# Chapter 2
# The Great Probability Debate

*All nature is but Art, unknown to thee;.*
*All Chance, Direction, which thou canst not see.*
– Alexander Pope, 1734.

## The Legacy of Leibniz

The eighteenth century had been in a very real sense the century of mathematics. Leibniz (1677; Fig. 2.1) reduced "the essence of things to numbers" and wanted "to establish a specific calculation to account for every truth." He entered the University of Leipzig in 1661, where he was "influenced by the mathematician and *philosophe* Erhard Weigel (1625–1699), whose work was focused on reconciling Aristotle with contemporary philosophers through a mathematical method of demonstration patterned on Euclid" (Pattanayak).

Leibniz himself praised the fourteenth-century English mathematician Richard Swyneshed for introducing mathematics into philosophy (North 2013: 49). "Leibniz believed that mathematical demonstration freed philosophy from its verbal limitations and embraced the field, believing its systematic coherence could unify the sciences" (Pattanayak). But one of the greatest proponents of Scottish Common Sense Realism, Dugald Stewart (1753–1828) at Edinburgh University, weighed in heavily against the legacy of Leibniz. Writing in 1821, Stewart regarded Leibniz as a prime obstacle to truth. Stewart (1829: 262) wrote of "the disregard manifested in his writings to the simple and obvious conclusions of experience and common sense." Stewart enlists the aid of the Roman historian Tacitus (56–117) with this quote from his tract *Agricola*: "But it was soon mellowed by reason and experience, and he retained from his learning that most difficult of lessons—moderation." He was not referring here to Leibniz but rather the English philosopher John Locke (1632–1704), whose philosophy directly influenced the Common Sense school of philosophy that in turn influenced Brougham in his 1802 polemic against the word 'asteroid.' Employing a classical allusion to the battle for supremacy between the giants and the gods of Olympus, and in language reminiscent of the high baroque period, Stewart excoriates Leibniz. Compare his employment of the term 'harmony'



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**Fig. 2.1** Gottfried Leibniz. His principle of pre-established harmony was inspired by the work of Kepler and his harmonic relations including the 'music of the spheres' (Bussotti 2015)

with that used by Capel Lofft in his tract about a missing planet between Mars and Jupiter, written just a year before the discovery of Ceres (See Chap. 3 for a discussion of Brougham and Lofft.).

> *How happily does this last expression* [of Tacitus] *characterize the temperate wisdom of Locke, when contrasted with that towering, but impotent ambition, which, in the Theories of Optimism and of Pre-established Harmony, seemed to realize the fabled revolt of the giants against the sovereignty of the gods!* (Stewart 1829: 262)

Leibniz would use these very theories in virtually every field he studied, including law.

> *Leibniz had learned from the law that probability is a relation between hypotheses and evidence. But he learned from the doctrine of chances that probabilities are a matter of physical propensities. Even now no philosopher has satisfactorily combined these two discoveries* (Hacking 1975: 139).

In his legal study Leibniz "assigned values of 0, 1, and ½ to conditions of laws that were impossible, necessary (absolute), or conditional" (Pattanayak). Leibniz's 1697 design of a medallion for Ernst August, duke of Brunswick (1629–1698), is the first example of the binary system used by Gauss in his anagram about Pallas (see Chap. 4). Leibniz first published its invention in 1703 (Fig. 2.2).

Mathematics became fashionable, and the European public took part. At the courts the reigning families were Leibniz's correspondents, and every prince had


**Fig. 2.2** Design of a medallion by Leibniz, incorporating the first use of the binary system

his court mathematician. Leibniz was the archivist of the House of Brunswick; his influence with King Frederick I led to the creation of the Prussian (Berlin) Society of Science in 1700 (reconstituted as the Academy in 1746; see Chap. 3). A century later Karl Wilhelm Ferdinand, the duke of Brunswick (1735–1806), perpetuated the ducal interest in science and mathematics of his ancestor Ernst August by being the patron of Gauss. “Through mathematics, so it was believed, the confusion in the world had been reduced to order; by it the Age of Enlightenment was inaugurated” (Friedenthal 1965: 398; Greene 2004).

For Descartes (Fig. 2.3), whose influence on Leibniz cannot be overestimated, mathematics was still clothed in religious garb, as evidenced by this letter he wrote on April 15, 1630, to the French philosopher and mathematician Marin Mersenne (1588–1648).

> *I would not allow myself in my physics to touch upon metaphysical questions, and particularly this one: that mathematical truths, which you call eternal, have been established by God and depend entirely on him, as does the rest of all creation* (Aczel 2005: 139).

But his reasoned claim in the *Meditations on First Philosophy* (1641) that God insures the truth of clear and distinct ideas, including geometry, has obscured the importance of his optical theory in his foundation of a mathematical science of nature. Descartes claims to have proved that we can know physical objects insofar as they are the objects of applied geometry, which he terms ‘pure mathematics.’ In his *Dioptrics* (1637), he shows that to formulate a science of nature, we can obtain reliable information about bodies by measuring them (Maull 1978). Kepler published two works on optics, both in the service of astronomy. Descartes was intimately acquainted with these works—he even referred to Kepler as his first master in optics (Hatfield 2015).

**Fig. 2.3** René Descartes, Louvre Museum

Descartes used his knowledge of optics to solve a problem Galileo Galilei (1564–1642) could not—why stars appear no larger through a telescope than to the naked eye. He discerned that stars are an intense dagger of light that agitate the nerves in the eye to generate an image that cannot be smaller—the *minimum visible* —which is the smallest the visual system can manage (George 2006). Descartes (1637) notes "that the stars, even though they appear quite small, nevertheless appear very much larger than their extreme distance should cause them to appear." This same reasoning applies to the asteroids, a word which itself means star-like, and this should be kept in mind when reading Herschel's investigations of the physical size of Ceres and Pallas in Chap. 10. The mathematical work of both Descartes and Leibniz had roots much deeper in the past than Kepler or Galileo.

Although it presaged a mathematical way of looking at nature, it was Newton who wrote the mathematical laws that ushered in the modern scientific age.

> *Like Descartes before him, Leibniz was attracted to the work of the thirteenth- century mystic Ramon Lull* (1235–1316; Fig. 2.4). *Lull's* Great Art of combinations*, which used rotating wheels within wheels to create a large number of combinations of concepts coded by letters on these wheels, took a new and deeper meaning in Leibniz's eyes. Leibniz saw in these efforts more than just mystical play, but rather a mathematical attempt to study combinations. Leibniz developed these same concepts into a mathematical theory, and published it in a 1666 treatise titled* Dissertatio de Arte Combinatoria (Fig. 2.5)*. This work developed the mathematical foundations of combinations* (Aczel 2005: 209).

