \gamma_i(\|w_i\|_{[0,\infty)})\right\} \tag{2.56}$$

for any initial state Z_{i0} and any w_i, for all $t \geq 0$.

Using the time-delay version of the cyclic-small-gain theorem, we can still guarantee the IOS of the closed-loop multiagent system with y_i' as the outputs and w_i as the inputs, following analysis similar to that for the proof of Theorem 2.1.

2.4 Formation Control of Nonholonomic Mobile Robots

Formation control of autonomous mobile agents is aimed at forcing agents to converge toward, and maintain, specific relative positions. Distributed formation control of multiagent systems based on available local information, e.g., relative position measurements, has attracted tremendous attention from the robotics and control communities.

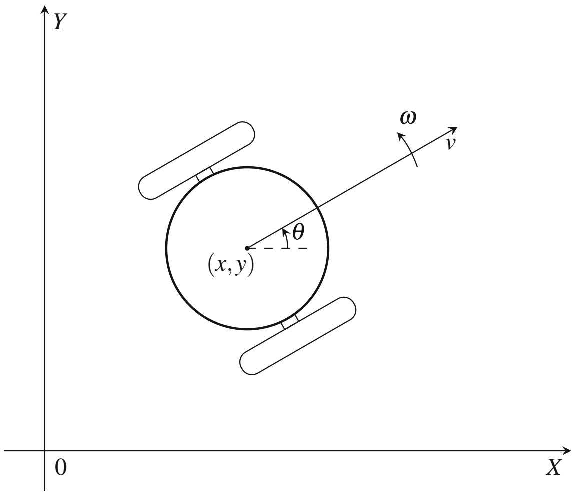

Fig. 2.4 Kinematics of the unicycle robot, where (x, y) represents the Cartesian coordinates of the center of mass of the robot, v is the linear velocity, θ is the heading angle, and ω is the angular velocity

Motivated by the cyclic-small-gain design for distributed output-feedback control of nonlinear systems in Sect. 2.3, this section proposes a class of distributed controllers for leader-following formation control of unicycle robots using the practically available relative position measurements. The kinematics of the unicycle robot are demonstrated by Fig. 2.4.

For this purpose, the formation control problem is first transformed into a state agreement problem of double integrators through dynamic feedback linearization. The nonholonomic constraint causes a singularity for the dynamic feedback linearization when the linear velocity of the robot is zero. This issue should be well taken into consideration for the validity of the transformed double integrator models. Then, distributed formation control laws are developed. To avoid the singularity problem caused by the nonholonomic constraint, saturation functions are introduced to the control design to restrict the linear velocities of the robots to be larger than zero. It should be noted that linear analysis methods may not be directly applicable due to the employment of the saturation functions. Then, the closed-loop system is transformed into a dynamic network of IOS systems. The cyclic-small-gain result in digraphs is used to guarantee the IOS of the dynamic network and thus the achievement of formation control.

With the effort mentioned above, the proposed design has three advantages:

1. The proposed distributed formation control law does not use global position measurements or assumes tree position sensing structures.
2. The formation control objective can be practically achieved in the presence of position measurement errors.
3. The linear velocities of the robots can be designed to be less than certain desired values, as practically required.

This section considers the formation control problem of a group of $N + 1$ mobile robots. For $i = 0, 1, \ldots, N$, the kinematics of the ith robot are described by the unicycle model:

$$\dot{x}_i = v_i \cos\theta_i \tag{2.57}$$
$$\dot{y}_i = v_i \sin\theta_i \tag{2.58}$$
$$\dot{\theta}_i = \omega_i, \tag{2.59}$$

where $[x_i, y_i]^T \in \mathbb{R}^2$ represent the Cartesian coordinates of the center of mass of the ith robot, $v_i \in \mathbb{R}$ is the linear velocity, $\theta_i \in \mathbb{R}$ is the heading angle, and $\omega_i \in \mathbb{R}$ is the angular velocity.

The robot with index 0 is the leader robot, and the robots with indices $1, \ldots, N$ are follower robots. The linear velocity v_i and the angular velocity ω_i are considered as the control inputs of the ith robot for $i = 1, \ldots, N$. For this system, the formation control objective is to control each ith follower robot such that

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = d_{xij} \tag{2.60}$$
$$\lim_{t\to\infty} (y_i(t) - y_j(t)) = d_{yij} \tag{2.61}$$

with d_{xij}, d_{yij} being appropriate constants representing the desired relative positions, and

$$\lim_{t\to\infty} \big((\theta_i(t) - \theta_j(t)) \bmod 2\pi\big) = 0 \tag{2.62}$$

for any $i, j = 0, \ldots, N$, where mod represents the modulo operation. For convenience of notation, let $d_{xii} = d_{yii} = 0$ for any $i = 0, \ldots, N$. We assume that $d_{xij} = d_{xik} - d_{xkj}$ and $d_{yij} = d_{yik} - d_{ykj}$ for any $i, j, k = 0, \ldots, N$.

Assumption 2.3 on v_0 is made throughout this section.

Assumption 2.3 The linear velocity v_0 of the leader robot is differentiable with bounded derivative, i.e., $\dot{v}_0(t)$ exists and is bounded on $[0, \infty)$, and has upper and lower constant bounds $\bar{v}_0, \underline{v}_0 > 0$ such that $\underline{v}_0 \le v_0(t) \le \bar{v}_0$, for all $t \ge 0$.

One technical problem of controlling groups of mobile robots is that accurate global positions of the robots are usually not available for feedback, and relative position measurements should be used instead. A digraph can be employed to represent the relative position sensing structure between the robots. The position sensing digraph $\mathscr{G}$ has $N + 1$ vertices with indices $0, 1, \ldots, N$ corresponding to the robots. If the relative position between robot i and robot j is available to robot j, then $\mathscr{G}$ has a directed edge from vertex i to vertex j; otherwise $\mathscr{G}$ does not have such an edge.

The goal of this section is to present a class of distributed formation controllers for mobile robots using local relative position measurements as well as the velocity and acceleration information of the leader. The basic idea of the design is to first transform the unicycle model into two double integrators through dynamic feedback linearization under constraints, and at the same time, to reformulate the formation control problem as a stabilization problem. Then, distributed control laws are designed to make each controlled mobile robot IOS. Finally, the cyclic-small-gain theorem is used to guarantee the achievement of the formation control objective.

2.4.1 Dynamic Feedback Linearization

In this subsection, the distributed formation control problem is reformulated with the dynamic feedback linearization technique. For details of dynamic feedback linearization and its applications to nonholonomic systems, please consult [6, 14].

