Structural Importance and Local Importance in Network Reliability

P. Tittmann and S. Kischnick

Abstract Network reliability analysis has interesting applications in areas such as computer and mobile networks. However, the computation of many important reliability measures (all-terminal reliability, reachability) turns out to be NP-hard. This statement applies to the computation of relevant reliability importance measures, too. In this paper we introduce *local importance measures* that describe the importance of an edge or vertex of the network in its local network neighborhood. Suitable scaling of the local neighborhood renders the computation of generally intractable reliability measures possible.

1 Introduction

The *importance* of an element x (vertex or edge) of a network (graph) G is a measure that describes the significance of the element for proper functionality. In order to make this idea more precise we assume that a given function $f : \mathcal{G} \to \mathbb{R}^+$ assigns a nonnegative real number $f(G)$ to any given graph G. This function is supposed to be monotone increasing with the performance (redundancy, reliability) of the network. Examples for those functions are edge and vertex connectivity, the number of vertex pairs that are reachable from each other, or the number of spanning trees of the graph. Let $G - x$ be the graph obtained from G by removal of element x (edge or vertex). Then

$$\frac{f(G) - f(G - x)}{f(G)}$$

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is a measure for the loss of performance when x has failed and hence also a measure for the importance of x. An importance measure is referred to as *structural importance* if it is solely dependent on the graph but not on availabilities of edges or vertices.

The concept of reliability importance was introduced by Birnbaum, [4]. Alternative structural reliability importance measures are presented in [12, 15, 16, 17]. Reliability importance measures have been applied for general binary and multi-state systems, see [14]. A game theoretic approach to reliability importance is presented in [8]. The interrelation of the importance of two different components has been investigated in [2] and [6]. For a more detailed introduction to system reliability and reliability importance, see [9, 19].

Unfortunately, it is shown that the computation of many reliability importance measures that are essential to systems and network analysis is **NP**-hard. A classical example is the Birnbaum importance given by

$$I_B(G, e) = \frac{\partial R(G)}{\partial p_e} = R(G/e) - R(G - e) \tag{1}$$

where $R(G)$ denotes the all-terminal reliability of a graph $G = (V, E)$, whose edges fail independently with given probability $p_e, e \in E$. As the computation of the all-terminal reliability is **NP**-hard (in fact it belongs to the class #P-complete, [18]) the computation of the Birnbaum importance is **NP**-hard, too. Also many structural importance measures that are based on counting all path sets or all cut sets of a graph are computationally intractable.

2 Structural Importance Measures

Structural importance measures provide an essential tool to rank the importance of components (edges or vertices) of a network in case that there are no reliability data available for edges and vertices of the network.

2.1 The Structural Birnbaum Importance

The measure of structural importance considered here is very likely the first measure that appeared in the literature; it was introduced by Birnbaum in [4]. Let $G = (V, E)$ be an undirected graph with m edges. We define a function $\Psi(G)$ by

$$\Psi(G) = \begin{cases} 1 & \text{if } G \text{ is connected,} \\ 0 & \text{if } G \text{ is not connected.} \end{cases} \tag{2}$$

If $F \subseteq E$ then we write $\Psi(V, F)$ instead of $\Psi((V, F))$. We consider a subgraph (V, F) of G operating if and only if (V, F) is connected. Consequently, a *path set* is in this context an edge subset F such that $\Psi(V, F) = 1$. Now let $e \in E$ be a fixed edge of G. An edge set $F \subseteq E$ is critical with respect to the edge e if

$$\Psi(V, F \cup \{e\}) - \Psi(V, F \setminus \{e\}) = 1. \tag{3}$$

Conversely, given a subset $F \subseteq E$ of edges, an edge $e \in E \setminus F$ is *essential* with respect to F if Eq. (3) is satisfied. The *structural Birnbaum importance* of an edge e of G is defined by

$$I_b(G, e) = \frac{1}{2^m} \sum_{F \subseteq E} [\Psi(V, F \cup \{e\}) - \Psi(V, F \setminus \{e\})]. \tag{4}$$

