

Fascination of Making Models Truth-Reality-Illusion?

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1 Introduction

Evidently, human thinking is based on an imagination of the surrounding world which one is going to explore, an abstraction, a model. Such a model, if successful, enables to understand what is really going on – reality, that is what we see and it might be the truth, but observation and imagination can also lead to confusion and illusion.

The basic aim of modeling is explanation. Explanation leads to reconstruction and prediction. The probably easiest way of reconstruction is to build a hardware model and to see what is going on under certain circumstances – like a ship model in a stormy sea pool: the Pamir [42]. This is enough to make a movie, to create an illusion, and perhaps to shed light on the one or the other reason of its accident. But is not enough to generalize these results for further applications and predictions. Here, mathematics comes into play. The hardware model then serves for experimental verification of theoretical physical modeling – like Prandtl’s water channel, or wind tunnels, indispensable till today. Verification is the last step in modeling: arbiter is always nature.

Hardware modeling is sometimes not possible, like Kepler’s satellite system. Then, physical modeling is pure abstraction, based on observation. The physical model states the ongoing process, but not yet its explanation. Here, Hooke commissioned Newton to calculate the assumption that gravity acts reciprocal to the square of the mass center distances – probably the best model ever – and with Euler’s momentum theorem, the problem became generalizable. Physical modeling and mathematical modeling go hand in hand.

Modeling, however, contains its danger. This is, for instance, the case in politics and economics when a model, unconsciously or consciously, misguides people (how did pope Urban VIII get Galileo into prison?) But also

in technical applications when one believes on a-priori known results where they actually do not arise, like chaotic motions in a double pendulum with small amplitudes – and worse: to conclude that a double pendulum can therefore not be calculated [11]. If so, with the double pendulum as a basic robot model, this presentation would already have come to an end.

Nearer inspection of the double pendulum, however, and assuming slender arms, leads to the question if elasticity will play a significant role.

2 Elasticity – Modeling Problems

The effect of elasticity is impressively demonstrated with a linear robot (i.e. only translational rigid body motions) where the aim (pick and place) cannot be achieved even allusively with common joint control [28].

To obtain insight into the problems of modeling, the spatial (fast) rigid body motion is decomposed into uniaxial ones. Most problems arise from rotations (linear motions are considered later). This reveals a list of model inaccuracies and/or faults in modeling elastic systems.

One common procedure to deal with elasticity is based on Hamilton's principle (which, according to Felix Klein [30], was already known to Lagrange):

$$\int_{t_o}^{t_1} [\delta T - \delta V] dt = 0 \quad (1)$$

(T : kinetic energy, V : potential energy).

2.1 Rotating Beams

Considering a rotating bending beam one has

$$\begin{aligned} \delta T &= \int_o^L (\mathbf{v}_c^T \rho A \delta \mathbf{v}_c + \boldsymbol{\omega}_c^T \rho \mathbf{I} \delta \boldsymbol{\omega}_c) dx \\ \delta V &= \int_o^L (EI_z v'' \delta v'' + EI_y w'' \delta w'') dx \end{aligned} \quad (2)$$

where ρ : mass density, A : cross sectional area, $\mathbf{I}$: tensor of area moments of inertia, $\mathbf{v}_c, \boldsymbol{\omega}_c$: translational and rotational velocity w.r.t. the mass center c of the beam element; EI_z, EI_y : bending rigidities referred to the deflection functions $v(x, t), w(x, t)$. Prime indicates differentiation: $(\)' = \partial(\)/\partial x$.

Longitudinal rotation. Rotation around the extensional axis x with $\dot{\alpha}(t)$ yields the velocities (up to second order)

$$\mathbf{v}_c = \begin{pmatrix} 0 \\ \dot{v} - \dot{\alpha}w \\ \dot{w} + \dot{\alpha}v \end{pmatrix}, \quad \boldsymbol{\omega}_c = \begin{bmatrix} \left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) & v' & 0 \\ -v' & \left(1 - \frac{v'^2}{2}\right) & 0 \\ -w' & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{\alpha} \\ -\dot{w}' \\ \dot{v}' \end{pmatrix} \quad (3)$$

where $\mathbf{v}_c$ refers to the rotating reference frame while $\boldsymbol{\omega}_c$ is resolved in an element fixed frame due to the herein constant inertia tensor. Cardan angles (Tait-Bryan-angles) $[\alpha \rightarrow \beta = -w' \rightarrow \gamma = v']$ are chosen for description. The dot represents time derivation.

The calculation is (although a bit cumbersome) as usual: carry out variation, integrate by parts w.r.t. time

$$\begin{aligned} & \int_{t_o}^{t_1} [\delta T - \delta V] dt = \int_{t_o}^{t_1} \left[\int_o^L \{ \delta \alpha [-\rho I_x \ddot{\alpha}] \right. \\ & + \delta v [\rho A (-\ddot{v} + 2\dot{\alpha}\dot{w} + \ddot{\alpha}w + \dot{\alpha}^2 v)] \\ & + \delta w [\rho A (-\ddot{w} - 2\dot{\alpha}\dot{v} - \ddot{\alpha}v + \dot{\alpha}^2 w)] \\ & + \frac{\delta v' [-\rho I_z \ddot{v}' + \rho(I_z + I_y - I_x)\dot{\alpha}\dot{w}' + \rho(I_y - I_x)\dot{\alpha}^2 v' + \rho I_z w' \ddot{\alpha}]}{ } \\ & + \frac{\delta w' [-\rho I_y \ddot{w}' - \rho(I_z + I_y - I_x)\dot{\alpha}\dot{v}' + \rho(I_z - I_x)\dot{\alpha}^2 w' + \rho(I_x - I_y)v' \ddot{\alpha}]}{ } \\ & \left. - \frac{\delta v'' E I_z v'' - \delta w'' E I_y w''}{ } \right\} dx \Big] dt + \left[\dots \right]_{t_o}^{t_1} = 0. \quad (4) \end{aligned}$$

Next, set the δ 's at the time boundaries equal to zero. The time integral is then obsolete. Integrate by parts the underlined part once and the double underlined part twice w.r.t. space and order according to the δ 's to come out with an integral part and a boundary part

$$\begin{aligned}
& \int_o^L \left[\delta\alpha [-\rho I_x \ddot{\alpha}] \right. \\
+ \quad & \delta v \quad \left\{ \rho A (-\ddot{v} + 2\dot{\alpha}\dot{w} + \ddot{\alpha}w + \dot{\alpha}^2 v) \right. \\
& + \rho I_z \ddot{v}'' - \rho(I_z + I_y - I_x) \dot{\alpha}\dot{w}'' - \rho(I_y - I_x) \dot{\alpha}^2 v'' - \underline{\rho I_z w'' \ddot{\alpha}} \\
& \left. \left. - (EI_z v'')'' \right\} \right. \\
+ \quad & \delta w \quad \left\{ \rho A (-\ddot{w} - 2\dot{\alpha}\dot{v} - \ddot{\alpha}v - \dot{\alpha}^2 w) \right. \\
& + \rho I_y \ddot{w}'' + \rho(I_z + I_y - I_x) \dot{\alpha}\dot{v}'' - \rho(I_z - I_x) \dot{\alpha}^2 w'' - \rho(I_x - I_y) v'' \ddot{\alpha} \\
& \left. \left. - (EI_y w'')'' \right\} \right] dx, \tag{5}
\end{aligned}$$

$$\begin{aligned}
& + \left[\left\{ \begin{array}{ll} \delta v & [-\rho I_z \ddot{v}' + \rho(I_z + I_y - I_x) \dot{\alpha}\dot{w}' + \rho(I_y - I_x) \dot{\alpha}^2 v' + \underline{\rho I_z w' \ddot{\alpha}} \\ & + (EI_z v'')'] \end{array} \right\} \right. \\
& + \left\{ \begin{array}{ll} \delta w & [-\rho I_y \ddot{w}' - \rho(I_z + I_y - I_x) \dot{\alpha}\dot{v}' + \rho(I_z - I_x) \dot{\alpha}^2 w' + \rho(I_x - I_y) v' \ddot{\alpha} \\ & + (EI_y w'')'] \end{array} \right\} \\
& + \left\{ \begin{array}{ll} \delta v' & (-EI_z v'') \end{array} \right\} \\
& \left. + \left\{ \begin{array}{ll} \delta w' & (-EI_z w'') \end{array} \right\} \right]_o^L = 0. \tag{6}
\end{aligned}$$

Assuming $\delta v, \delta w, \delta\alpha$ independent yields eq.(5) for motion equations along with eq.(6) for boundary conditions. In order to simplify matters, let $I_y = I_z = I$, $I_x = 2I$ (circular or square cross section) to obtain

$$\begin{aligned}
& \left[\rho A (\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v) - \rho I \ddot{v}'' - \rho I \dot{\alpha}^2 v'' + \underline{\rho I w'' \ddot{\alpha}} + (EI_z v'')'' \right] = 0, \\
& \left[\rho A (\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w) - \rho I \ddot{w}'' - \rho I \dot{\alpha}^2 w'' + \underline{\rho I v'' \ddot{\alpha}} + (EI_y w'')'' \right] = 0. \tag{7}
\end{aligned}$$

(Notice that the sign is changed to obtain $+\rho A \ddot{v}$. Furthermore, the α -equation is decoupled from the remainder. It reads, replenished with the drive torque, $\left[\int_o^L \rho I_x dx \right] \ddot{\alpha} - M_x = A \ddot{\alpha} - M_x = 0$.) The result according to

eq.(7) is a bit strange: same as for the (underlined) translational part one would expect skew symmetry for the (double underlined) rotational motion.

The question is: are $\delta v, \delta w, \delta \alpha$ really independent?

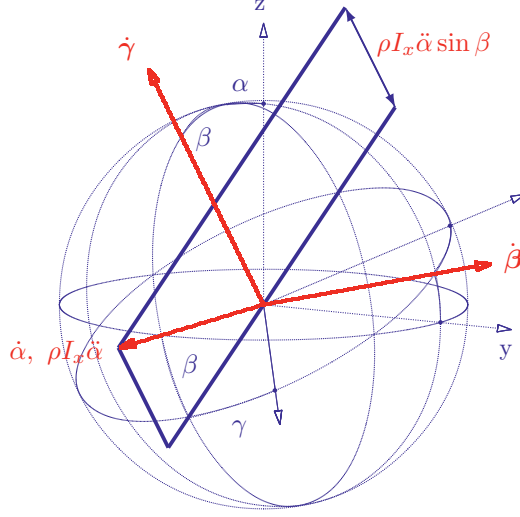


Figure 1. On orthogonality

As Figure 1 shows, the $\dot{\alpha}$ -axis is not perpendicular to the $\dot{\beta}$ - $\dot{\gamma}$ -plane. Assuming small deflection amplitudes, one has $\rho I_x \ddot{\alpha}(\sin \beta) \simeq \rho I_x \ddot{\alpha}(-w')$ and $\rho I_x \ddot{\alpha}(\cos \beta) \simeq \rho I_x \ddot{\alpha}$. Eq.(5) has thus to be replenished with $\delta W_{add} = \int_o^L \rho I_x \ddot{\alpha}(-w') \delta v' dx = \int_o^L \rho I_x \ddot{\alpha} w'' \delta v dx - [\rho I_x \ddot{\alpha} w' \delta v]_o^L$. The underlined term in eq.(5) thus changes to $\int_o^L \delta v [-\rho(I_z - I_x) \ddot{\alpha} w''] dx$, or, for $I_z = I, I_x = 2I$, to $\int_o^L \delta v [+ \rho I \ddot{\alpha} w''] dx$, yielding for eq.(7)

$$\begin{aligned} \left[\rho A (\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v) - \rho I \ddot{v}'' - \rho I \dot{\alpha}^2 v'' - \underline{\underline{\rho I w'' \ddot{\alpha}}} + (EI_z v'')'' \right] &= 0, \\ \left[\rho A (\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w) - \rho I \ddot{w}'' - \rho I \dot{\alpha}^2 w'' + \underline{\underline{\rho I v'' \ddot{\alpha}}} + (EI_y w'')'' \right] &= 0, \end{aligned} \quad (8)$$

along with the corrected boundary terms.

Usually, Hamilton's Principle is well established for the calculation of elastic bodies, and one would never doubt the corresponding results. What is going on here, however, is due to the fact that the elastic deflections are

assumed small but the rigid body motions are not. One has thus to be careful in linearization.

The beam under consideration is called a Rayleigh beam. One gets rid of such kind of difficulties when passing over to a so called Euler-Bernoulli-beam, where the inertia moments are negligible:

$$\begin{aligned} \left[\rho A (\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v) + (EI_z v'')'' \right] &= 0, \\ \left[\rho A (\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w) + (EI_y w'')'' \right] &= 0. \end{aligned} \quad (9)$$

However, do these equations still represent reality? *Physical model needs intuition.*

“The Swedish engineer Laval (1845–1913) was the first one to show, by practical experiments in 1889, that a shaft can rotate much faster than people at that time believed. When Laval’s results came to light the reaction was disbelief everywhere” [18].

Changing notations to those usually used in rotor dynamics one can easily deduce from eq.(9) the equations for the first bending mode [$\ddot{v} \rightarrow \ddot{x}$, $\ddot{w} \rightarrow \ddot{y}$ etc.) along with $\dot{\alpha} = \Omega = \text{const.}$ (rotational symmetry provided)]

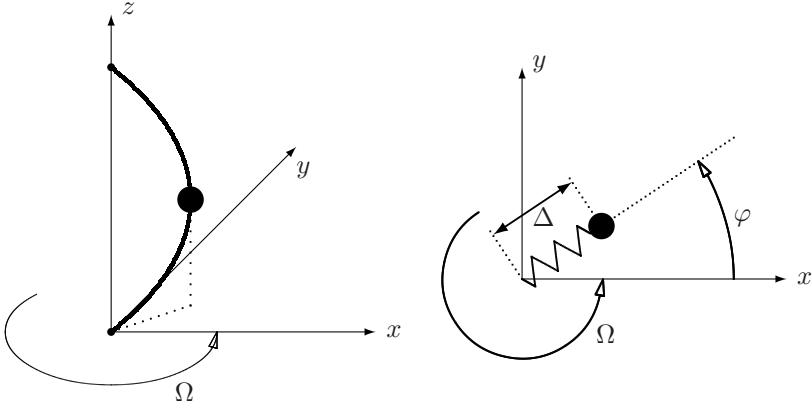


Figure 2. (a) Laval-rotor; (b) top view

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{bmatrix} 0 & -2\Omega \\ 2\Omega & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{bmatrix} (\omega^2 - \Omega^2) & 0 \\ 0 & (\omega^2 - \Omega^2) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (10)$$

August Föppl (1854–1924) in a contribution from 1895 gave an explanation: W. Rankine (1820 – 1872) had used an “unfortunate model”, i.e. some kind of a rotating massless guide mechanism with a point mass inside which is attached to the origin by a spring. Because the guide prevents relative velocity perpendicular to the spring elongation, the number of degrees of freedom is reduced by one, e.g. $\varphi = 0$ in Figure 2. The remaining equation $\ddot{x} + (\omega^2 - \Omega^2)x = 0$ then indicates that once Ω passes the first bending frequency ω , the motion becomes unstable. In reality, however, the Coriolis (1792 – 1843) forces stabilize motion.

Föppl’s contribution had been published in a journal which was probably not well known by contemporary rotor dynamicists. In 1916, the Royal Society of London therefore commissioned Henry H. Jeffcott (1877-1937) to resolve the conflicts between theoretical investigations and practical results. Three years later, Jeffcott published his results in the *Philosophical Magazine* [40]. Since then, the technical term *Jeffcott-Rotor* is used in english speaking areas, while in other countries the *Laval-Rotor* is more common.

Laval’s experimental setup, together with a stack of conically shaped disks inserted into the separator bowl of a milk centrifuge in around 1890 (so-called alpha-disks, patented by Clemens von Mauchenheim) was to revolutionize the dairy industry. In 2008, the Alpha-Laval-Company celebrated its 125th birthday. *Physical model needs intuition*

In autumn 1957, the former USSR launched its first statellite Sputnik. Three months later, the US Satellite Explorer successfully entered orbit. It consisted of a rigid body and some nearly massless hence negligible elastic antennas, and it was spin stabilized. One can see, in a foto from that time, how happy they were, Pickering, van Allen, von Braun. However, the satellite became unstable [35].

Considering the antimetric mode of a free rigid body in space, the equations of motion are obtained from eq.(4) by setting $\delta v, \delta w$ equal to zero and neglecting the restoring forces:

$$\begin{aligned} & \delta v' [\rho I_z \ddot{v}' - \rho(I_z + I_y - I_x) \dot{\alpha} \dot{w}' - \rho(I_y - I_x) \dot{\alpha}^2 v' - \underline{\rho I_z w' \ddot{\alpha}}] \\ & + \delta w' [\rho I_y \ddot{w}' + \rho(I_z + I_y - I_x) \dot{\alpha} \dot{v}' - \rho(I_z - I_x) \dot{\alpha}^2 w' - \underline{\rho(I_x - I_y) v' \ddot{\alpha}}] = 0. \end{aligned}$$

The cumbersome correction for the underlined terms becomes obsolete for $\ddot{\alpha} = 0$. Then, $\delta v'$ and $\delta w'$ are independent. Changing the axes notations ($x \rightarrow z, y \rightarrow x, z \rightarrow y, \dot{\alpha} \rightarrow \Omega$) along with $I_y = I_z \rightarrow A, I_x \rightarrow C$, one obtains the equations of motion in a co-rotating frame $\mathbf{M}\ddot{\mathbf{y}} + \mathbf{G}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0$ which, after dividing by A , read

$$\begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} + (\frac{C}{A}-2)\Omega \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + (\frac{C}{A}-1)\Omega^2 \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (11)$$

Here, $\mathbf{J} = \text{diag}\{A, A, C\}$ represents the inertia tensor. Spin takes place around the z -axis and $\mathbf{y} = (\alpha, \beta)^T$ considers small angular deviations.

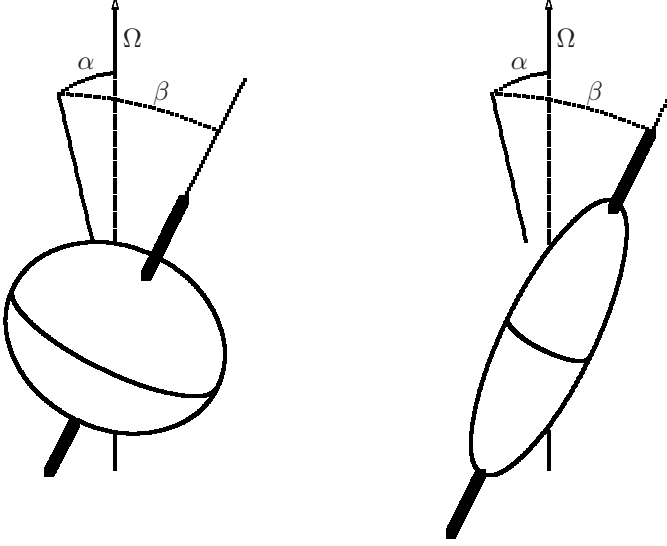


Figure 3. Spinning body, (a) $C/A > 1$; (b) $C/A < 1$

The characteristic polynomial for eq.(11) reads

$$P(\lambda) = [\lambda^2 + (\frac{C}{A}-1)\Omega^2]^2 + [\lambda(\frac{C}{A}-2)\Omega]^2 = 0 \Rightarrow \lambda = \begin{cases} \pm i(\frac{C}{A}-1)\Omega \\ \pm i\Omega \end{cases}. \quad (12)$$

Because $0 < \frac{C}{A} \leq 2 \forall \frac{C}{A}$ holds, the motion is always stable. This is also stated by the stability theorem of W.Thomson (1824 – 1907) and P.G.Tait (1831 – 1901) from 1879: stability is warranted for $\mathbf{K} > 0, \mathbf{K} < 0$ as long as $\det \mathbf{G} \neq 0, \|\mathbf{G}\|$ sufficient high.

When damping takes place, the equations of motion $\mathbf{M}\ddot{\mathbf{y}} + \mathbf{D}\dot{\mathbf{y}} + \mathbf{G}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0$ read:

$$\begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} + \left(\frac{d}{A}\right) \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + \left(\frac{C}{A} - 2\right)\Omega \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + \left(\frac{C}{A} - 1\right)\Omega^2 \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \quad (13)$$

Then, P.G.L. Dirichlet's (1805 – 1859) result from 1846 holds in the form: If $\mathbf{D} > 0$, then $\mathbf{K} > 0$ yields stability, otherwise $\mathbf{K} < 0$: instability, independent from $\mathbf{G}$. This was reported by N.G. Chetaev (1902 – 1959) (english translation 1961, [12]). Hence, as $\mathbf{K}$ from eq.(13) indicates, $(C/A) < 1$ reveals instability. This corresponds to the Explorer shape (Figure 3 (b)), but where should dampig come from? The breaktrough came with [38]: The reason of instability is (mainly) due to the elastic antennas with its material damping. Here, only $\mathbf{D} \geq 0$ holds, of course. Thus, separate the damping within the state equation,

$$\frac{d}{dt} \begin{pmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{E} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{G} \end{bmatrix} \begin{pmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{pmatrix} + \begin{bmatrix} 0 \\ -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \dot{\mathbf{y}}, \quad (14)$$

and treat it like a control input, one comes out with

$$\begin{aligned} & \mathbf{D} \geq 0 \bigwedge \left\{ \begin{bmatrix} 0 & \mathbf{E} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{G} \end{bmatrix}, \begin{bmatrix} 0 \\ -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \right\} \text{controllable} \\ & \Rightarrow \begin{cases} \mathbf{K} > 0 : & \text{stable} \\ \mathbf{K} < 0 : & \text{unstable} \end{cases} . \end{aligned} \quad (15)$$

Clearly, the material damping is very small. But, however, not negligible at all: instability means that also the smallest parameter will unevodingly lead to amplitude growing after a short while. An advice to rotor designers: let $(C/A) > 1$! *Physical model needs intuition.*

Perpendicular rotation. Considering a beam which rotates around a perpendicular axis, one has

$$\begin{aligned} & [C_{\text{Hub}} + \int_0^L (\rho A x^2)] \ddot{\gamma} + \int_0^L \rho A x \ddot{v} dx = M_z, \\ & \rho A (\ddot{v} + x \ddot{\gamma} - v \dot{\gamma}^2) + (EI_z v'')'' = 0, \end{aligned} \quad (16)$$

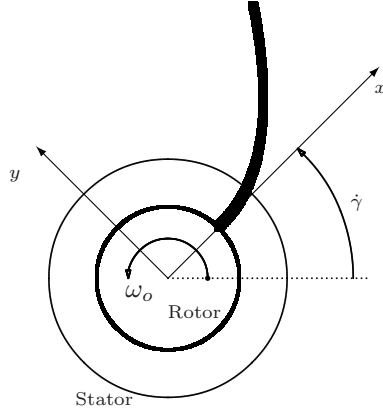


Figure 4. Rotating beam

where $v(x, t)$ denotes deflection and the prime represents differentiation w.r.t. x . In order to find a solution, one may use a Ritz ansatz [49]

$$v(x, t) = \mathbf{v}(x)^T \mathbf{q}(t) \quad (17)$$

along with Galerkin's method [20] which leads to

$$\begin{aligned}
 & \begin{bmatrix} C^o & \int \rho A x \mathbf{v}^T dx \\ \int \rho A x \mathbf{v} dx & \int \rho A \mathbf{v} \mathbf{v}^T dx \end{bmatrix} \begin{pmatrix} \ddot{\gamma} \\ \ddot{\mathbf{q}} \end{pmatrix} \\
 & + \begin{bmatrix} 0 & 0 \\ 0 & \int EI_z \mathbf{v} \mathbf{v}'''^T dx - \int \rho A \mathbf{v} \mathbf{v}^T dx \dot{\gamma}^2 \end{bmatrix} \begin{pmatrix} \gamma \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} M_M \\ 0 \end{pmatrix}. \quad (18)
 \end{aligned}$$

We will use clamped-free beam functions for shape functions. Integrating by parts yields $\int_o^L EI_z \mathbf{v} \mathbf{v}'''^T dx = \int_o^L EI_z \mathbf{v}'' \mathbf{v}''^T dx$ because of $EI_z \left[\mathbf{v} \mathbf{v}''' - \mathbf{v}' \mathbf{v}''^T \right]_o^L = 0$ due to boundary conditions.

The shape functions obey the orthogonality conditions

$$v_i(x) = \cosh k_i x - \beta_i \sinh k_i x - \cos k_i x + \beta_i \sin k_i x, \quad \int_0^L v_i v_j dx = \delta_{ij} L, \quad (19)$$

$$\frac{v_i(x)''}{k_i^2} = \cosh k_i x - \beta_i \sinh k_i x + \cos k_i x - \beta_i \sin k_i x, \quad \int_0^L v_i'' v_j'' dx = \delta_{ij} k_i^4 L \quad (20)$$

(δ_{ij} : Kronecker's symbol), with k_i from $\cosh(k_i L) \cos(k_i L) + 1 = 0$. Hence, the only term which remains unknown in eq.(18) reads $\int_0^L \rho A x v dx$. Inserting v_i from the differential equation $v_i'''' - k_i^4 v_i = 0$ yields, with an integration by parts,

$$\int_0^L x v_i dx = \frac{1}{k_i^4} \int_0^L x v_i'''' dx = \frac{1}{k_i^4} [L v_i'''(L) + v_i''(0) - v_i''(L)] = \frac{2}{k_i^2}, \quad (21)$$

see eq.(20), where $v''(L) = 0, v'''(L) = 0$ due to the boundary conditions. Assuming EI and ρA constant, the equations of motion $\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{Q}$ read

$$\mathbf{y} = \begin{pmatrix} \gamma \\ q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \quad \mathbf{M} = m_B \begin{bmatrix} \frac{L^2}{3} + \frac{C_{\text{Hub}}}{m_B} & \frac{2}{k_1^2 L} & \frac{2}{k_2^2 L} & \cdots & \frac{2}{k_n^2 L} \\ \frac{2}{k_1^2 L} & & & & \\ \frac{2}{k_2^2 L} & & & & \\ \vdots & & & & \\ \frac{2}{k_n^2 L} & & & & \end{bmatrix}, \quad (22)$$

$\mathbf{E}$

functions] would yield a nonzero value for eq.(24) – and thus totally unreliable results.

The reason for this is, by nearer inspection, quite clear. Consider, for instance, plane motion of a beam element represented by the deformation functions v, w . For the rotating beam one has then

$$\mathbf{p} \in \mathbb{R}^2, \quad \mathbf{q} = \begin{pmatrix} v \\ w \\ \alpha \end{pmatrix} \in \mathbb{R}^3, \quad (26)$$

where $\mathbf{p}$ denotes momentum and $\mathbf{q}$ consists of the minimal coordinates. Hence, the problem is kinematically overdetermined. For correct representation one would either additionally need the angular momentum $\mathbf{L}$ (here: $\mathbf{L} \in \mathbb{R}^1$, i.e. considering C_{Hub} along with some additional assumptions). Or: $\mathbf{q} = (v \ w)^T \in \mathbb{R}^2$ along with $\alpha = \alpha(t)$ as a (rheonomic) constraint.

Instability. Concerning the latter we may assume $\gamma = \Omega t$, $\Omega = \text{const.}$ The equations of motion from eq.(22) et seq. are then

$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{pmatrix} + \left[\frac{EI}{\rho A} \text{diag}(k_i^4) - \omega_o^2 \mathbf{E} \right] \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (27)$$

and the corresponding motor torque (which leads to $\Omega = \text{const.}$) reads

$$M_M = m_B \begin{bmatrix} \frac{2}{k_1^2 L} & \frac{2}{k_2^2 L} & \cdots & \frac{2}{k_n^2 L} \end{bmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{pmatrix}. \quad (28)$$

$\Omega = \text{const}$ is a common assumption in rotor dynamics –mostly without asking where it comes from. The question arises whether a motor torque according to eq.(28) is possible. In 1902, Sommerfeld observed that under certain circumstances the drive torque did not longer accelerate the rotor speed but instead excited shaft bending oscillations [51].

But even if, in our case, the corresponding torque is available – is the equation of motion correct? Obviously not: Applying a constant torque leads to bending instability after a certain while.

