Organizing Families of Aggregation Operators into a Cube of Opposition

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Abstract The cube of opposition is a structure that extends the traditional square of opposition originally introduced by Ancient Greek logicians in relation with the study of syllogisms. This structure, which relates formal expressions, has been recently generalized to non Boolean, graded statements. In this paper, it is shown that the cube of opposition applies to well-known families of idempotent, monotonically increasing aggregation operations, used in multiple criteria decision making, which qualitatively or quantitatively provide evaluations between the minimum and the maximum of the aggregated quantities. This covers weighted minimum and maximum, and more generally Sugeno integrals on the qualitative side, and Choquet integrals, with the important particular case of Ordered Weighted Averages, on the quantitative side. The main appeal of the cube of opposition is its capability to display the various possible aggregation attitudes in a given setting and to show their complementarity.

1 Introduction

The application of fuzzy sets [1] to multiple criteria decision making [2] has led to the continued blossoming of a vast amount of studies on different classes of aggregation operators for combining membership grades. This includes in particular triangular norms and co-norms [3] on the one hand, and Sugeno and Choquet integrals [4, 5] on the other hand. Ronald Yager, in his vast amount of important contributions to fuzzy set theory on many different topics, has been especially at the forefront of

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creativeness regarding aggregation operators, with in particular the introduction of a noticeable family of triangular norms and co-norms [6], of uninorms [7], and of Ordered Weighted Averages (OWA) [8–10].

Sugeno and Choquet integrals are well-known families of idempotent, monotonically increasing aggregation operators, used in multiple criteria decision making, with a qualitative and a quantitative flavor respectively. They include weighted minimum and maximum, and weighted average respectively, as particular cases, and provide evaluations lying between the minimum and the maximum of the aggregated quantities. In such a context, the gradual properties corresponding to the criteria to fulfill are supposed to be positive, i.e., the global evaluation increases with the partial ratings. But some decisions or alternatives can be found acceptable because they do not satisfy some (undesirable) properties. So, we also need to consider negative properties, the global evaluation of which increases when the partial ratings decreases. This reversed integral is a variant of Sugeno integrals, called desintegrals [11, 12]. Their definition is based on a decreasing set function called anti-capacity. Then, a pair of evaluations made of a Sugeno integral and a reversed Sugeno integral is useful to describe acceptable alternatives in terms of properties they must have and of properties they must avoid.

Besides, we can distinguish the optimistic part and the pessimistic part of any capacity [13]. It has been recently indicated that Sugeno integrals associated to these capacities and their associated desintegrals form a cube of opposition [14], the integrals being present on the front facet and the desintegrals on the back facet of the cube (each of these two facets fit with the traditional views of squares of opposition [15]). As this cube exhausts all the evaluation options, the different Sugeno integrals and desintegrals present on the cube are instrumental in the selection process of acceptable choices. We show in this paper that a similar cube of opposition exists for Choquet integrals, which can then be particularized for OWA operators.

The paper is organized as follows. Section 2 provides a brief reminder on the square of opposition, and introduces the cube of opposition and its graded extension in a multiple criteria aggregation perspective. Section 3 restates the cube of opposition for Sugeno integrals and desintegrals. Section 4 presents the cube for Choquet integrals and then for OWA operators, and discusses the different aggregation attitudes and their relations.

2 Background and Notations

We first recall the traditional square of opposition originally introduced by Ancient Greek logicians in relation with the study of syllogisms. This square relates universally and existentially quantified statements. Then its extension into a cube of opposition is presented, together with its graded version, in a qualitative multiple criteria aggregation perspective.

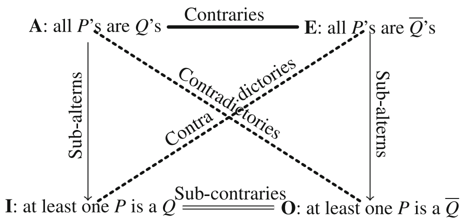


Fig. 1 Square of opposition

2.1 The Square and Cube of Opposition

The traditional square of opposition [15] is built with universally and existentially quantified statements in the following way. Consider a statement (**A**) of the form "all P's are Q's", which is negated by the statement (**O**) "at least one P is not a Q", together with the statement (**E**) "no P is a Q", which is clearly in even stronger opposition to the first statement (**A**). These three statements, together with the negation of the last statement, namely (**I**) "at least one P is a Q" can be displayed on a square whose vertices are traditionally denoted by the letters **A**, **I** (affirmative half) and **E**, **O** (negative half), as pictured in Fig. 1 (where $\overline{Q}$ stands for "not Q").

As can be checked, noticeable relations hold in the square:

(i) **A** and **O** (resp. **E** and **I**) are the negation of each other;
(ii) **A** entails **I**, and **E** entails **O** (it is assumed that there is at least one P, to avoid existential import problems);
(iii) together **A** and **E** cannot be true, but may be false;
(iv) together **I** and **O** cannot be false, but may be true.