Thus the structural Birnbaum importance $I_b(G, e)$ counts critical sets of G with respect to e. We define

$$I_1(G, e) = \frac{1}{2^{m-1}} \sum_{F : e \in F \subseteq E} [\Psi(V, F) - \Psi(V, F \setminus \{e\})], \tag{5}$$

$$I_0(G, e) = \frac{1}{2^{m-1}} \sum_{F \subseteq E \setminus \{e\}} [\Psi(V, F \cup \{e\}) - \Psi(V, F)]. \tag{6}$$

Then we obtain by splitting the sum in Eq. (4)

$$I_0(G, e) = \frac{1}{2}(I_1(G, e) + I_0(G, e)). \tag{7}$$

A closer look at the Eqs. (5) and (6) shows that

$$I_1(G, e) = I_0(G, e) \tag{8}$$

as both sums count the same critical sets. This also shows that we can define the structural Birnbaum importance by Eq. (5).

2.1.1 Connected Spanning Subgraphs and the Tutte Polynomial

Let $G = (V, E)$ be an undirected graph. The graphs obtained from G by removal and contraction of an edge e are denoted by $G - e$ and G/e, respectively. Let $\tau(G)$ be a number of connected spanning subgraphs of G. Then we can easily conclude from Eq. (4) that

$$I_b(G,e) = \frac{\tau(G/e) - \tau(G-e)}{2^m}. \tag{9}$$

We call an edge of G that is neither a loop nor a bridge of G a *link*. The function $\tau(G)$ can be recursively computed by

$$\tau(G) = \begin{cases} 0 & \text{if } G \text{ is disconnected,} \\ 1 & \text{if } G \text{ is a tree,} \\ 2\tau(G-e) & \text{if } e \text{ is a loop,} \\ \tau(G/e) & \text{if } e \text{ is a bridge,} \\ \tau(G-e) + \tau(G/e) & \text{if } e \text{ is a link.} \end{cases} \tag{10}$$

The last representation follows from the well-known fact that $\tau(G)$ equals the evaluation $T(G;1,2)$ of the Tutte polynomial of G, see [5].

Theorem 1 *The computation of the structural Birnbaum importance is #P-hard.*

Proof It has been shown in [21] that the evaluation of $T(G;1,2)$ belongs to the class of #P-hard problems. Now let $G = (V,E)$ be a given graph and $v \in V$ an arbitrarily chosen vertex of G. We construct the new graph $G' = (V',E')$ by inserting a new vertex u and an edge $e = \{u,v\}$ in G. Then clearly a subset $F \subseteq E'$ is critical for e in G' if and only if (V,F) is a spanning subgraph of G. Hence we obtain $\tau(G) = 2^{|E'|} I_b(G',e)$.

Figure 1 shows an example graph with edges weighted by 100 times the structural Birnbaum importance. We see, as expected, that the bridges in the graph are the edges with maximum importance.

2.1.2 Modified Structural Birnbaum Importance

The normalizing factor, 2^{-m}, in Eq. (4) accounts for the powerset of E, which implies that in large sparse networks $I_b(G,e) \ll 1$ for all edges. The modified structural Birnbaum importance, defined by

$$I'_b(G,e) = \frac{\tau(G) - \tau(G-e)}{\tau(G)}, \tag{11}$$

provides a measure of structural importance with the property $I'_b(G,e) = 1$ for any bridge e of G. As $I'_b(G,\cdot)$ is a constant multiple of $I_b(G,\cdot)$ the importance ranking of edges is the same for both measures.