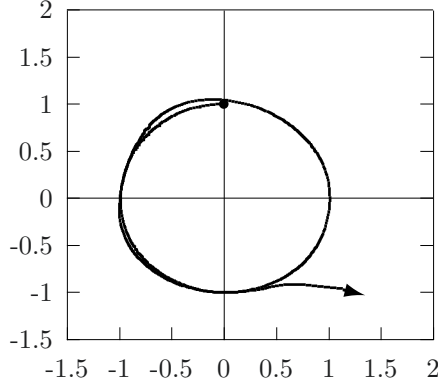


Figure 5. Unstable beam (tip deflection)

The reason is quite simple: the centrifugal force which acts on the beam element reads

$$d\mathbf{f}_{\text{cent}} = \rho A \omega_o^2 dx \begin{pmatrix} x \\ v \end{pmatrix} \quad (29)$$

while the deflection, for a linear approach, only takes the y -direction into account. In reality, however, the element does not move on a straight line but on some kind of circular arc. There is thus also a deflection in (negative) x -direction which reads $\int_o^x v'^2 dx/2$. The effect of longitudinal forces was already known to Euler [14]. It may be calculated via the corresponding virtual work (along with a Ritz ansatz)

$$\delta W_{\text{cent},x} = - \int_o^L \left[\int_o^x v' \delta v' d\xi \right] \rho A \omega_o^2 x dx = - \delta \mathbf{q}^T \int_o^L \mathbf{v}' \mathbf{v}'^T \frac{\rho A}{2} \omega_o^2 (L^2 - x^2) dx \mathbf{q}. \quad (30)$$

Considering eq.(30) along with eq.(27), the motion equation reads

$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{pmatrix} + \left[\frac{EI}{\rho A} \text{diag}(k_i^4) - \omega_o^2 \left\{ \mathbf{E} - \frac{1}{2L} \int_o^L (L^2 - x^2) \mathbf{v}' \mathbf{v}'^T dx \right\} \right] \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (31)$$

Applying a constant motor torque, the beam motion now remains stable.

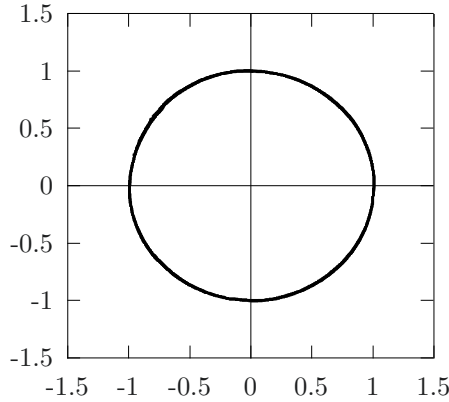


Figure 6. Rotating beam (tip deflection)

This behaviour reflects reality: a helicopter blade will never lead to instability when the rotor speed increases. Just in contrary: the blade is somehow more and more stiffened. The effect which results from second order displacement is therefore sometimes called “dynamical stiffening”. Such a notation, however, should not lead to believe in a new theory. Neglecting such effects is, on the contrary, a severe mistake when linearizing the equations for small bending amplitudes. *Mathematical modeling needs interpretation.* There are more such effects which result from second order displacements. These are obtained in two steps:

- From Cauchy (fluctuating reference during motion) to Trefftz (fixed reference) where zero-order unit forces/torques are correctly applied, [26],
- Fubini type integration by parts, $\int_0^L g(x) \int_0^x h(\xi) d\xi dx = \int_0^L h(x) \left[\int_x^L g(\xi) d\xi \right] dx$, yields independency of load assumptions, [8].

The second order displacements [superscript (2)] are then

$$\begin{aligned} \mathbf{r}^{(2)} &= -\frac{1}{2} \int_0^x \begin{pmatrix} v'^2 + w'^2 \\ -2[(x-\xi)\vartheta'w' - \vartheta w'] \\ +2[(x-\xi)\vartheta'v' - \vartheta v'] \end{pmatrix} d\xi, \\ \varphi^{(2)} &= +\frac{1}{2} \int_0^x \begin{pmatrix} +(v'w'' - w'v'') d\xi \\ +2\vartheta'v' d\xi \\ +2\vartheta'w' d\xi \end{pmatrix} d\xi \end{aligned} \quad (32)$$

[compare $\mathbf{r}_1^{(2)}$ (first component) with eq.(30)].

First attempts to consider flexible multilink robots considered a rotating beam with a heavy end load m_E , representing the neighbouring subsystem in a first approximation [56]. Then, torsional deflection has additionally to be taken into account. The equations read

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{Q}, \quad \mathbf{y} = \begin{pmatrix} \gamma(t) \\ \mathbf{q}_v(t) \\ \mathbf{q}_\vartheta(t) \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} M_{mot} \\ 0 \\ 0 \end{pmatrix} \quad (33)$$

$$\text{where, for } m_E \gg \rho AL, \quad (34)$$

$$\mathbf{K} = \int_0^L \begin{bmatrix} 0 & 0 & 0 \\ 0 & EI_z \mathbf{v}'' \mathbf{v}''^T + \dot{\gamma}^2 m_E [L \mathbf{v}' \mathbf{v}'^T - \mathbf{v}_E \mathbf{v}_E^T] & 0 \\ 0 & 0 & GI_D \vartheta' \vartheta'^T \end{bmatrix} dx \quad (35)$$

(index E for “end”). Here, torsion appears decoupled. However, the corresponding (first) frequency differed by around 30% from the experimental data. This cannot be explained by numerical unconciseness.

The reason is once more a second order effect: as eq.(32) shows, there is a severe coupling between bending and torsion (which was already known to Prandtl, [46]). Taking that into account, the restoring matrix becomes

$$\mathbf{K} = \int_0^L \begin{bmatrix} 0 & 0 & 0 \\ 0 & EI_z \mathbf{v}'' \mathbf{v}''^T + \dot{\gamma}^2 m_E [L \mathbf{v}' \mathbf{v}'^T - \mathbf{v}_E \mathbf{v}_E^T] & m_E g(L-x) \mathbf{v}'' \vartheta'^T \\ 0 & m_E g(L-x) \vartheta \mathbf{v}''^T & GI_D \vartheta' \vartheta'^T \end{bmatrix} dx. \quad (36)$$

The centrifugal effect (underlined in eq.(36)) will probably not play a significant role in robot maneuvers. However, from physical interpretation, it will not make sense to cancel the “dynamical stiffening” part and leave the divergent one. For non-permanent rotations, the whole centrifugal reaction may be neglected, coming out with a *linear* motion equation. These assumptions have to be verified, of course.

Small forces are usually neglected, but caution with nonlinear effects! A T-shaped elastic beam system for instance shows – in a certain parameter range – out-of-plane motions which correspond to parameter resonance, although resonance will of course not really take place due to limited energy: Same as in the resonance case, the beginning of motion is exponentially unstable, but then, at a certain amplitude, the effect turns the other way round. This leads to beating waves for β and q . Figure 7 shows a simplified model.

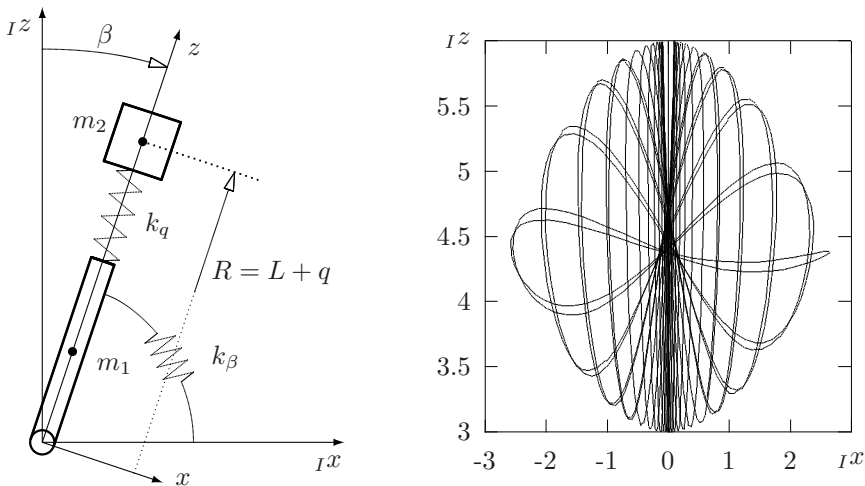


Figure 7. (a) a 2-dof oscillator; (b) tip mass motion

Physical modeling needs intuition, mathematical modeling needs interpretation.

2.3 Modeling Questions

- The-Helicopter-Blade-Example: Where is Coriolis?
 - Excursion: the tremendous Coriolis effects,
 - * e.g. weather forecast (remember hurricane Carrie and the Pamir)

- * e.g. rotor dynamics (remember Laval)
- Caution with nonlinear effects!
 - * e.g. weather forecast (see Lorenz and Chaos)
 - * e.g. T-shaped beam (reveals *typical* effects as often observed)
- elastic robot slewing maneuver.
 - Experiment: Coriolis negligible? yes.
 - Experiment: Torsion a decoupled motion? mostly not.
 - Experiment: Centrifugal effects negligible? mostly yes.

What does modeling tell us – truth, reality or just illusion? To make things true one may use assumptions. The truth is, that Carthago has been destroyed; the assumption may be that it owned weapons of mass destruction to justify deletion. This corresponds to *political correctness* which has nothing in common with truth.

René Descartes – one of our philosophers’ protagonists – formulated an impact rule: when a smaller body hits a bigger one, then the smaller one is reflected while the bigger remains at rest. And he continues: “if experience seems to contradict, then we must trust reason more than perception” [52]. This might be called *scientific correctness* for use in philosophy which has nothing in common with truth.

Truth may perhaps never be obtained but at least reality. What is needed here is *experimental verification*, a postulation since Galileo’s days.

3 Modeling – Basic Requests

Basic request in modeling is its fundamental law “*as simple as possible, as complicated as necessary*”. The aim is to evaluate the easiest straight-on procedure(s).

The *physical model* consists of bodies and interaction forces. As simple as possible for instance means: the hub motion of a car should not include the shell model of the cabin. The *mathematical model* may either be considered on the basis of Euler’s results or on the Lagrangian approach. The angular velocity ω hereby turns out nonholonomic – a headache for generations of scientists. To obtain ω via Poisson’s equation – as sometimes proposed – is by far too complicated. Mostly, in holonomic systems, ω is directly inserted as a function of the minimal coordinates $\mathbf{q}$ and its derivatives – this complicates the mathematical model, even more if ω is (as a whole) available from measurements.

The Euler approach uses the momentum and the moment of momentum theorems along with constraint forces. The Lagrangian approach (Lagrange-Principle) combines both as, so to say, a “flat rate” (all in one) but needs

some explanation (how to obtain two axioms out of one?). Its evaluation needs functional matrices (Jacobians) which sometimes calculated via Poisson's equation (with an indication for numerical codes) – this is a complicated and unnecessary detour: the Jacobians are a-priori known from kinematics without any extra calculation.

The use of inertial reference frames is frequently recommended. This is by far too restrictive. Instead, moving frames adapt physical and mathematical modeling for the actual problem under consideration.

When elastic deformations take place, then Galerkin's approach is often used. Its shape functions have to fulfill the dynamical boundary conditions. The latter, however, can be considered with an extended Galerkin procedure (as often recommended). With an integration by parts, the boundary terms are shifted into the integral – which brings us back to the beginning of investigation: the direct Ritz approach (virtual work). A re-integration from Galerkin is obsolete.

If nonholonomic constraints arise: are there better suited Principles – as for instance Jourdain's one ([27], as often asserted)? This question leads to an interpretation of the Lagrangian foregoing which simultaneously builds bridges to optimization.

Considering the physical model as interacting rigid and beamlike elastic bodies, *the Euler-Bernoulli-Beam is a powerful tool.*

3.1 The Physical Model

Let us continue with physical modeling. Two typical robot assemblies are depicted in Figure 8: a motor-gear-arm unit and a telescoping arm mounted at the gear output.

Inspecting these figures shows that the physical model consists of

- rigid and elastic bodies, interconnected by
- impressed forces
 - passive: springs, dampers, ...
 - active: control

Everybody knows that a rigid body does not exist in reality. It is, however, a brilliant basic model.

3.2 The Mathematical Model

Eulerian approach.

$$\begin{aligned}
 \mathbf{p} &= m\mathbf{v} \in \mathbb{R}^3, \quad \dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}}\mathbf{p} = \mathbf{f} \quad (\text{translation (1736), [13]}) \\
 \mathbf{L} &= \mathbf{J}\boldsymbol{\omega} \in \mathbb{R}^3, \quad \dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}\mathbf{L} = \mathbf{M} \quad (\text{rotation (1750), [15]})
 \end{aligned}
 \tag{37}$$

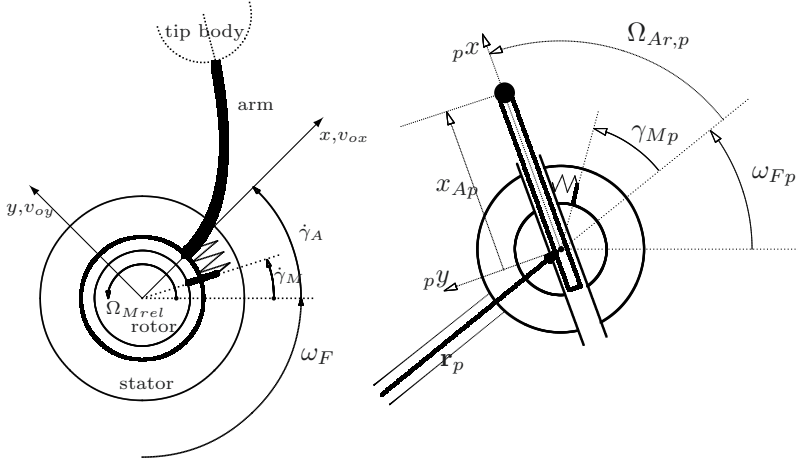


Figure 8. (a) motor-gear-arm model; (b) telescoping arm

(m : mass, $\mathbf{v}$: velocity (body-fixed frame), $\mathbf{J}$: inertia tensor (constant in body-fixed frame), $\boldsymbol{\omega}$: angular velocity (body-fixed frame), $\tilde{\boldsymbol{\omega}}$: spin tensor, $\mathbf{f}$: force, $\mathbf{M}$: moment).

The crucial point is $\boldsymbol{\omega}$. How to obtain $\boldsymbol{\omega}$? Let ${}_B\mathbf{r} := \mathbf{r}$ denote a point within the considered body, represented in a body-fixed frame B . Let furthermore $\mathbf{A} = \mathbf{A}_{IB}$ be the transformation matrix which transforms ${}_B\mathbf{r}$ into an inertial representation: ${}_I\mathbf{r} = \mathbf{A}_{IB} {}_B\mathbf{r}$. Thereby, $\mathbf{A}_{IB}$ is considered orthonormal: $\mathbf{A}_{BI}^T = \mathbf{A}_{IB} := \mathbf{A}$, $\mathbf{A}^T \mathbf{A} = \mathbf{E}$ (unity). To obtain the absolute velocity in a body-fixed representation one has first to transform vector $\mathbf{r}$ into the inertial base, then differentiate w.r.t. time (to get the absolute measure) and eventually to transform into the (desired) body-fixed frame:

$$\mathbf{v} = \mathbf{A}^T \frac{d}{dt} [\mathbf{A} \mathbf{r}] = \dot{\mathbf{r}} + \left[\mathbf{A}^T \frac{d\mathbf{A}}{dt} \right] \mathbf{r}. \quad (38)$$

From eq.(38) one obtains

$$\left[\mathbf{A}^T \frac{d\mathbf{A}}{dt} \right] = \tilde{\boldsymbol{\omega}} \quad (39)$$

where $\tilde{\boldsymbol{\omega}} \in \mathbb{R}^{3,3}$ is the skew symmetric spin tensor assigned to $\boldsymbol{\omega} \in \mathbb{R}^3$. Eq.(39) is often called “Poisson Equation” [Simeon Poisson (1781 – 1840)]. Is eq.(39) the way to obtain $\boldsymbol{\omega}$ – as often proposed in literature? Obviously not. Once disposing of $\mathbf{A}$, the angular velocity $\boldsymbol{\omega}$ is in advance known. Let for instance $\mathbf{A}$ be

$$\mathbf{A}^T = \mathbf{A}_{BI} = \mathbf{A}_\alpha \mathbf{A}_\beta \mathbf{A}_\gamma, \quad (40)$$

i.e. consisting of consecutive elementary transformations γ (z -axis), β (y -axis), α (x -axis). The angular velocity (in body-fixed representation) is then simply

$$\boldsymbol{\omega} = [\mathbf{e}_1 \mid \mathbf{A}_\alpha \mathbf{e}_2 \mid \mathbf{A}_\alpha \mathbf{A}_\beta \mathbf{e}_3] \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} \quad (41)$$

which means that the first rotation ($\dot{\gamma}\mathbf{e}_3$, $\mathbf{e}_3$: 3rd unit vector) is transformed with the following rotations β, α ; the second one ($\dot{\beta}\mathbf{e}_2$, $\mathbf{e}_2$: 2nd unit vector) is transformed with the following rotation α while the last one ($\dot{\alpha}\mathbf{e}_1$, $\mathbf{e}_1$: 1st unit vector) is represented in the final (body-fixed) frame. Extracting $\boldsymbol{\omega}$ from eq.(39) is by far too tedious.

Eq.(41) reads explicitly

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{bmatrix} 1 & 0 & -\sin \beta \\ 0 & \cos \alpha & \sin \alpha \cos \beta \\ 0 & -\sin \alpha & \cos \alpha \cos \beta \end{bmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}. \quad (42)$$

Let $\boldsymbol{\omega} dt := d\boldsymbol{\pi}$. One has then, for the first component,

$$\begin{aligned} d\pi_x &= [1]d\alpha + [0]d\beta + [-\sin \beta]d\gamma \\ &= \left(\frac{\partial \pi_x}{\partial \alpha} \right) d\alpha + \left(\frac{\partial \pi_x}{\partial \beta} \right) d\beta + \left(\frac{\partial \pi_x}{\partial \gamma} \right) d\gamma \end{aligned} \quad (43)$$

which shows, with $\left(\frac{\partial^2 \pi_x}{\partial \beta \partial \gamma} \right) = 0$ and $\left(\frac{\partial^2 \pi_x}{\partial \gamma \partial \beta} \right) = -\cos \beta$, that $\boldsymbol{\omega}$ (and, consequently $\mathbf{v}$ for a noninertial representation) is not (elementary) integrable [H.A. Schwarz (1843 – 1921)].

$\boldsymbol{\omega}$ is called a nonholonomic (velocity) variable. Nonholonomicity seems somewhat rare and caused a lot of problems over decades. This might also be the reason that mostly Lagrange's (1736 – 1813) equations of the second kind [33] have been used by directly inserting eq.(42) into the kinetic energy where then $\dot{\mathbf{q}} = (\dot{\alpha} \ \dot{\beta} \ \dot{\gamma})^T$ is, of course, holonomic. Consequently one was helpless with nonholonomic constraints.

A simple example will enlighten the background (greetings from “the red baron” with his Fokker Dr.I; Dr=“Dreidecker”=tri-plane): Let the plane model be a rigid body in space. Its motion is represented by eq.(37) the

angular part of which (the so-called Euler Equation) reads explicitly

$$\begin{aligned} A\dot{\omega}_x - (B - C)\omega_y\omega_z &= M_x \\ B\dot{\omega}_y - (C - A)\omega_x\omega_z &= M_y \\ C\dot{\omega}_z - (A - B)\omega_x\omega_y &= M_z \end{aligned} \quad (44)$$

The Pilot applies simultaneously torques M_x (aileron), M_y (rudder), M_z (elevator) as depicted in Figure 9(a).

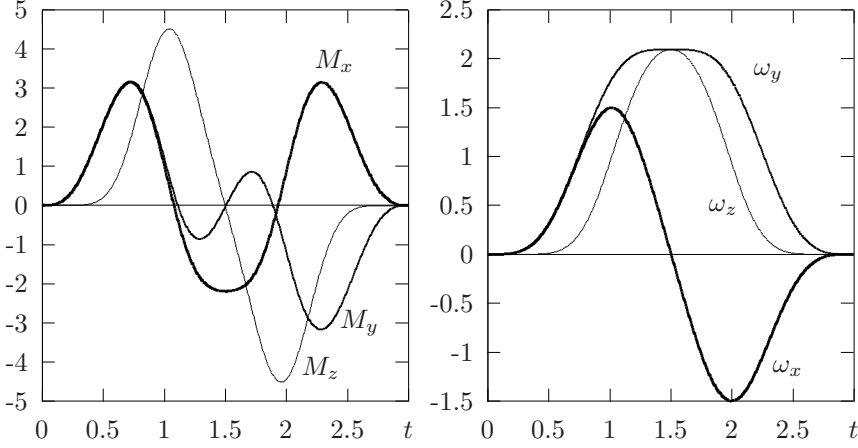


Figure 9. (a) Applied torque over time; (b) on board measurement

The aim is to hold γ equal to zero and to turn α and β around 180 degrees (Immelmann maneuver). However, as the basic gyro effects tell (“parallelism”), the w.r.t. $\dot{\gamma}$ perpendicular rotations will cause a precession around the $\mathbf{e}_3$ -axis which forces the pilot to counteract. The (simultaneous) Immelmann maneuver challenges the pilot a lot.

The applied torques according to Figure 9(a) lead to angular velocities as depicted in Figure 9(b). These are on board measured by means of rate gyros for instance.

Because they are not elementary integrable a parametrization is needed, carried out with eq.(42) for instance. Resolving eq.(42) for $\dot{\mathbf{q}} = (\dot{\alpha} \ \dot{\beta} \ \dot{\gamma})^T$ leads to the ODE

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{bmatrix} 1 & \sin \alpha \tan \beta & \cos \alpha \tan \beta \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha / \cos \beta & \cos \alpha / \cos \beta \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad (45)$$

which has to be solved to obtain the orientation, see Figure 10.

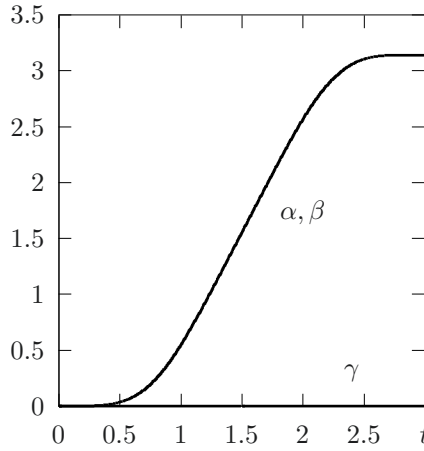


Figure 10. Resulting Orientation

Remark: Max Immelmann (1890 – 1916) flew his maneuver with a Fokker E.I. (E.=“Eindecker”=mono-plane) which did not have ailerons. Immelmann could only apply wing torsion by means of cables. It is therefore quite assumable that he operated his plane not by a simultaneous maneuver but consecutively, first β ($0 \rightarrow 180^\circ$) and then α ($0 \rightarrow 180^\circ$), without any γ -motion. Such a maneuver is, of course, elementary integrable.

Lagrangian approach. The Lagrange Principle: It was in 1764 when Lagrange won the big prize of the french academy [32]. The title of his paper was *récherches sur la libration de la lune*, and on page 9 we find his principle (in modern notation)

$$\int_{(S)} (d\mathbf{m}\ddot{\mathbf{r}} - d\mathbf{f}^e)^T \delta \mathbf{r} = 0 \quad (46)$$

which nowadays is mostly referred to d’Alembert. However, d’Alembert’s principle did not have much in common with eq.(46): he inspected equilibrium conditions of the so called lost forces [24], a procedure, which is hard to apply for multi degree systems.

Lagrange concentrated his research on analytical methods; his famous second kind equations came out in 1780 (*Théorie de la libration de la lune*),

and in his famous *Méchanique Analytique* from 1788 he used eq.(46) (*the general formula of dynamics*) as the basis of all his investigations.

The Flat Rate: All in One. Eq.(46) needs an inertial representation to obtain the absolute acceleration via $\ddot{\mathbf{r}}$. However, considering a rigid body, one may decompose $\mathbf{r}$ into $\mathbf{r} = {}_I\mathbf{r}_c + \mathbf{A}_{IBB}\mathbf{r}_p := \mathbf{r}_c + \mathbf{A}\mathbf{r}_p$ where $\mathbf{r}_c$ denotes the mass center position and $\mathbf{r}_p$ represents an arbitrary point, being constant in a body fixed representation. Then, $\mathbf{A}$ transforms $\mathbf{r}_p$ onto the inertial frame.

Along with $\dot{\mathbf{r}} = \dot{\mathbf{r}}_c + \dot{\mathbf{A}}\mathbf{r}_p \Rightarrow \delta\mathbf{r} = \delta\mathbf{r}_c + \delta\mathbf{A}\mathbf{r}_p$, $\mathbf{A}^T\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}} \Rightarrow \mathbf{A}^T\delta\mathbf{A} = \delta\tilde{\boldsymbol{\pi}}$ and $\tilde{\mathbf{r}}_p\tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\tilde{\mathbf{r}}_p + \tilde{\boldsymbol{\omega}}\tilde{\mathbf{r}}_p\tilde{\boldsymbol{\omega}} = 0$, having $\int_{(B)} \mathbf{r}_p dm = 0$ (mass center) in mind, one obtains

$$\delta\boldsymbol{\pi}^T [\dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}\mathbf{L} - \mathbf{M}^e] + \delta\boldsymbol{\rho}^T [\dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}}\mathbf{p} - \mathbf{f}^e] = 0 \quad (47)$$

where

- $\dot{\boldsymbol{\pi}} = \boldsymbol{\omega}$ (angular velocity), $\dot{\mathbf{p}} = \mathbf{v}$ (translational velocity),
- $\mathbf{L} = \mathbf{J}\boldsymbol{\omega}$ (momentum of momentum), $\mathbf{J} = \int_B \tilde{\mathbf{r}}\tilde{\mathbf{r}}^T dm$ (inertia tensor),
- $\mathbf{p} = m\mathbf{v}$ (momentum), m (mass),

all vector terms represented in the body fixed frame. Eq.(47) holds for a single rigid body; because it represents a scalar equation, the multi body system simply needs summation. One has thus found the “Flat Rate” [compare eq.(37)].

Consolidation. But something is seemingly going wrong. It was in 1775 (39 years after Euler’s first contribution on dynamics) when Leonhard Euler proposed the linear and the rotational momentum to be independent axioms (*nova methodus motum corporum rigidorum determinandi* [17]). Then, however, the question arises how to construe these two just from one in eq.(46) – a wondrous amplification of axioms?

The answer is simple but worth to be reflected carefully: A certain number of particles (or mass elements dm) will never form a solid unless they are glued together. In a rigid body, these gluing forces do not come at sight because they are constraint forces (rigidity: they do not perform work because of missing deflection). But they are nevertheless present in order to hold the body together.

Applying the momentum of momentum theorem then yields the constraint torques which, in terms of stress components σ_{ij} , reveals a symmetric stress tensor $\sigma_{ij} = \sigma_{ji} \forall i \neq j$. One can argue the other way round: Assuming a symmetric stress tensor as a second axiom (so to say in the background) allows to calculate the momentum of momentum via a simple vector product which then defines the rigid body correctly. However,

symmetry has then to be seen axiomatic, mostly referred to as Boltzmann's (1844–1906) axiom (the namig is possibly due to Georg Hamel (1877 – 1954) [53]).