Changing P into $\neg P$, and Q in $\neg Q$ leads to another similar square of opposition **aeoi**, where we also assume that the set of "not-P's" is non-empty. Then the 8 statements, **A**, **I**, **E**, **O**, **a**, **i**, **e**, **o** may be organized in what may be called a *cube of opposition* as in Fig. 2.

This cube first appeared in [16] in a renewed discussion of syllogisms, and was reintroduced recently in an information theoretic perspective [17]. The structural properties of the cube are:

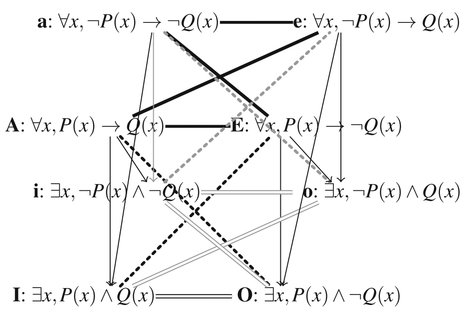


Fig. 2 The cube of opposition for quantified statements

- **AEOI** and **aeoi** are squares of opposition,
- **A** and **e**; **a** and **E** cannot be true together,
- **I** and **o**; **i** and **O** cannot be false together,
- **A** entails **i**, **E** entails **o**, **a** entails **I**, **e** entails **O**.

In the cube, if we also assume that the sets of "Q's" and "not-Q's" are non-empty, then the thick non-directed segments relate contraries, the double thin non-directed segments sub-contraries, the diagonal dotted non-directed lines contradictories, and the vertical uni-directed segments point to subalterns, and express entailments.

Stated in set-theoretic notation, **A**, **I**, **E**, **O**, **a**, **i**, **e**, **o**, respectively mean $P \subseteq Q$, $P \cap Q \neq \emptyset$, $P \subseteq \overline{Q}$, $P \cap \overline{Q} \neq \emptyset$, $\overline{P} \subseteq \overline{Q}$, $\overline{P} \cap \overline{Q} \neq \emptyset$, $\overline{P} \subseteq Q$, $\overline{P} \cap Q \neq \emptyset$. In order to satisfy the four conditions of a square of opposition for the front and the back facets, we need $P \neq \emptyset$ and $\overline{P} \neq \emptyset$. In order to have the inclusions indicated by the diagonal arrows in the side facets, we need $Q \neq \emptyset$ and $\overline{Q} \neq \emptyset$, as further normalization conditions.

Suppose P denotes a set of important properties, Q a set of satisfied properties (for a considered object). Vertices **A**, **I**, **a**, **i** correspond respectively to 4 different cases:

(i) all important properties are satisfied,
(ii) at least one important property is satisfied,
(iii) all satisfied properties are important,
(iv) at least one non satisfied property is not important.

Note also the cube is compatible with a bipolar understanding [18]. Suppose that among possible properties for the considered objects, some are desirable (or requested) and form a subset R and some others should be excluded (forbidden or undesirable) and form a subset E. Clearly, one should have $E \subseteq \overline{R}$. The set of properties of a given object is partitioned into the subset of satisfied properties S and the subset $\overline{S}$ of not satisfied properties. Then vertex **A** corresponds to $R \subseteq S$ and **a** to $\overline{R} \subseteq \overline{S}$. Then **a** also corresponds to $E \subseteq \overline{S}$.

2.2 A Gradual Cube of Opposition

It has been recently shown that the structure of the cube of opposition underlies many knowledge representation formalisms used in artificial intelligence, such as first order logic, modal logic, but also formal concept analysis, rough set theory, abstract argumentation, as well as quantitative uncertainty modeling frameworks such as possibility theory, or belief function theory [14, 19]. In order to accommodate quantitative frameworks, a graded extension of the cube has been defined in the following way.

Let $\alpha, \iota, \varepsilon, o$, and $\alpha', \iota', \varepsilon', o'$ be the grades in $[0, 1]$ associated to vertices **A**, **I**, **E**, **O** and **a**, **i**, **e**, **o**. Then we consider an involutive negation n, a symmetrical conjunction $*$ that respects the law of contradiction with respect to this negation, and we interpret entailment in the many-valued case by the inequality $\leq$: the conclusion is at least as true as the premise. The constraints satisfied by the cube of Fig. 2 can be generalized in the following way [14]:

(i) $\alpha = n(o)$, $\varepsilon = n(\iota)$ and $\alpha' = n(o')$ and $\varepsilon' = n(\iota')$;
(ii) $\alpha \leq \iota$, $\varepsilon \leq o$ and $\alpha' \leq \iota'$, $\varepsilon' \leq o'$;
(iii) $\alpha * \varepsilon = 0$ and $\alpha' * \varepsilon' = 0$;
(iv) $n(\iota) * n(o) = 0$ and $n(\iota') * n(o') = 0$;
(v) $\alpha \leq \iota'$, $\alpha' \leq \iota$ and $\varepsilon' \leq o$, $\varepsilon \leq o'$;
(vi) $\alpha' * \varepsilon = 0$, $\alpha * \varepsilon' = 0$;
(vii) $n(\iota') * n(o) = 0$, $n(\iota) * n(o') = 0$.