For each $i = 0, \ldots, N$, introduce a new input $r_i \in \mathbb{R}$ such that

$$\dot{v}_i = r_i. \tag{2.63}$$

Define $v_{xi} = v_i \cos\theta_i$ and $v_{yi} = v_i \sin\theta_i$. Then, $\dot{x}_i = v_{xi}$ and $\dot{y}_i = v_{yi}$. Take the derivatives of v_{xi} and v_{yi}, respectively. Then,

$$\begin{pmatrix} \dot{v}_{xi} \\ \dot{v}_{yi} \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -v_i \sin\theta_i \\ \sin\theta_i & v_i \cos\theta_i \end{pmatrix} \begin{pmatrix} r_i \\ \omega_i \end{pmatrix}. \tag{2.64}$$

In the case of $v_i \neq 0$, by designing

$$\begin{pmatrix} r_i \\ \omega_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\frac{\sin\theta_i}{v_i} & \frac{\cos\theta_i}{v_i} \end{pmatrix} \begin{pmatrix} u_{xi} \\ u_{yi} \end{pmatrix}, \tag{2.65}$$

the unicycle model (2.57)–(2.59) can be transformed into two double integrators with new inputs u_{xi} and u_{yi}:

$$\dot{x}_i = v_{xi}, \quad \dot{v}_{xi} = u_{xi}, \tag{2.66}$$

$$\dot{y}_i = v_{yi}, \quad \dot{v}_{yi} = u_{yi}. \tag{2.67}$$

Define $\tilde{x}_i = x_i - x_0 - d_{xi}$, $\tilde{y}_i = y_i - y_0 - d_{yi}$, $\tilde{v}_{xi} = v_{xi} - v_{x0}$, $\tilde{v}_{yi} = v_{yi} - v_{y0}$, $\tilde{u}_{xi} = u_{xi} - u_{x0}$, and $\tilde{u}_{yi} = u_{yi} - u_{y0}$. Then,

$$\dot{\tilde{x}}_i = \tilde{v}_{xi}, \quad \dot{\tilde{v}}_{xi} = \tilde{u}_{xi}, \tag{2.68}$$

$$\dot{\tilde{y}}_i = \tilde{v}_{yi}, \quad \dot{\tilde{v}}_{yi} = \tilde{u}_{yi}. \tag{2.69}$$

The formation control problem is solvable, if we can design control laws for system (2.68)–(2.69) with $\tilde{u}_{xi}$ and $\tilde{u}_{yi}$ as the control inputs, so that $v_i \neq 0$ is guaranteed, and at the same time,

$$\lim_{t\to\infty} \tilde{x}_i(t) = 0, \tag{2.70}$$

$$\lim_{t\to\infty} \tilde{y}_i(t) = 0. \tag{2.71}$$

It should be noted that the validity of (2.66)–(2.67) (and thus (2.68)–(2.69)) for the unicycle model is under the condition that $v_i \neq 0$. Such requirement is basically caused by the nonholonomic constraint of the mobile robot. This leads to the major difference between this problem and the distributed control problem for double integrators.

To use (2.68)–(2.69) for control design, each follower robot should have access to u_{x0}, u_{y0}, which represent the acceleration of the leader robot. This requirement can be fulfilled if the leader robot can calculate u_{x0}, u_{y0} using $r_0, \omega_0, \theta_0, v_0$ according to (2.65) and transmit them to the follower robots. Note that ω_0, θ_0, v_0 are usually measurable, and r_0 is normally available as it is the control input of the leader robot.

2.4.2 A Class of IOS Control Laws

As an ingredient for the distributed control design, this subsection presents a class of nonlinear control laws for the following double integrator system with an external input, such that the closed-loop system is UO and IOS:

$$\dot{\eta} = \zeta \tag{2.72}$$

$$\dot{\zeta} = \mu \tag{2.73}$$

$$\hat{\eta} = \eta + w, \tag{2.74}$$

where $[\eta, \zeta]^T \in \mathbb{R}^2$ is the state, $\mu \in \mathbb{R}$ is the control input, $w \in \mathbb{R}$ represents an external input, $\hat{\eta}$ can be considered as a measurement of η subject to w, and only $(\hat{\eta}, \zeta)$ are used for feedback. As shown later, each controlled robot can be transformed into the form of (2.72)–(2.74) with w representing the interaction between the robots.

Lemma 2.2 *For system (2.72)–(2.74), consider a control law taking the form:*

$$\mu = -k_\mu(\zeta - \phi(\hat{\eta})). \tag{2.75}$$

For any constant $\overline{\phi} > 0$, one can find an odd, strictly decreasing, continuously differentiable function $\phi : \mathbb{R} \to [-\overline{\phi}, \overline{\phi}]$ and a positive constant k_μ satisfying

$$-\frac{k_\mu}{4} < \frac{d\phi(r)}{dr} < 0 \tag{2.76}$$

for all $r \in \mathbb{R}$, such that the closed-loop system (2.72)–(2.75) is UO with zero offset, and is IOS with the identity function as the gain, i.e., the following properties hold:

$$|\eta(t)| \leq \overline{\beta}(|[\eta(0), \zeta(0)]^T|, t) + \|w\|_t \tag{2.77}$$

$$|\zeta(t)| \leq |\zeta(0)| + \alpha_{UO}(\|\eta\|_t + \|w\|_t) \tag{2.78}$$

for some $\overline{\beta} \in \mathcal{KL}$, $\alpha_{UO} \in \mathcal{K}_\infty$, *and all* $t \geq 0$.

It is necessary to note that condition (2.76) is easily checkable for practical implementation of the control law (2.75).

2.4.3 Distributed Formation Controller Design and Small-Gain Analysis

As discussed in Sect. 2.4.1, for the validity of (2.68)–(2.69) of the formation control design, v_i should be guaranteed to be nonzero. For a specified λ_* satisfying $0 < \lambda_* < \underline{v}_0$, by designing a control law for the ith robot such that

$$\max\left\{|\tilde{v}_{xi}|, |\tilde{v}_{yi}|\right\} \leq \frac{\sqrt{2}}{2}(\underline{v}_0 - \lambda_*) \leq \frac{\sqrt{2}}{2}(v_0 - \lambda_*), \tag{2.79}$$

it can be guaranteed that $|v_i| = \sqrt{v_{xi}^2 + v_{yi}^2} = \sqrt{(v_{x0} + \tilde{v}_{xi})^2 + (v_{y0} + \tilde{v}_{yi})^2} \geq \lambda_* > 0$ and thus $v_i \neq 0$. In this way, singularity is avoided.