$\dot{\omega}$ is the time derivative of the corresponding quasi-coordinate $\pi(\mathbf{q})$. [The same holds for $\mathbf{v} = \dot{\boldsymbol{\rho}}$ where $\rho(\mathbf{q})$]. In order to calculate eq.(47) in minimal representation one has $(\partial\pi/\partial\mathbf{q})\delta\mathbf{q}$ and $(\partial\rho/\partial\mathbf{q})\delta\mathbf{q}$, respectively. Because $\delta\mathbf{q}$ is arbitrary, one obtains

$$\left(\frac{\partial\pi}{\partial\mathbf{q}}\right)^T [\dot{\mathbf{L}} + \tilde{\omega}\mathbf{L} - \mathbf{M}^e] + \left(\frac{\partial\rho}{\partial\mathbf{q}}\right)^T [\dot{\mathbf{p}} + \tilde{\omega}\mathbf{p} - \mathbf{f}^e] = 0. \quad (48)$$

How to calculate $(\partial\pi/\partial\mathbf{q})$ and $(\partial\rho/\partial\mathbf{q})$? Some authors postulate to extract $\partial\tilde{\pi} = \mathbf{A}^T\partial\mathbf{A}$ from the Poisson-equation $\tilde{\omega} = \dot{\tilde{\pi}} = \mathbf{A}^T\dot{\mathbf{A}}$ where $\mathbf{A} = \mathbf{A}_{IB}$ consists of unit vectors ${}_I\mathbf{e}_{Bi}$, $i = 1, 2, 3$ or, in short, $\mathbf{A} = [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3]$. Extracting $\partial\pi$ from the skew symmetric spin tensor then yields

$$\begin{bmatrix} 0 & -\partial\pi_z & +\partial\pi_y \\ +\partial\pi_z & 0 & -\partial\pi_x \\ -\partial\pi_y & +\partial\pi_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{e}_1^T\partial\mathbf{e}_2 & \mathbf{e}_1^T\partial\mathbf{e}_3 \\ \mathbf{e}_2^T\partial\mathbf{e}_1 & 0 & \mathbf{e}_2^T\partial\mathbf{e}_3 \\ \mathbf{e}_3^T\partial\mathbf{e}_1 & \mathbf{e}_3^T\partial\mathbf{e}_2 & 0 \end{bmatrix} \\ \Rightarrow (\partial\pi_x \ \partial\pi_y \ \partial\pi_z) = [\mathbf{e}_3^T\partial\mathbf{e}_2 \ \mathbf{e}_1^T\partial\mathbf{e}_3 \ \mathbf{e}_2^T\partial\mathbf{e}_1]. \quad (49)$$

Considering the antimetric motions of a rotor according to Figure 3, with $\mathbf{q}^T = (\alpha, \beta)$, the transformation matrix (40) reads

$$\mathbf{A} = [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3] = \begin{bmatrix} \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \cos\beta\sin\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma \\ \sin\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta \end{bmatrix}. \quad (50)$$

One obtains for instance

$$\begin{aligned}
\partial\pi_x = & \left[\cos^2 \alpha \sin^2 \beta \cos^2 \gamma + \sin \alpha \cos \alpha \sin \beta \sin \gamma \cos \gamma \right. \\
& + \sin \alpha \cos \alpha \sin \beta \sin \gamma \cos \gamma + \sin^2 \alpha \sin^2 \gamma \\
& + \cos^2 \alpha \sin^2 \beta \sin^2 \gamma - \sin \alpha \cos \alpha \sin \beta \sin \gamma \cos \gamma \\
& - \sin \alpha \cos \alpha \sin \beta \sin \gamma \cos \gamma + \sin^2 \alpha \cos^2 \gamma \\
& + \cos^2 \alpha \cos^2 \beta \left. \partial\alpha \right. \\
& + \left[\sin \alpha \cos \alpha \sin \beta \cos \beta \cos^2 \gamma + \sin^2 \alpha \cos \beta \sin \gamma \cos \gamma \right. \\
& + \sin \alpha \cos \alpha \sin \beta \cos \beta \sin^2 \gamma - \sin^2 \alpha \cos \beta \sin \gamma \cos \gamma \\
& \left. - \sin \alpha \cos \alpha \sin \beta \cos \gamma \right] \partial\beta.
\end{aligned} \tag{51}$$

Applying the well known trigonometric formulas reduces eq.(51) to $\partial\pi_x = [1] \partial\alpha + [0] \partial\beta$ – and motivates to take a few minutes to review the procedure. At first we must state that π itself is not obtained but only the components of $(\partial\pi/\partial\mathbf{q})$. But π does not enter the calculation anywhere, only the corresponding differentials, see eq.(48). These are, however, already known. Decomposing eq.(41) yields

$$\boldsymbol{\omega} = \dot{\boldsymbol{\pi}} = [\mathbf{e}_1 \mid \mathbf{A}_\alpha \mathbf{e}_2] \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + [\mathbf{A}_\alpha \mathbf{A}_\beta \mathbf{e}_3] \dot{\gamma} = \left(\frac{\partial\boldsymbol{\pi}}{\partial\mathbf{q}} \right) \dot{\mathbf{q}} + \left(\frac{\partial\boldsymbol{\pi}}{\partial t} \right) \tag{52}$$

or explicitly, along with $\dot{\gamma} = \Omega$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{bmatrix} \frac{1}{0} & \frac{0}{\cos \alpha} \\ 0 & -\sin \alpha \end{bmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + \begin{pmatrix} -\sin \beta \\ \sin \alpha \cos \beta \\ \cos \alpha \cos \beta \end{pmatrix} \Omega \tag{53}$$

where the underlined components refer to eq.(51). Once $\boldsymbol{\omega}$ is known, there is no additional calculation required (nor the use of a corresponding computer code as sometimes proposed to solve eq.(51)). Because the (cartesian) velocity is always linear in $\dot{\mathbf{q}}$, the requested Jacobian is nothing but the coefficient matrix w.r.t. the minimal velocities $\dot{\mathbf{q}}$: eq.(52) yields

$$\left(\frac{\partial\boldsymbol{\pi}}{\partial\mathbf{q}} \right) = \left(\frac{\partial\boldsymbol{\omega}}{\partial\dot{\mathbf{q}}} \right) \left[\text{and, analogously } \left(\frac{\partial\boldsymbol{\rho}}{\partial\mathbf{q}} \right) = \left(\frac{\partial\mathbf{v}}{\partial\dot{\mathbf{q}}} \right) \right]. \tag{54}$$

Thus, replacing the Jacobians in eq.(48) by eq.(54), one obtains

$$\left(\frac{\partial\boldsymbol{\omega}}{\partial\dot{\mathbf{q}}} \right)^T [\dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}\mathbf{L} - \mathbf{M}^e] + \left(\frac{\partial\mathbf{v}}{\partial\dot{\mathbf{q}}} \right)^T [\dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}}\mathbf{p} - \mathbf{f}^e] = 0 \tag{55}$$

where the unbeloved quasi-coordinates do not appear any more.

Up to here, the calculation corresponds to a body fixed coordinate system. This is far too restrictive. Some authors prefer an inertial frame I (since then, at least the translational part is integrable). This is also far too restrictive. However, eq.(55) contains a simple access to arbitrary (orthonormal) frames: just insert a unit matrix in the form $\mathbf{A}_{RB}^T \mathbf{A}_{RB}$:

$\left(\frac{\partial \boldsymbol{\omega}}{\partial \dot{\mathbf{q}}}\right)^T \mathbf{A}_{RB}^T \mathbf{A}_{RB} [\dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}} \mathbf{L} - \mathbf{M}^e] + \left(\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}}\right)^T \mathbf{A}_{RB}^T \mathbf{A}_{RB} [\dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}} \mathbf{p} - \mathbf{f}^e] = 0$.
 $\mathbf{A}_{RB}^T$ can not depend on velocities, it may be shifted to $(\partial[\mathbf{A}_{RB} \boldsymbol{\omega}]/\partial \dot{\mathbf{q}})^T = (\partial_R \boldsymbol{\omega}/\partial \dot{\mathbf{q}})^T$. The remaining part then reads $\mathbf{A}_{RB}[\dot{\mathbf{L}} + \mathbf{A}_{BI} \mathbf{A}_{IB} \mathbf{L}]$ which yields the representation $[_R \dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}_{IR} \mathbf{L}]$, and $\mathbf{A}_{RB} \mathbf{M}^e$ transforms to $_R \mathbf{M}^e$ (translational parts analogously). One obtains thus, for a representation in an arbitrary coordinate system R ,

$$\left(\frac{\partial \boldsymbol{\omega}}{\partial \dot{\mathbf{q}}}\right)^T \left[\dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}_{IR} \mathbf{L} - \mathbf{M}^e\right] + \left(\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}}\right)^T [\dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}}_{IR} \mathbf{p} - \mathbf{f}^e] = 0 \quad (56)$$

where

- all vector terms represented in reference frame R
- $\boldsymbol{\omega}, \mathbf{v}$: mass center velocities; $\mathbf{L} = \mathbf{J} \boldsymbol{\omega}, \mathbf{p} = m\mathbf{v}$: momenta
- $\boldsymbol{\omega}_{IR}$: angular velocity of the reference frame w.r.t. inertial frame
 - e.g. $R = I$: inertial representation ($\boldsymbol{\omega}_{IR} = \boldsymbol{\omega}_{II} = 0$),
 - e.g. $R = B$: body fixed representation ($\boldsymbol{\omega}_{IR} = \boldsymbol{\omega}_{IB} = \boldsymbol{\omega}$),
 - e.g. $R \neq I \neq B$: arbitrary representation

Moving reference frames. The advantages of moving frames are clearly demonstrated with a simple example: the edge mill (already in use in the ancient Greece, later copied by the romans, still in use today). It consists of two mill stones (Figure 11, only one is sketched).

We are interested in the reaction torque at the contact point. The use of an inertial frame then needs a post-transformation: $\mathbf{A}_\alpha \dot{\mathbf{L}} = -\mathbf{M}^{react} =$

$$\begin{pmatrix}
 (A \cos^2 \alpha + C \sin^2 \alpha) \dot{\gamma}^2 \sin \alpha \cos \alpha - (A - C) \dot{\gamma}^2 \sin \alpha \cos^3 \alpha \\
 + (A - C) \dot{\gamma}^2 \sin^3 \alpha \cos \alpha - (A \sin^2 \alpha + C \cos^2 \alpha) \dot{\gamma}^2 \sin \alpha \cos \alpha \\
 (A \cos^2 \alpha + C \sin^2 \alpha) \dot{\alpha} \dot{\gamma} \cos^2 \alpha - (A - C) \dot{\alpha} \dot{\gamma} \sin^2 \alpha \cos^2 \alpha \\
 + A \dot{\alpha} \dot{\gamma} \cos^2 \alpha + (A - C) \dot{\alpha} \dot{\gamma} \sin^2 \alpha \cos^2 \alpha - (A \sin^2 \alpha + C \cos^2 \alpha) \dot{\alpha} \dot{\gamma} \cos^2 \alpha \\
 - (A \sin^2 \alpha + C \cos^2 \alpha) \dot{\alpha} \dot{\gamma} \sin^2 \alpha - (A - C) \dot{\alpha} \dot{\gamma} \sin^2 \alpha \cos^2 \alpha \\
 + A \dot{\alpha} \dot{\gamma} \sin^2 \alpha + (A - C) \dot{\alpha} \dot{\gamma} \sin^2 \alpha \cos^2 \alpha - (A \cos^2 \alpha + C \sin^2 \alpha) \dot{\alpha} \dot{\gamma} \sin^2 \alpha \\
 - (A \cos^2 \alpha + C \sin^2 \alpha) \dot{\alpha} \dot{\gamma} \cos \alpha \sin \alpha + (A - C) \dot{\alpha} \dot{\gamma} \sin^3 \alpha \cos \alpha \\
 - A \dot{\alpha} \dot{\gamma} \cos \alpha \sin \alpha - (A - C) \dot{\alpha} \dot{\gamma} \sin^3 \alpha \cos \alpha \\
 - (A \sin^2 \alpha + C \cos^2 \alpha) \dot{\alpha} \dot{\gamma} \sin \alpha \cos \alpha - (A - C) \dot{\alpha} \dot{\gamma} \sin \alpha \cos^3 \alpha \\
 + A \dot{\alpha} \dot{\gamma} \sin \alpha \cos \alpha + (A - C) \dot{\alpha} \dot{\gamma} \sin \alpha \cos^3 \alpha \\
 (A \sin^2 \alpha + C \cos^2 \alpha) \dot{\alpha} \dot{\gamma} \cos \alpha \sin \alpha - (A \cos^2 \alpha + C \sin^2 \alpha) \dot{\alpha} \dot{\gamma} \sin \alpha \cos \alpha
 \end{pmatrix} \quad (57)$$

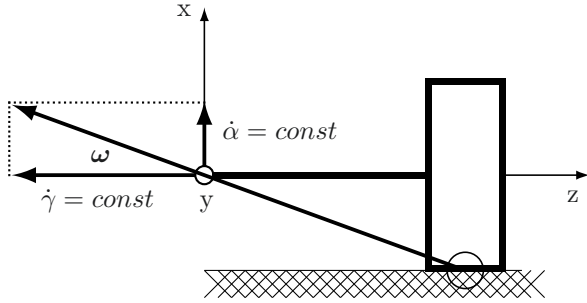


Figure 11. The edge mill

yielding, with the use of trigonometric formulas, $-\mathbf{M}^{react} = (0 \ C \dot{\gamma} \dot{\alpha} \ 0)^T$. On the other hand, using the frame wich rotates with $\dot{\alpha}$, one has $\dot{\alpha} \tilde{\mathbf{e}}_1 \mathbf{L} = -\mathbf{M}^{react}$ which leads directly to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\alpha} \\ 0 & +\dot{\alpha} & 0 \end{bmatrix} \begin{pmatrix} A \dot{\alpha} \\ 0 \\ -C \dot{\gamma} \end{pmatrix} = \begin{pmatrix} 0 \\ C \dot{\gamma} \dot{\alpha} \\ 0 \end{pmatrix}. \quad (58)$$

Elasticity/Nonholonomicity. It is quite obvious that nor a body fixed frame nor an inertial frame will meet the aim for the general case. Considering an elastic robot, for instance, no one would use element fixed reference frames for calculation. Also, referring motions to an inertial frame is by far

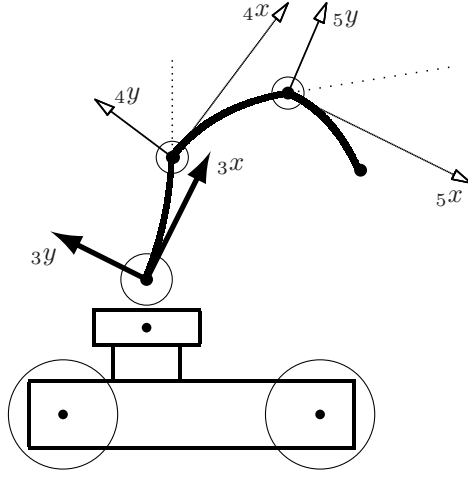


Figure 12. Sketch of a nonholonomic elastic robot

too tedious. As Figure 12 shows, the natural choice here is the undeformed moving frame for every link under consideration. We are thus coming back to elastic systems and will structurize the problem step by step.

The moving beam from eq.(8) will serve as an example for demonstration how to proceed:

Equations of Motion:

$$\begin{aligned} \left[\rho A (\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v) \right] - \left[\rho I (\ddot{v}'' + \dot{\alpha}^2 v'' + \ddot{\alpha}w'') \right] + EI_z v'''' &= 0, \\ \left[\rho A (\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w) \right] - \left[\rho I (\ddot{w}'' + \dot{\alpha}^2 w'' - \ddot{\alpha}v'') \right] + EI_y w'''' &= 0. \end{aligned}$$

Boundary Conditions:

$$\begin{aligned} \left[\rho I \ddot{v}' + \rho I \dot{\alpha}^2 v' + \rho I \ddot{\alpha} w' - EI_z v''' \right]_o^L &= 0, \quad \left[EI_z v'' \right]_o^L = 0, \\ \left[\rho I \ddot{w}' + \rho I \dot{\alpha}^2 w' - \rho I \ddot{\alpha} v' - EI_y w''' \right]_o^L &= 0, \quad \left[EI_y w'' \right]_o^L = 0. \end{aligned} \tag{59}$$

Obviously, it will (mostly) be impossible to find an analytical solution for such kind of partial differential equations (PDEs).

Galerkin – extended Galerkin – Ritz. For an approximative solution, the methods of Rayleigh-Ritz-Bubnov-Galerkin-Petrov (chronologically ordered) are at our disposal.

Boris Galerkin published his method in [20]. For a special case, the procedure had already been anticipated by I.G. Bubnov in 1913. The method is therefore sometimes referred to as the Bubnov-Galerkin method. Its original application field is mainly characterized by his original paper entitled *series solution of some problems in elastic equilibrium of rods and plates*. The procedure is, however, not restricted to linear elasticity problems but obviously reveals its main power in the field of nonlinear systems, thus extended by G.I. Petrov to nonlinear problems in gas dynamics, hydrodynamics and aerodynamics in 1940 (referred to then as the Galerkin-Petrov method). It is sometimes also called the Ritz-Galerkin method, based on a comparison with W. Ritz's procedure [49]. Ritz in his contribution expressly refers to J.W.S. Rayleigh [48] when he uses the exact (but unknown) eigenfrequency for lower bound w.r.t. the approximated one, in order to guarantee convergence. Therefore, his method is sometimes called the Rayleigh-Ritz method. However, Rayleigh approximated *one* frequency while Ritz solves the problem for a set of eigenfrequencies (46 eigenvalues and eigenvectors approximated with 49 shapefunctions for the plate problem within an accuracy of 2,2%). Although Lord Rayleigh claimed the basic idea his own literary property, it is reported that he was very astonished about Ritz's powerful procedure.

Bubnov-Galerkin's procedure makes use of a series expansion ("Ritz series") which will of course not fulfil the PDEs:

$$\begin{aligned} & \left[\rho A(\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v) \right] - \left[\rho I(\ddot{v}'' + \dot{\alpha}^2 v'' + \ddot{\alpha}w'') \right] + EI_z v'''' \neq 0, \\ & \left[\rho A(\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w) \right] - \left[\rho I(\ddot{w}'' + \dot{\alpha}^2 w'' - \ddot{\alpha}v'') \right] + EI_y w'''' \neq 0 \end{aligned} \quad (60)$$

for $v = \mathbf{v}^T \mathbf{q}_v$, $w = \mathbf{w}^T \mathbf{q}_w$. Hence, nullify the "weighted residua":

$$\int_0^L \mathbf{v} \left\{ \left[\rho A(\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v) \right] - \left[\rho I(\ddot{v}'' + \dot{\alpha}^2 v'' + \ddot{\alpha}w'') \right] + EI_z v'''' \right\} dx = 0,$$

$$\int_o^L \mathbf{w} \left\{ \left[\rho A (\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w) \right] - \left[\rho I (\ddot{w}'' + \dot{\alpha}^2 w'' - \ddot{\alpha}v'') \right] + EI_y w'''' \right\} dx = 0$$

iff $v = \mathbf{v}^T \mathbf{q}_v$, $w = \mathbf{w}^T \mathbf{q}_w$ fulfill the BCs (Galerkin-Approximation) (61)

Although correct so far, it will be nearly impossible to find shape fuctions which fulfill the BCs (see eq.(59) for instance). However, one might add the BCs which are premultiplied by $\mathbf{v}$, $\mathbf{v}'$ and $\mathbf{w}$, $\mathbf{w}'$, resp., to obtain an extended approximation scheme:

$$\begin{aligned} & \int_o^L \mathbf{v} \left\{ \left[\rho A (\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v) \right] - \left[(\rho I \ddot{v}'' + \dot{\alpha}^2 v'' + \ddot{\alpha}w'') \right] + EI_z v'''' \right\} dx + \\ & \int_o^L \mathbf{w} \left\{ \left[\rho A (\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w) \right] - \left[\rho I (\ddot{w}'' + \dot{\alpha}^2 w'' - \ddot{\alpha}v'') \right] + EI_y w'''' \right\} dx + \\ & \left[\mathbf{v} \left[\rho I \ddot{v}' + \rho I \dot{\alpha}^2 v' + \rho I \ddot{\alpha}w' - EI_z v'''' \right] + \mathbf{v}' \left[EI_z v'' \right] \right]_o^L + \\ & \left[\mathbf{w} \left[\rho I \ddot{w}' + \rho I \dot{\alpha}^2 w' - \rho I \ddot{\alpha}v' - EI_y w'''' \right] + \mathbf{w}' \left[EI_y w'' \right] \right]_o^L = 0. \end{aligned} \quad (62)$$

(extended Galerkin). This equation may be integrated by parts yielding

$$\begin{aligned} & \int_o^L \left\{ \mathbf{v} \left[\rho A (\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v) \right] + \mathbf{v}' \left[\rho I (\ddot{v}' + \dot{\alpha}^2 v' + \ddot{\alpha}w') \right] + \right. \\ & \quad \left. \mathbf{w} \left[\rho A (\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w) \right] + \mathbf{w}' \left[\rho I (\ddot{w}' + \dot{\alpha}^2 w' - \ddot{\alpha}v') \right] + \right. \\ & \quad \left. \mathbf{v}'' EI_z v'' + \mathbf{w}'' EI_y w'' \right\} dx = 0. \end{aligned} \quad (63)$$

This kind of integration is, by nearer inspection, a-priori known as can be seen by premultiplying with $\delta \mathbf{q}_v^T, \delta \mathbf{q}_w^T$, respectively. Having

$$\begin{aligned} \delta \mathbf{q}_v^T \mathbf{v} &= \delta v, & \delta \mathbf{q}_v^T \mathbf{v}' &= \delta v', & \delta \mathbf{q}_v^T \mathbf{v}'' &= \delta v'' \\ \delta \mathbf{q}_w^T \mathbf{w} &= \delta w, & \delta \mathbf{q}_w^T \mathbf{w}' &= \delta w', & \delta \mathbf{q}_w^T \mathbf{w}'' &= \delta w'' \end{aligned}$$

in mind, one obtains

$$\begin{aligned} \delta W = \int_0^L \bigg\{ & \delta v [\rho A (\ddot{v} - 2\dot{\alpha}\dot{w} - \ddot{\alpha}w - \dot{\alpha}^2 v)] + \delta v' [\rho I (\ddot{v}' + \dot{\alpha}^2 v' + \ddot{\alpha}w')] + \\ & \delta w [\rho A (\ddot{w} + 2\dot{\alpha}\dot{v} + \ddot{\alpha}v - \dot{\alpha}^2 w)] + \delta w' [\rho I (\ddot{w}' + \dot{\alpha}^2 w' - \ddot{\alpha}v')] + \\ & \delta v'' EI_z v'' + \delta w'' EI_y w'' \bigg\} dx = 0. \end{aligned} \quad (64)$$

Eq.(64) represents the starting point of investigation, see eq.(4) (corrected according to Figure 1): after an integration by parts w.r.t. time we carried out a subsequent integration by parts w.r.t. space in order to bring the boundary conditions at light, and these are now re-integrated. This is a totally useless calculation. The main difference in consideration hereby is that previously space was an independent variable ($\rightarrow$ PDEs) and now it is not any more ($\rightarrow$ ODEs). The shape functions are prechosen and have only to fulfill geometrical demands (and form an complete functional set, of course). We will call the procedure according to eq.(64) along with $v = \mathbf{v}^T \mathbf{q}_v, w = \mathbf{w}^T \mathbf{q}_w$ a “direct Ritz approach” and come to the following conclusion:

- Bubnov-Galerkin-Petrov’s procedures drop out, because their ansatz functions have to fulfill the boundary conditions which is mostly impossible.
- Rayleigh’s (original) contribution does not fit the problem.
- As Poincaré puts it: “Seulement Ritz est parvenu dans deux cas, celui du problème de Dirichlet et celui de l’élasticité” (– only Ritz succeeded in two cases: that one of Dirichlet and that one of elasticity).

Because the shape functions are prechosen, the only remaining variable is $\mathbf{q}$. Therefore, Hamilton’s principle yields

$$\int_{t_o}^{t_1} [\delta T - \delta V] dt \Rightarrow \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \left(\frac{\partial T}{\partial \mathbf{q}} \right) + \left(\frac{\partial V}{\partial \mathbf{q}} \right) \right] \delta \mathbf{q} = 0. \quad (65)$$

We are thus, for holonomic systems, left with Lagrange's equations (of the second kind). On the other hand, we have eq.(56) which yields, for moving beams,

$$\int_B \left\{ \left(\frac{\partial \omega}{\partial \dot{\mathbf{q}}} \right)^T \left[d\dot{\mathbf{L}} + \tilde{\omega}_{IR} d\mathbf{L} - d\mathbf{M}^e \right] + \left(\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{q}}} \right)^T \left[d\dot{\mathbf{p}} + \tilde{\omega}_{IR} d\mathbf{p} - d\mathbf{f}^e \right] \right\} = 0 \quad (66)$$

(as already mentioned, eq.(56) needs, for a multi body system, summation. This summation here passes to an integral (B : body under consideration) while the momenta $d\mathbf{L}$ and $d\mathbf{p}$, resp., refer to the mass element dm).

3.3 Are there better suited Principles?

Here, the question arises: which method will be best? and what about nonholonomicity (as Figure 12 indicates)? Up to here, the whole consideration was based on Lagrange's principle (46). Are there better suited principles – can the Lagrangean one also be used for nonholonomic systems or do we need Jourdain's Principle as has often been stated? To answer this question we will have to have a look on Lagrange's $\delta \mathbf{r}$ – often rejected as obscure (Poincot), esoteric (Angeles), nearest to black magic (Kane).

What does $\delta \mathbf{r}$ mean? In the introduction of his paper from 1764 Lagrange refers to Johann Bernoulli and the Principle of Virtual Velocities (in statics). Lagrange reports that Bernoulli had communicated that principle to Varignon in a letter from 1717. Varignon stated its equivalence with the parallelogram of forces, thus forming a solid basis for statics, [57]. Lagrange started his investigations in 1760 with *essai d'une nouvelle méthode...*, [31], (essay on a new method...) considering the maxima and the minima of undetermined integral forms, followed one year later by *application de la méthode...* where he applied it to e.g. the brachistochrone problem. He had communicated that new method (*nouvelle méthode*) to Euler already in 1755 when he was nineteen years old, very much appreciated by Euler. Lagrange later reports (in the *Miscellanea Taurinensis*): “I have to emphasize, because this method needs to vary the same terms in two different ways, in order not to confuse these variations I have introduced a new characteristic δ . Thus, δZ represents a differential with respect to Z , which is not the same as dZ , but which is nevertheless calculated by the same rules, such that, if any $dZ = m dx$ exists, one has simultaneously $\delta Z = m \delta x$, and this holds for any other equation.” It might be this expression which later – up to the time being – caused a lot of misunderstanding not at least due to the

naming “virtual displacement” instead of “virtual velocity”. It is not clear who introduced it (at the latest Ernst Mach used it already 1883, [36]) and why (may be because statics does not know velocities).

Comparing eq.(46) with dW yields

$$\delta W = \int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T \delta \mathbf{r} = 0, \quad dW = \int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T d\mathbf{r} = 0, \quad (67)$$

which states that $\delta \mathbf{r}$ has to be infinitesimal. And it is, in this interpretation, indeed a displacement. However, what is this comparison good for? Isn't there an integration missing?

Integration yields the (real, finite) work which will be accomplished under the action of impressed forces

$$W = \int_o^{\mathbf{r}} \int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T d\mathbf{r} = \int_o^{t(\mathbf{r})} \int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T \frac{d\mathbf{r}}{dt} dt = 0. \quad (68)$$

Differentiating w.r.t. time yields the (real) work rate (or power)

$$\dot{W} = \int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T \dot{\mathbf{r}} = 0. \quad (69)$$

Looking for the (unknown) motion we are next going to replace $\dot{\mathbf{r}}$ by a comparison function:

$$\begin{aligned} \dot{\mathbf{r}} \rightarrow \dot{\mathbf{r}} + \delta \dot{\mathbf{r}} : \int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T \delta \dot{\mathbf{r}} + \int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T \dot{\mathbf{r}} &= 0 \\ \stackrel{eq.(69)}{\Rightarrow} \int_{(S)} (dm\ddot{\mathbf{r}} - d\mathbf{f}^e)^T \delta \dot{\mathbf{r}} &= 0, \end{aligned} \quad (70)$$

and we are back with Lagrange's virtual velocity ansatz in dynamics. This goes along with Lagrange's own explanations:

1. “ δZ shall express a differential w.r.t. Z which is not the same as dZ but built with the same rules” [31]
2. “The virtual velocity is the velocity which will really take place when the equilibrium is disturbed” [34].

How to calculate $\delta \mathbf{r}$? “The same rules” means that $\dot{\mathbf{r}}$ and $\delta \mathbf{r}$ belong to the same vector space. If the system under consideration is constrained via an implicit function $\Phi(\mathbf{r}) = 0$, then

$$\left(\frac{\partial \Phi}{\partial \mathbf{r}}\right) \dot{\mathbf{r}} = 0, \quad \left(\frac{\partial \Phi}{\partial \mathbf{r}}\right) \delta \mathbf{r} = 0 \quad \text{and} \quad d\mathbf{r} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right) d\mathbf{q}, \quad \delta \mathbf{r} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right) \delta \mathbf{q}. \quad (71)$$

$\mathbf{q} \in \mathbb{R}^f$ is totally unconstrained (minimal coordinates). Hence, $\delta \mathbf{q}$ may be looked at in a variational sense

$$\delta \mathbf{q} = \left. \frac{\partial \mathbf{q}}{\partial \varepsilon} \right|_{\varepsilon=0} \cdot \varepsilon = \varepsilon \cdot \boldsymbol{\eta} \quad (72)$$

which corresponds to a Taylor expansion. Due to the linear approach in ε , the expansion is complete (no higher terms) and thus $\delta \mathbf{q}$ is arbitrary in magnitude. Then, due to eq.(71) $\delta \mathbf{r}$ is also arbitrary in magnitude. This builds bridges to optimization theory which has been started with the same approach (Lagrange, Euler, Legendre ...) until Weierstrass’ new ideas (around 1900, see section 6.1).