In the paper, we restrict to the numerical setting and let $n(a) = 1 - a$. It leads to define $* = \max(0, \cdot + \cdot - 1)$ (the Łukasiewicz conjunction). In the sequel, we show that the (gradual) cube of opposition is relevant for describing different families of multiple criteria aggregation functions. We first illustrate this fact by considering weighted minimum and maximum, together with related aggregations.

3 A Cube of Simple Qualitative Aggregations

In multiple criteria aggregation objects are evaluated by means of criteria i where $i \in \mathscr{C} = \{1, \dots, n\}$. The evaluation scale L is a totally ordered scale with top 1, bottom 0, and the order-reversing operation is denoted by $1 - (\cdot)$. For simplicity, we take $L = [0, 1]$, or a subset thereof, closed under the negation and the conjunction.

An object x is represented by a vector $x = (x_1, \dots, x_n)$ where x_i is the evaluation of x according to the criterion i. We assume that $x_i = 1$ means that the object fully satisfies criterion i and $x_i = 0$ expresses a total absence of satisfaction. Let $\pi_i \in [0, 1]$ represent the level of importance of criterion i. The larger π_i the more important the criterion. We note $\pi = (\pi_1, \dots, \pi_n)$.

In such a context, simple qualitative aggregation operators are the weighted min and the weighted max [20]:

- The weighted min measures the extent to which all important criteria are highly satisfied; it corresponds to the expression $\min_{i=1}^n \max(1 - \pi_i, x_i)$,
- the weighted max, $\max_{i=1}^n \min(\pi_i, x_i)$, is optimistic and only requires that at least one important criterion be highly satisfied.

The weighted min and weighted max correspond to vertices **A** and **I** of the cube on Fig. 3. As it can be noticed, the cube of Fig. 3 is just a multiple-valued counterpart of the initial cube of Fig. 2.

Under the hypothesis of the double normalization ($\exists i, \pi_i = 1$ and $\exists j, \pi_j = 0$) and the hypothesis $\exists r, x_r = 1$ and $\exists s, x_s = 0$, which correspond to the non-emptiness of P, $\overline{P}$, Q, and $\overline{Q}$ in cube of Fig. 2, it can be checked that all the constraints (i–vii) of the gradual cube hold. For instance, the entailment from **A** to **I** translates into

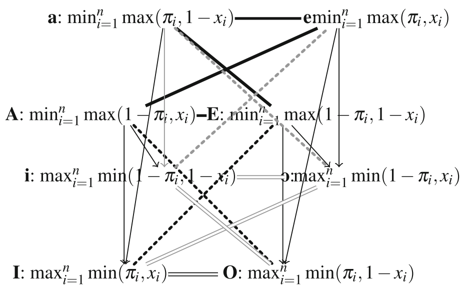


Fig. 3 The cube of weighted qualitative aggregations

$\min_{i=1}^n \max(1-\pi_i, x_i) \leq \max_{i=1}^n \min(\pi_i, x_i)$, which holds as soon as $\exists i, \pi_i = 1$. Formally speaking, in terms of possibility theory [21, 22], it is nothing but the expression that the necessity of a fuzzy event $N_\pi(x)$ is less or equal to the possibility $\Pi_\pi(x)$ of this event, provided that the possibility distribution π is normalized. While **A** and **I** are associated with $N_\pi(x)$ and $\Pi_\pi(x)$ respectively, **a** is associated with a guaranteed possibility $\Delta_\pi(x)$ (which indeed reduces to $\Delta_\pi(x) = \min_{i \mid x_i=1} \pi_i$ in case $\forall i, x_i \in \{0,1\}$). Note also that $\Delta_\pi(x) = N_{\overline{\pi}}(1-x)$, where $\overline{\pi} = 1 - \pi_i$; lastly **i** is associated with $\nabla_\pi(x) = 1 - \Delta_\pi(1-x)$. Moreover there is a correspondence between the aggregation functions on the right facet of the cube and those on the left facet, replacing x with $1-x$.