The reliability polynomial of a graph $G = (V,E)$ can be represented by

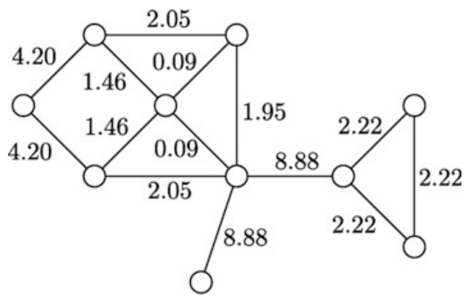


Fig. 1 Structural Birnbaum importance

$$R(G,p) = (1-p)^m \sum_{F \subseteq E} \Psi(V,F) \left(\frac{p}{1-p} \right)^{|F|}. \tag{12}$$

see, for instance, [3]. Consequently, $2^m R(G, 1/2)$ gives the number of connected spanning subgraphs of G, which implies

$$I_b(G,e) = R\left(G/e, \frac{1}{2}\right) - R\left(G - e, \frac{1}{2}\right) = \left. \frac{\partial R(G,p)}{\partial p_e} \right|_{p=\frac{1}{2}}, \tag{13}$$

which is the evaluation of the Birnbaum importance, given in Eq. (1) at $p = \frac{1}{2}$.

2.2 *Spanning Trees and Electrical Resistance*

The practical consequence of Theorem 1 is that the calculation of structural importance in large networks is impossible within reasonable time. There are several approaches to overcome this problem:

- The problem might be solvable in polynomial time when restricted to special graph classes. In fact it can be shown, see [1], that $I_b(G,e)$ can be efficiently calculated in graphs of bounded tree-width.
- Often polynomial-time algorithms can provide lower and upper bounds for the structural importance.
- An estimation for the desired measure can be obtained by Monte-Carlo simulation.
- In some cases another importance measure that is efficiently computable provides (almost) the same information.
- The desired importance measure can be *locally* calculated, i.e. with respect to a part of the network that is close to the edge (or vertex) to be investigated.

We will focus in the following on the latter two methods.

2.2.1 Spanning Trees

Let us consider a second importance measure in more detail. We choose the number $t(G)$ of spanning trees of a graph G as performance measure. A spanning tree is a minimal spanning subgraph of a given graph that ensures connectedness. Hence we may assume that a graph with many spanning trees is *more reliable* than a graph with a fewer number of spanning trees. This argument might be questionable when comparing completely different graphs. However, we can show that it is a suitable measure to study the effect of edge removal. We define

$$I_r(G,e) = \frac{t(G) - t(G-e)}{t(G)} = \frac{t(G/e)}{t(G)}. \tag{14}$$

The second equality results from the well-known decomposition formula $t(G) = t(G-e) + t(G/e)$, which is valid for any edge $e \in E(G)$. If e is a bridge of G then $G - e$ does not have any spanning trees, $t(G-e) = 0$, which implies $I_r(G,e) = 1$. Hence a bridge has maximum importance. Any edge of a series system (a tree) has importance 1. If G is a graph with two vertices that are linked with each other by m parallel edges then each edge has importance $1/m$. However, in both cases (series and parallel system), all edges have the same importance.

Equation (14) resembles the definition of structural Birnbaum importance by Eq. (6), whereas now the number of spanning trees of G is used to normalize the measure. The similarity of the two measures becomes even more obvious if we compare the two definitions given in Eqs. (14) and (11). The only difference is that we count in $I_r(G,e)$ only *minimum* connected spanning subgraphs. Hence we expect the importance measure I_r to provide a similar edge importance ranking than the one defined by I_b.

2.2.2 Electrical Resistance

There is a completely different interpretation of the fraction

$$\frac{t(G/e)}{t(G)}.$$

According to Kirchhoff laws, [11], this fraction is also the effective resistance that we measure between the endpoints u and v of the edge $e = \{u, v\}$ in the network $G = (V, E)$ where all the edges of E represent unit resistors, see also [5]. Hence we might call $I_r(G,e)$ the *resistance importance* of e. Interestingly, this new interpretation supports the understanding of $I_r(G,e)$ as an importance measure. The effective resistance between the end vertices u and v is 1 if and only if there is no other path than the edge e in G that connects u with v. On the other hand, if the resistance is small then there must be many other (short) paths that connect the end vertices of e making e less important for the connectivity of G. Figure 2 shows a