We can draw the following conclusions:

- “Virtual displacements” (if one prefers this notation) are arbitrary tangent vectors of the constraint plane. They are well defined and by no means obscure. There is no need to assume them infinitesimal.
- There is no obstacle to impose “non-holonomic” constraints (see below). The above mentioned (very much overestimated) Jourdain-Principle is identical (and so is Gauss’ [22] in its “differential” form). There is no need to add any other “Principle” to the Lagrangean one.
- The (frequent) statement “*in case of scleronomic* (i.e. time-independent) *constraints, $d\mathbf{r}$ and $\delta \mathbf{r}$ coincide*” is definitely wrong. (It led Poincot [45] to severe mistakes).
- The (frequent) statement “*virtual displacements are infinitesimally small and happen infinitely fast*” is incomprehensible, hence rejectable.

3.4 The Euler-Bernoulli-Beam – A Powerful Model

Eq.(66) leads to several model considerations. If the (relative) elastic deflection is modeled by just one deflection function for each bending direction, then eq.(66) represents the so called Rayleigh beam. If the basic motion contains bending and shear, then the beam is called a Timoshenko beam ([54] although already established by Bresse [10]). The simplest and most powerful model is obtained when neglecting the angular momentum

[first summation term in eq.(66)]. One is then left with the so called Euler–Bernoulli beam. This neglect has carefully to be considered. In a simple interpretation it refers to slender beams (rotational inertia negligible). A basic rule of thumb here indicates a L/D –ratio of about 10 (length to diameter). However, one should also have in mind that neighbouring systems (or tip masses) may have influence, such as frequency breakdown and shear deflection.

4 Solutions, Aims, Methods

Considering elastic bodies, the exact solution in Daniel Bernoulli’s notation is represented by an infinite series. Infinity does not make sense in a physical interpretation. The number of requested eigenfunctions may be averaged via the influence of material damping.

Approximating the eigenfunctions needs adequate shape functions. How many? What kind? The latter is answered through convergence considerations, the first depends on the actual model (e.g. controlled system) and remains open in general.

In fast moving controlled systems, the effect of second order displacement fields are reconsidered. Reaction forces from moderately moving models may be neglected, otherwise acceleration reactions need assignment to the mass matrix (because of later inversion).

One basic task is frequency tuning. In noncontrolled systems it leads to the harmony model in occidental music (from Huygens via Bach to the time being). In controlled systems, however, it is evident for open-loop the result of which serves for closed-loop interpretations.

All these considerations go hand in hand with the mathematical method to be applied. A comparison of the up to here evaluated procedures gives already valuable hints. However, the real problem arises with nonholonomic constraints.

4.1 Exact solutions – how many eigenfunctions?

Exact solutions are obtained for a reduced class of models. These are, for instance, single Euler–Bernoulli beams with various boundary conditions as well as strings or, as a combination, tightened beams. The tightening force f_x acts at $x = L$. The corresponding elongation reads, with eq.(32), $r_x^{(2)} = -\frac{1}{2} \int_0^L w'^2 dx$ for plane motion in z-direction. Hence, f_x contributes via $\delta W_f = - \int_0^L f_x w' \delta w' dx = \int_0^L f_x w'' \delta w - [f_x w' \delta w]_0^L$ which has to be

considered within $\int_o^t (\delta T - \delta V + \delta W_f) dt = 0$. One obtains then

$$\begin{aligned} &\text{the motion equation} \quad \rho A \ddot{w} + EI w'''' - f_x w'' = 0 \\ &\text{the boundary conditions} \quad \begin{cases} [w]_o^L = 0 & \vee & [f_x w' - EI w''']_o^L = 0 \\ [w']_o^L = 0 & \vee & [EI w'']_o^L = 0 \end{cases} \end{aligned} \quad (73)$$

(compare eq.(5) and (6) for $\alpha = 0$ and $\rho I = 0$). The ansatz

$$w(x, t) = \sum_{n=1}^{\infty} w_n(x) q_n(t) \quad (74)$$

yields for the n th summation term of the motion equation

$$\frac{\ddot{q}_n}{q_n} = -\frac{EI}{\rho A} \left(\frac{w_n''''}{w_n} \right) + \frac{f_x}{\rho A} \left(\frac{w_n''}{w_n} \right) = -\omega_n^2 \quad (75)$$

where – because q_n depends on time and w_n on space, respectively – both terms in eq.(75) can only be fulfilled by a constant which is denoted $-\omega_n^2$. One has thus $\ddot{q}_n + \omega_n^2 q_n = 0$ for the time functions.

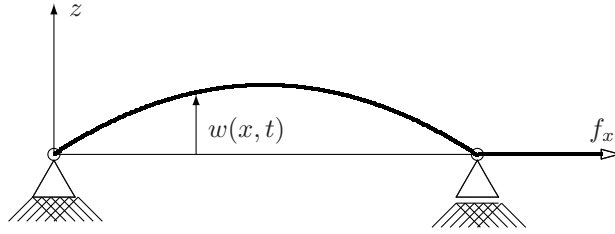


Figure 13. Tightened beam

Considering the pinned-pinned beam as representative model, the ansatz $w_n = \sin(n\pi x/L)$ yields $w_n'' = -(n\pi/L)^2 w_n$ which fulfills the boundary conditions $w(0) = w(L) = 0, w''(0) = w''(L) = 0$. Along with $w_n'''' = +(n\pi/L)^4 w_n$, one obtains from eq.(75)

$$\omega_n^2 = \frac{EI}{\rho A} \left(\frac{n\pi}{L} \right)^4 + \frac{f_x}{\rho A} \left(\frac{n\pi}{L} \right)^2. \quad (76)$$

The n th solution term in eq.(74) then reads

$$w_n(x) q_n(t) = \sin \left(\frac{n\pi x}{L} \right) \left[q_{on} \cos \omega_n t + \frac{\dot{q}_{no}}{\omega_n} \sin \omega_n t \right] \quad (77)$$

Bernoulli's String Vibrations. The breakthrough came 1753 with D. Bernoulli's *Reflexions et Eclaircissement sur les nouvelles vibrations des cordes* (Considerations on and explanation of the new chord vibrations), [52]. Considering a string, the bending stiffness becomes negligible. The solution (77) still holds, while the frequency reduces to

$$\omega_n = \sqrt{\frac{f_x}{\rho A}} \left(\frac{n\pi}{L} \right). \quad (78)$$

It was Bernoulli who noticed the simultaneous existence of different tones which lead him to the solution in form of a (Fourier) series [see eq.(74) with eq.(77)]. His friend Euler first doubted this result but later, in 1777, he himself anticipated Fourier's results.

Considering eq.(74) leads to the question: How many modes exist? As eq.(74) shows, there are infinite modes which independently fulfill the equations of motion (73). However, this would indicate that, considering higher modes, the string in Bernoulli's solution could move faster than the speed of light. Here, modeling comes to its borders.

One possibility is to adjust modeling to the actual aim. Let for instance the basic tone of a piano string be ω_o . If one wants to model the quint w.r.t. the first octave ($\omega_5 = \frac{3}{2}(2\omega_o)$) and the third w.r.t. the second octave ($\omega_3 = \frac{5}{4}(4\omega_o)$), then $\sqrt{f_x/\rho A}(n\pi/L) \Rightarrow n = 5$ fulfills the task. This is quite trivial: one needs modeling up to the fifth overtone. Then, the vibrating basic string (Note C) with its third overtone excites the quint (Note G) from the first octave and the third (Note E) from the second octave, respectively. It is reported that this effect can be demonstrated experimentally by exciting the basic tone, such that the correspondig G- and E-strings remain resonantly vibrating for a while (only for a short while due to structural damping, and only for just tuning, see section 4.1).

Euler's beam vibrations. In real systems the always existing structural bending cannot be neglected. Introducing damping in the form $d(\omega)\dot{w}''''$ (which is quite pragmatic and corresponds to the "hypothesis of convenience") one has

$$\rho A \ddot{w} + d(\omega) \dot{w}'''' + EI w'''' = 0 \quad (79)$$

for an Euler-Beam which may serve as a span bridge model for instance. Along with eq.(73) one obtains for the pinned-pinned case

$$\ddot{q}_n + \frac{d(\omega)}{\rho A} \left(\frac{n\pi}{L} \right)^4 \dot{q}_n = -\frac{EI}{\rho A} \left(\frac{n\pi}{L} \right)^4 = -\omega_n^2. \quad (80)$$

Let $d(\omega) = 2\zeta \frac{EI}{\omega_n}$. Then, eq.(74) yields for the n th summation term

$$\begin{aligned} \ddot{q}_n + 2\zeta\omega_n \dot{q}_n &= -\frac{EI}{\rho A} \left(\frac{n\pi}{L} \right)^4 = -\omega_n^2 \\ \Rightarrow \ddot{q}_n + 2\zeta\omega_n \dot{q}_n + \omega_n^2 q_n &= 0, \quad \omega_n = \sqrt{\frac{EI}{\rho A}} \left(\frac{n\pi}{L} \right)^2, \quad w_n = \sin \left(\frac{n\pi}{L} x \right). \end{aligned} \quad (81)$$

The eigenfunctions $w_n(x)$ are not influenced, the Ritz-coefficients $q_n(t)$ are obtained from a damped linear oscillator where damping increases for higher modes (which reflects daily experience):

$$w(x, t) = \sum_{i=1}^n w_i(x) q_i(t) = \sum_{i=1}^n \sin \left(\frac{i\pi x}{L} \right) e^{-\zeta\omega_i t} \left[q_{oi} \cos \nu_i t + \frac{\dot{q}_{io}}{\nu_i} \sin \nu_i t \right], \quad (82)$$

$\nu_i = \omega_i \sqrt{1 - \zeta^2}$. Since the eigenmodes are decoupled, one may proceed in a similar way as above: excite the structure experimentally (with a hammer, for instance) to estimate the frequencies via signal analysis. Or: look at the time history simulation for different n and estimate its minimum requested number. Adding a force f_z which acts at $x = \xi$, one obtains the corresponding virtual work via eq.(79)

$$\delta W = \int_0^L [\rho A \ddot{w} + d(\omega) \dot{w}'''' + EI w'''' - f_z \delta(x - \xi)] \delta w dx = 0 \quad (83)$$

(δ : dirac distribution). The separation statement (74) yields

$$\begin{aligned} \int_0^L \left[\rho A \sum w_i \ddot{q}_i + d(\omega) \sum w_i'''' \dot{q}_i + EI \sum w_i q_i \right] w_j dx \delta q_j \\ = f_z \int_0^L \left[w_j \delta(x - \xi) \right] dx \delta q_j \quad \forall j. \end{aligned} \quad (84)$$

($w_i = \sin(i\pi x/L)$ is a priori known and does not undergo variations any more). Hence, one has $\int w_i w_j dx = 2L$ for $j = i$ and $\int w_i w_j dx = 0$ for $j \neq i$. Regarding $w_i'''' = (i\pi/L)^4 w_i$, one obtains, along with $\int [w_j \delta(x - \xi)] dx = w_i(\xi)$

$$\begin{aligned} \ddot{q}_i + 2\zeta\omega_i \dot{q}_i + \omega_i^2 q_i &= \frac{f_z}{2m} w_i(\xi), \quad m = \rho AL \\ \Rightarrow q_i(t) &= w_i(\xi) e^{-\zeta\omega_i t} \frac{p}{2m\nu_i} \sin(\nu_i t), \quad \nu_i = \omega_i \sqrt{1 - \zeta^2} \end{aligned} \quad (85)$$

if f_z is modeled as $p\delta(t - t_o)$ (impulse at time $t = t_o = 0$). The solution

$$w(x, t) = \sum_{i=1}^n e^{\zeta\omega_i t} \frac{p}{2m\nu_i} \sin(\nu_i t) \sin[(i\pi/L)\xi] \sin[(i\pi/L)x] \quad (86)$$

shows that every eigenmode $w_i(x)$ will take part as long as $w_i(\xi) \neq 0$ (controllability). However, exciting higher modes needs much energy. To give an impression: Figure 14 is initially calculated with $n = 50$. A recalculation with $n = 20$ does not show perceivable differences.

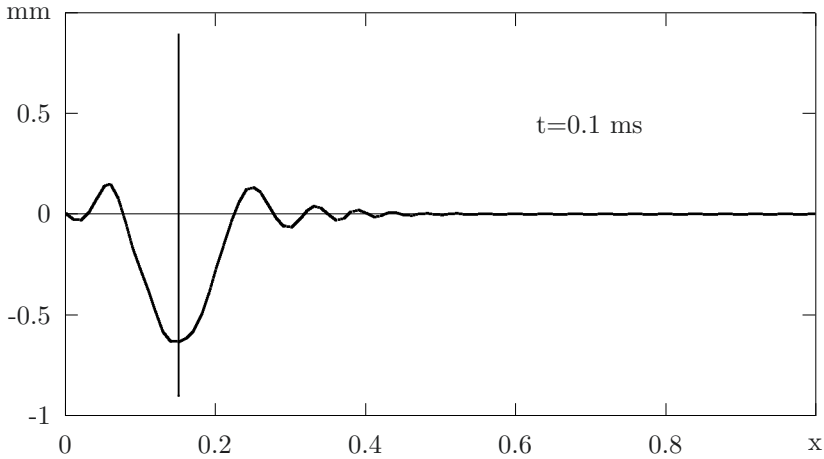


Figure 14. Time response after impact at $x = 0,15$

The piano chord – a tightened beam. In reality, a piano chord is more a beam than a string, especially for the lower tones. Then the modes are calculated with eq.(76): Let the string frequency (78) be ω_{sn} . One has then

$$\frac{\omega_n}{\omega_{s1}} = \sqrt{\left[\frac{EI \left(\frac{n\pi}{L} \right)^2}{f_x} + 1 \right]} \cdot n$$

To obtain the string overtones $(\omega_{sn}/\omega_{s1}) = n$, $n = 1, 2, 3 \dots$ one needs quite high forces f_x . In a piano, these range up to 1,6 kN.

The Euler-Bernoulli-beam is a powerful model

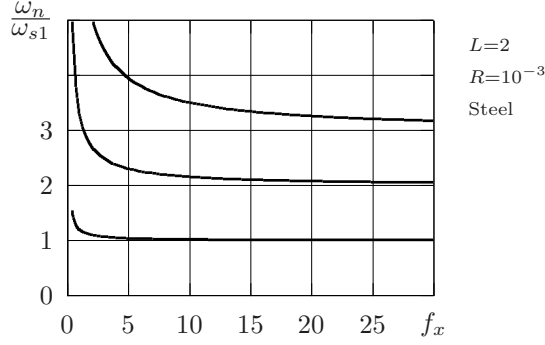


Figure 15. Tightened beam frequencies

4.2 Approximative solutions: how many shape functions?

Once the *number* of requested eigenfunctions is estimated, the eigenfunctions themselves remain unknown (for the general case). One may ask oneself how to represent these by a series expansion (the Ritz ansatz in this context). This task refers to expand an unknown function $f(x)$ in terms of $\mathbf{a}^T \mathbf{g}(x)$ with known functions $g_i(x)$, for instance to expand a periodic function into a Fourier series. The well known procedure is to minimize the (quadratic) error $\Delta = \frac{1}{2} \int [\mathbf{a}^T \mathbf{g}(x) - f(x)]^2 dx$ w.r.t. $\mathbf{a}$ yielding $\int [\mathbf{a}^T \mathbf{g}(x) - f(x)] \mathbf{g}(x)^T dx = 0$, thus

$$\left[\int \mathbf{g}(x) \mathbf{g}(x)^T dx \right] \mathbf{a} = \int f(x) \mathbf{g}(x) dx. \quad (87)$$

Using orthogonal beam functions $g_i(x) = v_i(x)$ with $v_i'''' - k_i^4 v_i = 0$, one obtains with an integration by parts

$$a_i = \int_o^L f(x) \frac{v_i''''}{k_i^4} dx = \frac{1}{k_i^4} \left[(f'' v_i' - f' v_i'' + f v_i''') \Big|_o^L - \int_o^L f''' v_i' dx \right]. \quad (88)$$

For free-free boundaries, $v_i'' \Big|_o^L$ and $v_i''' \Big|_o^L$ vanish, hence

$$a_i = \frac{1}{k_i^3} \left[f'' \left(\frac{v_i'}{k_i} \right) \Big|_o^L - \int_o^L f''' \left(\frac{v_i'}{k_i} \right) dx \right] \leq \kappa_i \Rightarrow a_i \leq \frac{\kappa_i}{k_i^3} \quad (89)$$

since $(v'_i/k_i) \in \{-2, +2\}$ and f'', f''' are assumed finite and steady. Because $k_i \simeq (i + \frac{1}{2})\pi$, the series $\mathbf{a}^T \mathbf{v}$ converges with $\simeq (1/i^3)$ for increasing i to $f(x)$ as a lower bound. An expansion of $f(x, y) = \mathbf{a}_{ij}^T u_i(x) v_j(y)$ with beam functions $u_i(x), i = 1 \cdots n$, and $v_j(y), j = 1 \cdots m$, resp., led Ritz to his series representation for a plate with free borders which converges rapidly, with $(1/(ij)^3)$, to its lower bound the existence of which is warranted with Rayleigh's quotient.

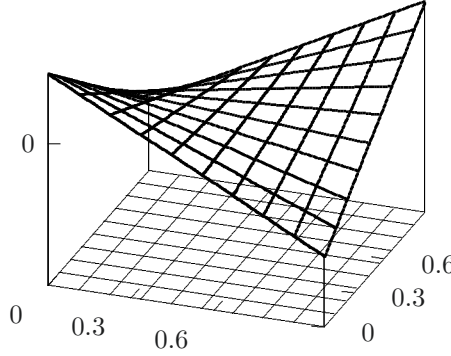


Figure 16. 1st plate eigenfunction

Let us consider Figure 17(a) which corresponds to Figure 8(a) with a stiff gear. The restoring matrix $\mathbf{K} = 0$ is known with eq.(36). We address to the mass matrix calling attention to modeling aspects.

We assume the tip tensor of inertia $\mathbf{J}_{\text{tip}}$ negligible. Along with $\rho I_y \rightarrow 0$ and $\rho I_z \rightarrow 0$ (Euler beam) one obtains the mass matrix for a rotating beam

$$\mathbf{M} = \begin{bmatrix} \int \frac{\partial m}{\partial x} x^2 dx & \int x \mathbf{v}^T \frac{\partial m}{\partial x} dx & 0 \\ \int x \mathbf{v} \frac{\partial m}{\partial x} dx & \int \mathbf{v} \mathbf{v}^T \frac{\partial m}{\partial x} dx & 0 \\ 0 & 0 & \int \rho I_x \boldsymbol{\vartheta} \boldsymbol{\vartheta}^T dx \end{bmatrix} \quad (90)$$

where the sequence of coordinates is

$$\mathbf{y} = (\gamma \quad \mathbf{q}_v^T \quad \mathbf{q}_\vartheta^T)^T. \quad (91)$$

Is the Euler-Bernoulli beam really a powerful model? C_{Hub} does not arise anywhere which in eq.(24) led to singularity. However, inserting

$$\frac{\partial m}{\partial x} = m_{\text{hub}} \tilde{\delta}(x - 0) + m_{\text{tip}} \tilde{\delta}(x - x|_e) + \rho A \quad (92)$$

functions will remain transparent. However, convergence here depends on the beam bending functions with its fourth order differential equation. The situation will change dramatically for torsional deflections with its second order differential equation.

Quality of shape functions – increasing the convergence. Let us consider a torsional shaft with disks attached at its ends. The shaft shall be clamped at $x = 0$ ($\gamma = \text{const.}$)

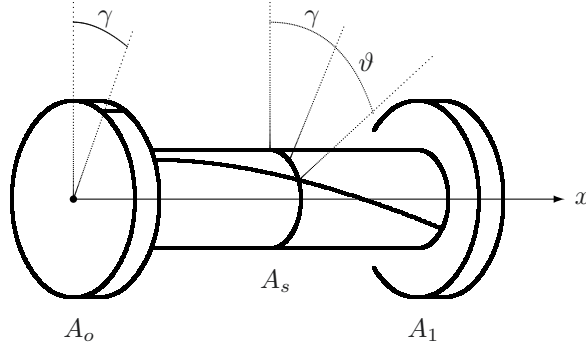


Figure 18. torsional shaft

The eigenfunctions are known with $\vartheta = \sin\left(\frac{k_i x}{L}\right)$ where $\tan k_i = \frac{A_s}{A_1} \frac{1}{k_i}$ with $A_s = \rho I_x L$. (They are easily obtained from Hamilton's principle for instance). The first three eigenfunctions are sketched in Figure 19(a). From eq.(87) we obtain

$$\mathbf{a} = \left[\int \mathbf{g}(x) \mathbf{g}(x)^T dx \right]^{-1} \int f(x) \mathbf{g}(x) dx. \quad (95)$$

Using *orthogonal* beam functions $g_i(x) = \vartheta_{bi}(x)$ with $\vartheta_{bi}'' + k_i^2 \vartheta_{bi} = 0$, one obtains with an integration by parts

$$a_i = - \int_0^L f(x) \frac{\vartheta_{bi}''}{k_i^2} dx = \frac{1}{k_i} \left[-f \left(\frac{\vartheta_{bi}'}{k_i} \right) \Big|_0^L + \int_0^L f' \left(\frac{\vartheta_{bi}}{k_i} \right) dx \right] \leq \frac{\kappa}{k_i}. \quad (96)$$

Since $(\vartheta_{bi}'/k_i) = \cos(k_i x) \in \{-1, +1\}$, $k_i \simeq (i - \frac{1}{2})\pi$, and f, f' are assumed finite and steady, the series $\mathbf{a}^T \boldsymbol{\vartheta}_b$ converges *slowly* with $\simeq (1/i)$ for increasing i .

Figure 19(b) shows the right end (magnified) of the second eigenfunction ϑ_{2e} , approximated with $n=50, 100, 500$ beam functions

$\vartheta_b^T = \left(\sin\left(\frac{\pi}{2}x\right), \sin\left(\frac{3\pi}{2}x\right), \dots, \sin\left(\frac{(2n-1)\pi}{2}x\right) \right)$. It seems obvious that a very high number of shape functions is needed, since none of these show a slope at the right end. This leads to the well known effect that the approximation forms some kind of oscillation around the true solution.

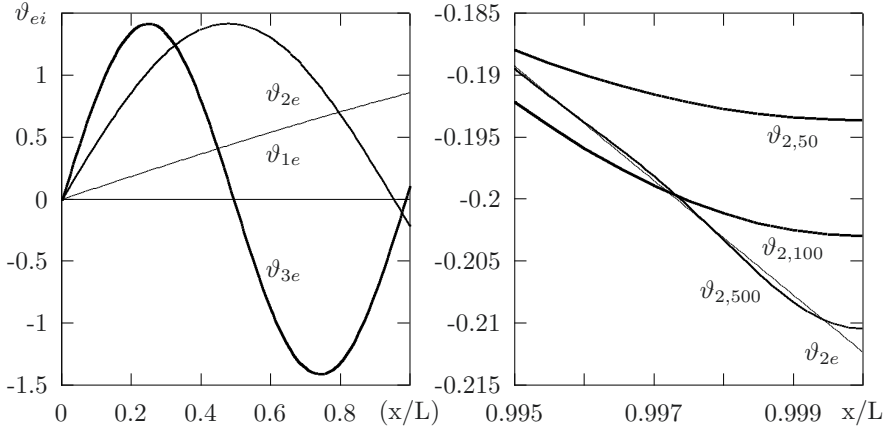


Figure 19. (a) EF (exact), (b) 2.EF approximations (magnified)

Being aware of this, one might add an auxiliary function with a slope at the right end: $\vartheta_{ab}^T = \left(\sin(\pi x), \sin\left(\frac{\pi}{2}x\right), \sin\left(\frac{3\pi}{2}x\right), \dots, \sin\left(\frac{(2n-1)\pi}{2}x\right) \right)$. These functions are *not orthogonal*, thus the inverse in eq.(95) enters the consideration and the convergence is not so easy to check. However, the result shown in Figure 20(b) is tremendous. Here, only four shape functions are used. Figure 20(a) depicts the first four approximated eigenfunctions for this case; the deviations become apparent with the highest (the fourth) eigenfunction.

How to construe auxiliary functions? Here, a physical interpretation will be helpful. Considering rapidly moving higher modes, the inertia of the disk comes more and more into play and will act in the limiting case as a clamped support (notice that $\sin(\pi x)$ is the first eigenmode for this case). This tendency can already be seen in Figure 19 (a) where the higher modes tend more and more to zero at the right end.

One has thus also the recipe at hand how to treat the bending beam with

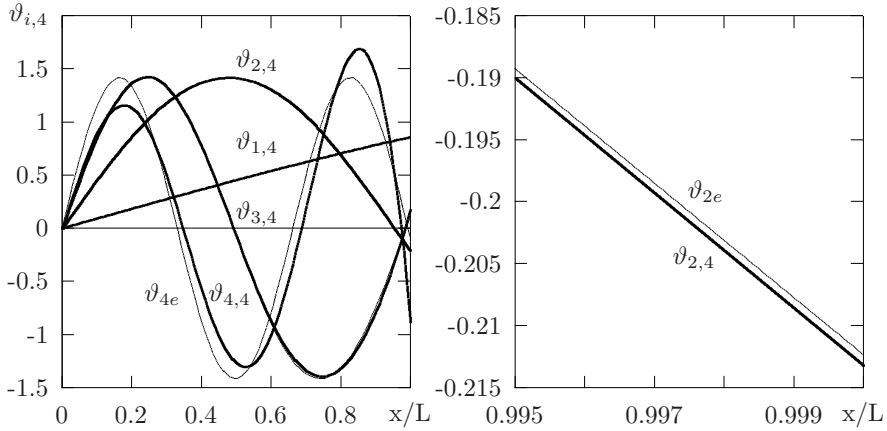


Figure 20. (a) EF (approximated), (b) 2.EF approximations (magnified)

a tip mass according to Figure 17(a): for a point mass (rotational inertia neglected), the auxiliary function corresponds to the clamped-pinned case, for a rigid body (with rotational inertia), it is represented by the clamped-clamped case.

Being aware of the quality of shape functions, one could start to calculate the motions of a rotating beam, see Figure 17, for instance. The mass matrix is known from eq.(94) and the restoring matrix from eq.(36), respectively, which contains the influences of the second order elastic displacements. However, the latter need reconsideration.

Second order field effects. A simple case will give insight into the problem: the “linear robot” (only linear guided motions, no rotation) according to Figure 21.

Same as in the case of centrifugal reaction forces $df_x = \rho A \omega^2 dx$ which led to

$$\int_0^L \int_x^L \rho A (\omega^2 \xi) d\xi \mathbf{v}' \mathbf{v}'^T dx \mathbf{q} \quad (97)$$

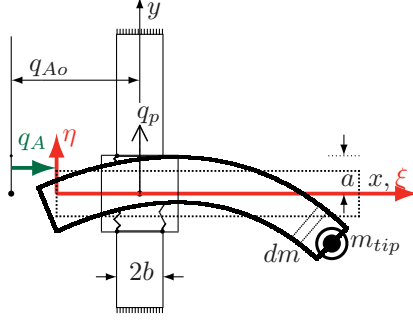


Figure 21. A “linear robot”

$$\Rightarrow \mathbf{K}_{cent} \mathbf{y} = \begin{bmatrix} 0 & 0 \\ 0 & \int_o^L \rho A \left[\frac{1}{2} \omega_o^2 (L^2 - x^2) \right] \mathbf{v}' \mathbf{v}'^T dx \end{bmatrix} \begin{pmatrix} \gamma \\ \mathbf{q} \end{pmatrix} \quad (98)$$

(see eq.(30)), we now have a reaction force $df_x = -\rho A \ddot{q}_A dx$ which leads to

$$- \int_o^L \int_o^L \rho A (\ddot{q}_A \xi) d\xi \mathbf{v}' \mathbf{v}'^T dx \mathbf{q} \quad (99)$$

and consequently

$$\mathbf{K}_{acc} \mathbf{y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \int_o^L \rho A \ddot{q}_A (x - L) \mathbf{v}' \mathbf{v}'^T dx \end{bmatrix} \begin{pmatrix} q_p \\ q_A \\ \mathbf{q} \end{pmatrix} \quad (100)$$

However, eq.(100) may be reorganized to

$$\mathbf{M}_{acc} \ddot{\mathbf{y}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \int_o^L \rho A (x - L) \mathbf{v}' \mathbf{v}'^T dx \mathbf{q} & 0 \end{bmatrix} \begin{pmatrix} \ddot{q}_p \\ \ddot{q}_A \\ \ddot{\mathbf{q}} \end{pmatrix} \quad (101)$$

This is essential. $\mathbf{M}_{acc}$ has to be assigned to the mass matrix which later needs inversion! We can thus draw the following conclusions:

- reaction forces:
 - accelerations to mass matrix (mandatory)

- consequently coriolis and centrifugals to gyroscopic matrix (not necessary)
- impressed forces
 - to restoring matrix

In case of moderate maneuvers the influence of the reaction forces may be negligible, but that one of impressed forces not (see buckling, for instance).