Practically, the linear velocity of each robot is usually required to be less than a desired value. For any given $\lambda^* > \overline{v}_0$, we can also guarantee $|v_i| \leq \lambda^*$ by designing a control law such that

$$\max\left\{|\tilde{v}_{xi}|, |\tilde{v}_{yi}|\right\} \leq \frac{\sqrt{2}}{2}(\lambda^* - \overline{v}_0). \tag{2.80}$$

For specified constants $\lambda_*, \lambda^*, \underline{v}_0, \overline{v}_0$ satisfying $0 < \lambda_* < \underline{v}_0 < \overline{v}_0 < \lambda^*$, we define

$$\lambda = \min\left\{\frac{\sqrt{2}}{2}(\underline{v}_0 - \lambda_*), \frac{\sqrt{2}}{2}(\lambda^* - \overline{v}_0)\right\}. \tag{2.81}$$

Then, conditions (2.79) and (2.80) can be satisfied if

$$\max\left\{|\tilde{v}_{xi}|, |\tilde{v}_{yi}|\right\} \leq \lambda. \tag{2.82}$$

The proposed distributed control law is composed of two stages: (a) initialization and (b) formation control. The initialization stage is employed because the formation control stage cannot solely guarantee $v_i \neq 0$ if (2.79) is not satisfied at the beginning of the control procedure. With the initialization stage, the linear velocity and the heading direction of each follower robot can be controlled to satisfy (2.82) within some finite time. Then, the formation control stage is triggered, and thereafter, the satisfaction of (2.82) is guaranteed, and at the same time, the formation control objective is achieved.

Initialization Stage

For this stage, we design the following control law:

$$\omega_i = \phi_{\theta i}(\theta_i - \theta_0) + \omega_0 \tag{2.83}$$

$$r_i = \phi_{vi}(v_i - v_0) + \dot{v}_0 \tag{2.84}$$

for each ith follower robot, where $\phi_{\theta i}, \phi_{vi} : \mathbb{R} \to \mathbb{R}$ are nonlinear functions.

Define $\tilde{\theta}_i = \theta_i - \theta_0$ and $\tilde{v}_i = v_i - v_0$. Taking the derivatives of $\tilde{\theta}_i$ and $\tilde{v}_i$, respectively, and using (2.83) and (2.84), we have

$$\dot{\tilde{\theta}}_i = \phi_{\theta i}(\tilde{\theta}_i), \tag{2.85}$$

$$\dot{\tilde{v}}_i = \phi_{vi}(\tilde{v}_i). \tag{2.86}$$

By designing $\phi_{\theta i}, \phi_{vi}$ such that $-\phi_{\theta i}(s), \phi_{\theta i}(-s), -\phi_{vi}(s), \phi_{vi}(-s)$ are positive definite for $s \in \mathbb{R}_+$, we can guarantee the asymptotic stability of systems (2.85) and (2.86). Moreover, there exist $\beta_{\tilde{\theta}}, \beta_{\tilde{v}} \in \mathscr{KL}$ such that $|\tilde{\theta}(t)| \le \beta_{\tilde{\theta}}(|\tilde{\theta}(0)|, t)$ and $|\tilde{v}(t)| \le \beta_{\tilde{v}}(|\tilde{v}(0)|, t)$.

By directly using the property of continuous functions, there exist $\overline{\delta}_{v0} > 0$ and $\overline{\delta}_{\theta 0} > 0$ such that, for all $v_0 \in [\underline{v}_0, \overline{v}_0]$, $\theta_0 \in \mathbb{R}$, $|\delta v_0| \le \overline{\delta}_{v0}$ and $|\delta_{\theta 0}| \le \overline{\delta}_{\theta 0}$,

$$|(v_0 + \delta_{v0})\cos(\theta_0 + \delta_{\theta 0}) - v_0 \cos\theta_0| \le \lambda, \tag{2.87}$$

$$|(v_0 + \delta_{v0})\sin(\theta_0 + \delta_{\theta 0}) - v_0 \sin\theta_0| \le \lambda. \tag{2.88}$$

Recall that for any $\beta \in \mathscr{KL}$, there exist functions $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ such that $\beta(s,t) \le \alpha_1(s)\alpha_2(e^{-t})$, for all $s, t \in \mathbb{R}_+$ according to [52, Lemma 8]. With control law (2.83)–(2.84), there exists a finite time T_{Oi} for the ith robot such that $|\theta_i(T_{Oi}) - \theta_0(T_{Oi})| \le \overline{\delta}_{\theta 0}$ and $|v_i(T_{Oi}) - v_0(T_{Oi})| \le \overline{\delta}_{v0}$, and thus condition (2.82) is satisfied at time T_{Oi}.

It should be noted that if $v_i(0) \le \lambda^*$, then control law (2.84) guarantees $v_i(t) \le \lambda^*$ for $t \in [0, T_{Oi}]$ because of $v_0(t) \le \overline{v}_0 < \lambda^*$.

Formation Control Stage

At time T_{Oi}, the distributed control law for the ith follower robot is switched to the formation control stage.

In this stage, we design

$$\tilde{u}_{xi} = -k_{xi}(\tilde{v}_{xi} - \phi_{xi}(z_{xi})) \tag{2.89}$$

$$\tilde{u}_{yi} = -k_{yi}(\tilde{v}_{yi} - \phi_{yi}(z_{yi})), \tag{2.90}$$

where $\phi_{xi}, \phi_{yi} : \mathbb{R} \to [-\lambda, \lambda]$ are odd, strictly decreasing, and continuously differentiable functions and k_{xi}, k_{yi} are positive constants satisfying

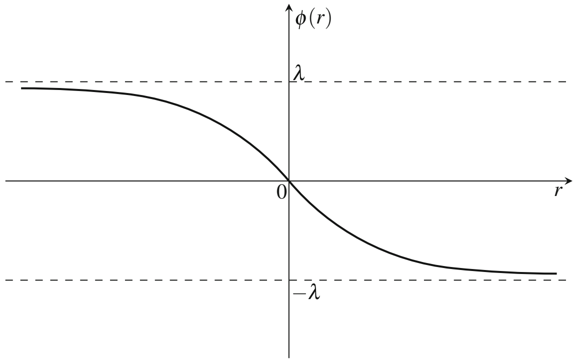

Fig. 2.5 An example for ϕ_{xi} and ϕ_{yi}

$$-k_{xi}/4 < d\phi_{xi}(r)/dr < 0 \tag{2.91}$$
$$-k_{yi}/4 < d\phi_{yi}(r)/dr < 0 \tag{2.92}$$

for all $r \in \mathbb{R}$. An example for ϕ_{xi} and ϕ_{yi} is shown in Fig. 2.5.

The variables z_{xi} and z_{yi} are defined as

$$z_{xi} = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} (x_i - x_j - (d_{xi} - d_{xj})) \tag{2.93}$$
$$z_{yi} = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} (y_i - y_j - (d_{yi} - d_{yj})), \tag{2.94}$$

where N_i is the size of $\mathcal{N}_i$ with $\mathcal{N}_i$ representing the position sensing structure. If $j \in \mathcal{N}_i$, then the position sensing digraph $\mathcal{G}$ has a directed edge (j, i) from vertex j to vertex i. Note that $d_{xi} - d_{xj}, d_{yi} - d_{yj}$ in (2.93) and (2.94) represent the desired relative position between the ith robot and the jth robot. By default, $d_{x0} = d_{y0} = 0$.

In the formation control stage, the control inputs r_i and ω_i are defined as (2.65) with $u_{xi} = \tilde{u}_{xi} + u_{x0}$ and $u_{yi} = \tilde{u}_{yi} + u_{y0}$.