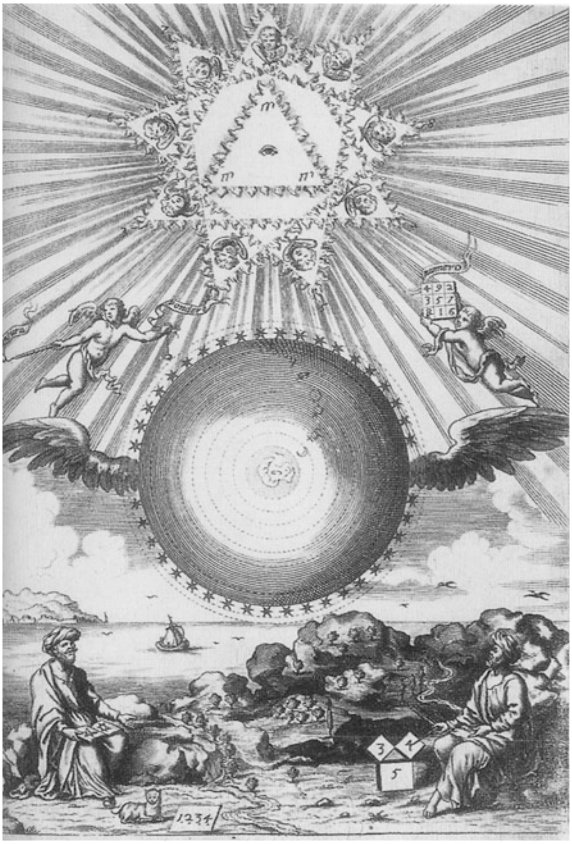


**Fig. 2.4** This engraving of the planets from Athansius Kircher (1665), just 22 years before Newton published his *Principia*, is suffused with mysticism. Above is the Perpetuum Mobile of the Solar System, an enneagram from Lull's second combinatory figure that ascends into the super-celestial sphere. Although it presaged a mathematical way of looking at nature, it was Newton who wrote the mathematical laws that ushered in the modern scientific age (courtesy Google Books)

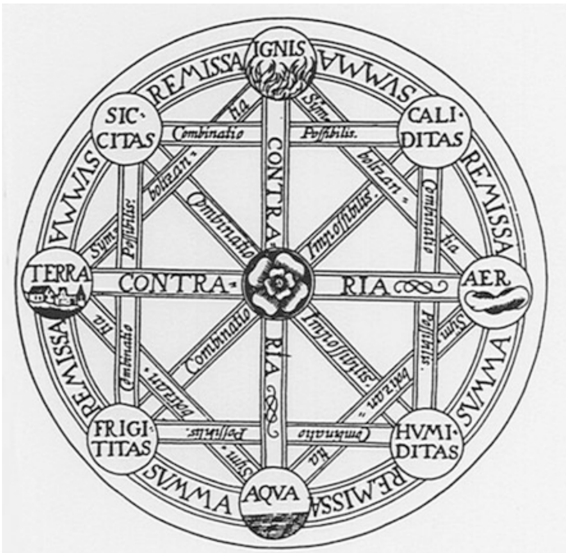


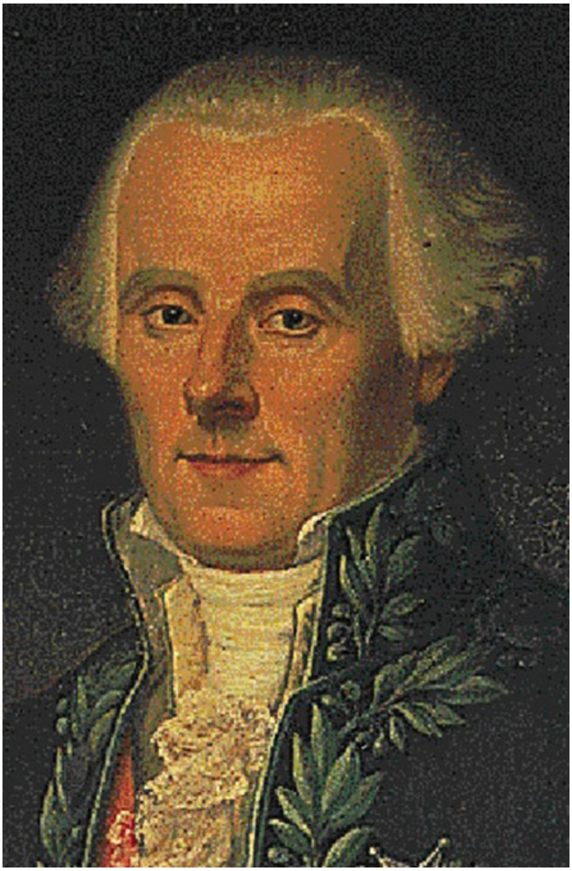

**Fig. 2.5** A diagram of the *Ars combinatorial* by Leibniz is based on four elements (earth, air, fire, water), four material states (heat, cold, dryness, moistness), and two possible rotational directions. For Leibniz, the ars combinatoria was nothing less than the "key to all the sciences." (Antognazza 2009: 63)

Lull's system represents the beginnings of formal logic, the foundation of modern science. This in turn led to work in probability theory. By the nineteenth century the role of mathematics had been transformed and largely shorn of superstition. "All the mathematical sciences are founded on relations between physical laws and laws of numbers, so that the aim of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers." Maxwell (1856) When Laplace published his *Philosophical Essay on Probability* in 1814, he had firmly established the theory of probabilities as a branch of mathematics and science with none of the dissolute connotations of the doctrine of chances, as we shall see in the next chapter (Plackett 1989).

## The Most Delicate of Mathematical Theories

Throughout the first two decades of the nineteenth century there was a great clash between Wilhelm Olbers' planetary explosion hypothesis and Laplace's application of probability to physical astronomy (Fig. 2.6). How this intertwined with the

**Fig. 2.6** Pierre-Simon Laplace



mathematical work of Carl Gauss in relation to the early study of the asteroids is explored in this chapter.

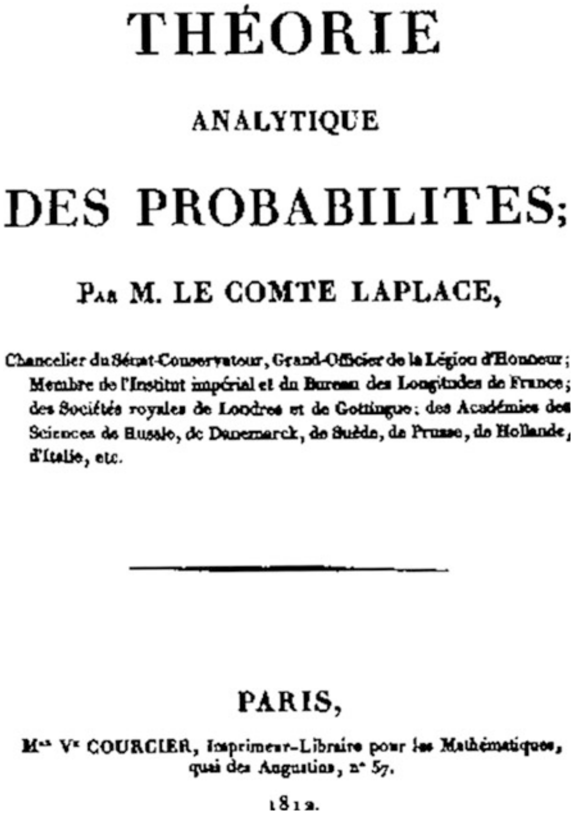
In addition to his study of the Solar System—being the first to offer orbital elements for the missing planet—Jacob Bernoulli was also a crucial founding figure in the mathematics of probability. In the 1680s he showed it was possible to "extend the application of probability theory from games of chance to other fields where stable relative frequencies exist." (Hald 2004: 11) When this finally got published in 1713, "probability came before the public with a brilliant portent of all the things we know about it now: its mathematical profundity, its unbounded practical applications, its squirming duality and its constant invitation for philosophizing." (Hacking 1975: 143)