4.3 Frequency tuning

Frequency tuning plays an important role in modeling. It starts already with the above mentioned piano: the harmony model is based on the string overtones which might be achieved with strongly tightened beams. Thereby, at least for occidental ears, the third ($\omega/\omega_o = 5/4$) and the quint ($\omega/\omega_o = 3/2$) are the most harmonic sequences. Then, after a certain sequence of quints, one should meet the basic tone (or rather one of its octaves). However, with 12 quints one has $(3/2)^{12} = 129,75$, while 7 octaves yield $2^7 = 128$. Hence, since the sequence of octaves is mandatory, one has to adjust the tone intervals anyhow. This busied a lot of scientists, from Huygens and Euler up to d'Alembert [1], just to mention a few of them. Some people believe that the great J.S. Bach gave a cryptographic rule on the first page of his welltempered clavier – nobody knows, it remains a mystery.

Mysteries are not allowed in mechanical modeling. Setting $q_A = \text{const.}$ in Figure 21, one arrives, along with stiff gears, to a gantry robot model, or the crane problem, Figure 22. Here, an open loop control can easily be modeled: Once the beam frequency ω_{beam} is known, then a desired q -motion for an end position q_e with maneuver time T can be calculated as

$$q(t)_d = \frac{q_e}{2\pi}(\Omega t - \sin \Omega t) \quad (102)$$

where $\Omega = \frac{2\pi}{T} = \frac{1}{2}\omega_{\text{beam}}$.

The payload motion results then in

$$y(t) = \frac{q_e}{6\pi} \left[\frac{1}{2} \sin \omega_{\text{beam}} t - \sin \Omega t \right] \quad (103)$$

where the vanishing slope at the beginning and at the end of the maneuver yield a soft placement of the payload. The force f_q which is needed for q is then proportional to $\sin \Omega t$. This scenario is typical for any point-to-point maneuver: at the beginning, the payload motion y stays behind the driver's position as seen from the driver's cabin. After a while (at half the maneuver time in the present), the driver changes from acceleration to deceleration.

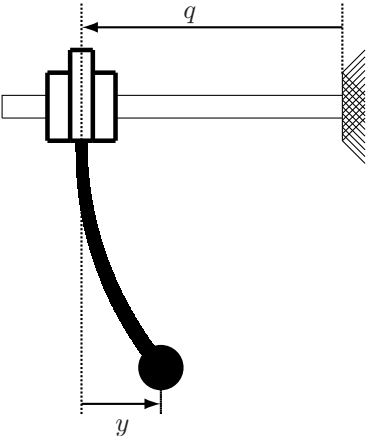


Figure 22. The crane model

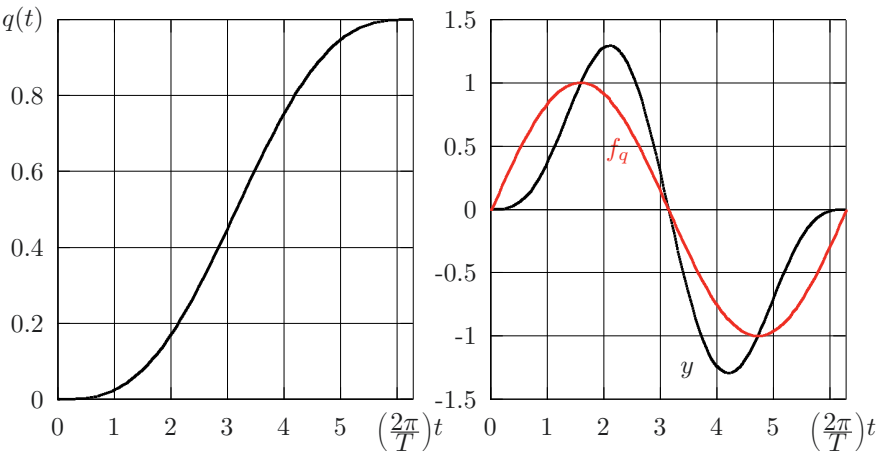


Figure 23. (a) driver motion (b) force and payload motion

The payload gets ahead such that at the end of the maneuver it comes smoothly to rest, see Figure 23.

This is, in principal, the behaviour of any point-to-point maneuver, also in closed loop controlled system. The crane problem consideration is helpful to interpret the results. Also the slewing maneuver of the Truckenbrodt

model (see eq.(33) et seq.) obeys these rules; any model assumptions there have been experimentally verified. *Verification is the last step in modeling.*

4.4 Procedures

Having a look on the fore arm (partial differential) equation of a plane elastic double pendulum (see Figure 24),

$$\begin{aligned}
 & \frac{\partial m_2}{\partial x_2} \ddot{v}_2 - \frac{\partial}{\partial x_2} \left[\frac{\partial J_2}{\partial x_2} \ddot{v}'_2 \right] + \frac{\partial^2}{\partial x_2^2} \left[(EI)_2 v''_2 \right] \\
 & + \frac{\partial m_2}{\partial x_2} \left\{ \left[L_1 \cos \gamma_2 - (L_1 v'_{1L} - v_{1L}) \sin \gamma_2 + x_2 \right] \ddot{\gamma}_1 + x_2 \ddot{\gamma}_2 \right. \\
 & + \left[L_1 \sin \gamma_2 + (L_1 v'_{1L} - v_{1L}) \cos \gamma_2 - v_2 \right] \dot{\gamma}_1^2 - v_2 \dot{\gamma}_2^2 \\
 & + \left[2\dot{v}_{1L} \sin \gamma_2 - 2v_2 \dot{\gamma}_2 \right] \dot{\gamma}_1 + g \left[\sin(\gamma_1 + \gamma_2) + v'_{1L} \cos(\gamma_1 + \gamma_2) \right] \\
 & + \left. (\ddot{v}_{1L} \cos \gamma_2 + x_2 \ddot{v}'_{1L}) \right\} - \frac{\partial}{\partial x_2} \left[(\ddot{\gamma}_1 + \ddot{\gamma}_2 + \ddot{v}'_{1L}) \frac{\partial J_2}{\partial x_2} \right] \\
 & - \frac{\partial}{\partial x_2} \left[\int_{x_2}^{L_2} \left[g \cos(\gamma_1 + \gamma_2) - L_1 \ddot{\gamma}_1 \sin \gamma_2 + L_1 \dot{\gamma}_1^2 \cos \gamma_2 + \xi (\dot{\gamma}_1 + \dot{\gamma}_2)^2 \right] \frac{\partial m_2}{\partial \xi} d\xi v'_2 \right] \\
 & = 0,
 \end{aligned} \tag{104}$$

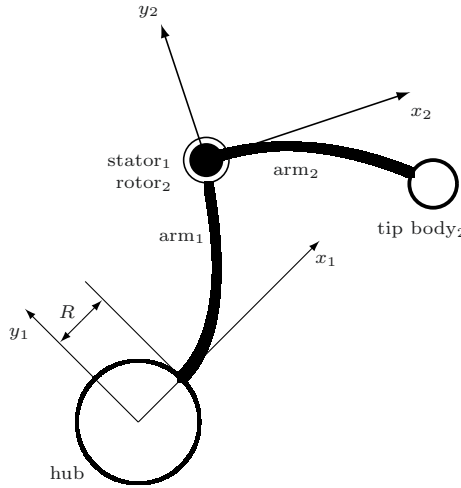


Figure 24. An articulated flexible robot

it becomes obvious that effective procedures are needed. Without going into detail, one interprets the first line of eq.(104) as the Rayleigh beam

equation, and the remainder comes from gravity and dynamical reactions from the upper arm [9]. To find a solution, Galerkin's procedure drops out due to the corresponding boundary conditions. Considering the boundary conditions within the approximation procedure (the "extended Galerkin") is tedious. But integrating these one obtains the "direct Ritz" as pointed out in eq.(64). This was the very first step in direction of effectivity.

Eq.(64) considers the elastic rotor which will serve for demonstration: One has simply to set $v(x, t) = \mathbf{v}(x)^T \mathbf{q}_v(t)$, $\delta v(x, t) = \delta \mathbf{q}_v(t)^T \mathbf{v}(x)$ etc. If one concentrates on the $\ddot{\alpha}$ -depending terms,

$$\begin{aligned} \delta W &= \int_o^L \ddot{\alpha} \rho \left\{ -\delta v A w + \delta v' I w' + \delta w A v - \delta w' I v' \right\} dx + \dots \\ &= (\delta \mathbf{q}_v^T \delta \mathbf{q}_w^T) \int_o^L \ddot{\alpha} \rho \left[\begin{array}{cc} -(A \mathbf{v} \mathbf{w}^T - I \mathbf{v}' \mathbf{w}'^T) \mathbf{q}_w & \\ + (A \mathbf{w} \mathbf{v}^T - I \mathbf{w}' \mathbf{v}'^T) \mathbf{q}_v & \end{array} \right] dx + \dots = 0, \end{aligned} \quad (105)$$

one obtains the $\mathbf{N}$ -matrix (nonconservative restoring matrix) which reads

$$\mathbf{N} \mathbf{q} = \ddot{\alpha} \int_o^L \rho \left[\begin{array}{cc} 0 & -[A \mathbf{v} \mathbf{w}^T - I \mathbf{v}' \mathbf{w}'^T] \\ +[A \mathbf{w} \mathbf{v}^T - I \mathbf{w}' \mathbf{v}'^T] & 0 \end{array} \right] dx \begin{pmatrix} \mathbf{q}_v \\ \mathbf{q}_w \end{pmatrix}. \quad (106)$$

Notice, however, that eq.(105) needed a tedious correction (underlined in eq.(105)).

On the other hand, we have eq.(66) for deriving the motion equations. We define v and w as functions w.r.t. the undeformed system (moving frame). The mass center velocities are, for the moving frame representation,

$$\mathbf{v}_c = \begin{pmatrix} 0 \\ \dot{v} - \dot{\alpha} w \\ \dot{w} + \dot{\alpha} v \end{pmatrix}, \quad \boldsymbol{\omega}_c = \begin{pmatrix} \dot{\alpha} \\ -\dot{w}' \\ +\dot{v}' \end{pmatrix}. \quad (107)$$

The inertia tensor for a moving frame representation reads $d\mathbf{J} = [\mathbf{E} - \tilde{\boldsymbol{\varphi}}]^T \rho I dx [\mathbf{E} - \tilde{\boldsymbol{\varphi}}]$ where $\mathbf{I} = I \text{diag}\{2, 1, 1\}$ (circular cross section). The bending angles are $\boldsymbol{\varphi} = (0 \quad -w' \quad v')^T$. Thus, the first order momenta read

$$d\mathbf{p} = \begin{pmatrix} 0 \\ \dot{v} - \dot{\alpha} w \\ \dot{w} + \dot{\alpha} v \end{pmatrix} \rho A dx, \quad d\mathbf{L} = \begin{pmatrix} 2\dot{\alpha} \\ \dot{\alpha} v' - \dot{w}' \\ \dot{\alpha} w' + \dot{v}' \end{pmatrix} \rho I dx. \quad (108)$$

Inserting the Ritz ansatz into eq.(107) yields

$$\mathbf{v}_c = \begin{bmatrix} 0 & 0 \\ +\mathbf{v}^T & 0 \\ 0 & +\mathbf{w}^T \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 0 & 0 \\ 0 & -\dot{\alpha}\mathbf{w}^T \\ +\dot{\alpha}\mathbf{v}^T & 0 \end{bmatrix} \mathbf{q}, \quad (109)$$

$$\boldsymbol{\omega}_c = \begin{bmatrix} 0 & 0 \\ 0 & -\mathbf{w}'^T \\ +\mathbf{v}'^T & 0 \end{bmatrix} \dot{\mathbf{q}} + \begin{pmatrix} \dot{\alpha} \\ 0 \\ 0 \end{pmatrix}, \quad (110)$$

the functional matrices (Jacobians) are thus

$$\left(\frac{\partial \mathbf{v}_c}{\partial \dot{\mathbf{q}}} \right) = \begin{bmatrix} 0 & 0 \\ +\mathbf{v}^T & 0 \\ 0 & +\mathbf{w}^T \end{bmatrix}, \quad \left(\frac{\partial \boldsymbol{\omega}_c}{\partial \dot{\mathbf{q}}} \right) = \begin{bmatrix} 0 & 0 \\ 0 & -\mathbf{w}'^T \\ +\mathbf{v}'^T & 0 \end{bmatrix}. \quad (111)$$

Along with $\boldsymbol{\omega}_R = (\dot{\alpha} \ 0 \ 0)$, the motion equation reads

$$\begin{aligned} & \int_o^L \begin{bmatrix} 0 & 0 & +\mathbf{v}' \\ 0 & -\mathbf{w}' & 0 \end{bmatrix} \begin{pmatrix} 2\ddot{\alpha} \\ -\ddot{w}' - \dot{\alpha}^2 w' + \ddot{\alpha} v' \\ +\ddot{v}' + \dot{\alpha}^2 v' + \ddot{\alpha} w' \end{pmatrix} \rho I dx + \\ & \int_o^L \begin{bmatrix} 0 & +\mathbf{v} & 0 \\ 0 & 0 & +\mathbf{w} \end{bmatrix} \begin{pmatrix} 0 \\ \ddot{v} - 2\dot{\alpha}\dot{w} - \dot{\alpha}^2 v - \ddot{\alpha} w \\ \ddot{w} + 2\dot{\alpha}\dot{v} - \dot{\alpha}^2 w + \ddot{\alpha} v \end{pmatrix} \rho A dx. \end{aligned} \quad (112)$$

Once more concentrating on $\mathbf{N}$, one has

$$\begin{aligned} \mathbf{N} \mathbf{q} = & \int_o^L \begin{bmatrix} 0 & 0 & +\mathbf{v}' \\ 0 & -\mathbf{w}' & 0 \end{bmatrix} \begin{pmatrix} 0 \\ \ddot{\alpha} v' \\ \ddot{\alpha} w' \end{pmatrix} \rho I dx + \\ & \int_o^L \begin{bmatrix} 0 & +\mathbf{v} & 0 \\ 0 & 0 & +\mathbf{w} \end{bmatrix} \begin{pmatrix} 0 \\ -\ddot{\alpha} w \\ +\ddot{\alpha} v \end{pmatrix} \rho A dx. \end{aligned} \quad (113)$$

One obtains automatically the correct signs. The procedure (66) is easy (matrix-vector multiplications, no directional derivatives any more). After all, eq.(66) shows clear cut advantages, but how to deal with nonholonomic constraints?

5 Nonholonomicity and Redundancy

Including nonholonomic constraints leads to the Projection Equation as most powerful and easy to handle procedure. A fair comparison of methods is obtained through the Central Equation of Dynamics. Focusing on modeling aspects, the decomposition into subsystems leads to their topology, the interchangeability of submodels, and their interactions. The corresponding equations are either obtained in minimal form or as a recursive scheme which saves a tremendous amount of CPU time. Disposing of modern computers: should one confine oneself with rigid body segmentation (or lumped mass) models? A simple example: to obtain the first mode of a clamped beam with a certain accuracy needs one hundred spring interconnected rigid links – this is not convincing. Or the other way round: Why not use commercial FE-codes, also for beamlike (sub-) structures? An example: the motions of a rolling mill turned out inexecutable. Reconsideration saved the CPU time with a factor of one year to half an hour – isn't this convincing?

5.1 Analytical methods

In case of nonholonomic variables s_m one can not conclude from

$$\sum_n \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_n} - \frac{\partial T}{\partial q_n} - \mathbf{Q}_n \right] \delta q_n = 0 \quad (114)$$

on the Lagrange equations because δq_n is not arbitrary any more. Hence

$$\sum_{n=1}^f \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_n} - \frac{\partial T}{\partial q_n} - \mathbf{Q}_n \right] \sum_{m=1}^g \left(\frac{\partial q_n}{\partial s_m} \right) \delta s_m = 0 \quad (115)$$

(Maggi, [37]). Hereby, the relationship $(\partial \mathbf{q} / \partial \mathbf{s})$ is always known: Assume $\mathbf{q} = \mathbf{q}(\mathbf{s})$. Then $\dot{\mathbf{q}} = (\partial \mathbf{q} / \partial \mathbf{s}) \dot{\mathbf{s}}$ yields by differentiation w.r.t. $\dot{\mathbf{s}}$ simply $(\partial \dot{\mathbf{q}} / \partial \dot{\mathbf{s}}) = (\partial \mathbf{q} / \partial \mathbf{s})$ (same as in eq.(52) et seq.). In eq.(115), f denotes the positional degree of freedom and g the velocity degree of freedom, $g = f$ for the holonomic case. Notice that the Lagrangean part in eq.(115) needs a full range calculation and, afterwards, replacement of $\dot{\mathbf{q}} \rightarrow \dot{\mathbf{s}}$ etc. This is extremely laborious.

Hamel [23] derived his famous *Euler-Lagrange-Equations* which in its variational form reads (for derivation see for instance eq.(128))

$$\sum_n \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{s}_n} - \frac{\partial T}{\partial s_n} - \mathbf{Q}_n \right] \delta s_n + \sum_\nu \frac{\partial T}{\partial \dot{s}_\nu} \left[\frac{d\delta s_\nu - \delta ds_\nu}{dt} \right] = 0. \quad (116)$$

Of course, the kinetic energy T does not depend on s_n directly but $T = T(\mathbf{q}, \dot{\mathbf{s}})$. What is meant by $(\partial T / \partial s_n)$ is simply $(\partial T / \partial \mathbf{q})(\partial \mathbf{q} / \partial s_n)$ with known $(\partial \mathbf{q} / \partial s_n)$ (see above). Hamel says “I found them [the equations] when I asked myself: which equations replace the Lagrangean ones, when

I insert, besides the n position variables $\mathbf{q}$, some other n independent linear combinations of $d\mathbf{q}/dt$?" Thus, at first, nonholonomic constraints are not considered (same number n). In the first term of eq.(116), δs_n is factored out while in the second it is somehow hidden. But along with $\delta s_\nu = \sum_i (\partial s_\nu / \partial q_i) \delta q_i \Rightarrow d\delta s_\nu = \sum_{i,k} (\partial^2 s_\nu / \partial q_i \partial q_k) \delta q_i dq_k + \sum_i (\partial s_\nu / \partial q_i) d\delta q_i$ one obtains

$$\frac{d\delta s_\nu - \delta d s_\nu}{dt} = \sum_{i,k} \left[\frac{\partial^2 s_\nu}{\partial q_i \partial q_k} - \frac{\partial^2 s_\nu}{\partial q_k \partial q_i} \right] \delta q_i \dot{q}_k + \sum_i \frac{\partial s_\nu}{\partial q_i} (d\delta q_i - \delta dq_i). \quad (117)$$

Using a variational approach $\delta \mathbf{q} = (\partial \mathbf{q} / \partial \varepsilon)|_{\varepsilon=0} \cdot \varepsilon = \varepsilon \cdot \boldsymbol{\eta}$ one obtains $d\delta \mathbf{q} - \delta d\mathbf{q} = \varepsilon \cdot d\boldsymbol{\eta} - \varepsilon \cdot d\boldsymbol{\eta} = 0$: Then (and only then), the last term in eq.(117) vanishes and the remainder [the brackets to be more precise] represents Schwarz' integrability rule – which is, preconditionally, not fulfilled for nonholonomic variables. However, $\delta q_i = \sum_n (\partial q_i / \partial s_n) \delta s_n$ and $\dot{q}_k = \sum_\mu (\partial q_k / \partial s_\mu) \dot{s}_\mu$ are, of course, known. Insertion into eq.(117) and eq.(116), δs_n is also factored out for the second term. Since δs_n is arbitrary, one obtains the n th motion equation

$$\sum_n \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{s}_n} - \frac{\partial T}{\partial s_n} - \mathbf{Q}_n \right] + \sum_n \sum_{\nu, \mu} \left[\frac{\partial T}{\partial \dot{s}_\nu} \dot{s}_\mu \beta_\nu^{\mu, n} \right] = 0 \quad (118)$$

where $\beta_\nu^{\mu, n} = \sum_{i,k} \frac{\partial q_k}{\partial s_\mu} \frac{\partial q_i}{\partial s_n} \left(\frac{\partial^2 s_\nu}{\partial q_i \partial q_k} - \frac{\partial^2 s_\nu}{\partial q_k \partial q_i} \right)$.

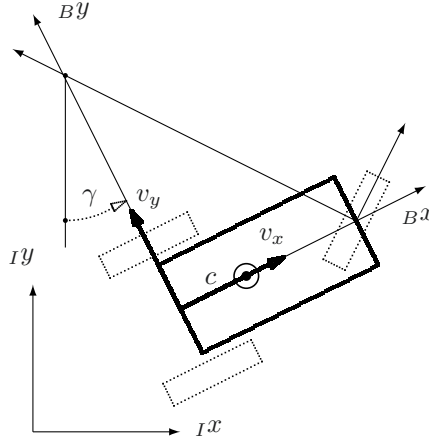


Figure 25. Carriage

Example. Carriage: The reference frame origin lies on the rear axle; the rear wheels are not allowed to slide sideways (nonholonomic constraint). The car is able to reach every point within the inertial x - y -plane and orient itself arbitrarily (angle γ). Using the inertial coordinates $\mathbf{q}^T = (x \ y \ \gamma)$, the rear axle velocities (represented in the body fixed reference frame) are

$$\dot{\mathbf{s}} = \begin{pmatrix} v_x \\ v_y \\ \omega_z \end{pmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\gamma} \end{pmatrix} := \begin{pmatrix} \partial \mathbf{s} \\ \partial \mathbf{q} \end{pmatrix} \dot{\mathbf{q}} \in \mathbb{R}^{f=3}. \quad (119)$$

The kinetic energy refers to the mass center velocities

$$T = \frac{m}{2} [v_x^2 + (v_y + c\omega_z)^2] + \frac{C^c}{2} \omega_z^2 = \frac{m}{2} (v_x^2 + v_y^2) + \underline{mcv_y \omega_z} + \frac{C^o}{2} \omega_z^2. \quad (120)$$

Since $(\partial \mathbf{s} / \partial \mathbf{q}) \in \mathbb{R}^{3,3}$ is orthonormal for this example, its inverse $(\partial \mathbf{q} / \partial \mathbf{s})$ is simply the transpose. The Hamel-coefficients (118) are

$$\begin{aligned} \beta_1^{1,2} &= 0 \\ \beta_1^{1,3} &= -\frac{\partial q_1}{\partial s_1} \frac{\partial^2 s_1}{\partial q_1 \partial q_3} - \frac{\partial q_2}{\partial s_1} \frac{\partial^2 s_1}{\partial q_2 \partial q_3} = (-\cos \gamma)(-\sin \gamma) - (\sin \gamma)(\cos \gamma) = 0 \\ \beta_1^{2,3} &= -\frac{\partial q_1}{\partial s_2} \frac{\partial^2 s_1}{\partial q_1 \partial q_3} - \frac{\partial q_2}{\partial s_2} \frac{\partial^2 s_1}{\partial q_2 \partial q_3} = (\sin \gamma)(-\sin \gamma) - (\cos \gamma)(\cos \gamma) = -1. \end{aligned} \quad (121)$$

Analogously, one obtains for $\nu = 2$: $\beta_2^{1,2} = 0, \beta_2^{1,3} = 1, \beta_2^{2,3} = 0$ etc. Hence, the motion equations read

$$\begin{aligned} \left[\frac{d}{dt} \frac{\partial T}{\partial v_x} + \underbrace{\beta_2^{3,1}}_{-1} \dot{\gamma} \frac{\partial T}{\partial v_y} - Q_1 \right] \delta s_1 &= \left[m\dot{v}_x - \underline{mc\dot{\gamma}^2} - \underline{m\dot{\gamma}v_y} - f_x \right] \delta s_1 = 0 \\ \left[\frac{d}{dt} \frac{\partial T}{\partial v_y} + \underbrace{\beta_1^{3,2}}_{+1} \dot{\gamma} \frac{\partial T}{\partial v_x} - Q_2 \right] \delta s_2 &= \left[\underline{m\dot{v}_y} + \underline{mc\ddot{\gamma}} + \underline{m\dot{\gamma}v_x} - f_y \right] \delta s_2 = 0 \\ \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{\gamma}} + \underbrace{\beta_1^{2,3}}_{-1} v_y \frac{\partial T}{\partial v_x} + \underbrace{\beta_2^{1,3}}_{+1} v_x \frac{\partial T}{\partial v_y} - Q_3 \right] \delta s_3 &= \\ &= \left[C^o \ddot{\gamma} + \underline{mc\dot{v}_y} + \underline{msv_x \dot{\gamma}} - M_z \right] \delta s_3 = 0 \end{aligned} \quad (122)$$

Considering the nonholonomic constraint $\dot{s}_2 = v_y = 0$ yields $\delta s_2 = 0$ (null!), while δs_1 and δs_3 are arbitrary. One has thus

$$\Rightarrow \begin{bmatrix} m & 0 \\ 0 & C^o \end{bmatrix} \begin{pmatrix} \dot{v}_x \\ \dot{\gamma} \end{pmatrix} + \begin{bmatrix} 0 & -mc\dot{\gamma} \\ +mc\dot{\gamma} & 0 \end{bmatrix} \begin{pmatrix} v_x \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} f_x \\ M_z \end{pmatrix} \quad (123)$$

Also Hamel's equations need a full range calculation at first ($f=3$). The nonholonomic constraint has to be considered afterwards. Setting $v_y = 0$ already in T [eq.(120), underlined] would yield a loss of information because then, due to the required partial derivatives, the underlined terms in eq.(122) vanish and the gyroscopic matrix in eq.(123) gets lost. The procedure is laborious. (We should, however, have in mind that our scientific ancestors were much more trained in calculus – and perhaps in thinking.)

5.2 Synthetical method(s)

On the other hand we have eq.(66) (where $\omega = \dot{\pi} \wedge \pi$ is a quasicordinate – so what? And $\mathbf{v} = \dot{\rho} \wedge \rho$ is a quasicordinate – so what?) Replacement of $(\partial \rho / \partial \mathbf{q}) = (\partial \mathbf{v} / \partial \dot{\mathbf{q}})$ and $(\partial \pi / \partial \mathbf{q}) = (\partial \omega / \partial \dot{\mathbf{q}})$ makes the quasicordinates disappear and simultaneously shows that the Jacobians are simply coefficient matrices which are known from kinematics, obtained thus as a by-product without any extra calculation. This holds also if we introduce new minimal velocities $\dot{\mathbf{s}} = \mathbf{H}(\mathbf{q})\dot{\mathbf{q}}$ where $\mathbf{s}$ is a quasicordinate – so what? We call this final result

$$\sum_{i=1}^N \left[\left(\frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{s}}} \right)^T \left(\frac{\partial \omega_i}{\partial \dot{\mathbf{s}}} \right)^T \right] \begin{pmatrix} \dot{\mathbf{p}} + \tilde{\omega}_R \mathbf{p} - \mathbf{f}^e \\ \dot{\mathbf{L}} + \tilde{\omega}_R \mathbf{L} - \mathbf{M}^e \end{pmatrix}_i = 0 \quad (124)$$

the *Projection Equation* since it projects the momentum balances into the unconstrained directions. (Other projections are possible, $\dot{\mathbf{s}} \rightarrow \dot{\mathbf{z}}$, which characterizes either a congruence transformation ($\dot{\mathbf{z}} \in \mathbb{R}^g$) or any other but constrained vector space ($\dot{\mathbf{z}} \in \mathbb{R}^n$, $n > g$) which needs to consider the corresponding constraint forces, of course). Eq.(124) refers to moving frames indicated by ω_R . However, it enables to choose different coordinate systems for different bodies, yet different systems for momentum and momentum of momentum of the same body. This makes the procedure very flexible. There are no directional derivatives any more; the equation holds for nonholonomic variables and nonholonomic constraints: $\dot{\mathbf{s}} \in \mathbb{R}^g$ considers the constraints in advance.