Let us discuss the different aggregation attitudes displayed on the cube. Suppose that a fully satisfactory object x is an object with a global rating equal to 1. Then, vertices **A**, **I**, **a** and **i** correspond respectively to 4 different cases: x is such that

(i) **A**: all properties having some importance are fully satisfied (if $\pi_i > 0$ then $x_i = 1$ for all i),
(ii) **I**: there exists at least one important property i fully satisfied ($\pi_i = 1$ and $x_i = 1$),
(iii) **a**: all somewhat satisfied properties are fully important (if $x_i > 0$ then $\pi_i = 1$ for all i),
(iv) **i**: there exists at least one unimportant property i that is not satisfied at all ($\pi_i = 0$ and $x_i = 0$).

These cases are similar to those encountered in the cube of Fig. 2.

Example 1 We consider $\mathscr{C} = \{1, 2, 3\}$ and $\pi_1 = 0$, $\pi_2 = 0.5$ and $\pi_3 = 1$; see Fig. 4.

- on vertex A (resp. I) a fully satisfied object is such that $x_2 = x_3 = 1$ (resp. $x_3 = 1$),
- on vertex a (resp. i) a fully satisfied object is such that $x_1 = x_2 = 0$ (resp. $x_1 = 0$).

The operations of the front facet of the cube of Fig. 3 merge positive evaluations that focus on the high satisfaction of important criteria, while the local ratings x_i on the back could be interpreted as negative ones (measuring the intensity of faults). Then, aggregations yield global ratings evaluating the lack of presence of important faults. In this case, weights are tolerance levels forbidding a fault to be too strongly present. Then, the vertices a and i in the back facet are interpreted differently:

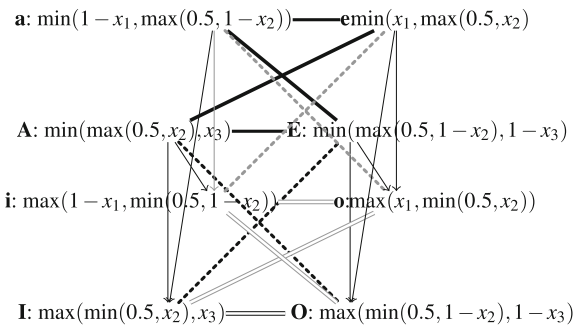


Fig. 4 Example of a cube of weighted qualitative aggregations

- the evaluation associated to a is equal to 1 if all somewhat intolerable faults are fully absent;
- the evaluation associated to i is equal to 1 if there exists at least one intolerable fault that is absent.

This framework thus involves two complementary points of view, recently discussed in a multiple criteria aggregation perspective [11].

4 The Cube of Sugeno Integrals

Weighted minimum and maximum (as well as ordered weighted minimum and maximum [23]) are particular cases of Sugeno integrals. The cube on Fig. 3 can indeed be extended to Sugeno integrals and its associated so-called desintegrals. Before presenting the cube associated with Sugeno integrals, let us recall some definitions used in the following, namely the notions of capacity, conjugate capacity, qualitative Moebius transform, and focal sets.

In the definition of a Sugeno integral the relative weights of the set of criteria are represented by a capacity (or fuzzy measure) which is a set function $\mu : 2^{\mathscr{C}} \to L$ that satisfies $\mu(\emptyset) = 0$, $\mu(\mathscr{C}) = 1$ and $A \subseteq B$ implies $\mu(A) \leq \mu(B)$. The conjugate capacity of μ is defined by $\mu^c(A) = 1 - \mu(\overline{A})$ where $\overline{A}$ is the complement of A.

The inner qualitative Moebius transform of a capacity μ is a mapping $\mu_\# : 2^{\mathscr{C}} \to L$ defined by

$$\mu_\#(E) = \mu(E) \text{ if } \mu(E) > \max_{B \subset E} \mu(B) \text{ and 0 otherwise.}$$

A set E such that $\mu_\#(E) > 0$ is called a focal set. The set of the focal sets of μ is denoted by $\mathscr{F}(\mu)$.

The Sugeno integral of an object x with respect to a capacity μ is originally defined by [24, 25]:

$$S_\mu(x) = \max_{\alpha \in L} \min(\alpha, \mu(\{i \mid x_i \geq \alpha\})). \quad (1)$$

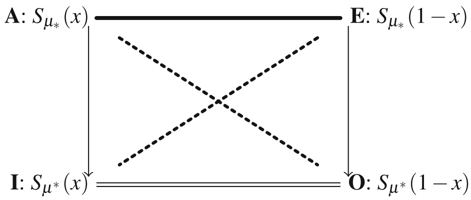
When Sugeno integrals are used as aggregation functions to select acceptable objects, the properties of which are assumed to have a positive flavor: namely, the global evaluation increases with the partial ratings. But generally, we may also have negative properties, as already described in the introduction. In such a context we can use a desintegral [11, 12] associated to the Sugeno integral. We now present this desintegral.