Laplace was not only the first use of the trigonometric series in celestial mechanics, but went further by linking his work on infinite series with problems of chance, a topic earlier studied by Abraham de Moivre (1667–1754). One of the great pioneers of classical probability and a disciple of both Newton and Halley, de Moivre wrote: "Another Method I have made use of, is that of Infinite Series, which in many cases will solve the Problems of Chance more naturally than Combinations." (Moivre 1756: vii) The unsuccessful application of infinite series to the orbital problem posed by Ceres was explored in *Early Investigations of Ceres and the Discovery of Pallas*. In the Leibniz-Newton dispute over the invention of calculus, Moivre sided with Newton while Bernoulli sided with Leibniz. The contemporary reach of Moivre's work in probability, including his two major books on the subject, *The Doctrine of Chances* (1718) and *Miscellanea Analytica* (1730), has been revealed by Bellhouse et al. (2009).

Laplace's interest in probability can be traced directly to Euler of the Prussian Academy of Science. His colleague Lagrange, whose 1776 work in probability theory enhanced de Moivre's efforts, was also a great admirer of Euler. Lagrange accepted Euler's ideas, believing that "algebraic operations apply to infinite series, but other mathematicians soon began to doubt the rigor of Lagrange's algebraic analysis. Among these were Abel Burja (1752–1816) in Germany and Jan Sniadecki (1756–1830)" in Poland, one of the most avid observers of the asteroids and a regular contributor to Zach's journal *Monthly Correspondence* (Synowiec 2000: 14). (Letters to Sniadecki from Zach, and his observations of Ceres, are included in this book.) Lagrange later used mathematics to study Olbers' planetary explosion hypothesis to explain the origin of the asteroids, a topic that also engaged the mind of Laplace, as shown later in this chapter, but it was all grounded in work that began in the 1770s.

> *Soon after gaining election to the* Académie des Sciences *in 1773, Laplace became seriously interested in mathematical astronomy, which was to become his dominant concern. By then two major areas of activity were evident: in celestial mechanics, the fine details of the motions of the heavenly bodies as analyzed using especially Newton's laws and allowing for perturbations; and in planetary mechanics the analysis of their shapes* (Grattan-Guinness 2005: 243).

The calculation of probabilities must have appealed to Napoleon Bonaparte (1769–1821), to whom Laplace dedicated the first edition of his work in 1812


THÉORIE

ANALYTIQUE

DES PROBABILITES;

PAR M. LE COMTE LAPLACE,

Chancelier du Sénat-Conservateur, Grand-Officier de la Légion d'Honneur; Membre de l'Institut impérial et du Bureau des Longitudes de France; des Sociétés royales de Londres et de Gottingue; des Académies des Sciences de Russie, de Danemarck, de Suède, de Prusse, de Hollande, d'Italie, etc.

PARIS,

Mme Ve COURCIER, Imprimeur-Libraire pour les Mathématiques, quai des Augustins, n° 57.

1812.

**Fig. 2.7** Title page of Laplace's 1812 book on probability

(Fig. 2.7). Here is a letter dated August 12, 1812 from Witepsk in the Russian Empire, which Napoleon wrote to Laplace when he received the treatise:

> *There was a time where I would have read with interest your* Treatise on the Calculation of Probabilities. *Today I must limit myself to conveying to you the satisfaction I feel each time I see you give new works which perfect and extend the first of the Sciences, and contribute to the illustriousness of the Nation. The advancement, the perfecting of Mathematics are linked to the prosperity of the State.*

## The Historical Development of Probability

In *Prior Analytics* (70a1–10), "Aristotle defines *eikos* as that which is generally approved" (Hoffman 2008: 7). This is roughly "equivalent to the pre-Enlightenment sense of the Latin word *probabilitas,*" which was used in a sense that was true to its root verb *probo*, which means "to approve or commend." Thus anything worthy of approval could be called *probabilitas*, and this use persisted into the eighteenth century. For example, the Scottish philosopher David Hume (1711–1776):

> *...was aware of the connection between mathematical reasoning and reasoning from probabilities, but evidently unaware of the possibility of expressing probabilities mathematically. He preferred to conceive probability in terms of the 'agreeing', or concurrence, of images together to form a 'superior' idea* (Raynor 1964: 104).

The modern usage of probability arose in tandem with inductive logic. "It is connected with the degree of belief warranted by evidence, and it is connected with the tendency, displayed by some chance devices, to produce relative frequencies." (Hacking 1975: 1) Dugald Stewart made explicit what demarcated the eighteenth century from earlier times when he wrote:

> [the] *growing disposition to weigh scrupulously the* probability *of alleged* facts *against the faith due to the testimonies brought to attest them, and, even in some cases, against the apparent evidence of our own senses, enters largely and essentially into the composition of that philosophical* spirit *or temper, which so strongly distinguishes the eighteenth century from all those which preceded it* (Stewart 1829: 263).

This new divide that existed between geometrical proofs and the sort of subjects treated by moralists was highlighted by chemist and rhetorician Joseph Priestley (1733–1804; Fig. 2.8) Priestley's fame today rests chiefly in his research as a chemist, but his views raised the ire of Robison, who denounced both him and Laplace as enemies of humanity. For him, Priestley prepared the "minds of his

**Fig. 2.8** Joseph Priestley painting by Ellen Sharples. National Portrait Gallery, London

readers for Atheism by his theory of the mind." (Heilbron 2015: 104) Not surprisingly, Priestley had no use for Hume, Reid, or Common sense philosophy.

> *The geometrician, when he hath laid down his proposition, proceeds, by a series of steps which terminate in a single proof, to show the agreement or coincidence of the terms of it… But in this the moralist and divine must content themselves with following them at a great and very humble distance. As the subjects they treat of are not always capable of strict* demonstration, *they are obliged to have recourse to a variety of arguments, each of which may add something to* probability *(which in its own nature admits of degrees) till the united strength of them all be sufficient to determine the assent* (Priestley 1777: 50).

Laplace explained what he meant by probability and its application to physical astronomy in 1773 (it was published in 1776). Even though it was written nearly 30 years before the discovery of Ceres and Pallas, it applies to them as well. Even though Laplace is noted for not explicitly introducing God into his scientific works, he did in fact preface his remarks about physical astronomy and probability with the concept of a universal Intelligence.

> *The present state of the system of nature is evidently the result of what it was in the preceding instant, and if we imagine an Intelligence who, for a given moment, encompasses all the relations of beings in this Universe, it will also be able to determine for any instant of the past or future their respective positions, motions, and generally their disposition* (Hahn 2005: 56).