Example. Consider once more Figure 25. The rear axle velocities are $(v_x \ v_y \ \omega_z)$. The nonholonomic constraint $v_y = 0$ is in advance taken into account:

$$\dot{\mathbf{s}} = \begin{pmatrix} v_x \\ \omega_z \end{pmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\gamma} \end{pmatrix} \in \mathbb{R}^{g=2}. \quad (125)$$

The mass center velocities $(v_{cx} \ v_{cy} \ \omega_{cz})$ are

$$\begin{pmatrix} v_{cx} \\ v_{cy} \\ \omega_{cz} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & c \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ \omega_z \end{pmatrix} \Rightarrow \begin{pmatrix} (\partial v_{cx}/\partial \dot{s}) \\ (\partial v_{cy}/\partial \dot{s}) \\ (\partial \omega_{cz}/\partial \dot{s}) \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & c \\ 0 & 1 \end{bmatrix}. \quad (126)$$

Along with the momenta $(p_x \ p_y \ L_z) = (mv_{cx} \ mv_{cy} \ C^c\omega_{cz})$ one obtains

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 1 \end{bmatrix} \left[\begin{bmatrix} m & 0 \\ 0 & mc \\ 0 & C^c \end{bmatrix} \begin{pmatrix} \dot{v}_x \\ \dot{\omega}_z \end{pmatrix} + \begin{bmatrix} 0 & -\omega_z & 0 \\ +\omega_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & mc \\ 0 & C^c \end{bmatrix} \begin{pmatrix} v_x \\ \omega_z \end{pmatrix} - \begin{pmatrix} f_x \\ f_y \\ M_z \end{pmatrix} \right] \quad (127)$$

which leads directly to eq.(123) without any detour.

5.3 The horn of plenty?

Collecting procedures (Euler, Hamel, Hamilton, Kane, Lagrange, Maggi, Mangeron, Tzenoff ...) reveals the beautiful variety of methods, the horn of plenty – really? Kane’s method for instance turns out to be a restrictive special case, along with misleading abbreviations, although Kane (“the man who mastered motion”, [47]) is keen enough to call Lagrange’s results black magic. And Poincot called Lagrange’s δ obscure [45]. Are the procedures interconnected – and if, how are they connected? It is time to bring some order into the variety of methods, just to enable a fair comparison and a motivated choice.

The Central Equation of Dynamics. Lagrange is indeed sometimes a bit vague: “because this method needs to vary the same terms in two different ways ... I have introduced a new characteristic δ .” And he was probably not aware of the independency of the momentum and the momentum of momentum theorems. The background is (hopefully) clarified with the above interpretation of $\delta \mathbf{r}$ and with Boltzmann’s axiom in the background. So let us continue with Lagrange’s $\int d\mathbf{m} \dot{\mathbf{r}}^T \delta \mathbf{r} - \delta W = 0$ and extract one time derivative (which needs a correction term due to the product rule): $\int d\mathbf{m} \dot{\mathbf{r}}^T \delta \mathbf{r} = \frac{d}{dt} [\int d\mathbf{m} \dot{\mathbf{r}}^T \delta \mathbf{r}] - \int d\mathbf{m} \dot{\mathbf{r}}^T \frac{d}{dt} \delta \mathbf{r} = 0$. Within the last term, the d - and the δ -operation may be interchanged: Consider eq.(117) for $s_\nu \rightarrow r_\nu$. Along with the variational ansatz for δq_i , the last term vanishes and one is left with Schwarz’ rule which for r_ν has to be fulfilled just from modeling assumptions. One has then $\int d\mathbf{m} \dot{\mathbf{r}}^T \delta \frac{d\mathbf{r}}{dt} = \delta \int \frac{1}{2} \dot{\mathbf{r}}^T d\mathbf{m} \dot{\mathbf{r}}$. Using the same quadratic form for the penultimate term yields $\frac{d}{dt} \int \frac{\partial}{\partial \mathbf{r}} [\frac{1}{2} \dot{\mathbf{r}}^T d\mathbf{m} \dot{\mathbf{r}}] \delta \mathbf{r}$. From $\mathbf{r} = \mathbf{r}(\mathbf{q}(s)) := \mathbf{r}(s)$ one has $\delta \mathbf{r} = (\partial \mathbf{r}/\partial s) \delta s$. Same as above $(\partial \mathbf{r}/\partial s) =$

$(\partial \dot{\mathbf{r}}/\partial \dot{\mathbf{s}})$ holds, yielding $\frac{d}{dt} \int \frac{\partial}{\partial \dot{\mathbf{r}}} \left[\frac{1}{2} \dot{\mathbf{r}}^T d\mathbf{m} \dot{\mathbf{r}} \right] \delta \mathbf{r} = \frac{d}{dt} \int \frac{\partial}{\partial \dot{\mathbf{r}}} \left[\frac{1}{2} \dot{\mathbf{r}}^T d\mathbf{m} \dot{\mathbf{r}} \right] \frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{s}}} \delta \mathbf{s} = \frac{d}{dt} \frac{\partial}{\partial \dot{\mathbf{s}}} \left[\int \frac{1}{2} \dot{\mathbf{r}}^T d\mathbf{m} \dot{\mathbf{r}} \right] \delta \mathbf{s}$ (chain rule). Along with $T = \int \frac{1}{2} \dot{\mathbf{r}}^T d\mathbf{m} \dot{\mathbf{r}}$ (kinetic energy) one obtains *the Central Equation of Dynamics* [4]:

$$\frac{d}{dt} \left[\left(\frac{\partial T}{\partial \dot{\mathbf{s}}} \right) \delta \mathbf{s} \right] - \delta T - \delta W = 0. \quad (128)$$

Methodology. From eq.(128) one obtains by elementary calculations a considerable body of methods, see Table 1. If the aim is to derive the motion equations – and this is the aim in the present context – then the choice of procedure is without any doubt the Projection Equation. This is simply because here any step of calculation is carried out as far as possible such that one only has to insert the problem specific quantities.

5.4 Structurizing the problem

Subsystems. Let the total system consist of a number of subsystems N_{sub} each of which contains N_n bodies. Splitting the sum in eq.(124) into a double sum, and inserting $\dot{\mathbf{y}}_n$ which describes the n th subsystem yields, along with the chain rule,

$$\sum_{n=1}^{N_{sub}} \sum_{i=1}^{N_n} \left(\frac{\partial \dot{\mathbf{y}}_n}{\partial \dot{\mathbf{s}}} \right)^T \left\{ \left[\left(\frac{\partial \mathbf{v}_c}{\partial \dot{\mathbf{y}}_n} \right)^T \left(\frac{\partial \boldsymbol{\omega}_c}{\partial \dot{\mathbf{y}}_n} \right)^T \right] \left[\begin{array}{c} \dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}}_{IR} \mathbf{p} - \mathbf{f}^e \\ \dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}_{IR} \mathbf{L} - \mathbf{M}^e \end{array} \right] \right\}_i = 0. \quad (129)$$

Shifting the inner sum,

$$\sum_{n=1}^{N_{sub}} \left(\frac{\partial \dot{\mathbf{y}}_n}{\partial \dot{\mathbf{s}}} \right)^T \sum_{i=1}^{N_n} \left\{ \left[\left(\frac{\partial \mathbf{v}_c}{\partial \dot{\mathbf{y}}_n} \right)^T \left(\frac{\partial \boldsymbol{\omega}_c}{\partial \dot{\mathbf{y}}_n} \right)^T \right] \left[\begin{array}{c} \dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}}_{IR} \mathbf{p} - \mathbf{f}^e \\ \dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}_{IR} \mathbf{L} - \mathbf{M}^e \end{array} \right] \right\}_i = 0, \quad (130)$$

yields, in a well known notation for the subsystems,

$$\sum_{n=1}^{N_{sub}} \left(\frac{\partial \dot{\mathbf{y}}_n}{\partial \dot{\mathbf{s}}} \right)^T [\mathbf{M} \ddot{\mathbf{y}} + \mathbf{G} \dot{\mathbf{y}} - \mathbf{Q}]_n = 0. \quad (131)$$

Flexible subsystem are obtained with the limitig case

$$\sum_{n=1}^{N_{sub}} \int_{B_n} \left(\frac{\partial \dot{\mathbf{y}}_n}{\partial \dot{\mathbf{s}}} \right)^T \left\{ \left[\left(\frac{\partial \mathbf{v}_c}{\partial \dot{\mathbf{y}}_n} \right)^T \left(\frac{\partial \boldsymbol{\omega}_c}{\partial \dot{\mathbf{y}}_n} \right)^T \right] \left[\begin{array}{c} d\dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}}_{IR} d\mathbf{p} - d\mathbf{f}^e \\ d\dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}_{IR} d\mathbf{L} - d\mathbf{M}^e \end{array} \right] \right\}_n = 0. \quad (132)$$

$$\begin{aligned}
(1) \quad & T + V = H \\
(2) \quad & \left(\frac{\partial G}{\partial \dot{\mathbf{s}}} \right)^T = \mathbf{Q} \\
(3) \quad & \delta \int_{t_o}^{t_1} (T - V) dt = 0 \\
(4) \quad & \dot{\mathbf{p}}^T = - \left(\frac{\partial H}{\partial \mathbf{q}} \right); \quad \dot{\mathbf{q}}^T = \left(\frac{\partial H}{\partial \mathbf{p}} \right) \\
(5) \quad & \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} - \mathbf{Q}^T = 0 \\
(6) \quad & \frac{1}{m} \frac{\partial T^m}{\partial \mathbf{q}^m} - \frac{m+1}{m} \frac{\partial T}{\partial \mathbf{q}} - \mathbf{Q}^T = 0 \\
(7) \quad & \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} - \mathbf{Q}^T \right] \left(\frac{\partial \dot{\mathbf{q}}}{\partial \dot{\mathbf{s}}} \right) = 0 \\
(8) \quad & \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{s}}} - \frac{\partial T}{\partial \mathbf{s}} - \mathbf{Q}^T \right] \delta \mathbf{s} + \frac{\partial T}{\partial \dot{\mathbf{s}}} \left(\frac{d\delta \mathbf{s}}{dt} - \delta d\mathbf{s} \right) = 0
\end{aligned}$$

Central-	$\frac{d}{dt} \left[\left(\frac{\partial T}{\partial \dot{\mathbf{s}}} \right) \delta \mathbf{s} \right] - \delta T - \delta W = 0$	Equation
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$$(9) \quad \sum_{i=1}^N \left\{ \left(\frac{\partial \mathbf{v}}{\partial \dot{\mathbf{s}}} \right)^T (\dot{\mathbf{p}} + \tilde{\omega}_R \mathbf{p} - \mathbf{f}^e) + \left(\frac{\partial \omega}{\partial \dot{\mathbf{s}}} \right)^T (\dot{\mathbf{L}} + \tilde{\omega}_R \mathbf{L} - \mathbf{M}^e) \right\}_i = 0$$

Table 1. Methodology

The special structure of equations in this case is

$$\sum_{n=1}^{N_{sub}} \int_{B_n} \left(\frac{\partial \dot{\mathbf{y}}_n}{\partial \dot{\mathbf{s}}} \right)^T [d\mathbf{M}\ddot{\mathbf{y}} + d\mathbf{G}\dot{\mathbf{y}} - d\mathbf{Q}]_n = 0. \quad (133)$$

Shifting the integral (like the sum in eq.(130)) is not yet possible because $\mathbf{y}_n$ contains for instance the deflections and its spatial derivatives while $\mathbf{s}$ refers to the deflections only. The way out here is to define spatial operators which eventually leads to the partial differential equations along with the boundary conditions, see [9]. But this is clearly not the aim of investigation here. To obtain the ordinary (approximative) equations one has to insert the Ritz ansatz $\dot{\mathbf{y}}(\mathbf{x}, t) = \mathbf{\Psi}(\mathbf{x})^T \dot{\mathbf{y}}_R(t)$ and make use of the chain rule once more:

$$\sum_{n=1}^{N_{sub}} \int_{B_n} \left(\frac{\partial \dot{\mathbf{y}}_{Rn}}{\partial \dot{\mathbf{s}}} \right)^T \left(\frac{\partial \dot{\mathbf{y}}_n}{\partial \dot{\mathbf{y}}_{Rn}} \right)^T [d\mathbf{M}\ddot{\mathbf{y}} + d\mathbf{G}\dot{\mathbf{y}} - d\mathbf{Q}]_n = 0. \quad (134)$$

Since then $(\partial \dot{\mathbf{y}}_{Rn} / \partial \dot{\mathbf{s}})$ depends on time and $(\partial \dot{\mathbf{y}}_n / \partial \dot{\mathbf{y}}_{Rn}) = \mathbf{\Psi}^T$ on space only, the integral may be shifted, and, inserting the Ritz ansatz for $\mathbf{y}$, $\dot{\mathbf{y}}$, $\ddot{\mathbf{y}}$, one obtains

$$\sum_{n=1}^{N_{sub}} \left(\frac{\partial \dot{\mathbf{y}}_{Rn}}{\partial \dot{\mathbf{s}}} \right)^T \left[\int_B \mathbf{\Psi} d\mathbf{M} \mathbf{\Psi}^T \right] \ddot{\mathbf{y}}_R + \left[\int_B \mathbf{\Psi} d\mathbf{G} \mathbf{\Psi}^T \right] \dot{\mathbf{y}}_R - \int_B \mathbf{\Psi} d\mathbf{Q} \Big|_n = 0 \quad (135)$$

which has the same structure as eq.(131). The general subsystem representation results in (with $N_{sub} \rightarrow N$ for simplicity)

$$\left[\left(\frac{\partial \dot{\mathbf{y}}_1}{\partial \dot{\mathbf{s}}} \right)^T \left(\frac{\partial \dot{\mathbf{y}}_2}{\partial \dot{\mathbf{s}}} \right)^T \cdots \left(\frac{\partial \dot{\mathbf{y}}_N}{\partial \dot{\mathbf{s}}} \right)^T \right] \begin{bmatrix} \mathbf{M}_1 \ddot{\mathbf{y}}_1 + \mathbf{G}_1 \dot{\mathbf{y}}_1 - \mathbf{Q}_1 \\ \mathbf{M}_2 \ddot{\mathbf{y}}_2 + \mathbf{G}_2 \dot{\mathbf{y}}_2 - \mathbf{Q}_2 \\ \vdots \\ \mathbf{M}_N \ddot{\mathbf{y}}_N + \mathbf{G}_N \dot{\mathbf{y}}_N - \mathbf{Q}_N \end{bmatrix} = 0. \quad (136)$$

The kinematic chain. Starting with the first subsystem, one has $\dot{\mathbf{y}}_1 = \mathbf{F}_1 \dot{\mathbf{s}}_1$. Next, calculate the velocities at the coupling point and transform the result into the successor frame to obtain the guidance velocity of the successor subsystem in the form $\mathbf{T}_{21} \dot{\mathbf{y}}_1$. (Notice that $\dot{\mathbf{y}}_i$ refers to the corresponding frame origin). This guidance velocity is superimposed by the relative velocity which the neighbouring subsystem contributes, hence $\dot{\mathbf{y}}_2 = \mathbf{T}_{21} \dot{\mathbf{y}}_1 + \mathbf{F}_2 \dot{\mathbf{s}}_2$,

etc. One obtains, with p as predecessor of i , $\dot{\mathbf{y}}_i = \mathbf{T}_{ip}\dot{\mathbf{y}}_p + \mathbf{F}_i\dot{\mathbf{s}}_i$ (where $\mathbf{F}_i$ are the local functional matrices, or Jacobians) explicitly

$$\begin{pmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \\ \vdots \\ \dot{\mathbf{y}}_N \end{pmatrix} = \begin{bmatrix} \mathbf{F}_1 & & & \\ \mathbf{T}_{21}\mathbf{F}_1 & \mathbf{F}_2 & & \\ \vdots & \vdots & \ddots & \\ \mathbf{T}_{N1}\mathbf{F}_1 & \mathbf{T}_{N2}\mathbf{F}_2 & \cdots & \mathbf{F}_N \end{bmatrix} \begin{pmatrix} \dot{\mathbf{s}}_1 \\ \dot{\mathbf{s}}_2 \\ \vdots \\ \dot{\mathbf{s}}_N \end{pmatrix} := \mathbf{F}\dot{\mathbf{s}}. \quad (137)$$

$\mathbf{F}$ is the global functional matrix (nonmarked elements are zero).

The equations of motion. Using eq.(137) for eq.(136) yields

$$\begin{bmatrix} \mathbf{F}_1^T & \mathbf{F}_1^T \mathbf{T}_{21}^T & \cdots & \mathbf{F}_1^T \mathbf{T}_{N1}^T \\ & \mathbf{F}_2^T & \cdots & \mathbf{F}_2^T \mathbf{T}_{N2}^T \\ & & \ddots & \vdots \\ & & & \mathbf{F}_N^T \end{bmatrix} \begin{bmatrix} \mathbf{M}_1 \ddot{\mathbf{y}}_1 + \mathbf{G}_1 \dot{\mathbf{y}}_1 - \mathbf{Q}_1 \\ \mathbf{M}_2 \ddot{\mathbf{y}}_2 + \mathbf{G}_2 \dot{\mathbf{y}}_2 - \mathbf{Q}_2 \\ \vdots \\ \mathbf{M}_N \ddot{\mathbf{y}}_N + \mathbf{G}_N \dot{\mathbf{y}}_N - \mathbf{Q}_N \end{bmatrix} = 0. \quad (138)$$

This gives either the access to the minimal form or to a recursive algorithm. The minimal form is

$$\mathbf{M}\ddot{\mathbf{s}} + \mathbf{G}\dot{\mathbf{s}} - \mathbf{Q} = 0 \in \mathbb{R}^g,$$

$$\begin{aligned} \mathbf{M} &= \mathbf{F}^T \text{blockdiag}\{\mathbf{M}_1 \cdots \mathbf{M}_N\} \mathbf{F}, \\ \mathbf{G} &= \mathbf{F}^T \text{blockdiag}\{\mathbf{G}_1 \cdots \mathbf{G}_N\} \mathbf{F} + \mathbf{F}^T \text{blockdiag}\{\mathbf{M}_1 \cdots \mathbf{M}_N\} \dot{\mathbf{F}}, \\ \mathbf{Q} &= \mathbf{F}^T [\mathbf{Q}_1^T \cdots \mathbf{Q}_N^T]^T. \end{aligned} \quad (139)$$

with $\mathbf{F}$ from eq.(137), while the recursive form is already given with eq.(138): solve in a Gaussian sense.

Example. Consider Figure 26: at distance a from the carriage frame origin (index 1) a robot arm is mounted (index 2). Its motion in z -direction (perpendicular to the drawing plane) is decoupled from the remainder and will not be considered.

The carriage (see eq.(125)/eq.(123)):

$$\dot{\mathbf{s}}_1 = \begin{pmatrix} v_x \\ \omega_z \end{pmatrix}_1 = \begin{bmatrix} \cos \gamma_1 & \sin \gamma_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\gamma} \end{pmatrix}_1 = \mathbf{H}_1(\mathbf{q}_1) \dot{\mathbf{q}}_1. \quad (140)$$

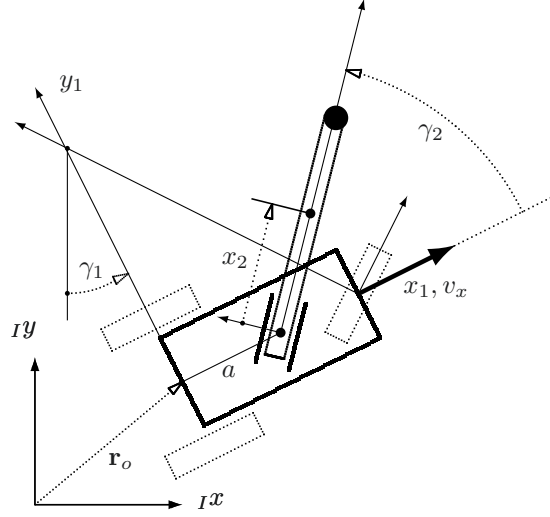


Figure 26. Nonholonomic robot model, top view

$$\mathbf{M}_1 = \begin{bmatrix} m & 0 \\ 0 & C^o \end{bmatrix}_1, \quad \mathbf{G}_1 = \begin{bmatrix} 0 & -mc\dot{\gamma} \\ +mc\dot{\gamma} & 0 \end{bmatrix}_1. \quad (141)$$

The robot arm: mass center velocities in terms of the “describing velocities” $\dot{\mathbf{y}}$ (frame origin o)

$$\begin{pmatrix} v_{cx} \\ v_{cy} \\ \omega_{cz} \end{pmatrix}_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_2 \begin{pmatrix} v_{ox} \\ v_{oy} \\ \omega_{oz} \\ \dot{x} \end{pmatrix}_2 := \bar{\mathbf{F}}_2 \dot{\mathbf{y}}_2 \quad (142)$$

(Notice that $\bar{\mathbf{F}}$ depends on time due to x_2 .) Along with $\bar{\mathbf{M}} = \text{diag}\{m, m, C^c\}$ one obtains the mass matrix $\mathbf{M} = \bar{\mathbf{F}}^T \bar{\mathbf{M}} \bar{\mathbf{F}}$ and the gyroscopic matrix $\mathbf{G} = \bar{\mathbf{F}}^T \dot{\bar{\mathbf{M}}} \bar{\mathbf{F}} + \bar{\mathbf{F}}^T \tilde{\omega}_R \bar{\mathbf{M}} \bar{\mathbf{F}}$ which read

$$\mathbf{M}_2 = \begin{bmatrix} m & 0 & 0 & m \\ 0 & m & mx & 0 \\ 0 & mx & C^o & 0 \\ m & 0 & 0 & m \end{bmatrix}_2, \quad \mathbf{G}_2 = \begin{bmatrix} 0 & -m & -mx & 0 \\ m & 0 & -0 & 2m \\ mx & 0 & 0 & 2mx \\ 0 & -m & -mx & 0 \end{bmatrix}_2 \omega_{oz2}, \quad (143)$$

($\omega_{oz2} = \dot{\gamma}_1 + \dot{\gamma}_2$). The kinematic chain: the carriage needs $\dot{\mathbf{y}}_1 = (v_x \ \omega_z)_1^T = \dot{\mathbf{s}}_1$. Thus, $\mathbf{F}_1$ is simply the unit matrix. The arm frame origin is guided

with the velocity at distance a : For a representation in frame 1 one has $(v_x \ a \dot{\gamma}_1 \ 0)$. Transformed into reference frame 2 and superimposed with the relative angular velocity $\dot{\gamma}_2$ one has

$$\begin{pmatrix} v_{ox} \\ v_{oy} \\ \omega_{oz} \end{pmatrix}_2 = \begin{pmatrix} \cos \gamma_2 & \sin \gamma_2 & 0 \\ -\sin \gamma_2 & \cos \gamma_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ \dot{\gamma}_1 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\gamma}_2. \quad (144)$$

To obtain $\dot{\mathbf{y}}_2$, eq.(144) has simply to be augmented with $\dot{x}_2$:

$$\dot{\mathbf{y}}_2 = \begin{pmatrix} v_{ox} \\ v_{oy} \\ \omega_{oz} \\ \dot{x} \end{pmatrix}_2 = \underbrace{\begin{bmatrix} \cos \gamma_2 & a \sin \gamma_2 \\ -\sin \gamma_2 & a \cos \gamma_2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{T}_{21}\mathbf{F}_1} \underbrace{\begin{pmatrix} v_x \\ \dot{\gamma}_1 \end{pmatrix}}_{\dot{\mathbf{s}}_1} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{F}_2} \underbrace{\begin{pmatrix} \dot{x}_2 \\ \dot{\gamma}_2 \end{pmatrix}}_{\dot{\mathbf{s}}_2}, \quad (145)$$

the (global) functional matrix thus reads

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & 0 \\ \mathbf{T}_{21}\mathbf{F}_1 & \mathbf{F}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \cos \gamma_2 & a \sin \gamma_2 & 0 & 0 \\ -\sin \gamma_2 & a \cos \gamma_2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]. \quad (146)$$

From eq.(139) one eventually obtains, for $\dot{\mathbf{s}}^T = (v_x \ \dot{\gamma}_1 \ \dot{x}_2 \ \dot{\gamma}_2)$,

$$\mathbf{M} = \begin{bmatrix} m_1 + m_2 & -m_2 x_2 \sin \gamma_2 & m_2 \cos \gamma_2 & -m_2 x_2 \sin \gamma_2 \\ -m_2 x_2 \sin \gamma_2 & C_1^o + C_2^o + m_2(a^2 + 2ax_2 \cos \gamma_2) & m_2 a \sin \gamma_2 & m_2 a x_2 \cos \gamma_2 \\ m_2 \cos \gamma_2 & m_2 a \sin \gamma_2 & m_2 & 0 \\ -m_2 x_2 \sin \gamma_2 & C_2^o + m_2 a x_2 \cos \gamma_2 & 0 & C_2^o \end{bmatrix}, \quad (147)$$

$$\mathbf{G} = \begin{bmatrix} 0 & -m_2 a \omega_1 - m_1 c_1 \omega_1 & -m_2 x_2 \omega_2 \cos \gamma_2 & -2m_2 \omega_2 \sin \gamma_2 - m_2 x_2 \omega_2 \cos \gamma_2 \\ m_2 x_2 \omega_1 \cos \gamma_2 + & & 2m_2 x_2 \omega_2 + \\ m_2 a \omega_1 + m_1 c_1 \omega_1 & -m_2 a x_2 \dot{\gamma}_2 \sin \gamma_2 & 2m_2 a \omega_2 \cos \gamma_2 - m_2 a x_2 \omega_2 \sin \gamma_2 \\ & -m_2 x_2 \omega_2 & & \\ m_2 \omega_1 \sin \gamma_2 & -m_2 a \omega_1 \cos \gamma_2 & 0 & -m_2 x_2 \omega_2 \\ m_2 x_2 \omega_1 \cos \gamma_2 & m_2 a x_2 \omega_1 \sin \gamma_2 & 2m_2 x_2 \omega_2 & 0 \end{bmatrix}. \quad (148)$$

The state vector reads $\mathbf{x}^T = (\mathbf{q}^T \ \dot{\mathbf{s}}^T)$ where $\dot{q}_1 = \dot{x} = v_x \cos \gamma_1$, $\dot{q}_2 = \dot{y} = v_x \sin \gamma_1$ (the remainder is holonomic). The state equations need to invert eq.(147). This is, with $\mathbf{M} \in \mathbb{R}^{4,4}$, of course unproblematic. However, concerning large systems, the Gaussian form gains in importance. For a plane chain with twenty links, the time saving factor is already more than ten when solving the equations with an elimination process instead of eq.(139).

It was about thirty years ago when Brandl et.al. proposed a recursive procedure (“order-n-algorithm”) for the numerical simulation of multi body systems without inversion of the (entire) mass matrix [3]. The probably first attempt in this field is already due to Vereshchagin [58] as pointed out by Schwertassek [50]; quite an amount of procedures followed in the eighties till the late nineties and sporadically till today, see e.g. [41]. This fact clearly demonstrates its general interest and importance concerning large (rigid) multibody systems.

These papers look at single rigid bodies for subsystems and pursue the idea to *eliminate* the (generalized) constraint forces step by step, starting with the last body, along with successively applying Euler’s cut principle. Clearly, this procedure leads to quite a big amount of non-structured equations/abbreviations. Over the time, several authors tried to simplify the equations, e.g. [59]. Bremer and Pfeiffer extended the procedure to elastic bodies (along with a Ritz series expansion) [6]. Such a procedure results in an extreme analytical effort which is susceptible to careless mistakes.

In the late nineties we decided to abandon the *Elimination*-process in favor of a *Projection*-process [7]. Elimination means to write down the momentum equations and then to eliminate line by line the (generalized) constraint forces (somehow comparable to d’Alembert’s principle, [2]). Projection, on the other hand, means to project the momentum equations into the unconstrained (“minimal”) space (as an application of Lagrange’s principle, or “virtual work”). Its mathematical evaluation then leads to the Gauss-form (139) where simultaneously, as a generalization, instead of single bodies

subsystems enter consideration along with nonholonomic variables and constraints. The simplest subsystem hereby is, of course, the single body itself but in general it characterizes an assembly group (like in Figure 8 for instance) . This foregoing yields clear cut advantages not only concerning modeling aspects but also due to the clear structured representation of the basic dynamics. Easy subsystem interchangeability without restarting the whole calculation is one of its advantages. (The “order-n-evaluation” needs three steps: kinematics from bottom to top, kinetics from top to bottom, resolution of minimal accelerations from bottom to top; for details see see e.g. [8]).

5.5 Redundancy

The robot model from Figure 26 reveals another aspect: the motion z_E of the end effector is somehow overdetermined. The position is well defined with $z_E = \bar{x}_1 + b + \bar{x}_2$ (if $\bar{x}_1 = x$ and $\bar{x}_2 = x_2 \cos \gamma_2$ are known, “forward kinematics”) but, on the other hand, if z_E is known, how much does $\bar{x}_1$ contribute, and how much $\bar{x}_2$ (“inverse kinematics”)?