Consider the $(\tilde{v}_{xi}, \tilde{v}_{yi})$-system defined in (2.68) and (2.69). With condition (2.82) satisfied at time T_{Oi}, the boundedness of ϕ_{xi} and ϕ_{yi} together with the control law (2.89) and (2.90) guarantees the satisfaction of (2.82) after T_{Oi}. For the proof of this statement, we can consider $\left\{(\tilde{v}_{xi}, \tilde{v}_{yi}) : \max\{|\tilde{v}_{xi}|, |\tilde{v}_{yi}|\} \le \lambda\right\}$ as an invariant set of the $(\tilde{v}_{xi}, \tilde{v}_{yi})$-system.

The main result of distributed formation control is summarized below.

Theorem 2.2 *Consider the multirobot model (2.57)–(2.59) and the distributed control laws defined by (2.63), (2.65), (2.83), (2.84), (2.89), and (2.90) with parameters k_{xi}, k_{yi} satisfying (2.91) and (2.92). Under Assumption 2.3, if the position sensing digraph $\mathcal{G}$ has a spanning tree with vertex 0 as the root, then for any constants $d_{xi}, d_{yi} \in \mathbb{R}$ with $i = 1, \ldots, N$, the coordinates $(x_i(t), y_i(t))$ and the angle $\theta_i(t)$ of each ith robot asymptotically converge to $(x_0(t) + d_{xi}, y_0(t) + d_{yi})$ and*

$\theta_0(t) + 2k\pi$ with $k \in \mathbb{Z}$, respectively. Moreover, given any $\lambda^* > \overline{v}_0$, if $v_i(0) \leq \lambda^*$ for $i = 1, \ldots, N$, then $v_i(t) \leq \lambda^*$, for all $t \geq 0$.

The two-stage distributed control law results in a switching incident of the control signal for each follower robot during the control procedure. The trajectories of such systems can be well defined in the spirit of Rademacher (see, e.g., [13]), and the performance of the system can be analyzed by considering the two stages one by one.

2.4.4 Small-Gain Analysis and Proof of Theorem 2.2

Recall the definition of λ in (2.81). With condition (2.82) satisfied after T_{Oi}, we have $v_i \neq 0$ and thus the validity of (2.65), for all $t \geq T_{Oi}$. Under the condition of $v_i(0) \leq \lambda^*$, the boundedness of $v_i(t)$, i.e., $v_i(t) \leq \lambda^*$, can also be directly proved based on the discussions in Sect. 2.4.3.

Denote $\tilde{x}_0 = 0$ and $\tilde{y}_0 = 0$. We equivalently represent z_{xi} and z_{yi} as

$$\begin{aligned} z_{xi} &= \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} (x_i - d_{xi} - x_0 - (x_j - d_{xj} - x_0)) \\ &= \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} (\tilde{x}_i - \tilde{x}_j) = \tilde{x}_i - \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \tilde{x}_j \end{aligned} \tag{2.95}$$

and similarly,

$$z_{yi} = \tilde{y}_i - \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \tilde{y}_j. \tag{2.96}$$

Denote

$$\omega_{xi} = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \tilde{x}_j, \tag{2.97}$$

$$\omega_{yi} = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \tilde{y}_j. \tag{2.98}$$

Then, control laws (2.89) and (2.90) are in the form of (2.75).

In the following proof, we only consider the $(\tilde{x}_i, \tilde{v}_{xi})$-system (2.68). The $(\tilde{y}_i, \tilde{v}_{yi})$-system (2.69) can be studied in the same way.

Define $T_O = \max_{i=1,\ldots,N}\{T_{Oi}\}$. Using Lemma 2.2, for each $i = 1, \ldots, N$, the closed-loop system composed of (2.68) and (2.89) has the following properties: for any $\tilde{x}_{i0}, \tilde{v}_{xi0} \in \mathbb{R}$, with $\tilde{x}_i(T_O) = \tilde{x}_{i0}$ and $\tilde{v}_{xi}(T_O) = \tilde{v}_{xi0}$,

$$|\tilde{x}_i(t)| \leq \beta_{xi}(|[\tilde{x}_{i0}, \tilde{v}_{xi0}]^T|, t - T_O) + \|\omega_{xi}\|_{[T_O,t]} \tag{2.99}$$

$$|\tilde{v}_{xi}(t)| \leq |\tilde{v}_{xi0}| + \alpha_{xi}(\|\tilde{x}_i\|_{[T_O,t]} + \|\omega_{xi}\|_{[T_O,t]}), \tag{2.100}$$

where $\beta_{xi} \in \mathscr{K}\mathscr{L}$ and $\alpha_{xi} \in \mathscr{K}_\infty$.

Notice that for any constants $a_1, \ldots, a_n > 0$ satisfying $\sum_{i=1}^n (1/a_i) \leq n$, it holds that $\sum_{i=1}^n d_i = \sum_{i=1}^n (1/a_i) a_i d_i \leq n \max_{1 \leq i \leq n}\{a_i d_i\}$, for all $d_1, \ldots, d_n \geq 0$. We have

$$|\omega_{xi}| \leq \delta_i \max_{j \in \overline{\mathscr{N}}_i} \{a_{ij}|\tilde{x}_j|\}, \tag{2.101}$$

where $\delta_i = (N_i - 1)/N_i$, $\overline{\mathscr{N}}_i = \mathscr{N}_i \backslash \{0\}$, and $\sum_{j \in \overline{\mathscr{N}}_i} (1/a_{ij}) \leq N_i - 1$ if $0 \in \mathscr{N}_i$; $\delta_i = 1$, $\overline{\mathscr{N}}_i = \mathscr{N}_i$, and $\sum_{j \in \overline{\mathscr{N}}_i} (1/a_{ij}) \leq N_i$ if $0 \notin \mathscr{N}_i$.

Then, properties (2.99) and (2.100) imply

$$|\tilde{x}_i(t)| \leq \beta_{xi}(|[\tilde{x}_{i0}, \tilde{v}_{xi0}]^T|, t - T_O) + \delta_i \max_{j \in \overline{\mathscr{N}}_i} \{a_{ij}\|\tilde{x}_j\|_{[T_O,t]}\}, \tag{2.102}$$

$$|\tilde{v}_{xi}(t)| \leq |\tilde{v}_{xi0}| + \alpha_{xi}(\|\tilde{x}_i\|_{[T_O,t]} + \delta_i \max_{j \in \overline{\mathscr{N}}_i} \{a_{ij}\|\tilde{x}_j\|_{[T_O,t]}\}). \tag{2.103}$$

Define the follower sensing digraph $\mathscr{G}_f$ as a subgraph of $\mathscr{G}$. Digraph $\mathscr{G}_f$ has N vertices with indices $1, \ldots, N$ corresponding to the vertices with indices $1, \ldots, N$ of $\mathscr{G}$ and representing the follower robots. From the definitions of $\overline{\mathscr{N}}_i$ and $\mathscr{G}_f$, it can be observed that, for $i = 1, \ldots, N$, if $j \in \overline{\mathscr{N}}_i$, then there is a directed edge (j, i) from the jth vertex to the ith vertex in $\mathscr{G}_f$. Clearly, $\mathscr{G}_f$ represents the interconnection topology of the network composed of the $(\tilde{x}_i, \tilde{v}_{xi})$-systems (2.68).