In the case of negative properties, fault-tolerance levels are assigned to sets of properties by means of an anti-capacity (or anti-fuzzy measure), which is a set function $\nu : 2^{\mathscr{C}} \to L$ such that $\nu(\emptyset) = 1$, $\nu(\mathscr{C}) = 0$, and if $A \subseteq B$ then $\nu(B) \leq \nu(A)$. The conjugate ν^c of an anti-capacity ν is an anti-capacity defined by $\nu^c(A) = 1 - \nu(\overline{A})$, where $\overline{A}$ is the complementary of A. The desintegral $S^{\downarrow}_{\nu}(x)$ is defined from the corresponding Sugeno integral, by reversing the direction of the local value scales (x becomes $1 - x$), and by considering a capacity induced by the anti-capacity ν, as follows:

$$S^{\downarrow}_{\nu}(x) = S_{1-\nu^c}(1 - x). \tag{2}$$

In order to present the square of Sugeno integrals, we need to define the pessimistic part and the optimistic part of a capacity. They are respectively called assurance and opportunity functions by Yager [26]. This need should not come as a surprise. Indeed the entailment from **A** to **I** requires that the expression in **A** have a universal flavor, i.e. here, is minimum-like, while the expression in **I** have an existential flavor, i.e. here, is maximum-like, but the capacity μ, on which the considered Sugeno integral is based, may have neither.

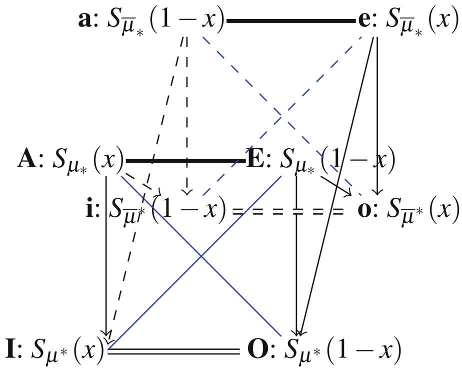
When we consider a capacity μ, its pessimistic part is $\mu_*(A) = \min(\mu(A), \mu^c(A))$ and its optimistic part is $\mu^*(A) = \max(\mu(A), \mu^c(A))$ [13]. Observe that $\mu_* \leq \mu^*$, ${\mu_*}^c = \mu^*$ and $\mu^{*c} = \mu_*$. So a capacity μ induces the following square of opposition (see [27] for more details).



Note that $S_{\mu_*}(1 - x) = S^{\downarrow}_{1-\mu^*}(x)$ and $S_{\mu^*}(1 - x) = S^{\downarrow}_{1-\mu_*}(x)$, where $1 - \mu^*$ and $1 - \mu_*$ are anti-capacities.

Lastly, in order to build the cube associated to Sugeno integrals, just as $\overline{\pi}$ is at work on the back facet of the cube associated with weighted min and max, we also need the opposite capacity $\overline{\mu}$, defined as follows: $\overline{\mu_\#}(E) = \mu_\#(\overline{E})$ and $\overline{\mu}(A) = \max_{\overline{E} \subseteq A} \mu_\#(E)$. A square of opposition **aieo** can be defined with the capacity $\overline{\mu}$. Hence, supposing $\exists i \in \mathscr{C}$ such that $x_i = 0$ and $\exists j \in \mathscr{C}$ such that $x_j = 1$, we can construct a cube of opposition **AIEO** and **aieo** as presented in Fig. 5 [14].

Fig. 5 Cube of opposition of Sugeno integrals associated to a capacity μ



The fact that all the constraints of a gradual cube hold in this case has been only established under a specific type of normalization for capacities [27], i.e., $\exists A \neq \mathscr{C}$ such that $\mu(A) = 1$ and $\exists B \neq \mathscr{C}$ such that $\mu^c(B) = 1$; note that in such a context there exists a non empty set, $\overline{B}$, such that $\mu(\overline{B}) = 0$. However, this does not cover another particular case where the constraints of the cube also hold, namely the one where μ is only non zero on singletons. Finding the most general condition on μ ensuring the satisfaction of all constraints (i–vii) in the cube of Sugeno integrals is still an open question.

Let us now present the aggregation attitudes expressed by the cube of Sugeno integrals. We can characterize situations where objects get a global evaluation equal to 1 using aggregations on the side facet.

The global evaluations at vertices **AIai** of a cube associated to a capacity μ are maximal respectively in the following situations pertaining to the focal sets of μ:

- **A** The set of totally satisfied properties contain a focal set with weight 1 and overlaps all other focal sets.
- **I** The set of satisfied properties contains a focal set with weight 1 or overlaps all other focal sets.
- **a** The set of totally violated properties contains no focal set and its complement is contained in a focal set with weight 1.
- **i** The set of totally violated properties contains no focal set or its complement is contained in a focal set with weight 1.