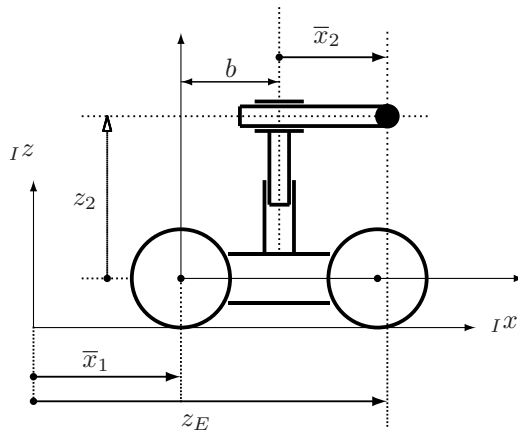


Figure 27. Nonholonomic robot model, side view

The problem reads in mathematical notation (for $\dot{\mathbf{s}} \rightarrow \dot{\mathbf{q}}$, holonomic)

$$\dot{\mathbf{z}}_E = \left[\begin{array}{c} \left(\frac{\partial \mathbf{v}_E}{\partial \dot{\mathbf{q}}} \right) \\ \left(\frac{\partial \boldsymbol{\omega}_E}{\partial \dot{\mathbf{q}}} \right) \end{array} \right] \dot{\mathbf{q}} := \mathbf{J}_E \dot{\mathbf{q}} \in \mathbb{R}^6, \quad \dot{\mathbf{q}} \in \mathbb{R}^{f>6}. \quad (149)$$

The Jacobian $\mathbf{J} := \mathbf{J}_E \in \mathbb{R}^{6,f}$ is not invertible, at least not directly (index E suppressed for simplicity; not to be confused with the end effector inertia tensor). How to solve for $\dot{\mathbf{q}}$? The answer, however, is simple and exciting: $f > 6$ offers the possibility to additionally impose conditions on the system. This has for instance masterfully been done since 1989 in walking machines [44]. How should one construe effective walking? Effectivity may have been detected by nature with its long time evolution. Hence, assuming minimum hip muscle tension (why should one apply torques while standing on a non-slippery ground?) and minimum bending potential in the legs (why should one load the legs unnecessarily?) etc. – and then contact biologists in order to find out if these model assumptions match reality.

Let the additional condition (on the velocity level) for example be

$$\frac{1}{2} \dot{\mathbf{q}} \mathbf{W} \dot{\mathbf{q}} \rightarrow \min. \quad (150)$$

Minimization with eq.(149) as a constraint leads to the Lagrange-function

$$L(\boldsymbol{\lambda}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}} \mathbf{W} \dot{\mathbf{q}} + \boldsymbol{\lambda}^T (\dot{\mathbf{z}}_E - \mathbf{J} \dot{\mathbf{q}}). \quad (151)$$

The solution reads

$$\dot{\mathbf{q}} = \mathbf{W}^{-1} \mathbf{J}^T [\mathbf{J} \mathbf{W}^{-1} \mathbf{J}^T]^{-1} \dot{\mathbf{z}}_E := \mathbf{J}^+ \dot{\mathbf{z}}_E \quad (152)$$

where $\mathbf{J}^+ \in \mathbb{R}^{f,6}$ is the weighted pseudo-inverse. (Notice $\mathbf{J} \mathbf{J}^+ = \mathbf{E} \in \mathbb{R}^{6,6}$ (unit matrix) while $\mathbf{J}^+ \mathbf{J} \in \mathbb{R}^{f,f}$ is not regular). In the last step, one obtains $\mathbf{q}$ by elementary integration. Eq.(152) distributes thus the prescribed $z_{Ei}(t)$ over the minimal coordinates q_j . It enables the robot according to Figure 27 for instance to catch a ball.

To fulfill the task $\mathbf{z}_E$, the system contains redundant minimal coordinates. One may preselect these within $\mathbf{q}_N$ to identify the null space. Eq.(150) is therefore changed to

$$\frac{1}{2} (\dot{\mathbf{q}} - \dot{\mathbf{q}}_N) \mathbf{W} (\dot{\mathbf{q}} - \dot{\mathbf{q}}_N) \rightarrow \min \quad (153)$$

with solution

$$\dot{\mathbf{q}} = \mathbf{J}^+ \dot{\mathbf{z}}_E + [\mathbf{E} - \mathbf{J}^+ \mathbf{J}] \dot{\mathbf{q}}_N, \quad \mathbf{E} \in \mathbb{R}^{f,f}, \quad (154)$$

[41]. Premultiplying eq.(154) by $\mathbf{J}$ yields eq.(149) and shows, with $\mathbf{J}\mathbf{J}^+ = \mathbf{E}$, that $\dot{\mathbf{z}}_E$ is not influenced by $\dot{\mathbf{q}}_N$. The null space dynamics can thus be operated separately and be used e.g. for collision avoidance.

Considering nonholonomic systems, the foregoing is almost the same: Instead of $\dot{\mathbf{z}}_E = \mathbf{J}\dot{\mathbf{q}}$ one has $\dot{\mathbf{z}}_E = \mathbf{J}\dot{\mathbf{s}} = \mathbf{J}\mathbf{H}(\mathbf{q})\dot{\mathbf{q}}$.

Example. Put the robot from Figure 26 on a straight rail and fix the coordinates $(y, \gamma_1, \gamma_2) = 0$ to come out with Figure 27 for $\bar{x}_1 = x := x_1$, $\bar{x}_2 = x_2$ (holonomic) where

$$\mathbf{q} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}, \quad \mathbf{G} = 0. \quad (155)$$

Eq.(152) yields, for $\mathbf{W} = \text{diag}\{w_1, w_2\}$,

$$\dot{\mathbf{z}}_E = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{1}{w_1 + w_2} \begin{pmatrix} w_2 \\ w_1 \end{pmatrix} \dot{\mathbf{z}}_E. \quad (156)$$

There exists a lot of solutions; $w_1 = w_2 = 1$ yields for instance $x_1 = x_2 = z_E/2$ which is, considering the mass distribution, not a very good decision. Intuitively one would rather set $w_1 = m_1$, $w_2 = m_2$ since m_2 is much smaller than m_1 just to balance the energetic flow etc.

After all one must state that all these considerations give valuable hints but need a lot of modeling intuition. Moreover, kinematics is not the whole story. To move the system needs kinetics. Prescribing the end effector motion in a desired manner and conclude on the corresponding forces belongs to open loop control – the success of which is not at all warranted when disturbances occur. We will therefore have a short look on control.

6 Control

The fundamental (historical) control scheme has not changed so much, nor did the (underlying) optimization process. Some basic considerations show problems especially in elastic modeling. Nonlinearity has carefully to be investigated. A path correction procedure for the open-loop part which had already been postulated 25 years ago [29] is still a powerful tool – with the exception that the formerly off-line calculated part can now on-line be computed, due to modern computers. The corresponding result is impressively demonstrated with a “linear elastic robot” (or gantry robot).

A (computer oriented) automated procedure to get rid of nonlinearities can not be recommended if the nonlinear part has stabilizing character – *the art of modeling* needs interpretation. Moreover, friction is critical. The latter can be overcome with a (high gain) observer. The electronic circuit for such procedures is quite simple. Being aware of the fact that, in view of separately considered motor units, the nonlinearities contain the disturbances as well as all the interaction forces leads to a decentralized control concept with the computer as a “black box”. This impressive technology does (nearly) not need any model assumption, hence one may shift the whole consideration towards the beautiful world of Big Data.

However, although computational data collection and statistical evaluation without any intermediate “physical law” may be (at least economically) successful in some applications, it will fail (and has already failed) in mechanical engineering. Concentrating on separated motors for control is indeed promising, but not without a mechanical model. Modeling remains indispensable. The computer cannot overtake thinking.

6.1 Classical Methods

One of the perhaps most instructive examples is from early rocket control (e.g. Fieseler Fi 103, a predecessor of the A4 (till A10) which formed the basis of astronautics/kosmonautics after WWII). Its centerpiece is a displacement gyro which holds its direction in space (air driven with an internal feedback via small pendulums: once the gyro deviates, the pendulums open small holes which act like nozzles to restore the gyro – patented by Elmer Ambrose Sperry (1860-1930), a Edison-like inventor who did not care much about patent holders and their patents). The outer gimbal is subjected to an external torque (electric motor) which comes from a precomputed flight program. The resulting gyro movement then leads compressed air via cam disks to pistons which operate the pitch elevator and the yaw rudder. This procedure represents the open loop control $\mathbf{u}_o$ in Fig.28.

As well known from daily life, nothing is perfect. Hence, it is necessary to compare the actual path $\mathbf{q}$ with the desired one $\mathbf{q}_d$ and feed the difference back (gain P , “proportional”, already achieved with the cam discs in the present). Doing so, however, one risks unstable oscillations (flutter) which is avoided by superimposing damping (gain D , “differential”) which is achieved with rate gyros. We have thus arrived at the classical closed loop control.

The desired path $\mathbf{q}_d$ is sometimes to be reconsidered along with a certain correction model. This will be done later. For the moment the question arises how to choose P, D adequately. Besides some rough rules of thumb

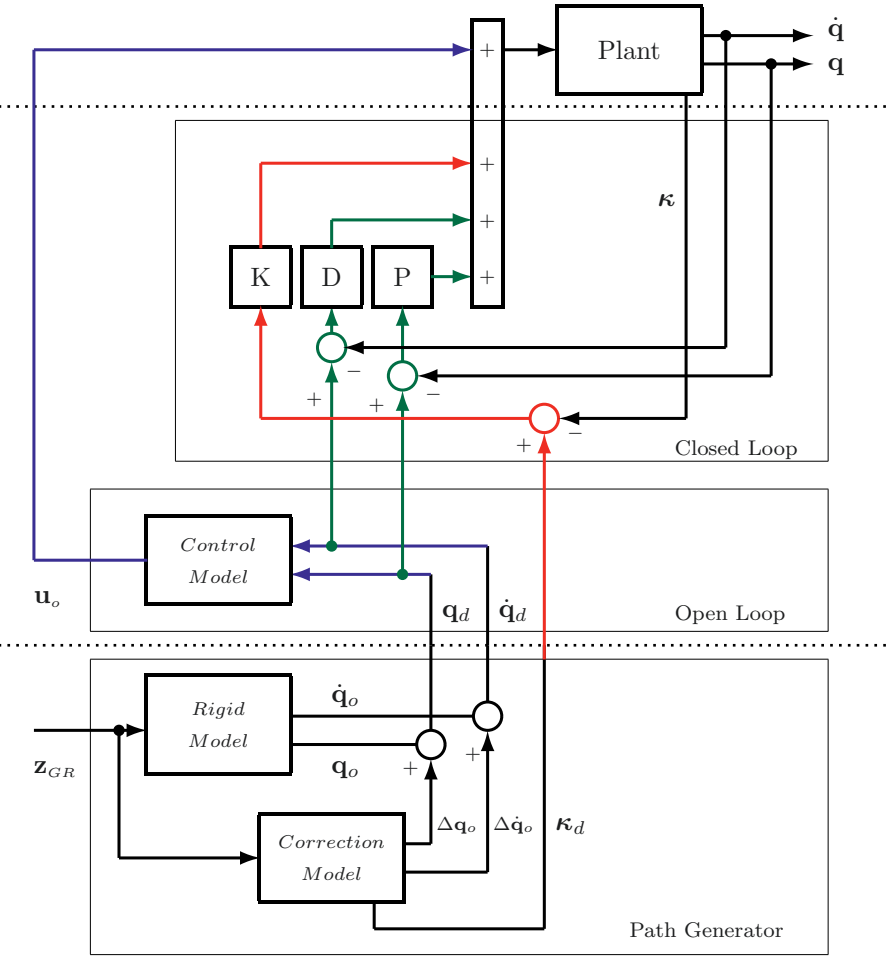


Figure 28. Classical control concept

one is often interested in an optimal choice. The corresponding criterion is then usually performed as a functional where optimality simultaneously needs to fulfill some constraints. A classical example is Dido's problem: Dido, a phoenician princess, had to flee from her brother Pygmalion after he had killed her husband Sychaeus (9th century BC). When she reached

the coast of (nowadays) Tunisia, the inhabitants there promised her as much land as she could span with a sliced cowhide. This became the foundation of Carthago [destroyed in 146 BC, remember Marcus Porcius Cato (234 - 149); “ceterum censeo carthaginem esse delendam”]. The [criterion](#) is $J = \int_{x_o}^{x_1} f(y, y') dx \rightarrow \max$. (Here, $f = y$: as much ground as possible) along with [constraint](#) $\int_{x_o}^{x_1} g(y, y') dx = \text{const.}$ (which, with $g = (1 + y'^2)^{1/2}$, represents the cowhide slice length). Thus, $\int_{x_o}^{x_1} [(\partial f / \partial y) \delta y + (\partial f / \partial y') \delta y'] dx = 0 \wedge \int_{x_o}^{x_1} [(\partial g / \partial y) \delta y + (\partial g / \partial y') \delta y'] dx = 0$ have simulatenously to be fulfilled. Lagranges brilliant idea is to combine $\int_{x_o}^{x_1} \{(\partial f / \partial y) \delta y + (\partial f / \partial y') \delta y'\} + \lambda [(\partial g / \partial y) \delta y + (\partial g / \partial y') \delta y'] dx = 0$
 $= \int_{x_o}^{x_1} [(\partial L / \partial y) \delta y + (\partial L / \partial y') \delta y'] dx$ [where $L = (f + \lambda g)$: [Lagrange-function](#)] which yields $\int_{x_o}^{x_1} [(\partial L / \partial y) - \frac{d}{dx}(\partial L / \partial y')] \delta y dx = 0$ (integration by parts). Then, δy is arbitrary if, by means of λ , the term $[(\partial L / \partial y) - \frac{d}{dx}(\partial L / \partial y')]$ is nullified ([Euler-equation](#), fundamental lemma). Moreover, since in the present L is independent from x , the corresponding [Hamilton-function](#) $H = (\partial L / \partial y') y' - L$ is constant because $dH/dx = [\frac{d}{dx}(\partial L / \partial y') - (\partial L / \partial y)] y' = 0$ (see Euler-equation); H can then be resolved for y' and integrated by separating the variables to come out with $(x - x_o)^2 + (y - y_o)^2 = \lambda^2 := R^2$. The biggest plot is obtained with (part of) a circle – as one would intuitively suggest. Hence, did Dido make such calculations? Probably not. But the above keywords sketch already the general results in optimization as shown in Table 2. One should, however, have in mind that up to here the optimality conditions are only necessary. Sufficient conditions have been obtained by Weierstrass (1815 - 1897): he concentrated on $(J - J_{\min}) > 0 \forall J$ to come out with his [Excess-function](#) E to be positive (semi-) definite – which is hard to check. Nevertheless, setting $\lambda = \lambda_{\text{opt}}$ (and $\mathbf{x} = \mathbf{x}_{\text{opt}}$) leads the condition $(\partial H / \partial \mathbf{u}) = 0$ to the more precise result $H_{\text{opt}} = H_{\text{max}}$ w.r.t. $\mathbf{u}$ – which also holds for bounded $\mathbf{u} \in \{U\}$ ([Maximum-principle](#), Pontryagin and co-workers around 1950). This is by no means trivial: an unrestricted time optimal control would of course yield $\mathbf{u} \rightarrow \infty$. However, one should still have in mind that this condition is only necessary. A numerical check will be unavoidable – which is of course automatically included in (usually performed) simulations. Here the computer comes into play.

6.2 PD-Control

PD-Control usually refers to small deviations, e.g. $\mathbf{q}_d - \mathbf{q}$ and $\dot{\mathbf{q}}_d - \dot{\mathbf{q}}$ in case of the rocket control (Figure 28), which is superimposed to the open loop control $\mathbf{u}_o$. Thus, the state equation $\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}, \mathbf{u})$ is subjected to $\mathbf{u} \rightarrow \mathbf{u}_o + \mathbf{u}$ which yields by linearization $\mathbf{x} \rightarrow \mathbf{x}_o + \mathbf{x}$ the open loop part

$\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}, \mathbf{u})$. Determine $\mathbf{u}$ such that $J = \int f(\mathbf{x}, \mathbf{u})dt \rightarrow \min$	
$H = \boldsymbol{\lambda}^T \mathbf{a}(\mathbf{x}, \mathbf{u}) - f$ (Lagrange) (Hamilton)	$E = -H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) + \mathbf{a}^T(\boldsymbol{\lambda} - \boldsymbol{\lambda}_{opt})$
necessary	sufficient
$H(\mathbf{x}_{opt}, \mathbf{u}_{opt}, \boldsymbol{\lambda}_{opt}) = 0$ “Canonical Equations” (Euler): $+\left(\frac{\partial H}{\partial \boldsymbol{\lambda}}\right) = \dot{\mathbf{x}}^T$ $-\left(\frac{\partial H}{\partial \mathbf{x}}\right) = \dot{\boldsymbol{\lambda}}^T$ $\pm\left(\frac{\partial H}{\partial \mathbf{u}}\right) = 0$	“Excess Function” (Weierstrass) : $E > 0$
$H_{opt} = \max_{u \in U} H$	$\Leftarrow$

Table 2. Results from classical optimization theory, e.g. [8]

$\dot{\mathbf{x}}_o = \mathbf{a}_o(\mathbf{x}_o, \mathbf{u}_o)$ and the linear plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}. \quad (157)$$

Let the control be optimized w.r.t.

$$J = \frac{1}{2} \int_0^\infty (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u}) dt \rightarrow \min \quad (158)$$

$[(\mathbf{A}, \mathbf{B}) \text{ controllable}, \mathbf{Q} \geq 0 \wedge (\mathbf{A}, \mathbf{Q}) \text{ observable}, \mathbf{R} > 0]$. The canonical equations from Table 2 yield the optimal PD-control

$$\mathbf{u} = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}_R \mathbf{x} = -[\mathbf{P} \quad \mathbf{D}] \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} \quad (159)$$

($\dot{\mathbf{q}} \rightarrow \dot{\mathbf{s}}$ for the nonholonomic case) with $\mathbf{P}_R$ being solution of the Riccati-type equation (J. F. Riccati (1676 – 1754))

$$\dot{\mathbf{P}}_R + \mathbf{A}^T \mathbf{P}_R + \mathbf{P}_R \mathbf{A} - \mathbf{P}_R \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}_R + \mathbf{Q} = 0. \quad (160)$$

Several procedures exist for solving eq.(160), see e.g. [5]. Here, the computer comes into play. However, the optimal feedback (159) refers to a linear plant according to eq.(157).

Example. Consider once more a robot according to Figure 26 but with an elastic arm. The carriage is hold at a certain position $\mathbf{q}_c = \text{const.}$ and the arm is clamped at its hub. Along with $\mathbf{q} = (\gamma \ \mathbf{q}_{\text{Ritz}}^T)^T$, the motion equations read $\mathbf{M}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{b}u$.

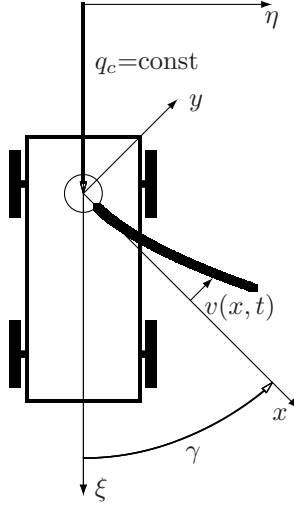


Figure 29. Carriage, elastic arm

Taking the second order field (32) into account, one has

$$\begin{aligned} \mathbf{M} &= \int_o^L \begin{bmatrix} \frac{C^o}{L} & \rho A x \mathbf{v}^T \\ \rho A x \mathbf{v} & \rho A \mathbf{v} \mathbf{v}^T \end{bmatrix} dx, \\ \mathbf{K} &= \int_o^L \begin{bmatrix} 0 & 0 \\ 0 & EI \mathbf{v}'' \mathbf{v}''^T - \dot{\gamma}^2 \rho A \left[\mathbf{v} \mathbf{v}^T - \frac{1}{2} (L^2 - x^2) \mathbf{v}' \mathbf{v}'^T \right] \end{bmatrix} dx, \\ \mathbf{b}^T &= [1 \quad 0], \quad u = M_\gamma. \end{aligned} \quad (161)$$

The arm can only be influenced by the hub motor $u = M_\gamma$, but controllability is fulfilled. We consider a 90° -slewing manoeuvre.

Due to the centrifugal effects in $\mathbf{K}$, the plant is nonlinear. However, linearization w.r.t. the end point $\mathbf{q} = \mathbf{q}_e + \mathbf{y}$ with $\mathbf{q}_e = (\gamma_e \ 0^T)^T$ yields a linear equation $\mathbf{M}_e \ddot{\mathbf{y}} + \mathbf{K}_e \mathbf{y} = \mathbf{b}_e u$ where

$$\mathbf{M}_e = \int_o^L \begin{bmatrix} \frac{C_o}{L} & \rho A x \mathbf{v}^T \\ \rho A x \mathbf{v} & \rho A \mathbf{v} \mathbf{v}^T \end{bmatrix} dx, \mathbf{K}_e = \int_o^L \begin{bmatrix} 0 & 0 \\ 0 & EI \mathbf{v}'' \mathbf{v}''^T \end{bmatrix} dx, \mathbf{b}_e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (162)$$

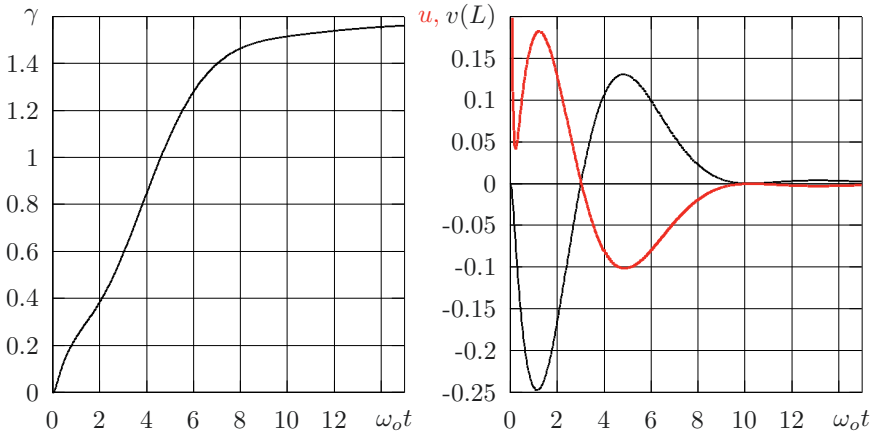


Figure 30. (a) $\gamma(t)$, (b) tip deflection $v(L,t)$ and control $u(t)$

Specifying the control torque as $u = M_\gamma = -P(\gamma - \gamma_e) - D(\dot{\gamma} - \dot{\gamma}_e)$ drives the arm asymptotically to the requested endposition, see Figure 30 (the abscissa is normalized with the first beam bending frequency ω_o).

The result looks quite good at a first glance. Control u and tip deflection $v(L)$ show the desired behaviour (compare Figure 23). However, the main drawback here is due to $u(0) = P\gamma_e$: the systems reacts like a step response ($\gamma_e = \pi/2$; in more general systems, $\mathbf{q}_e$ may be very far away from the initial state). This may perhaps be tolerated in rigid systems, but not at all in elastic ones. Such a step response causes jerks which initiate a lot of bending modes. These are not to be seen in Figure 30 which only takes the first bending mode into account.

One can thus not do without an open loop control, for instance by means of a “computed torque”

$$\mathbf{M}\ddot{\mathbf{q}}_d + \mathbf{K}\mathbf{q}_d = \mathbf{u}_{oL} \quad (163)$$

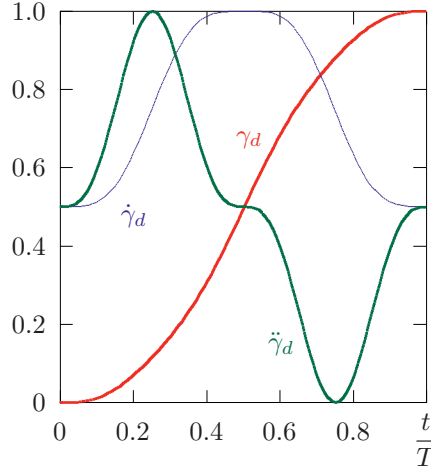


Figure 31. Sinusoidal transition trajectories (normalized)

e.g. using $\ddot{\gamma}_d = \left(\frac{3\pi}{T^2}\right) \sin^3\left(\frac{2\pi}{T}t\right)$ (“sinusoidal”) leading to γ_d at least being four times steady and steadily differentiable as a basic request (Figure 31). The open loop part smoothes the trajectory, and the closed loop control then needs less shape functions for simulation. This is due to its pervasive damping character (controllability). Nevertheless, we are once more left with a nonlinear plant. How to get rid of the nonlinearities?

6.3 Flatness based control

There exists an elegant procedure. Consider the nonlinear plant (single input u , single output y for simplicity):

$$\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}) + \mathbf{b}u \in \mathbb{R}^n, \quad (164)$$

$$y = \mathbf{c}^T \mathbf{x}. \quad (165)$$

Consecutive time differentiation yields

$$\begin{aligned}
 \dot{y} &= \frac{\partial y}{\partial \mathbf{x}}(\mathbf{a}(\mathbf{x}) + \mathbf{b}u) = \mathbf{c}^T \mathbf{a}(\mathbf{x}) &:= \mathbf{c}^T \mathbf{a}_1 = 0 \text{ if } &\mathbf{c}^T \mathbf{b} = 0 \\
 \ddot{y} &= \frac{\partial \dot{y}}{\partial \mathbf{x}}(\mathbf{a}(\mathbf{x}) + \mathbf{b}u) = \mathbf{c}^T \frac{\partial \mathbf{a}_1}{\partial \mathbf{x}} \mathbf{a}(\mathbf{x}) &:= \mathbf{c}^T \mathbf{a}_2 = 0 \text{ if } &\mathbf{c}^T \frac{\partial \mathbf{a}_1}{\partial \mathbf{x}} \mathbf{b} = 0 \\
 \ddot{\ddot{y}} &= \frac{\partial \ddot{y}}{\partial \mathbf{x}}(\mathbf{a}(\mathbf{x}) + \mathbf{b}u) = \mathbf{c}^T \frac{\partial \mathbf{a}_2}{\partial \mathbf{x}} \mathbf{a}(\mathbf{x}) &:= \mathbf{c}^T \mathbf{a}_3 = 0 \text{ if } &\mathbf{c}^T \frac{\partial \mathbf{a}_2}{\partial \mathbf{x}} \mathbf{b} = 0 \\
 &\vdots &&
 \end{aligned} \tag{166}$$

If, for the k th derivative, $\mathbf{c}^T(\partial \mathbf{a}_{k-1}/\partial \mathbf{x})\mathbf{b} \neq 0$, then

$$y^k = \mathbf{c}^T \frac{\partial \mathbf{a}_{k-1}}{\partial \mathbf{x}} [\mathbf{a}(\mathbf{x}) + \mathbf{b}u] := \bar{a}(\mathbf{x}) + \bar{b}u = 0. \tag{167}$$

If $k = n$, y is called a “flat output”. Let $u = u_{comp} + u_{CL}$ where $u_{comp} = -\frac{\bar{a}}{\bar{b}}$ compensates the nonlinearities (“exact linearization”). The closed loop part

$$u_{CL} = \frac{1}{\bar{b}} \left[\frac{d^n y_d}{dt^n} + \sum_{i=0}^{n-1} k_i \left(\frac{d^i y_d}{dt^i} - \frac{d^i y}{dt^i} \right) \right] \tag{168}$$

yields, with $z = y - y_d$, the Frobenius form

$$\frac{d}{dt} \begin{pmatrix} z \\ \dot{z} \\ \ddot{z} \\ \vdots \\ z^{n-1} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ -k_o & -k_1 & -k_2 & \cdots & -k_{n-1} \end{bmatrix} \begin{pmatrix} z \\ \dot{z} \\ \ddot{z} \\ \vdots \\ z^{n-1} \end{pmatrix} \tag{169}$$

[Georg Frobenius (1849 – 1917)] where k_i are the characteristic coefficients from $\lambda^n + k_{n-1}\lambda^{n-1} + \cdots + k_o = 0$ which can be used for pole assignment.

The differentiation process is somewhat tedious and will, of course, fail in case of backlash and friction. Moreover: The example according to eq.(161) contains a nonlinearity which stabilizes flexural oscillation. Should one really compensate positive effects with such an immense effort? Of course not. Looking at “exact linearization” procedures goes along with a strict warning not to rely on automated procedures without physical interpretation. In case of controlled systems, modeling refers to physical reality on the one hand side (which is needed for simulation) and to a control model which is not the same but reduced to minimal requirements (*– the art of modeling!*) We are thus left with the linear plant according to eq.(162) for control calculation. Its result has to be tested by simulation and, in the last step, verified in hardware.