Define $\mathscr{F}_0 = \{i \in \{1, \ldots, N\} : 0 \in \mathscr{N}_i\}$. Denote $\mathscr{C}_f$ as the set of all simple cycles of $\mathscr{G}_f$, and denote $\mathscr{C}_0 \subseteq \mathscr{C}_f$ as the set of all simple cycles through the vertices with indices belonging to $\mathscr{F}_0$.

For $i = 1, \ldots, N$, $j \in \overline{\mathscr{N}}_i$, we assign the positive value a_{ij} to edge (j, i) in $\mathscr{G}_f$. For a simple cycle $\mathscr{O} \in \mathscr{C}_f$, denote $A_\mathscr{O}$ as the product of the positive values assigned to the edges of the cycle.

Consider $\tilde{x}_i$ with $i = 1, \ldots, N$ as the outputs of the network composed of the $(\tilde{x}_i, \tilde{v}_{xi})$-systems (2.68). Using the IOS small-gain theorem for general nonlinear systems in [27, 28], $\tilde{x}_i(t)$ with $i = 1, \ldots, N$ converge to the origin if

$$A_\mathscr{O} \frac{N-1}{N} < 1 \quad \text{for } \mathscr{O} \in \mathscr{C}_0, \tag{2.104}$$

$$A_\mathscr{O} < 1 \quad \text{for } \mathscr{O} \in \mathscr{C}_f \backslash \mathscr{C}_0. \tag{2.105}$$

Note that $A_\mathscr{O}(N-1)/N < 1$ is equivalent to $A_\mathscr{O} < N/(N-1) = 1 + 1/(N-1)$.

If $\mathscr{G}$ has a spanning tree with vertex 0 as the root, then $\mathscr{G}_f$ has a spanning tree with the indices of the root vertices belonging to $\mathscr{F}_0$. According to Lemma 2.1, there exist positive constants a_{ij} such that both conditions (2.104) and (2.105) are satisfied. For system (2.68), with the convergence of each $\tilde{x}_i$ to the origin and the boundedness of $\tilde{u}_{xi}$, we can guarantee the convergence of $\tilde{v}_{xi}$ to the origin using Barbălat's lemma [29]. Similarly, we can prove the convergence of $\tilde{v}_{yi}$ to the origin. Using the definitions of $\tilde{v}_{xi}$ and $\tilde{v}_{yi}$, the convergence of θ_i to $\theta_0 + 2k\pi$ with $k \in \mathbb{Z}$ can be concluded. This ends the proof of Theorem 2.2.

2.4.5 Robustness to Relative Position Measurement Errors

Measurement errors can decrease the performance of a nonlinear control system. In this section, we discuss the robustness of our distributed formation controller in the presence of relative position measurement errors.

It can be observed that the initialization stage of the distributed control law defined in (2.83) and (2.84) is not affected by the position measurement errors. Also, condition (2.82) still holds for $t \geq T_{Oi}$ for $i = 1, \ldots, N$.

For the formation control stage, in the presence of relative position measurement errors, the z_{xi} and z_{yi} defined for the distributed control law (2.89) and (2.90) should be modified as

$$z_{xi} = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \left(x_i - x_j - (d_{xi} - d_{xj}) + \omega_{ij}^x \right) \tag{2.106}$$

$$z_{yi} = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \left(y_i - y_j - (d_{yi} - d_{yj}) + \omega_{ij}^y \right), \tag{2.107}$$

where N_i is the size of $\mathcal{N}_i$ and $\omega_{ij}^x, \omega_{ij}^y \in \mathbb{R}$ represent the relative position measurement errors corresponding to $(x_i - x_j)$ and $(y_i - y_j)$, respectively. Due to the boundedness of the designed ϕ_{xi} and ϕ_{yi} in (2.89) and (2.90), condition (2.82) is still satisfied in the presence of position measurement errors, which guarantees the validity of (2.65).

Here, we only consider each $\tilde{x}_i$-subsystem. The $\tilde{y}_i$-subsystems can be studied in the same way. By defining

$$\omega_{xi} = \frac{1}{N_i} \sum_{j \in \mathcal{N}_i} \left(\tilde{x}_j + \omega_{ij}^x \right), \tag{2.108}$$

we have $z_{xi} = \tilde{x}_i - \omega_{xi}$. With such definition, if the measurement errors ω_{ij}^x are piecewise continuous and bounded, then each $\tilde{x}_i$-subsystem still has the IOS and UO properties given by (2.99) and (2.100), respectively.

As in the discussion above (2.101), we have

$$
\begin{aligned}
|\omega_{xi}| &\leq \max\left\{\frac{\rho_i}{N_i}\sum_{j\in\mathcal{N}_i}(|\tilde{x}_j|), \frac{\rho_i'}{N_i}\sum_{j\in\mathcal{N}_i}(|\omega_{ij}^x|)\right\} \\
&:= \max\left\{\frac{\rho_i}{N_i}\sum_{j\in\mathcal{N}_i}(|\tilde{x}_j|), \omega_{xi}^e\right\} \\
&\leq \max_{j\in\overline{\mathcal{N}}_i}\left\{\delta_i a_{ij}|\tilde{x}_j|, \omega_{xi}^e\right\}, \qquad (2.109)
\end{aligned}
$$

where $\rho_i, \rho_i' > 0$ satisfying $1/\rho_i + 1/\rho_i' \leq 1$, and $\delta_i = \rho_i(N_i-1)/N_i$, $\overline{\mathcal{N}}_i = \mathcal{N}_i\backslash\{0\}$ and $\sum_{j\in\overline{\mathcal{N}}_i}(1/a_{ij}) \leq N_i - 1$ if $0 \in \mathcal{N}_i$; $\delta_i = \rho_i$, $\overline{\mathcal{N}}_i = \mathcal{N}_i$ and $\sum_{j\in\overline{\mathcal{N}}_i}(1/a_{ij}) \leq N_i$ if $0 \notin \mathcal{N}_i$.