Example 2 Assume $\mathscr{C} = \{1, 2, 3\}$ and the following capacities

Capacity	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
μ	0	0	0	1	1	0	1
μ^c	1	0	0	1	1	1	1
$\overline{\mu}$	0	1	1	1	1	1	1
$\overline{\mu}^c$	0	0	0	0	0	1	1

$\mu^c \geq \mu$ so $\mu_* = \mu$ and $\mu^* = \mu^c$
$\overline{\mu} \geq \overline{\mu}^c$ so $\overline{\mu}^* = \overline{\mu}$ and $\overline{\mu}_* = \overline{\mu}^c$
Note that $\overline{\mu}$ is a possibility measure.

The aggregation functions on the vertices are:

A: $S_\mu(x) = \max(\min(x_1, x_2), \min(x_1, x_3))$, **I**: $S_{\mu^c}(x) = \max(x_1, \min(x_2, x_3))$
a: $S_{\overline{\mu}^c}(1 - x) = \min(1 - x_2, 1 - x_3)$, **i**: $S_{\overline{\mu}}(1 - x) = \max(1 - x_2, 1 - x_3)$.

- For vertex **A**, the two focal sets overlap when $S_\mu(x) = 1$.
- For vertex **I**, one can see that $S_{\mu^c}(x) = 1$ when $x_1 = 1$ and $\{1\}$ does overlap all focal sets of μ; the same occurs when $x_2 = x_3 = 1$.
- For vertex **a**, $S_{\overline{\mu}^c}(1 - x) = 1$ when $x_2 = x_3 = 0$, and note that the complement of $\{2, 3\}$ is contained in a focal set of μ, while $\{2, 3\}$ contains no focal set of μ.
- For vertex **i**, $S_{\overline{\mu}}(1 - x) = 1$ when, $x_2 = 0$ or $x_3 = 0$, and clearly, neither $\{2\}$ not $\{3\}$ contain any focal set of μ, but the complement of each of them is a focal set of μ.

5 The Cube of Choquet Integrals

When criteria evaluations are quantitative, Choquet integrals often constitute a suitable family of aggregation operators, which generalize weighted averages, and which parallel, in different respects, the role of Sugeno integrals for the qualitative case. Although the evaluation scale can be taken as the real line $\mathbb{R}$, we use the unit interval $[0, 1]$ in the following.

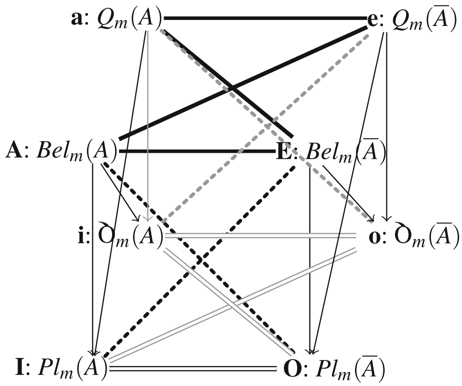
Belief and plausibility functions are particular cases of Choquet integrals, just as necessity and possibility measures are particular cases of Sugeno integrals. This is why we begin with the presentation of the cube of belief functions, before studying the cube of Choquet integrals, of which another noticeable particular case is the cube of ordered weighted averaging aggregation operators (*OWA*), which is then discussed, before concluding.

5.1 The Cube of Belief Functions

In Shafer's evidence theory [28], a belief function Bel_m is defined together with a dual plausibility function Pl_m from a mass function m, i.e., a real set function $m : 2^{\mathscr{C}} \to [0, 1]$ such that $m(\emptyset) = 0$, $\sum_{A \subseteq \mathscr{C}} m(A) = 1$. Then for $A \subseteq \mathscr{C}$, we have $Bel_m(A) = \sum_{E \subseteq A} m(E)$ and $Pl_m(A) = 1 - Bel_m(\overline{A}) = \sum_{E \cap A \neq \emptyset} m(E)$.

Viewing m as a random set, the complement $\overline{m}$ of the mass function m is defined as $\overline{m}(E) = m(\overline{E})$ [29]. The commonality function Q and its dual $\rotatebox[origin=c]{180}{Q}$ are then defined by $Q_m(A) = \sum_{A \subseteq E} m(E)$ and $\rotatebox[origin=c]{180}{Q}_m(A) = \sum_{\overline{E} \cap \overline{A} \neq \emptyset} m(E) = 1 - Q_m(\overline{A})$ respectively. The normalization $\overline{m}(\emptyset) = 0$ forces $m(\mathscr{C}) = 0$. Then, $Q_m(A) = Bel_{\overline{m}}(\overline{A})$ while $\rotatebox[origin=c]{180}{Q}_m(A) = Pl_{\overline{m}}(\overline{A})$. It can be checked that the transformation $m \to \overline{m}$ reduces to $\pi \to \overline{\pi} = 1 - \pi$ in case of nested focal elements. All these set functions can be put on the following cube of opposition [14]. See Fig. 6. Indeed, if $m(\emptyset) = 0$, we have $Bel_m(A) \leq Pl_m(A) \Leftrightarrow Bel_m(A) + Bel_m(\overline{A}) \leq 1 \Leftrightarrow Pl_m(A) + Pl_m(\overline{A}) \geq 1$, which gives birth to the square of