I suggest Laplace derived this in part from the work of Johann Heinrich Samuel Formey (1711–1797; Fig. 2.9), who wrote "Is it, in itself, more probable, that

**Fig. 2.9** Johann Heinrich Samuel Formey

things arranged with wonderful art, and constantly relative to evident and useful ends, are such, because they are such, than because an infinite Intelligence has presided over the arrangement of them?" (Formey 1748: 10; 1759: 279). Like Formey, Laplace capitalizes Intelligence, thus giving it the meaning 'infinite intelligence.' Formey (1759: 285) derives his interpretation of what constitutes "the logic of probabilities" directly from this. If, he writes:

> *...the enemies of Religion act like a man...who buys several tickets to a lottery in which there are 9999 white [draws] against one black one (these proportions are significantly lower than the dangers to which irreligion casts men), ...I ask that they save us from their ironic remarks and their endlessly repeated witticisms.*

How he determined the proportion of the dangers of irreligion are left unstated. So, who was Formey? None other than a professor at the French College in Berlin, perpetual secretary of the Berlin Academy from 1748, and a contributor to the French *Encyclopedie*. And with a reputed 17,000 letters flowing from his pen, he was a major figure in the Republic of Letters (Charrier-Vozel 2006: 118). Laplace takes his cue from Formey, but arrives at a very different conclusion. Instead of deriving probability and the nature of Intelligence from religion, Laplace turns to science, and finds at least a partial answer in physical astronomy. In this Laplace was directly following the views of the Marquis de Condorcet (1743–1794), who was writing in response to issues raised about "God's power to intervene in the natural laws God was credited with establishing" (Hahn, 2005: 51). The matter came to the fore in a 1756 prize question posed by none other than the Berlin Academy under the influence of Formey. Here is Condorcet (1768: 5):

> *One could conceive [of the universe] at any instant to be the consequence of the initial arrangement of matter in a particular order and left to its own devices. In such a case, an Intelligence knowing the state of all phenomena at a given instant, the laws to which matter is subjected, and their consequences at the end of any given time would have a perfect knowledge of the 'System of the World.'*

With the intellectual antecedents of the pillar of determinism in the Temple of Natural Philosophy now firmly in view, we can understand the continuation of Laplace's reasoning from his 1773 paper:

> *Physical astronomy, that subject of all our knowledge most worthy of the human intellect, offers an idea – albeit imperfect – of what such an Intelligence would be. The simplicity of the laws that move celestial bodies, and the relationship of their masses and their distances permit us to follow their motion up to a certain point with the use of calculus* [Analyze]. *To determine the state of the system of these large bodies in past or future centuries, it is enough that observations provide the mathematician with their position and speed at a given moment. Man derives this capacity from the power of the* [mathematical] *instrument he uses and the small number of parameters they include. But our ignorance of the various causes that produce these events as well as their complexities taken together with imperfections in the calculus, prevent him from making assertions about most phenomena with the same assurance. For him therefore there are things that are uncertain, and some that are more or less probable. Given the impossibility of* [total] *knowledge, man has compensated by determining their different degrees of likelihood; so that we owe to the frailty of the human mind one of the most delicate and ingenious of mathematical theories, namely the science of chance or probabilities* (quoted in Hahn 2005: 56).

He expanded on this in 1814 in the *Philosophical Essay on Probability*:

> *The regularity which astronomy shows us in the movements of the comets doubtless occurs in all phenomena. The curve described by a simple molecule of air or water vapor is regulated in a manner just as certain as the orbits of the planets; the only difference between these is that introduced by our ignorance. Probability is relative, in part to this ignorance, and in part to our knowledge* (Laplace 1814; 2007: 6).

As we saw in *Discovery of the First Asteroid, Ceres*, Laplace couched his analysis of the orbits of the planets and asteroids in terms of chance. Because the world is determined, Laplace implies, there can be no probabilities in things. "Indeed the probability isn't useful when we have at our disposal certain knowledge (p = 1), or when, on the contrary, we are in the presence of events of which the absolute impossibility is known (p = 0)." (Fortino 2002: 54) The theory of chances, then, "consists in reducing all events of the same kind to a certain number of equally possible cases, that is to say, those such that we are equally undecided about their existence." (Laplace 1814; 2007: 6) This concept of chance had a wide-ranging

**Fig. 2.10** Constantin-Francois Volney

influence (Daston 1992). We have encountered the 'doctrine of chances' from the pen of William Godwin in *Early Investigations of Ceres and the Discovery of Pallas*, but it had already been incorporated into French thought decades earlier. Witness this passage invoking harmony from Constantin-Francois de Chasseboeuf, count de Volney (1757–1820; Fig. 2.10), with his italics in Roman type:

> *When the* hidden power *which* animates *the universe, formed the globe which man inhabits, he implanted in the beings composing it,* essential properties *which became the* law *of their individual motion, the* bond *of their reciprocal relations, the cause of the harmony of the whole; he thereby established a regular order of causes and effects, of principles and consequences, which,* under an appearance of chance [sous une apparence de hasard], *governs the universe, and maintains the equilibrium of the world* (Volney 1791: 34).

## Equipossibility

The term *philosophe* will be employed several times in the pages that follow, so the question as to who a *philosophe* is must first be addressed. By the mid-eighteenth century the academies in France were in decline:

> *Internal instability left the institutions of French erudition vulnerable to external challenge, and that challenge came from the* philosophes*. To the* philosophes*, the relationship between erudition and authority was an obstacle to free inquiry and the progress of the sciences. Scholars in their own right, though excluded from the privileges of academic membership at least initially, the* philosophes *made extensive use of erudite scholarship even as they advanced a new intellectual and historical agenda exemplified by the historical essays of Voltaire and Montesquieu and ultimately by the* Encyclopédie *that appeared serially in the 1760s* (Carhart 2007: 155).

This great compendium of knowledge, the *Encyclopedie, Ou Dictionanaire Raisonne des Sciences, des Arts et des Metiers*, ran to 17 volumes and was published between 1751 and 1765. In its volume covering probability, dated 1765, we find a judicious survey of kinds of probability, but the only measurable probability is founded on the "equal possibility of several events." The author says of equipossibility that it "is to be employed when we suppose the several cases to be equally possible, and in effect it is only a supposition relative to our bounded knowledge, that we say, for example, that all the points on the die can occur equally."

As explained by Hacking (1975: 129), this concept was foreign to English writers, as they felt no need to equivocate by defining probability as possibility. "The French clearly perceived the need for both sides of probability but could not face up to it, thus taking refuge in the ambiguous concept of 'equipossibility.'" This reached its ultimate expression in Laplace, who defined probability in terms of equally possible cases. Lambert (1728–1777; 1761), who postulated the existence of a planet between Mars and Jupiter, also wrote on the matter. He maintained 'possibility' "in its old physical sense, determined by the laws of individual objects. In this early period equipossibility is not a feature of Laplace's work" either, but this changed as time went on.