6.4 Path correction

Considering a rigid articulated robot, the desired gripper position and orientation $\mathbf{z}_{Gd} \in \mathbb{R}^6$ is obvious. In case of an elastic articulated robot, the control model may be the purely rigid one, i.e. neglecting all structural and gear elasticities. The “real” elastic model then represents the simulation model. However, $\mathbf{z}_{Gd}$ does not more characterize the desired aim. Correcting the path is done in two steps.

Let $\mathbf{q} = (\mathbf{q}_M^T \ \mathbf{q}_A^T \ \mathbf{q}_e^T)^T$ (M : motor, A : arm, e : elastic (Ritz-coefficients)) and define $\mathbf{q}_M = \mathbf{q}_R$ (reference path). Along with $\mathbf{y}$ for deviation one has

$$\mathbf{q} = \mathbf{q}_o + \mathbf{y} = \begin{pmatrix} \mathbf{q}_R \\ \mathbf{q}_R \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{y}_M \\ \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix}, \quad \|\mathbf{y}\| < \|\mathbf{q}\|. \quad (170)$$

1. Reference motion control.

Decompose $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) = \bar{\mathbf{B}}(\mathbf{u}_{ol} + \mathbf{u}_{cl} + \mathbf{u}^c)$ and linearize,

$$\begin{bmatrix} \mathbf{M}_{MM} & \mathbf{M}_{MA} & \mathbf{M}_{Me} \\ \mathbf{M}_{MA}^T & \mathbf{M}_{AA} & \mathbf{M}_{Ae} \\ \mathbf{M}_{Me}^T & \mathbf{M}_{Ae}^T & \mathbf{M}_{ee} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{q}}_R \\ \ddot{\mathbf{q}}_R \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{g}_{0M} \\ \mathbf{g}_{0A} \\ \mathbf{g}_{0e} \end{pmatrix} - \begin{pmatrix} \mathbf{u}_{ol} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \mathbf{M}_o\ddot{\mathbf{y}} + \mathbf{P}_o\dot{\mathbf{y}} + \mathbf{Q}_o\mathbf{y} = \begin{pmatrix} \mathbf{u}_{cl} + \mathbf{u}^c \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad (171)$$

set $\mathbf{y}$ and $\mathbf{u}_{cl} + \mathbf{u}^c$ zero, formulate the virtual work for this case via $\delta\mathbf{q}^T = \delta\mathbf{q}_R^T [\mathbf{E} \ \mathbf{E} \ 0]$ to obtain the open loop control $\mathbf{u}_{ol}$,

$$\mathbf{u}_{ol} = [\mathbf{M}_{MM} + \mathbf{M}_{MA} + \mathbf{M}_{MA}^T + \mathbf{M}_{AA}] \ddot{\mathbf{q}}_R + (\mathbf{g}_{0M} + \mathbf{g}_{0A}). \quad (172)$$

Insert $\mathbf{u}_o$ (and $\mathbf{u}_{cl} = 0$) into eq.(171),

$$\begin{bmatrix} -\mathbf{f}_R \\ +\mathbf{f}_R \\ +\mathbf{f}_{eR} \end{bmatrix} + \mathbf{M}_o\ddot{\mathbf{y}} + \mathbf{P}_o\dot{\mathbf{y}} + \mathbf{Q}_o\mathbf{y} = \begin{pmatrix} \mathbf{u}^c \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad (173)$$

$$\text{where } \begin{bmatrix} -\mathbf{f}_R \\ +\mathbf{f}_R \\ +\mathbf{f}_{eR} \end{bmatrix} = \begin{bmatrix} -([\mathbf{M}_{MA}^T + \mathbf{M}_{AA}]) \ddot{\mathbf{q}}_R + \mathbf{g}_{0A} \\ +([\mathbf{M}_{MA}^T + \mathbf{M}_{AA}]) \ddot{\mathbf{q}}_R + \mathbf{g}_{0A} \\ +([\mathbf{M}_{Me}^T + \mathbf{M}_{Ae}^T]) \ddot{\mathbf{q}}_R + \mathbf{g}_{0e} \end{bmatrix}.$$

From eq.(173) one obtains the special “quasistatic” equilibrium $\mathbf{y}_{qs}^T = (\mathbf{0}^T \mathbf{y}_A^T \mathbf{q}_e^T)_{qs}^T$:

$$\begin{bmatrix} -\mathbf{f}_R \\ +\mathbf{f}_R \\ +\mathbf{f}_{eR} \end{bmatrix} + \mathbf{Q}_o \mathbf{y}_{qs} = \begin{pmatrix} \mathbf{u}^c \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \Rightarrow \begin{cases} \mathbf{u}^c = [\mathbf{Q}_{MA} \mathbf{Q}_{Me}] \begin{pmatrix} \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix}_{qs} \\ \begin{pmatrix} \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix}_{qs} = - [\mathbf{Q}_{AA} \mathbf{Q}_{Ae}]^{-1} \begin{bmatrix} \mathbf{f}_R \\ \mathbf{f}_{eR} \end{bmatrix} \end{cases} \quad (174)$$

where $\mathbf{Q}_o$ is decomposed into submatrices $\mathbf{Q}_{ij}$, $\{i, j\} \in \{M, A, e\}$ like in eq.(171). $\mathbf{u}^c$ is a constraint force which holds $\mathbf{y}_M$ zero within $\mathbf{y}_{qs}$. Presuming that $\mathbf{u}_o$ follows $\mathbf{q}_R$ in an ideal manner such that $\mathbf{y}_M$ vanishes all the time, $\mathbf{u}^c$ is implicitly taken into account by $\mathbf{u}_o$.

2. Path matching.

Calculate the resulting gripper deviation $\Delta \mathbf{z}_G$, and the deviation $\Delta \mathbf{z}_{GR}$ which would be obtained by modifying the reference path $\mathbf{q}_R$ (Taylor expansion)

$$\Delta \mathbf{z}_G = \underbrace{\begin{bmatrix} \frac{\partial \mathbf{v}_G}{\partial \dot{\mathbf{q}}_A} & \frac{\partial \mathbf{v}_G}{\partial \dot{\mathbf{q}}_e} \\ \frac{\partial \omega_G}{\partial \dot{\mathbf{q}}_A} & \frac{\partial \omega_G}{\partial \dot{\mathbf{q}}_e} \end{bmatrix}}_{\mathbf{F}_G} \begin{pmatrix} \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix}, \quad \Delta \mathbf{z}_{GR} = \underbrace{\begin{bmatrix} \frac{\partial \mathbf{v}_G}{\partial \dot{\mathbf{q}}_R} \\ \frac{\partial \omega_G}{\partial \dot{\mathbf{q}}_R} \end{bmatrix}}_{\mathbf{F}_{GR}} \Delta \mathbf{q}_R. \quad (175)$$

$(\Delta \mathbf{z}_G + \Delta \mathbf{z}_{GR}) = 0 \Rightarrow \Delta \mathbf{q}_R$ would compensate the deviation $\Delta \mathbf{z}_G$:

$$\Delta \mathbf{q}_R = -\mathbf{F}_{GR}^+ \mathbf{F}_G \begin{pmatrix} \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix} := [-\mathbf{F}_{A, \text{corr}} - \mathbf{F}_{e, \text{corr}}] \begin{pmatrix} \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix} \quad (176)$$

where $()^+$ is the generalized inverse. (Notice that the gripper motion does not depend on the “inner” coordinates $\mathbf{q}_M$ (motor coordinates) when defined according to Figure 8).

However, eq.(176) needs the solution of a differential equation [i.e. eq.(173) along with eq.(174): $\mathbf{M}_o \dot{\mathbf{y}} + \mathbf{P}_o \dot{\mathbf{y}} + \mathbf{Q}_o (\mathbf{y} - \mathbf{y}_{qs}) = \mathbf{0}$]. This is almost tedious and time consuming. Thus, assuming $\ddot{\mathbf{y}}$ and $\dot{\mathbf{y}}$ negligible to calculate eq.(176), one may use eq.(174) to come out with

$$\Delta \mathbf{q}_R = [-\mathbf{F}_{A, \text{corr}} - \mathbf{F}_{e, \text{corr}}] \begin{pmatrix} \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix}_{qs}. \quad (177)$$

As a result one has a corrected path $\mathbf{q}_R + \Delta \mathbf{q}_R$. One could think of using the new (corrected) path and restart calculations.

However, the correction may also be shifted to the deviations which means to retain the old reference and to define a new target

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_R + \Delta\mathbf{q}_R \\ \mathbf{q}_R + \Delta\mathbf{q}_R \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{q}_R \\ \mathbf{q}_R \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \Delta\mathbf{q}_R \\ \mathbf{y}_A + \Delta\mathbf{q}_R \\ \mathbf{q}_e \end{pmatrix} \quad (178)$$

yielding the correction model

$$\mathbf{y}_d = \begin{bmatrix} -\mathbf{F}_{A,\text{corr}} & -\mathbf{F}_{e,\text{corr}} \\ \mathbf{E} - \mathbf{F}_{A,\text{corr}} & -\mathbf{F}_{e,\text{corr}} \\ 0 & \mathbf{E} \end{bmatrix} \begin{pmatrix} \mathbf{y}_A \\ \mathbf{q}_e \end{pmatrix}_{qs} \quad (179)$$

to build a closed loop control $\mathbf{u}_{cl}$, compare Figure 28 (lower part).

Thereby, the elastic deflection itself can not be nullified itself via the hub motor (think of bending due to gravity, for instance). The deformation $\mathbf{q}_e$ which leads to the desired gripper position is, however, obtained from the correction model. It is thus recommendable to also feed $\mathbf{q}_e$ back, preferably with a curvature (output) control $\boldsymbol{\kappa} = (\partial\boldsymbol{\kappa}/\partial\mathbf{q}_e)\mathbf{q}_e$ which is easily accomplished with strain gauge measurements.

A lot of assumptions have been made to obtain the correction model. The result has therefore first to be checked by numerical simulations before going into hardware realization.

The outlined procedure was experimentally verified for an “elastic linear robot” (or gantry robot) with an overwhelming success, [28].

6.5 Observers

As already mentioned, friction plays an important role. An exact linearization according to eq.(166) ff. will fail due to the required differentiation process. This problem can be overcome with an observer.

A (state) observer is, in principle, nothing but an computerized reconstruction of the plant, but without any knowledge on the initial conditions. Even if these were known one would have a severe numerical drift-off after just a short time. Hence, if the real output $\mathbf{y} = \mathbf{C}\mathbf{x}$ differs from the observer signal $\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$, then the difference is fed back with an adequate gain matrix $\mathbf{L}_x$ to assure zero phase between the models.

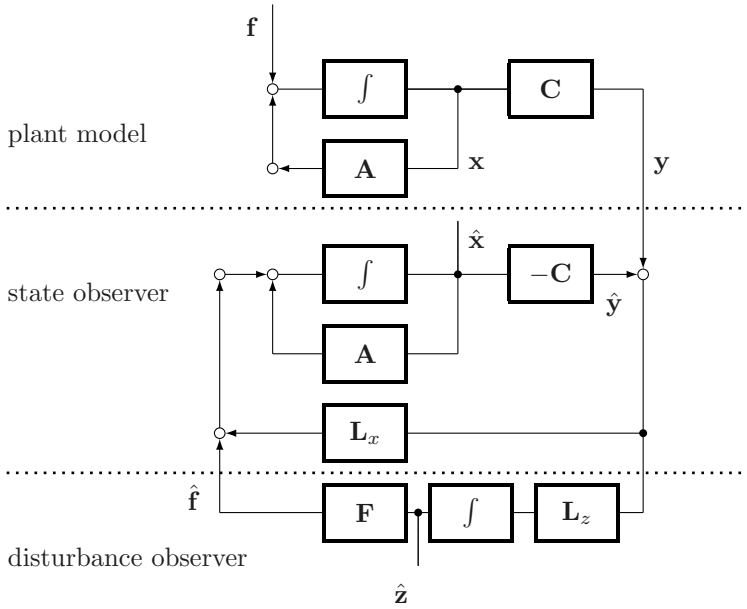


Figure 32. Observer scheme

A well known statement is that for sufficient high gains $\mathbf{L}_x$ (no disturbance reconstruction yet, $\mathbf{L}_z \equiv 0$) the estimated state suppresses the disturbance influence (“high gain observer”). This will be demonstrated with a single joint with friction in the gravity field, Figure 33.

The time histories in Figure 34 correspond to the initial conditions $(\mathbf{q}_o^T, \dot{\mathbf{q}}_o^T) = (0, 0)$ where $\mathbf{q}_o^T = (\gamma_M, \gamma_A)$. As (a) shows, the friction torque with its zero crossings at about $t = 1.8$ and $t = 3.4$, resp., puts hard conditions on the system. Nevertheless, the estimation (b) is perfect (notice the zero crossings for $\dot{\gamma}_M$). Of course, the gains $\mathbf{L}_x = (l_{x1}, l_{x2})$ need high values which are restricted due to measurement noise. The values used here conform with those which have successfully been implemented for biped walking machines [21], [44]. The hardware realization is almost simple:

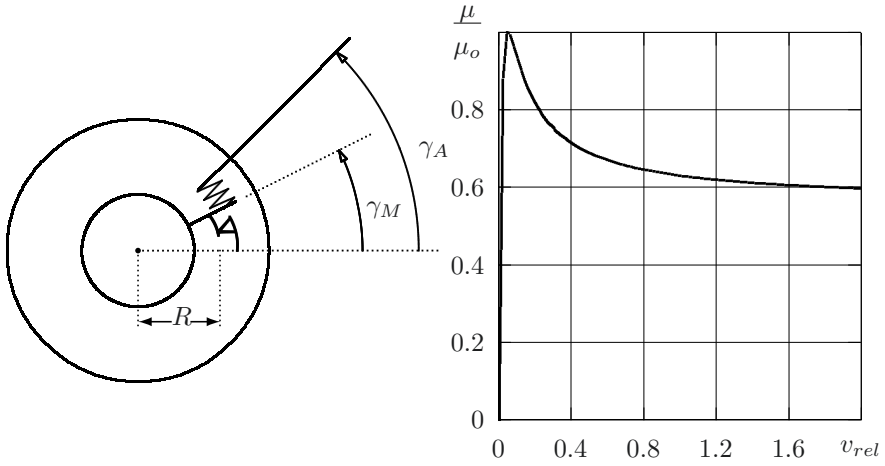


Figure 33. (a) single joint with friction, (b) Stribeck curve

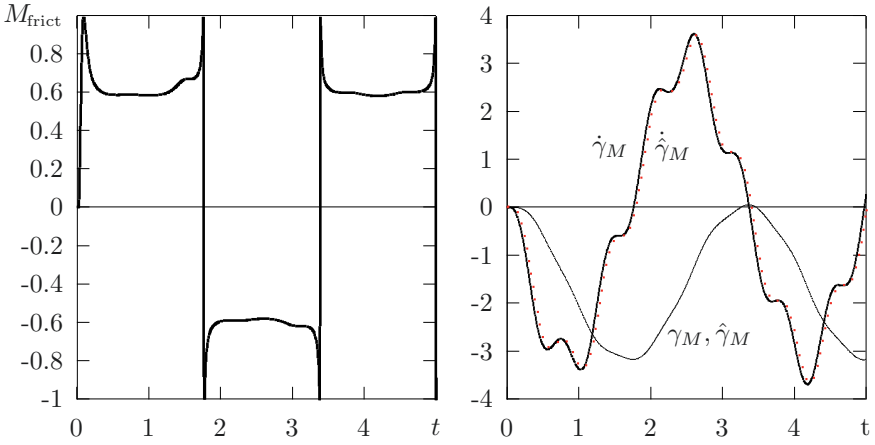


Figure 34. (a) M_{frict} normalized, (b) variables (solid), estimation (dotted)

from Figure 32 one has $\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{L}_x(\hat{\mathbf{y}} - \mathbf{y})$ which for $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ yields Figure 35.

$$\boxed{\frac{d}{dt} \begin{pmatrix} \hat{\gamma}_M \\ \hat{\gamma}_M \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \hat{\gamma}_M \\ \hat{\gamma}_M \end{pmatrix} - \begin{pmatrix} l_{x1} \\ l_{x2} \end{pmatrix} (\hat{\gamma}_M)} \quad \begin{matrix} \text{electrical board (black box)} \\ \text{input} \\ \text{(measurement)} \end{matrix}$$

Figure 35. Circuit scheme (state estimation)

The disturbance remains unknown. It contains the friction torque as well as the reactions from gear twisting $\Delta\gamma = (\gamma_M - \gamma_A)$. Modeling the disturbance via basic functions $\dot{\mathbf{z}} = 0$ ($\mathbf{z}$: step functions) such that $\mathbf{f} = \mathbf{F}\mathbf{z}$ leads to $\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{F}\hat{\mathbf{z}} - \mathbf{L}_x(\hat{\mathbf{y}} - \mathbf{y})$ and $\dot{\hat{\mathbf{z}}} = -\mathbf{L}_z(\hat{\mathbf{y}} - \mathbf{y})$ (lower part of Figure 32). Along with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{F} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{L}_z = l_z$ one obtains the hardware realization according to Figure 36.

$$\boxed{\frac{d}{dt} \begin{pmatrix} \hat{\gamma}_M \\ \hat{\gamma}_M \\ \hat{z} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \hat{\gamma}_M \\ \hat{\gamma}_M \\ \hat{z} \end{pmatrix} - \begin{pmatrix} l_{x1} \\ l_{x2} \\ l_z \end{pmatrix} (\hat{\gamma}_M)} \quad \begin{matrix} \text{electrical board (black box)} \\ \text{input} \\ \text{(measurement)} \end{matrix}$$

Figure 36. Circuit scheme (disturbance estimation)

The estimated disturbance once more contains friction as well as the interacting forces. Since $\mathbf{f} \in \mathbb{R}^n$ refers to the state equation, we call that part which enters the motion equations $\bar{\mathbf{f}} \in \mathbb{R}^f$. According to Figure 32, these are outer forces. The total outer and inner (reaction) forces are therefore denoted $\bar{\bar{\mathbf{f}}}$. Its estimation is depicted in Figure 37.

6.6 Decentralized Control

Disposing of the (estimated) state variables and the (estimated) disturbances, these may be used for control and for disturbance compensation.

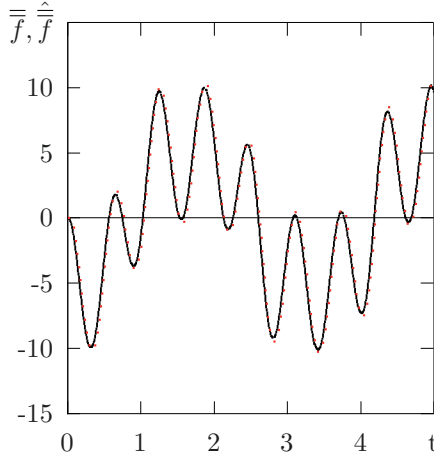


Figure 37. Disturbance (solid) and its estimation (dotted)

The motion equations

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} - \mathbf{Q} = \bar{\mathbf{f}}, \quad (180)$$

can be decomposed in the form

$$\text{diag}\{M_{oi}\}\ddot{\mathbf{q}} + \Delta\mathbf{M}\ddot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} - \mathbf{Q} = \bar{\mathbf{f}} \quad (181)$$

where $M_{oi} = \text{const}$ may be chosen such that it represents the desired (or nominal) value [39]. The i th component reads

$$M_{oi}\ddot{q}_i + \sum_{j=1}^f \Delta M_{ij}\ddot{q}_j + (\mathbf{G}\dot{\mathbf{q}} - \mathbf{Q})_i = \bar{f}_i \quad (182)$$

where now in

$$\ddot{q}_i = \frac{1}{M_{oi}} \left[- \sum_{j=1}^f \Delta M_{ij}\ddot{q}_j - (\mathbf{G}\dot{\mathbf{q}} - \mathbf{Q})_i + \bar{f}_i \right] := \bar{\bar{f}}_i, \quad (183)$$

$\bar{\bar{f}}_i$ contains all interaction terms (including, if need be, the parameter deviation of M_i from its desired (nominal) value) as well as all disturbances. One has thus

$$\frac{d}{dt} \begin{pmatrix} q \\ \dot{q} \end{pmatrix}_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix}_i + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\bar{f}}_i. \quad (184)$$

Hence, the simplified linear plant used above has by no means been a singular example but represents somehow generality: we have come to decentralized joint control. In Figure 38, the joint model is used for test. It represents a 90° maneuver in a vertical plane (acting against gravity) including joint friction and gear twisting.

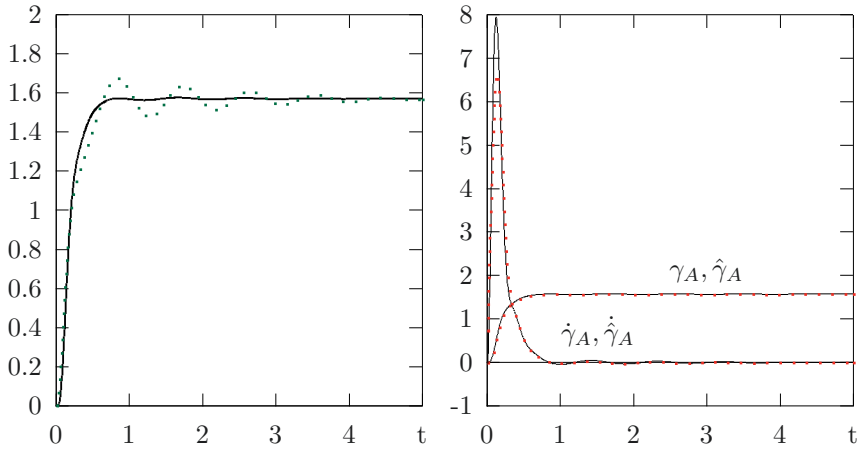


Figure 38. (a) slewing maneuver γ_A (dotted: γ_M), (b) “real values” (from simulation) and estimation (dotted)

Along with the closed loop control $u_{CL,i}$, the “exact linearization” from eq.(168) would require to eliminate the disturbances exactly,

$$\ddot{q}_i = u_{CL,i} + \underbrace{(\bar{\bar{f}}_i - \bar{\bar{f}}_i)}_{\equiv 0}, \quad (185)$$

which is hard to achieve. It needs exact knowlegde of parameter values and will fail in case of friction. The main difference now is, that the disturbances and the interactions are cancelled asymptotically,

$$\ddot{q}_i = u_{CL,i} + \underbrace{(\bar{\bar{f}}_i - \hat{\bar{\bar{f}}}_i)}_{\text{asympt.} \rightarrow 0}. \quad (186)$$

It gets along nearly without any parameter knowledge. This requires, of course, careful investigations primarily w.r.t. the gains $\mathbf{L}_x, \mathbf{L}_z$. Here, powerful computers come into play.

6.7 Big Data

Along with the fact that one does not need a model for the disturbances and interactions, and disposing of powerful computers, one may dispense of modeling at all and look at the corresponding hardware realization in the sense of a black box, which, for the practical engineer, is out of interest (buy, use, and don't care. The computer does the job).

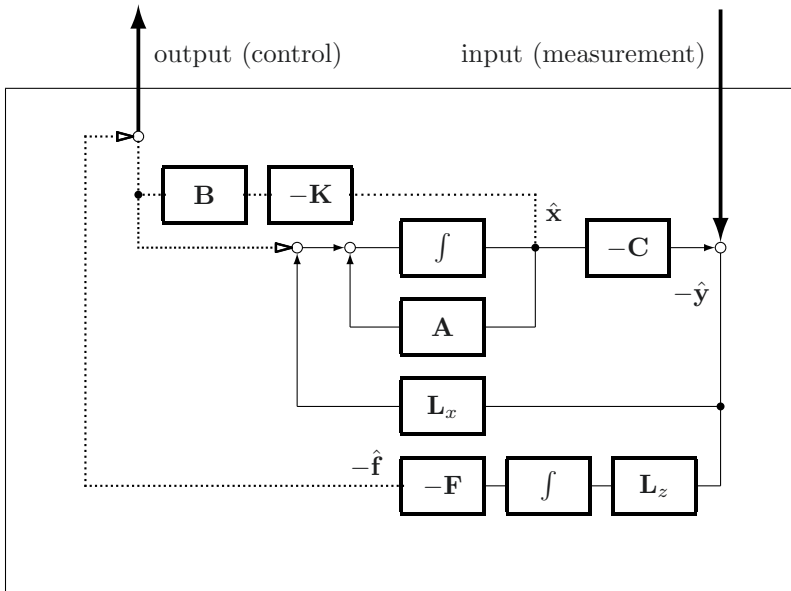


Figure 39. The black box

Data collection and statistics is indeed (economically) successful in a wide range. Statistics makes modeling obsolete. In 2009, two US scientists extracted the Hamilton function of a double pendulum just from measurement data [19]. However, does it belong to a rotational double pendulum or to a translational one (two masses with springs) which (for sufficient small amplitudes) have the same behaviour – or to a combination, and what about mass distribution and lengths? In mechanical engineering, attempts to do without modeling have a long tradition. To reconstruct a mechanical system just from measurement data has thereby definitely failed already fifty years ago. A simulation, and thereafter a control, cannot be done without a model.

7 Conclusion

Clearly, data and data collection may be helpful to find answers. But this is not the end of the story, just the beginning. An example for an overwhelming succes in modeling (may be the best model ever): until around 1600, Tycho Brahe collected observation data. 1619, Johannes Kepler postulated a physical model, 1686 Isaac Newton and fifty years later, Leonhard Euler formulated a mathematical model (gravity and momentum theorem, resp.). On this basis, the first satellite was launched in 1957, the moon landing followed in 1969, the first space station Saljut 1 was engineered 1971 and the international ISS from 1998 till today – and, on the same basis, what about internet, smart phone etc.? Thereby, step by step, more insight could be obtained.

Karl Popper: He who decides one day that scientific statements ... can be regarded as finally verified, retires from the game (F. Pfeiffer [43]).

Hence, in natural and engineering sciences,

- modeling remains indispensable,
- the computer is helpful, but nothing than a dummy slave.

Even more: everything what we see, what we believe, how we behave, in all the areas of life, is based on an abstraction process of thinking – what else are we dealing with than models? Models which describe reality, and which may perhaps converge to truth which, however, remains out of reach. From belief to truth with own capabilities: Münchhausen could not pull himself up by his own hair, this was a lie.

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List of Symbols

A, A	transformation, state matrix, area, inertia moment	r	(inertial) position vector
a, a	state function, min. coefficient	s	quasi coordinates (minimal)
B, B	input matrix, inertia moment	T, t	kinetic energy, time
C, C	output matrix, inertia moment	u	control vector
c, c	coefficient vector, coefficient, mass center distance	V	potential energy
D, D	damping matrix, gain (diff.)	v, v	vector of shape functions, bending function
E, E	identity matrix, Young's modulus, excess-function	W	work
e, e	unit vector, error	w, w	vector of shape functions, bending function
F, F	functional matrix (or Jacobian)	x, y, z	spatial variables
f, f	force, disturbance, frequency, function (general)	x, y, z	state vector, describing variables, disturbance functions
G, G	gyroscopic matrix, torsional elasticity, Gibbs-function		
g, g	approximation functions, gravitation vector, grav. constant	α, β, γ	Cardan (or Tait-Bryan) angles
H	Hamilton-function	Δ	difference, Laplace-operator
I, I	area moments of inertia	δ	variation, Dirac-distribution (δ)
J, J	mass moments of inertia, Jacobian, optimization criterion	ϵ	Cauchy strain, variational parameter
K	(conservative) restoring matrix, curvature feedback, state feedback matrix	ϑ, ϑ	vector of shape functions, torsional function
L, L	(angular) momentum, Lagrange-function, length	κ	(vector of) curvatures
M, M	mass matrix, moment	λ, λ	Lagrange-parameters, eigenvalue
m	mass, time derivation grade	ν	Poisson number, frequency
N	(nonconserv.) restoring matrix	π	quasi coordinates (rotation)
P, P	gain (proportional), power	ρ, ρ	quasi coordinates (translation), mass density
p	(linear) momentum	σ	Cauchy-stress
Q	(gen.) force, weighting matrix	τ	tangential plane
q	minimal (or generalized or Lagrangian) coordinates	Φ, φ	constraint, bending angles
R, R	weighting matrix, Rayleigh-quot.	Ψ	matrix of shape functions and derivatives
		Ω	angular velocity
		ω, ω	angular velocity, frequency

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