In the existence of the relative position measurement errors, we can still guarantee the IOS of the closed-loop distributed system by using the cyclic-small-gain theorem. In this case, the cyclic-small-gain condition is as follows:

$$
A_{\mathscr{O}}\frac{\rho(N-1)}{N} < 1 \quad \text{for } \mathscr{O} \in \mathscr{C}_0, \qquad (2.110)
$$

$$
A_{\mathscr{O}}\rho < 1 \quad \text{for } \mathscr{O} \in \mathscr{C}_f\backslash\mathscr{C}_0, \qquad (2.111)
$$

where $\rho := \max_{i\in\{1,\ldots,N\}}\{\rho_i\}$ is larger than one according to $\frac{1}{\rho_i} + \frac{1}{\rho_i'} \leq 1$, and can be chosen to be very close to one. Lemma 2.1 can guarantee (2.110) and (2.111) if $\mathscr{G}$ has a spanning tree with vertex 0 as the root. Thus, the proposed distributed control law is robust with respect to relative position measurement errors.

2.4.6 A Numerical Example

Consider a group of 6 robots with indices $0, 1, \ldots, 5$. Notice that the robot with index 0 is the leader. The neighbor sets of the robots are defined as follows: $\mathcal{N}_1 = \{0, 5\}$, $\mathcal{N}_2 = \{1, 3\}$, $\mathcal{N}_3 = \{2, 5\}$, $\mathcal{N}_4 = \{3\}$, $\mathcal{N}_5 = \{4\}$.

By default, the values of all the variables in this simulation are in SI units. For convenience, we omit the units. The desired relative position of the follower robots are defined by $d_{x1} = -\sqrt{3}d/2$, $d_{x2} = -\sqrt{3}d/2$, $d_{x3} = 0$, $d_{x4} = \sqrt{3}d/2$, $d_{x5} = \sqrt{3}d/2$, $d_{y1} = -d/2$, $d_{y2} = -3d/2$, $d_{y3} = -2d$, $d_{y4} = -3d/2$, $d_{y5} = -d/2$ with $d = 30$. Figure 2.6 shows the position sensing graph of the formation control system. Clearly, the position sensing graph has a spanning tree with vertex 0 as the root.

It should be noted that the control law for each follower robot also uses the velocity and acceleration information of the leader robot, the communication topology of which is not shown in Fig. 2.6.

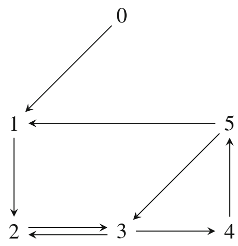

Fig. 2.6 The position sensing graph of the formation control system

The control inputs of the leader robot are $r_0(t) = 0.1\sin(0.4t)$ and $\omega_0(t) = 0.1\cos(0.2t)$. With such control inputs, the linear velocity v_0 with $v_0(0) = 3$ satisfies $\underline{v}_0 \le v_0(t) \le \overline{v}_0$ with $\underline{v}_0 = 3$ and $\overline{v}_0 = 3.5$.

Choose $\lambda_* = 0.45$ and $\lambda^* = 6.05$. The distributed control laws for the initialization stage are in the form of (2.83) and (2.84) with $\phi_{\theta i}(r) = \phi_{vi}(r) = -0.5(1 - \exp(-0.5r))/(1 + \exp(-0.5r))$ for $i = 1, \ldots, 5$. The distributed control laws for the formation control stage are in the form of (2.89) and (2.90) with $k_{xi} = k_{yi} = 2$ and $\phi_{xi}(r) = \phi_{yi}(r) = -1.8(1 - \exp(-0.5r))/(1 + \exp(-0.5r))$ for $i = 1, \ldots, 5$. With direct calculation, it can be verified that the designed $k_{xi}, k_{yi}, \phi_{xi}, \phi_{yi}$ satisfy (2.91) and (2.92). Also, $\phi_{xi}(r), \phi_{yi}(r) \in [-1.8, 1.8]$, for all $r \in \mathbb{R}$. With $\underline{v}_0 = 3$ and $\overline{v}_0 = 3.5$, the control laws can restrict the linear velocities of the follower robots to be in the range of $[3 - 1.8\sqrt{2}, 3.5 + 1.8\sqrt{2}] = [0.454, 6.046] \subset [\lambda_*, \lambda^*]$.

The initial states of the robots are chosen as

i	$(x_i(0), y_i(0))$	$v_i(0)$	$\theta_i(0)$
0	(0, 0)	3	$\pi/6$
1	(−40, 10)	4	π
2	(−20, −40)	3.5	$5\pi/6$
3	(5, −40)	2.5	0
4	(50, −10)	2	$-2\pi/3$
5	(50, 10)	3	0

The measurement errors are $\omega_{ij}^x(t) = 0.3(\cos(t + i\pi/6) + \cos(t/3 + i\pi/6) + \cos(t/5 + i\pi/6) + \cos(t/7 + i\pi/6))$ and $\omega_{ij}^y(t) = 0.3(\sin(t + i\pi/6) + \sin(t/3 + i\pi/6) + \sin(t/5 + i\pi/6) + \sin(t/7 + i\pi/6))$ for $i = 1, \ldots, N$, $j \in \mathcal{N}_i$.

The linear velocities and the angular velocities of the robots are shown in Fig. 2.7. The stage changes of the distributed controllers are shown in Fig. 2.8 with “0” representing initialization stage and “1” representing formation control stage. Figure 2.9 shows the trajectories of the robots. The simulation verifies the theoretical results.