**Fig. 2.11** Eugene Wigner

Laplace discerned that in astronomy observations of planetary positions and motions are distributed in a regular fashion, according to the law of error. "Error was the great enemy of enlightenment, the loathsome infamy the *philosophes* had journeyed forth to slay. In that battle, mathematical science was their most fearsome weapon." (Alder 2002: 213). Laplace concluded that astronomy "proceeded from the mathematical analysis of statistically regular empirical findings to an understanding of the constant causes that determined the phenomena of nature." He believed that probability theory provided grounds for concluding that constant causes were at work when, as in celestial mechanics, the multiplication of observations could be shown to lead to the elimination of 'strange effects.' (Buck 1981: 24) How does the Laplacian concept of probability theory mesh with our modern understanding of physics and mathematics? Nobel laureate Eugene Wigner (1902–1995; Fig. 2.11) believes the insights of Laplace are fundamentally correct although not as precise as he would have wished:

> *The laws of nature are all conditional statements and they relate only to a very small part of our knowledge of the world. Thus, classical mechanics, which is the best known prototype of a physical theory, gives the second derivatives of the positional coordinates of all bodies, on the basis of the knowledge of the positions,* etc., *of these bodies. It gives no information on the existence, the present positions, or velocities of these bodies. It should be mentioned, for the sake of accuracy, that we discovered about thirty years ago that even the conditional statements cannot be entirely precise: that the conditional statements are*

> *probability laws which enable us only to place intelligent bets on future properties of the inanimate world, based on the knowledge of the present state. They do not allow us to make categorical statements, not even categorical statements conditional on the present state of the world* (Wigner 1960: 5).

A related issue is unpredictability. "What is important about the systematic unpredictability of radical conceptual innovation is of course the consequent unpredictability of the future of science." (MacIntyre 1984) It is just such a radical conceptual innovation that Gauss produced to calculate the orbit of Ceres, an innovation that transformed the nascent science of celestial mechanics. Probability is also a factor in human genius itself. These words of the poet W.H. Auden (1907–1973) from 1940 are certainly applicable to Carl Gauss: "Any great man is a miracle of improbability." (Mendelson 2002: 56)

## Analogy and Probability

The concept of analogy is mentioned several times through this book. Olbers and Oriani use it (see Chap. 11), and it was invoked by Bode himself when writing of his 'law' of planetary distances. Analogy was also employed by Kepler throughout his scientific career (Gentner 1997). The use of analogy in the seventeenth century was expressed by Locke: "In things which Sense cannot discover, Analogy is the great Rule of Probability." (Schuurman 2003: 49) Kepler's use of analogy in the *Mysterium Cosmographicum* to relate the distances of the planets with geometrical entities (illustrated in Chap. 6) is something he later described as "admirable rather than happy or legitimate." (Kepler 1611) The importance of geometry to the study of the asteroids has been detailed in Chap. 1, and it was also at the heart of Kepler's agenda, where he uses the clockwork analogy (Peterson 1993) to explain the workings of the cosmos:

> *I am much occupied with the investigation of the physical causes. My aim in this is to show that the celestial machine is to be likened not to a divine organism but rather to a clockwork...Moreover, I show how this physical conception is to be presented through calculation and geometry* (Kepler 1605).

Kepler here uses analogy in the Humean sense, but this raises a peril of dire consequences. Hume (Fig. 2.12) says:

> *In the probability deriv'd from analogy, 'tis the resemblance only, which is affected. Without some degree of resemblance, as well as union, 'tis impossible there can be any reasoning: but as this resemblance admits of many different degrees, the reasoning becomes proportionally more or less firm and certain* (Garrett 1997: 144).

It is in the varying degrees admitted by the principle of resemblance that the problem arises. In the days of Newtonian physics and Laplacian certainty, all was well. But as celestial mechanics admitted chaotic orbits, and physics replaced certainty with uncertainty in the twentieth century, the analogies employed by Kepler and the asteroid researchers of the early nineteenth century became not only less firm but untenable.

**Fig. 2.12** David Hume painting by Allan Ramsey. National Galleries Scotland

In 1804 Olbers also raises the concept of doubt about the 'paradoxical results' found by Gauss in his study of Pallas (see Chap. 11). Here he follows the precept laid down eight years earlier by Pierre Samuel du Pont de Nemours (1739–1817) in his book *Philosophy of the Universe*. He urged one "not to definitively judge the planets and the suns. Let us doubt: it is the role of a man and a *philosophe*." (Du Pont 1796: 248)

## The Origin of the Asteroids and the Twilight of Probability

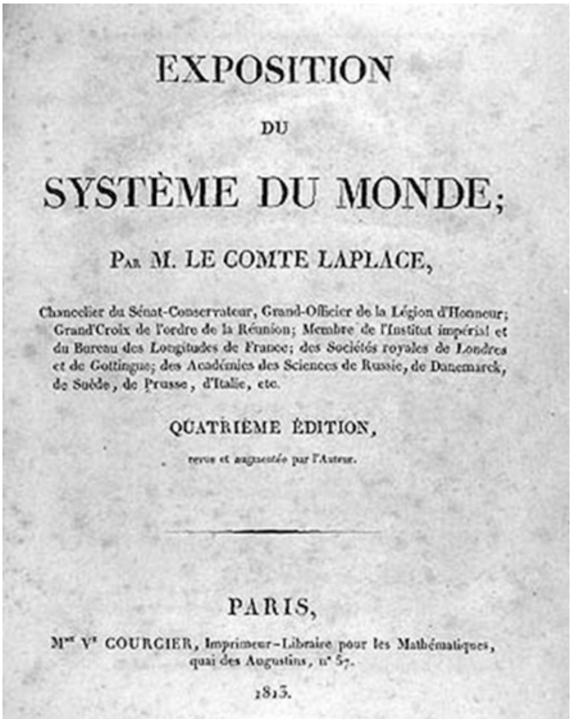
Leibniz observed that many judgments must be made in the "twilight of probability." (Daston 1988: 45) Nowhere was this more apparent than the discussion and debate between Olbers and Laplace about how the asteroids came to be. To put this in context, a consideration of contemporary opinion is in order. The following extract from Zach's *Monthly Correspondence* in July 1802 (pg. 71) sets the stage with the views of Olbers in that discovery year of Pallas.

> *In the previous issue (p. 598), we communicated Dr. Olbers' thought regarding both new planets to our readers, in which he is inclined to regard them as rubble of one single planet. This illustrious observer expresses himself in one of his later letters in the following way: 'Along with me, you are probably surprised at the curious position of the path of Pallas against that of Ceres. They are not interconnected like the links of a chain as was*

> *first suspected; rather, the path of Pallas is stuck in the path of Ceres like a ring in the other. With the descending node of the path of Pallas on the path of Ceres, both paths come very close together. Whether an actual intersection really occurs or has ever occurred before the perturbations move the position of both against the other is not clear at present since we do not yet know their paths well enough.*
>
> *From Dr. Gauss's 2nd path for Pallas and 7th for Ceres, I have found the distance of both paths in the descending node of Pallas to be = 0.07001. But if I combine the 2nd path of Pallas with Burckhardt's ellipse for Ceres, then the distance becomes only 0.06567. A small change in the elements of the path of Pallas, which is very possible, can continue to greatly reduce this distance. In this, the following question arises in me: whether Ceres and Pallas could be pieces and rubble of a formerly large planet destroyed either through natural forces working within it or through the collision of a comet? Which points to this suspicion which I offer as nothing more than a question for further investigation.*
>
> *That Ceres and Pallas are both of very changeable luminosity – this I explain in that both planetary fragments are not round, but rather of a very irregular shape. This idea has the advantage over other hypotheses that it will soon be able to be verified. If it is true, we will find yet more rubble of this destroyed planet and this is made all the more easy since all this rubble, which describes an elliptical path around the Sun (much may have flown off in parabolas and hyperbolas)* must cross the descending node of the path of Pallas on the path of Ceres. *In general, all of these* suspected *planetary fragments have the same nodal line on the plane of the paths of Ceres and Pallas.' Again in a more recent letter, Dr. Olbers repeats this opinion and writes: 'The same orbital time of Pallas and Ceres, the positions of these paths with each other, the proximity of same in the descending node of the path of Pallas on that of Ceres; this makes it all the more probable for me that both belong together, and again and again, I return to the conjecture which I have expressed to you, that both are perhaps pieces and rubble of a former large planet.'*