Fig. 6 Cube of opposition of evidence theory



opposition **AIEO**. We can check as well that $Bel_m(A) = \sum_{E\subseteq A} m(E) \leq \text{Ꝺ}_m(A) = 1 - \sum_{\overline{A}\subseteq E} m(E)$. Similar inequalities ensure that $Q_m(A) \leq Pl_m(A) = 1 - Bel_m(\overline{A})$, or $Bel_m(A) + Q_m(\overline{A}) \leq 1$, for instance, which ensures that the constraints of the cube hold.

Belief functions are a particular case of capacities. Note that the square can be extended replacing *Bel* and *Pl* by a capacity μ and its conjugate $\mu^c(A) = 1 - \mu(\overline{A})$, respectively. However, to build the cube, we also need inequalities such as $Q_m(A) \leq Pl_m(A)$ to be generalized to capacities.

5.2 Extension to Choquet Integrals

Considering a capacity μ on $\mathscr{C}$, the Moebius transform of μ, denoted by m_μ, is given by $m_\mu(T) = \sum_{K\subseteq T}(-1)^{|T\setminus K|}\mu(K)$. The Choquet integral with respect to μ is:

$$C_\mu(x) = \sum_{T\subseteq\mathscr{C}} m_\mu(T) \min_{i\in T} x_i. \tag{3}$$

Clearly, $Bel_m(A) = \sum_E m(E) \cdot \min_{u\in E} 1_A(u)$. We have the equality $Bel_{m_\mu}(A) = C_\mu(1_A) = \mu(A)$ if m_μ represents the Moebius transform of a capacity μ. This characterisation is presented in [30]. More precisely, a real set function m is the Moebius transform of a capacity μ if and only if $m(\emptyset) = 0$, $\sum_{K\subseteq S} m(i \cup K) \geq 0$ for all i and for all $S \subseteq \mathscr{C}\setminus i$ and $\sum_{K\subseteq\mathscr{C}} m(K) = 1$. And it is the Moebius transform of a belief function if and only if it is non-negative. In general, $m_\mu(E)$ can be negative for non-singleton sets.

Under these conditions, $Pl_m(A) = 1 - C_\mu(1_{\overline{A}})$. But we have $Q_m(A) = Bel_{\overline{m}}(\overline{A})$, so $Q_m(A) = C_{\overline{\mu}}(1_{\overline{A}})$ if $\overline{m}$ satisfies the conditions to be a Moebius transform of a capacity $\overline{\mu}$. In such a context, $\text{Ꝺ}_m(A) = Pl_{\overline{m}}(\overline{A}) = 1 - C_{\overline{\mu}}(1_A)$. It is worth noticing that there

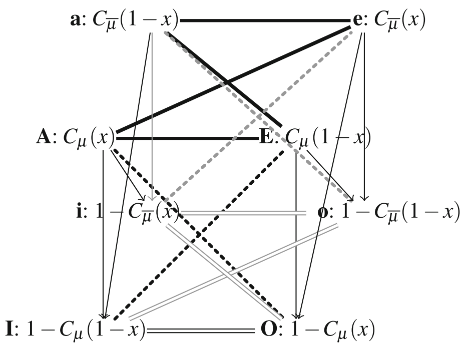


Fig. 7 Cube of opposition of Choquet integral

exist Moebius transforms m such that $\overline{m}$ is not a Moebius transform since we need to have the condition $m(\mathscr{C}) = 0$.

Hence one may consider the extension of the cube of Fig. 6 to general Choquet integrals. In order to understand the proof of the following proposition we need the other expression of the Choquet integral: $C_\mu(x) = \sum_{i=1}^{n}(x_i - x_{i-1})\mu(A_i)$ where we suppose that $x_1 \leq \cdots \leq x_n$, $A_i = \{i, \ldots, n\}$ and $x_0 = 0$.

With this expression, it is easy to check that the Choquet integral is increasing according to the capacity. Then the following holds (See Fig. 7).

Proposition 1 *The cube of Choquet integral is a cube of opposition if and only if* $\mu \leq \mu^c$, $\overline{\mu} \leq \overline{\mu}^c$ *and* $\mu + \overline{\mu} \leq 1$.

Proof We consider the evaluation scale $[0, 1]$. Without loss generality we can suppose that $x_1 \leq \cdots \leq x_n$.