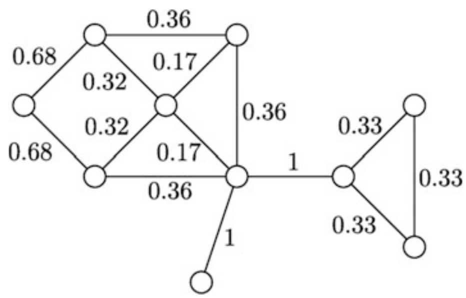


Fig. 2 A graph with edge weights $I_r(G, e)$

graph whose edges are weighted with $I_r(G, e)$. The total number of spanning trees of this graph is 198.

3 Flow and Distance Related Structural Importance Measures

We consider now a measure of importance that appeared first time in the context of social network analysis. The idea is that an edge is important if it is located on many shortest paths between different vertex pairs. A corresponding centrality measure has been introduced in [7] in order to determine the centrality of a vertex in a social network. An analog centrality measure with respect to the edges of a graph was employed in [10] in order to identify communities (dense subgraphs) in a graph. A nice overview about different kinds of centrality measures in social network analysis is given in [13].

3.1 Stress Centrality

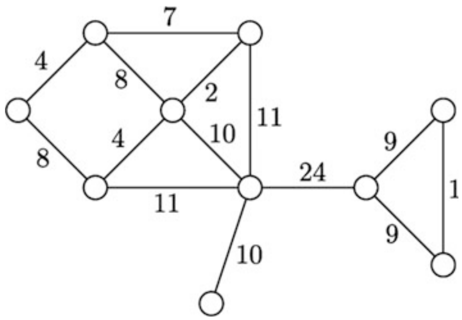
Let $G = (V, E)$ be a graph and $s, t \in V$. We denote by $\sigma_{st}(e)$ the number of shortest paths between s and t in G that contain the edge e and define

$$V^{(2)} = \{\{u, v\} | u, v \in V\}$$

as the set of all unordered pairs of vertices of V. The *stress centrality*, introduced in [20], of an edge e is defined by

$$c_s(e) = \sum_{\{s,t\} \in V^{(2)}} \sigma_{st}(e). \tag{15}$$

Fig. 3 Stress centrality as importance measure



Assume that traffic in a communication network is routed along shortest paths and the stress centrality $c_s(e)$ tries to evaluate the amount of flow that is routed through e. Hence we can consider the stress centrality of an edge as a measure of importance with respect to network reliability. Figure 3 shows the graph that has been also used in the preceding examples now weighted with the stress centralities of the edges. The stress centrality distinguishes the two bridges of the graph. Even more interesting, two non-bridge edges have a higher importance than one of the bridges.

3.2 *Modified Stress Centrality*

A slight modification of the stress centrality might yield a more appropriate importance measure for some applications. We define

$$\delta_{st}(e) = \begin{cases} 1 & \text{if } e \text{ is in a shortest } st\text{-path,} \\ 0 & \text{otherwise.} \end{cases}$$

The sum

$$I_p(G, e) = \sum_{\{s,t\} \in V^{(2)}} \delta_{st}(e) \tag{16}$$

gives the number of vertex pairs of G for which there exist a shortest path traversing the edge e.

3.3 Betweenness Centrality

We can refine the investigation of the importance of an edge by considering the possibilities of rerouting the flow in the network in case the edge fails. Suppose the flow between two vertices s and t is routed along a shortest path that traverses a given edge e. Then a failure of e is less important as long there are other paths of same length connecting s and t. The *betweenness centrality* takes this effect into account. It is defined by

$$c_b(e) = \sum_{\{s,t\} \in V^{(2)}} \frac{\sigma_{st}(e)}{\sigma_{st}}, \tag{17}$$

where σ_{st} denotes the number of shortest st-paths in G.