From the comet collision idea given to him by the German jurist Ferdinand von Ende (1760–1816), Olbers came to believe that the asteroids were created in an explosion (Cunningham and Orchiston 2013). Laplace disagreed, and this sparked a heated debate not only about Laplacian cosmology and the origin of the Solar System, but about his application of probability theory to the whole question. The opening shot was sounded in a letter Olbers received from Burckhardt in December 1804. This is how Olbers (1804) related the matter to Gauss:

> *Mr. De Laplace has communicated an objection to me* via *Burckhardt against my hypothesis on the origin of asteroids. It rests on the same orbital period as Ceres and Pallas and on the improbability that 2 fragments of a shattered planet should retain exactly the same velocity after the collision. Laplace, Burckhardt continues, explains the fact that several planets exist whose orbits have almost equal major axes very easily on the basis of his hypothesis on planet formation, namely: the planets formed as a result of the cooling of an immense atmosphere which enveloped the Sun.*
>
> The letter then continues with a direct quote from Laplace:
>
> *The complete volume of this atmosphere, located between Jupiter and Mars, had not been able to coalesce into a single planet for whatever reasons; it coalesced into several planets, all of which retained the same semi-major axis, namely, the distance of the Sun which the volume, had it become a solid body, would have had.*


EXPOSITION

DU

SYSTÈME DU MONDE;

PAR M. LE COMTE LAPLACE,

Chancelier du Sénat-Conservateur, Grand-Officier de la Légion d'Honneur; Grand'Croix de l'ordre de la Réunion; Membre de l'Institut impérial et du Bureau des Longitudes de France; des Sociétés royales de Londres et de Gottingue; des Académies des Sciences de Russie, de Danemarck, de Suède, de Prusse, d'Italie, etc.

QUATRIÈME ÉDITION,

revue et augmentée par l'Auteur.

PARIS,

Mme Ve COURCIER, Imprimeur-Libraire pour les Mathématiques, quai des Augustins, n° 57.

1813.

**Fig. 2.13** Title page of Laplace's book *Exposition du Système du Monde*

Laplace went public with his objections in his 1808 book *Exposition du Système du Monde*, which was a nonmathematical version of his *Celestial mechanics* (Fig. 2.13). He applied his probability theory to the orbits of the four asteroids, estimating the odds at 4 billion to one that such a planetary arrangement is not the effect of chance. Laplace addressed the issues raised by the asteroids in his 1808 edition, and expanded on their place in the formation of the Solar System in his 1813 edition:

> *There are many discoveries still to be made in our own system. The planet Uranus, and its newly recognized satellites, give grounds to suppose that other planets, as yet unobserved, exist. It has even been conjectured that there must be one between Mars and Jupiter, in order to satisfy the double progression, which obtains (g)* [the force of gravity] *very nearly, between the intervals of planetary orbits, to that of Mercury. This conjecture has been confirmed by the recent discovery of four small planets whose respective distance from the Sun is not much different than the distance assigned by this double progression to a planet intermediate between Jupiter and Mars. Jupiter's action on these planets, increased by the magnitude of the eccentricities and inclinations of the intersecting orbits, produces considerable inequalities in their motions, which will shed new light on the theory of celestial attractions, and will give rise to perfecting it even more. The arbitrary elements of this theory, and the convergence of its approximations, depend on the precision of observations and the progress of analysis, and this should thereby acquire every day more and more accuracy* (Laplace 1808: 396-397 French; 1830: 339-340 English).

> *If all the particles of a ring of vapours continued to condense without separating, they would at length constitute a solid or a liquid ring. But the regularity which this formation requires in all the parts of the ring, and in their cooling, ought to make this phenomenon very rare. Thus the Solar System presents but one example of it – namely, the rings of Saturn. In nearly every case the rings of vapor ought to be divided into several masses, which, being moved with velocities which differ little from each other, should continue to revolve at the same distance around the sun. These masses should assume a spheroidal form, with a rotary motion in the direction of their revolution, because their inferior particles have a less velocity than the superior; they have therefore constituted so many planets in a state of vapor. But if one of them was sufficiently powerful, to unite successively by its attraction, all the others around its center, the ring of vapors would thus be transformed into a single spheroidical mass, circulating about the Sun, with a motion of rotation in the same direction with that of revolution. This latter case has been the most common; however, the Solar System affords us an instance of the first case, in the four small planets which revolve between Jupiter and Mars, at least unless we suppose with Olbers, that they originally formed a single planet which a mighty explosion broke up into several portions each moving at different velocities* (Laplace 1813: 432-433 in French; [see reference for Laplace, 1808] 1830: 360, Note 7 in English).

Thus we see that Olbers disagreed with Laplace on the use of probability theory to explain the orbits of the asteroids, while Laplace disagreed with Olbers on the explosion theory to explain those very same orbits. In 1812 Olbers visited Paris, and had several personal discussions with Laplace about the origin of the asteroids. Olbers (1812a) must have been very persuasive, as witnessed in a letter to Bessel.

> *Laplace gave me his recently published* Théorie analytique des probabilités. *Until now I have only been able to look randomly at the embodied treasure. To read the book thoroughly is of course work and pleasure for several months.*
>
> *Laplace, who had always rejected my hypothesis of the origin of asteroids, appears to be more compliant. I have always emphasized the improbability that according to his idea of the development of our planetary system any planetary body might originally have an inclination of 35°. He meant against the solar equator the inclination of Pallas' orbit might be lesser. I replied that according to the current position of the nodes of Pallas' orbit exactly the contrary was occurring: Pallas' orbit still had 1¾ degree more inclination against the solar equator than against the ecliptic. The changing position of the line of nodes against each other and the small inclination against the solar equator against the ecliptic appears to me negligible. He then expressed his doubt whether Ceres' and Vesta's orbit might also intersect. I showed him that Vesta's distance from the sun at aphelion is larger than Ceres' distance at perihelion.*

Eight days later Olbers (1812b) wrote to Gauss with obvious satisfaction:

> *Laplace, who, as you know, has otherwise abandoned my hypothesis of the asteroids, appears to be favorably inclined towards it. We discussed it a lot. – Some days ago, when we were returning from a meeting of the institute, he said to me on his own free will: 'I further considered your argument, that you derive from the considerable inclination of Pallas' orbit, and now I, too, have to admit that your hypothesis is most likely.' – I had told him namely that the sum of the inclinations of the seven older planets against the equator of the sun was only 35.8889° (decimal degrees), but the inclination of Pallas' orbit alone 40.4098°. If the inclination of the older planetary orbits were up to 40.4098°, according to his own formulae, the probability that the sum of the inclinations of those seven orbits is included between 0° and 35.8889° would be*

$$\frac{1}{1.2.3\ldots.7}\left(\frac{35,8889}{40,4098}\right)^7 == 0,000086474 == \frac{1}{11564}.$$