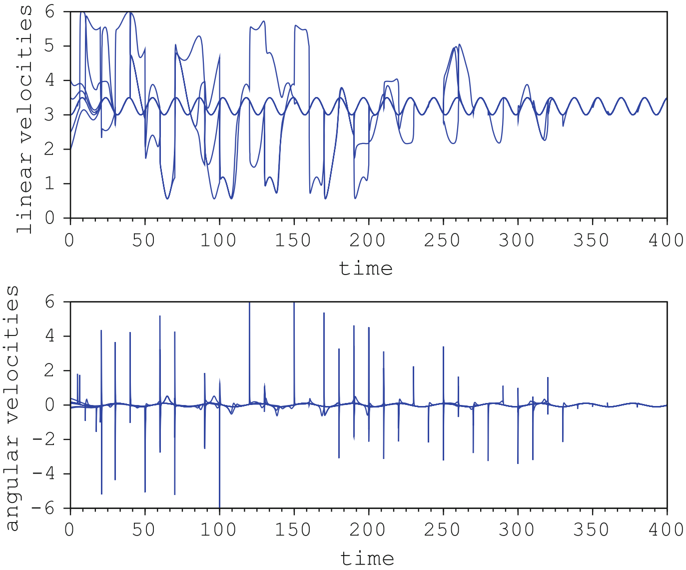

Fig. 2.7 The linear velocities and the angular velocities of the robots

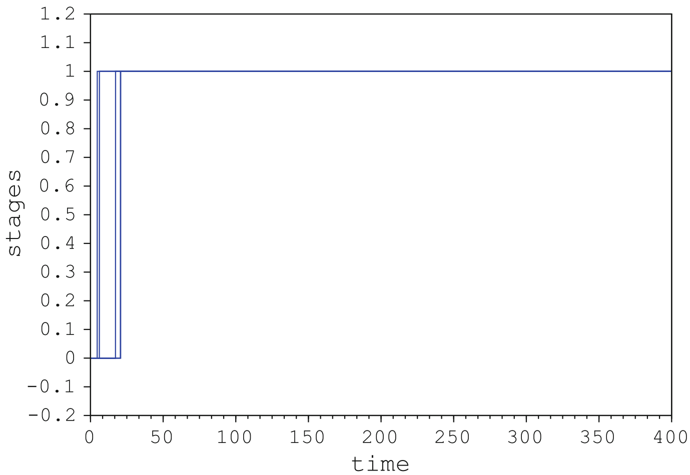

Fig. 2.8 The stages of the distributed controllers

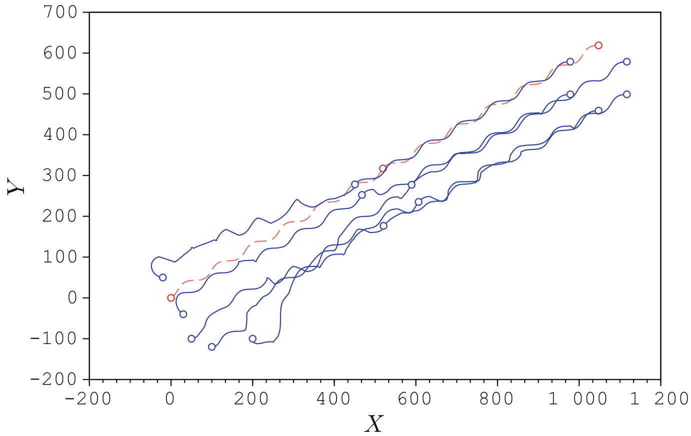

Fig. 2.9 The trajectories of the robots. The *dashed curve* represents the trajectory of the leader

2.5 Concluding Remarks

This chapter has presented cyclic-small-gain tools for distributed control of *nonlinear* multiagent systems.

In Sect. 2.3, with the proposed distributed observers and control laws, the outputs of the agents can be steered to within an arbitrarily small neighborhood of the desired agreement value under external disturbances. Asymptotic output agreement can be achieved if the system is disturbance-free. The robustness to bounded time delays of exchanged information can also be guaranteed. Section 2.3 only considers the case with time-invariant agreement value y_0. It is practically interesting to further study the distributed nonlinear control for agreement with a time-varying agreement value. Recent developments on the output-feedback tracking control of nonlinear systems (see, e.g., [22]) should be helpful for the research in this direction.

The distributed formation control law proposed in Sect. 2.4 uses relative position measurements without assuming a tree structure. For this purpose, the formation control problem is first transformed into a state agreement problem of double integrators with dynamic feedback linearization [6]. Then, a class of distributed nonlinear control laws is designed. With the proposed distributed nonlinear control law, the closed-loop system can be transformed into a network of IOS systems, and the achievement of the formation control objective can be guaranteed by using the cyclic-small-gain theorem. The special case in which there are only two robots and the desired relative positions are zero has been studied extensively in the past literature; see [8, 24] and the references therein.

By showing that a distributed control problem can be transformed into the stability/convergence problem of a dynamic network composed of IOS subsystems, Sects. 2.2–2.4 provide some partial answers to the question asked by Open Problem #5 in [28]: "Application of small-gain results for distributed feedback design of large-scale nonlinear systems." More discussions on the application of the cyclic-small-gain theorem to distributed control can be found in [36, 37, 39].

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References

1. Aguiar, A.P., Pascoal, A.M.: Coordinated path-following control for nonlinear systems with logic-based communication. In Proceedings of the 46th Conference on Decision and Control, pp. 1473–1479 (2007)
2. Arcak, M.: Passivity as a design tool for group coordination. IEEE Trans. Autom. Control **52**, 1380–1390 (2007)
3. Cao, M., Morse, A.S., Anderson, B.D.O.: Reaching a consensus in a dynamically changing environment: convergence rates, measurement delays, and asynchronous events. SIAM J. Control Optim. **47**, 601–623 (2008)
4. Cao, Y., Ren, W.: Distributed coordinated tracking with reduced interaction via a variable structure approach. IEEE Trans. Autom. Control **57**, 33–48 (2012)
5. Cortés, J., Martínez, S., Bullo, F.: Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. IEEE Trans. Autom. Control **51**, 1289–1298 (2006)
6. d'Andréa-Novel, B., Bastin, G., Campion, G.:Dynamic feedback linearization of nonholonomic wheeled mobile robots. In Proceedings of the 1992 IEEE International Conference on Robotics and Automation, pp. 2527–2532 (1992)
7. Desai, J.P., Ostrowski, J.P., Kumar, V.: Modeling and control of formations of nonholonomic mobile robots. IEEE Trans. Robot. Autom. **17**, 905–908 (2001)
8. Dixon, W.E., Dawson, D.M., Zergeroglu, E., Behal, A.: Nonlinear Control of Wheeled Mobile Robots. Springer, London (2001)
9. Do, K.D.: Formation tracking control of unicycle-type mobile robots with limited sensing ranges. IEEE Trans. Control Syst. Technol. **16**, 527–538 (2008)
10. Dong, W., Farrell, J.A.: Decentralized cooperative control of multiple nonholonomic dynamic systems with uncertainty. Automatica **45**, 706–710 (2009)
11. Everett, H.R.: Sensors for Mobile Robots: Theory and Application. A. K, Peters, Natick (1995)
12. Fax, J.A., Murray, R.M.: Information flow and cooperative control of vehicle formation. IEEE Trans. Autom. Control **49**, 1465–1476 (2004)
13. Filippov, A.F.: Differential Equations with Discontinuous Righthand Sides. Kluwer Academic Publishers, Dordrecht (1988)
14. Fliess, M., Lévine, J.L., Martin, P., Rouchon, P.: Flatness and defect of non-linear systems: introductory theory and examples. Int. J. Control **61**, 1327–1361 (1995)
15. Fridman, E., Bar Am, N.: Sampled-data distributed h_∞ control of a class of parabolic systems. In Proceedings of the 51st IEEE Conference on Decision and Control, pp. 7529–7534 (2012)
16. Gazi, V., Passino, K.M.: Stability analysis of swarms. IEEE Trans. Autom. Control **48**, 692–697 (2003)
17. Ghabcheloo, R., Aguiar, A.P., Pascoal, A., Silvestre, C., Kaminer, I., Hespanha, J.: Coordinated path-following in the presence of communication losses and time delays. SIAM J. Control Optim. **48**, 234–265 (2009)
18. Hirschhorn, J.: Kinematics and Dynamics of Plane Mechanisms. McGraw-Hill Book Company, New York (1962)
19. Hong, Y., Gao, L., Cheng, D., Hu, J.: Lyapunov-based approach to multiagent systems with switching jointly connected interconnection. IEEE Trans. Autom. Control **52**, 943–948 (2007)