3.4 An Importance Measure Based on Distances

Another way to measure the importance of an edge in a graph is obtained by considering the effect of edge removal on distances between vertices in a graph. We denote by $d(u, v)$ the *distance* of two vertices u and v in a graph $G = (V, E)$, that is the length of shortest path between u and v in G. The distance $d(u, v)$ is defined to be infinite if there is no path between u and v. The *total distance* of a vertex v is

$$\mathrm{td}(v) = \sum_{w \in V} d(v, w). \tag{18}$$

The *Wiener index* of G, defined by

$$W(G) = \sum_{v \in V} \mathrm{td}(v) = \sum_{\{u,v\} \subseteq V} d(u, v). \tag{19}$$

increases when an edge of G is removed. By

$$I_d(G, e) = 1 - \frac{W(G)}{W(G - e)} \tag{20}$$

we define the *distance importance* of the edge e in G. For communication networks, short paths between many vertex pairs are desirable. Hence the reciprocal of the Wiener index provides a suitable performance measure for networks, which explains the difference in Eq. (20). Figure 4 shows our example network this time with edges labeled according to Eq. (20), where $I_d(G, e)$ is multiplied with 100.

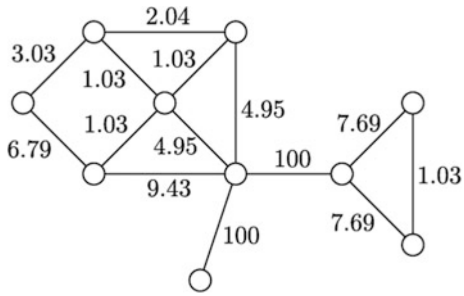


Fig. 4 A distance importance measure

4 Local Importance Measures

For large networks, the computation of structural importance measures might be time-consuming or even impossible. In order to obtain in this case at least good estimations of the importance of edges or vertices, we use the following approach. If an edge has a high importance in the network then it can be assumed that this edge also has a high importance in a certain *network neighborhood* of itself. Let $G = (V, E)$ be an undirected graph, $k \in \mathbb{N}$, and $v \in V$. The *k-neighborhood* of v in G is defined by

$$N_k(v) = \{w \in V | d(v, w) \leq k\}.$$

Consequently, the k-neighborhood of v consists of all vertices of G that have distance at most k from v. We define the k-neighborhood of an edge $e = \{u, v\}$ of G by

$$N_k(v) = N_k(u) \cup N_k(v).$$

Assume that $I(G, e)$ is any importance measure for the edge e of G. Then we define for any non-negative integer k the *local importance* $I(G, e, k)$ as

$$I(G, e, k) = I(G[N_k(e)], e).$$

Depending on the definition of the importance measure, $I(G, e, k)$ might be an approximation; it can also provide an upper or a lower bound for $I(G, e)$. Consider, as an example, the importance measure $I_r(G, e)$ defined in Eq. (14), which is the effective electrical resistance between the end vertices of e assuming all edges represent unit resistors. Building the neighborhood network $G[N_k(e)]$ means that we cut out a part of the network. This operation causes an increase of the resistance such that $I_r(G, e, k)$ is an upper bound for $I_r(G, e)$.

Table 1 shows the values of local importance measures for a grid graph of dimension 41×41 with respect to a central edge in the middle of the 21st column of the grid. Figure 5 shows the graph $G[N_3(e)]$, where G is a grid graph. The first column gives the parameter k that defines the local neighborhood of the edge. The

Table 1 Local importance measures

k	n	$I_r(G_s, e, k+1)$	$I_r(G, e, k)$	$I_p'(G, e, k)$	$I_b(G, e, k)$
1	8	0.4597701	0.6000000	0.5000000	8.66667
2	18	0.4786778	0.5402299	0.4117647	28.3555
3	32	0.4868770	0.5213222	0.3709677	65.8332
4	50	0.4911341	0.5131230	0.3469388	127.106
5	72	0.4936175	0.5088659	0.3309859	218.235
6	98	0.4951893	0.5063825	0.3195876	345.278
7	128	0.4962458	0.5048107	0.3110236	514.274
8	162	0.4969896	0.5037542	0.3043478	731.248
9	200	0.4975327	0.5030104	0.2989950	1002.21
10	242	0.4979412	0.5024673	0.2946058	1333.18
11	288	0.4982561	0.5020588	0.2909408	1730.15
12	338	0.4985041	0.5017438	0.2878338	2199.12
13	392	0.4987027	0.5014959	0.2851662	2746.10
14	450	0.4988643	0.5012972	0.2828508	3377.08
15	512	0.4989975	0.5011357	0.2808219	4098.06