*This makes it highly likely to me that Pallas was forced by an external force off its original path.*

The Dutch astronomer Jean Henri van Swinden (1746–1823; Fig. 2.14) had earlier issued a cautionary warning on this topic. Van Swinden (1803: 28), a professor at the Athenaeum Illustre of Amsterdam, made these remarks on the dangers of developing planetary laws based on incomplete data:

*Herschel nameth both the new Planets Asteroids, that is, who are looking like the shape of Stars, or Star-like, (just like Sphere, globe, Spheroid, globular). But this giveth no clearer notion of the matter. Let us learn from this large deviation* [inclination] *of the recently discovered Planet Pallas, of a law, which was considered to be certain (i.e., that the Planets move, all, inside the Zodiac), that it is always perilous, and hardly sage, to draw up general laws from only a few observations, if one is not brought there through basic Mathematical principles.*

Philosophically, Hume argues that a successful prediction based on induction (such as the one Olbers promoted), says nothing about the truth of the theory—for him, "inductive inference is not capable of being rationally justified." (Schurz 2014: 81). The origin of the asteroids is considered further in the next book of this series.

**Fig. 2.14** Jean Henri van Swinden. Universitetsmuseum Amsterdam

## The Invention of Least Squares

In 1775 the thin volume *Opera Inedita* appeared, containing several papers of the late German astronomer Tobias Mayer (1723–1762; Fig. 2.15). Among them was a catalog of the right ascensions and declinations of 998 stars, which may be occulted by the moon and planets; the places being adapted to the beginning of the year 1756. Mayer held the chair of mathematics at Goettingen University, and he became the superintendent of the Observatory in 1754. Many of the letters in this book refer to stars in Mayer's catalogue, and when Gauss moved to Goettingen, he began to observe the asteroids in the observatory established by Mayer (Forbes 1974). "He used Mayer's 6-foot mural quadrant, a pendulum clock made by John Shelton and an achromatic refracting telescope from the John and Peter Dollond firm in England. It was equipped with micrometers to measure small angular distances from neighboring stars with great precision." (Aubin 2005: 34).

Mayer has another crucial link with Gauss—his work of 1750 is "important in the history of statistics because it contains the first successful application of *equations of condition*, a method to solve an overdetermined system of equations" (Wepster). It became known as Mayer's method. Laplace (1788) generalized Mayer's method in his paper that explains the 'grand inequality' in the motions of Jupiter and Saturn that was detailed in Chap. 1. Mayer's method was in use for about half a century until it was replaced by the method of least squares (Forbes 1980), a method Gauss

**Fig. 2.15** Tobias Mayer

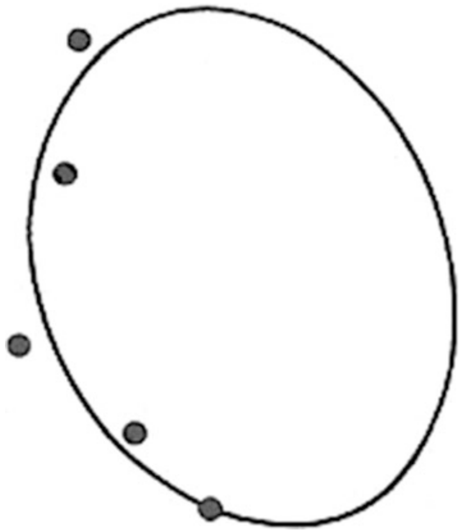


**Fig. 2.16** A best fitting curve to a given set of points

claims to have invented. Gauss himself attributes the principle of least squares to Mayer, as evidenced in this letter of January 24, 1812, to Olbers:

> *The only thing which is surprising is that this principle, which suggests itself to readily that no particular value at all can be placed on the idea alone, was not already applied 50 or 100 years earlier by others, e.g. Euler or Lambert or Halley or Tobias Mayer, although it may very easily be that the latter, for example, has applied that sort of thing without announcing it, just as every calculator necessarily invents a collection of devices and methods which he propagates by word of mouth only as occasion offers.*

The one person Gauss did not single out in this letter was Newton. But half a century before Mayer, in 1700, "Newton wrote down the first of the two so-called 'normal' equations known from the ordinary least squares method." (Belenky and Vila-Echague 2016) Newton's method forms the simplest kind of regression analysis, which was rediscovered and enhanced by Mayer.

Least squares is "a mathematical procedure for finding the best fitting curve to a given set of points by minimizing the sum of the squares of the offsets, known as "the residuals" of the points from the curve (Fig. 2.16). The sum of the *squares* of the offsets is used instead of the offset absolute values because this allows the sum to be treated as a continuous differentiable quantity. However, because squares of the offsets are used, outlying points can have a disproportionate effect on the fit, a property which may or may not be desirable depending on the problem at hand.

In practice, the *vertical* offsets from a line are almost always minimized instead of the *perpendicular* offsets (Fig. 2.17). This allows uncertainties of the data points along the *x*- and *y*-axes to be incorporated simply, and also provides a much simpler analytic form for the fitting parameters than would be obtained using a fit based on perpendicular distances. In addition, the fitting technique can be easily generalized

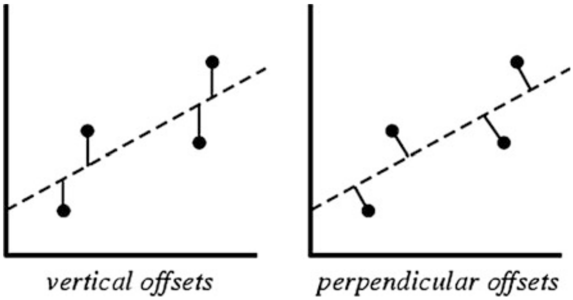



**Fig. 2.17** Vertical and perpendicular offsets from a line (Mathworld)

from a best-fit *line* to a best-fit *polynomial* when sums of vertical distances are used (which is not the case using perpendicular distances). For a reasonable number of noisy data points, the difference between vertical and perpendicular fits is quite small. The *linear* least squares fitting technique is the simplest and most commonly applied form of linear regression and provides a solution to the problem of finding the best fitting *straight* line through a set of points. In fact, if the functional relationship between the two quantities being graphed is known to within additive or multiplicative constants, it is common practice to represent the data in such a way that the resulting line *is* a straight line" (Mathworld).

The formulas for linear least squares fitting were independently derived by Gauss and Adrien-Marie Legendre (1752–1833; Fig. 2.18). The generally accepted version of events in the early nineteenth century was expressed by Bowditch (1820): "The method proposed by Professor Gauss and used by him since the year 1795 (and which was also invented by Legendre a few years afterwards) in the principle known by the name of the least squares."

This version has been the subject of heated dispute in recent decades. On the question of priority of discovery of the method, Stephen Stigler (1981) has questioned Gauss's veracity, something that has rightly been termed nothing short of libellous by Sheynin (1999). Stigler writes that while Legendre published the method in 1805, "Gauss had the temerity to claim that he had been using the method since 1795." In a letter to Laplace on January 30, 1812, Gauss wrote this about his application of least squares to the asteroids:

> *I have used the method of least squares since the year 1795 and I find in my papers, that the month of June 1798 is the time when I reconciled it with the principles of the calculus of probabilities. However my frequent applications of this method only date from the year 1802, since then I use it as you might say every day in my astronomical calculations on the new planets* (Plackett 1972).