20. Ihle, I.-A.F., Arcak, M., Fossen, T.I.: Passivity-based designs for synchronized path-following. Automatica **43**, 1508–1518 (2007)
21. Jadbabaie, A., Lin, J., Morse, A.: Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans. Autom. Control **48**, 988–1001 (2003)
22. Jiang, Z.P.: Decentralized and adaptive nonlinear tracking of large-scale systems via output feedback. IEEE Trans. Autom. Control **45**, 2122–2128 (2000)
23. Jiang, Z.P., Mareels, I.M.Y.: A small-gain control method for nonlinear cascade systems with dynamic uncertainties. IEEE Trans. Autom. Control **42**, 292–308 (1997)
24. Jiang, Z.P., Nijmeijer, H.: Tracking control of mobile robots: a case study in backstepping. Automatica **33**, 1393–1399 (1997)
25. Jiang, Z.P., Teel, A.R., Praly, L.: Small-gain theorem for ISS systems and applications. Math. Control Signals Syst. **7**, 95–120 (1994)
26. Jiang, Z.P., Wang, Y.: A converse Lyapunov theorem for discrete-time systems with disturbances. Syst. Control Lett. **45**, 49–58 (2002)
27. Jiang, Z.P., Wang, Y.: A generalization of the nonlinear small-gain theorem for large-scale complex systems. In Proceedings of the 7th World Congress on Intelligent Control and Automation, pp. 1188–1193 (2008)
28. Karafyllis, I., Jiang, Z.P.: Stability and Stabilization of Nonlinear Systems. Springer, London (2011)
29. Khalil, H.K.: Nonlinear Systems (3rd Edition). Prentice-Hall, New Jersey (2002)
30. Lan, Y., Yan, G., Lin, Z.: Synthesis of distributed control of coordinated path following. IEEE Trans. Autom. Control **56**, 1170–1175 (2011)
31. Li, Q., Jiang, Z.P.: Flocking control of multi-agent systems with application to nonholonomic multi-robots. Kybernetika **45**, 84–100 (2009)
32. Li, Z., Liu, X., Ren, W., Xie, L.: Consensus control of linear multi-agent systems with distributed adaptive protocols. In Proceedings of the 2012 American Control Conference, pp. 1573–1578 (2012)
33. Lin, Z., Francis, B., Maggiore, M.: Necessary and sufficient graphical conditions for formation control of unicycles. IEEE Trans. Autom. Control **50**, 121–127 (2005)
34. Lin, Z., Francis, B., Maggiore, M.: State agreement for continuous-time coupled nonlinear systems. SIAM J. Control Optim. **46**, 288–307 (2007)
35. Liu, T., Hill, D.J., Jiang, Z.P.: Lyapunov formulation of ISS cyclic-small-gain in continuous-time dynamical networks. Automatica **47**, 2088–2093 (2011)
36. Liu, T., Jiang, Z.P.: Distributed nonlinear control of mobile autonomous multi-agents. Automatica **50**, 1075–1086 (2014)
37. Liu, T., Jiang, Z.P.: Distributed control of nonlinear uncertain systems: a cyclic-small-gain approach. *Acta Automatica Sinica*, 2016. In press
38. Liu, T., Jiang, Z.P.: Distributed formation control of nonholonomic mobile robots without global position measurements. Automatica **49**, 592–600 (2013)
39. Liu, T., Jiang, Z.P.: Distributed output-feedback control of nonlinear multi-agent systems. IEEE Trans. Autom. Control **58**, 2912–2917 (2013)
40. Liu, T., Jiang, Z.P., Hil, D.J.: Nonlinear Control of Dynamic Networks. CRC Press, Boca Raton (2014)
41. Martensson, K., Rantzer, A.: A scalable method for continuous-time distributed control synthesis. In Proceedings of the 2012 American Control Conference, pp. 6308–6313 (2012)
42. Moreau, L.: Stability of multiagent systems with time-dependent communication links. IEEE Trans. Autom. Control **50**, 169–182 (2005)
43. Nersesov, S.G., Ghorbanian, P., Aghdam, A.G.: Stabilization of sets with application to multi-vehicle coordinated motion. Automatica **46**, 1419–1427 (2010)
44. Ögren, P., Egerstedt, M., Hu, X.: A control Lyapunov function approach to multiagent coordination. IEEE Trans. Robot. Autom. **18**, 847–851 (2002)
45. Ögren, P., Fiorelli, E., Leonard, N.: Cooperative control of mobile sensor networks: adaptive gradient climbing in a distributed network. IEEE Trans. Autom. Control **49**, 1292–1302 (2004)

46. Olfati-Saber, R.: Flocking for multi-agent dynamic systems: algorithms and theory. IEEE Trans. Autom. Control **51**, 401–420 (2006)
47. Qin, J., Zheng, W.X., Gao, H.: Consensus of multiple second-order vehicles with a time-varying reference signal under directed topology. Automatica **47**, 1983–1991 (2011)
48. Qu, Z., Wang, J., Hull, R.A.: Cooperative control of dynamical systems with application to autonomous vehicles. IEEE Trans. Autom. Control **53**, 894–911 (2008)
49. Sadowska, A., Kostic, D., van de Wouw, N., Huijberts, H., Nijmeijer, H.: Distributed formation control of unicycle robots. In Proceedings of the 2012 IEEE International Conference on Robotics and Automation, pp. 1564–1569 (2012)
50. Scardovi, L., Sepulchre, R.: Synchronization in networks of identical linear systems. Automatica **45**, 2557–2562 (2009)
51. Shi, G., Hong, Y.: Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies. Automatica **45**, 1165–1175 (2009)
52. Sontag, E.D.: Comments on integral variants of ISS. Syst. Control Lett. **34**, 93–100 (1998)
53. Stan, G.-B., Sepulchre, R.: Dissipativity and global analysis of oscillators. IEEE Trans. Autom. Control **52**, 256–270 (2007)
54. Su, Y., Huang, J.: Output regulation of a class of switched linear multi-agent systems: a distributed observer approach. In Proceedings of the 18th IFAC World Congress, pp. 4495–4500 (2011)
55. Tanner, H., Jadbabaie, A., Pappas, G.: Flocking in fixed and switching networks. IEEE Trans. Autom. Control **52**, 863–868 (2007)
56. Tanner, H.G., Pappas, G.J., Kummar, V.: Leader-to-formation stability. IEEE Trans. Robot. Autom. **20**, 443–455 (2004)
57. Wang, X., Hong, Y., Huang, J., Jiang, Z.P.: A distributed control approach to a robust output regulation problem for multi-agent systems. IEEE Trans. Autom. Control **55**, 2891–2895 (2010)
58. Wieland, P., Sepulchre, R., Allgöwer, F.: An internal model principle is necessary and sufficient for linear output synchronization. Automatica **47**, 1068–1074 (2011)
59. Yu, H., Antsaklis, J.: Distributed formation control of networked passive systems with event-driven communication. In Proceedings of the 51st IEEE Conference on Decision and Control, pp. 3292–3297 (2012)

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