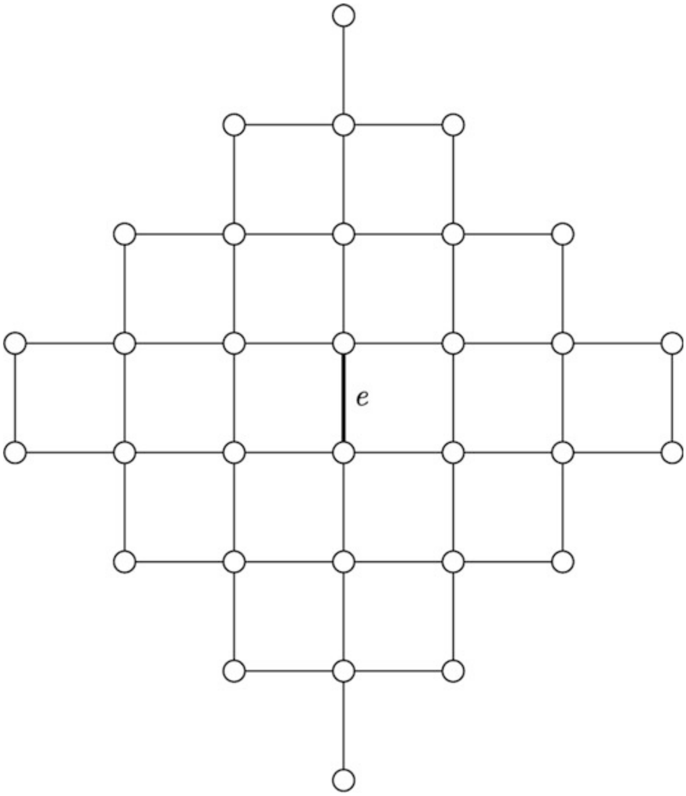


Fig. 5 The neighborhood graph of an edge in the grid for $k = 3$

second column shows the order of the neighborhood graph. The third column shows the resistance importance of $G_s[N_{k+1}(e)]$, which is the graph obtained from the edge neighborhood graph $G[N_{k+1}(e)]$ by merging all vertices that have distance $k+1$ from the edge e. This means that we shortcut all vertices of the "outer shell" of the neighborhood graph. The resulting value $I_r(G_s, e, k+1)$ is a lower bound for $I_r(G, e)$. The corresponding upper bound is given in the next column of Table 1. The values shown in column 5 of Table 1 present an importance measure obtained from I_p, see Eq. (16), by multiplication with $\frac{2}{n(n-1)}$, where n is the order of the neighborhood graph. This modification produces a normalized importance measure. The last column gives the betweenness centrality of the edge e in the respective neighborhood graph.

5 Summary and Conclusions

The structural importance of edges in graphs with respect to network reliability can be evaluated by a variety of local importance measures. Some of them, for instance the structural Birnbaum importance, are computationally intractable, whereas others, like the resistance importance, are computable in polynomially bounded time. An approach to overcome the computational hardness of the computation of importance measures is the introduction of local importance measures.

There remain, however, some interesting open questions:

- The importance ranking of edges depends on the selected structural importance measure. Which structural importance measures are closest to each other with respect to the ranking? Can we find bounds for the importance difference of edges?
- Numerical experiments suggests that the local importance quickly converges to the global structural importance when the neighborhood radius k grows. Is there a way to describe the quality of approximation in dependence on k?

All structural importance measures considered here are defined with respect to edges of a graph. The given importance measures can be easily generalized for vertices of a graph or even for components of monotone binary systems.

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