Stigler (1999: 330) reluctantly concedes that Gauss's claim is possible:

> *Let us grant that Gauss's later accounts were substantially accurate, and that he did devise the method of least squares between 1794 and 1799, independently of Legendre or any other discoverer. There still remains the question, what importance did he attach to the discovery? Here the answer must be that while Gauss himself may have felt the method useful, he was unsuccessful in communicating its importance to others before 1805.*

**Fig. 2.18** Adrien-Marie Legendre

*He may indeed have mentioned the method to Olbers, Lindenau, or von Zach before 1805, but the total lack of applications by others, despite ample opportunity, suggests the message was not understood.* [See also Stigler (1977).]

It was Lindenau who procured the 1802 outline of Gauss's first work on the determination of planetary orbits. The text of this important document is printed in Chap. 12.

When Gauss tried to determine the orbits of the asteroids, the curve he sought was an ellipse. From the point of view of the formalism of least squares, this was not more difficult than the straight line example on the previous page; he used his method to determine the ellipse for which the sum of the squares of the deviations from certain observed points is a minimum. This is illustrated in the generic diagram Fig. 2.15, showing observed points around an ellipse.

In 1823 Gauss published a systematic and generalized presentation of his earlier theory of observational errors. In it, he develops the method of least squares with mathematical rigor as, in general, "the best way of combining observations, independent of any hypothetical law concerning the probability of error." (Kim 2009: 1128) (Hall 1970; Brand 2003). Gauss (1811) himself explained it:

> *The method of determining the path to which all observations come the closest can be divided into two parts: first in the development of linear equations which give a difference between the observed and calculated positions through the functions of the sought corrections, and second in determining the most probable value of this correction through the method of the least squares, while here, at almost any time, more equations than unknessence, Gauss developown quantities occur, which therefore cannot be done strictly enough.*

There are indeed two parts in the method: (a) finding the best estimator (coefficients of the approximating curve); and (b) proving this estimator is the most efficient. So did Gauss really use least squares to compute the orbits of Ceres and Pallas as claimed by Marsden (1977)? The answer is yes.

Bessel states that in his opinion Legendre has the priority because he published the result first, but "he can testify that Gauss told him about the method some years earlier." (Krengel 2006) Also, Olbers testifies that Gauss told him about the method prior to 1805, and Wolfgang Bolyai knew of this too (Sheynin 1999). Baron von Zach (1813: 98) wrote this:

> *The famous Doctor Gauss was already since 1795 in possession of this method, and used it to advantage in the determination of the elements elliptical orbits of the four new Planets* [asteroids], *as can be seen in his beautiful work* [of 1809].

Other pieces of evidence are remarks in the diary of Gauss, which was found many years after Gauss's death. He did say he used least squares when computing the orbit of Ceres, but he threw away the pages with his calculations. Furthermore, in 1799 Gauss "applied his new method of least squares to the equation of time as given in Ulugh Beg's tables." (Neugebauer 1975: 11) Gauss's least squares graduation of the differences in the equation of time by the astronomer Ulugh Beg (1394–1449) is given in *Gauss Werke* 12, pp. 64-68, and has been confirmed by Dutka (1996). (The twelve-volume collection of Gauss's complete works, known as *Gauss Werke*, was published between 1863 and 1933.) The method of least squares is still being used to refine the ephemerides of Pallas (Hestroffer et al. 1998).

## Gauss and Laplace: The Mathematical Link

There was a synergistic relationship between Gauss and Laplace. "His first proof of the method of least squares is based on inverse probability inspired by Laplace's 1774 paper." (Hald 2004: 50) In this landmark paper, Laplace

> *…invented (or perfected) a calculus of statistical inference that allowed him to estimate the likelihood that a particular configuration of events would lead to a subsequent arrangement known through observation. Philosophically, its power stemmed from capturing game theory from the domain of conjecture (chance), and turning it into a method for calculating likelihood (probability)* (Hahn 1990: 379).

The full panoply of Gauss's work, presented in *Theoria Motus*, proved to be a revelation for Laplace, who then used it to bolster his own work in probability. Thus the great synthesizer of knowledge—Laplace—and the great mathematician—Gauss—came together to create the framework for understanding the orbits and perturbations of Ceres and Pallas (Plackett 1989).

> *The memoir Laplace presented in April 1810 was entitled* Memoir on Approximating Formulas That Are a Function of Large Numbers, and on Their Application to Probability. *There Laplace used the central limit theorem, but without realising that it was in any way related to the least squares method. All this fell neatly into place once Laplace had studied and absorbed Gauss's remarkable* Theoria Motus*, which appeared in 1809. At first, Laplace apparently did not fully grasp the work's relationship to his concern with finding mean values for a set of observations of comets. Gauss's treatise was studied again in December 1809, winning the Lalande medal awarded by the* Institut*. But it was only after July 1810 that Laplace became fully aware of the way it simplified his fundamental notions. In his book, not only did Gauss give a mathematically elegant explanation for finding positions of new bodies like Ceres and Pallas, but he also offered a new way of looking at the error curve – a way that became known as the Gaussian distribution. Moreover, he demonstrated how this error distribution was mathematically tied to the least square method. Seeing the light, Laplace penned a brief supplement to his 1810 memoir, in which he took full advantage of the Gaussian discovery, integrating it into his understanding of the theoretical basis for probability theory. By placing error theory on a Gaussian distribution, Laplace had integrated the principle of least squares into the now powerful tool of probability* (Hahn 2005: 112).

Gauss in turn used Laplace's work on perturbation theory, which was published in 1799 in the first volumes of his *Treatise on celestial mechanics*. But much of this work only became known in the twentieth century (Klein et al. 1919).

> *True to his principle that a work should have such a degree of completion 'that nothing more could be desired', Gauss published very little of his contributions to perturbation theory. But he communicated them to his students; for example the orbit computed for the comet named after Encke was computed by Gauss's method* [in 1818]. *Only when Gauss's posthumous papers were published in his* Collected Works *did one get a clear idea of his fundamental contributions to perturbation theory through his calculations for Ceres and Pallas. In his first calculations for the perturbations of Ceres, Gauss used his own research on the hypergeometric series and the arithmetic-geometric mean, and also made use of Laplace's work in an essential way. The calculations of the perturbations of Pallas within the bounds of accuracy of the observations consisted mostly of page after page of numerical calculations with no explanatory text. This difficult puzzle was put in order and published by the German astronomer Martin Brendel, who in his commentary points out that Gauss has almost completed an enormous problem, 'which even today an astronomer does not gladly set out upon.'*
> Hall (1970: 64)

In essence, Gauss developed "the theory of the normal distribution of the observation errors and the method of least squares to solve for orbits; the two concepts are related by a theorem, by which the probability distribution of the least square fit in the space of orbital elements is also normal in the linear approximation. He developed this theory to solve the problems of orbit determination of both Ceres and Pallas, and this method has been used by astronomers ever since" (Milani 1998: 271). How Gauss developed the fast Fourier transform and applied it to the orbital solution of the asteroids will be examined in a future book in this series.

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