A entails **I** iff $C_\mu(x) + C_\mu(1-x) \leq 1$. Considering $x = 1_A$ the characteristic function of A we need $\mu(A) \leq \mu^c(A)$. If $\mu \leq \mu^c$ then $C_\mu(x) \leq C_{\mu^c}(x)$. We have $C_\mu(1-x) = \sum_{T\subseteq\mathscr{C}} m_\mu(T)\min_{i\in T}(1-x_i) = \sum_{T\subseteq\mathscr{C}}(m_\mu(T) - m_\mu(T)\max_{i\in T} x_i) = 1 - C_{\mu^c}(x)$. So, $C_\mu(x) + C_\mu(1-x) \leq C_\mu(x) + C_{\mu^c}(1-x) = 1$.

By symmetry we have **E** entails **O**.

A and **E** cannot be equal to 1 together: $C_\mu(x) = 1$ entails $C_{\mu^c}(x) = 1$ since $\mu \leq \mu^c$, i.e., $C_\mu(1-x) = 0$. By duality **I** and **O** cannot be equal to 0 together.

So **AEIO** is a square of opposition.

Similarly $\overline{\mu} \leq \overline{\mu}^c$ is equivalent to making **aeio** a square of opposition.

If $C_\mu(x) \leq 1 - C_{\overline{\mu}}(x)$ then considering $x = 1_A$ we have $\mu(A) + \overline{\mu}(A) \leq 1$. Conversely if we suppose that $\mu + \overline{\mu} \leq 1$ then $C_\mu(x) + C_{\overline{\mu}}(x) = \sum_{i=1}^{n}(x_i - x_{i-1})(\mu(A_i) + \overline{\mu}(A_i)) \leq \sum_{i=1}^{n}(x_i - x_{i-1}) = x_n \leq 1$. This last equivalence permits to conclude that the considered cube is a cube of opposition.

The condition $\mu + \overline{\mu} \leq 1$ is valid for belief functions since $\overline{Bel_m}(A) = Bel_{\overline{m}}(A) = Q_m(\overline{A})$, and $Bel_m(A) + Q_m(\overline{A}) \leq 1$, but it needs to be investigated for more general capacities since some masses may be negative. Note that, in its back facet, the cube of Choquet integrals exhibits what maybe called desintegrals, associated to Choquet integrals. Namely, using $C_{\overline{\mu}}(1-x)$, the global evaluation increases when partial ratings decrease.

Let us discuss the aggregation attitudes when the evaluation scale is the real interval [0, 1] and μ is a belief function. More precisely we are going to characterize the situations where an object x gets a perfect global evaluation, i.e., a global evaluation equal to 1, for the different vertices **AIai**. We denote $\mathscr{F}_\mu$ the family of the sets having a Moebius transform not equal to 0.

- **A**: $C_\mu(x) = 1$ can be written $\sum_{F\subseteq\mathscr{C}} m_\mu(F)\cdot\min_{i\in F} x_i = 1$, which implies that $\forall F \in \mathscr{F}_\mu, \forall i \in F, x_i = 1$. So the focal sets of μ are included in the set of totally satisfied properties.
- **I**: $1 - C_\mu(1-x) = 1 = C_{\mu^c}(x)$ is equivalent to $\sum_{F\subseteq\mathscr{C}} m_\mu(F)\cdot\max_{i\in F} x_i = 1$. So in this case $\forall F \in \mathscr{F}_\mu$, $\exists i \in F$ such that $x_i = 1$. So each focal set of μ must intersect the set of totally satisfied properties.
- **a**: $C_{\overline{\mu}}(1-x) = \sum_{F\subseteq\mathscr{C}} \overline{m_\mu}(F)\cdot\min_{i\in F}(1-x_i) = \sum_{F\subseteq\mathscr{C}} m_\mu(F)\cdot\min_{i\in\overline{F}}(1-x_i)$. Then $C_{\overline{\mu}}(1-x) = 1$ is equivalent to $\forall F \in \mathscr{F}_\mu, \min_{i\in\overline{F}}(1-x_i) = 1$, or equivalently, $\forall F \in \mathscr{F}_\mu, \max_{i\in\overline{F}} x_i = 0$, which means $\forall F \in \mathscr{F}_\mu, \forall i \notin F, x_i = 0$.
  So all properties outside each focal set of μ are violated. The only properties that are satisfied are those in the intersection of the focal sets of μ.
- **i**: we have $1 - C_{\overline{\mu}}(x) = \sum_{F\subseteq\mathscr{C}} m_\mu(F)\cdot\max_{i\in\overline{F}}(1-x_i)$. Then $1 - C_{\overline{\mu}}(x) = 1$ is equivalent to $\forall F \in \mathscr{F}_\mu, \max_{i\in\overline{F}}(1-x_i) = 1$, i.e., $\forall F \in \mathscr{F}_\mu, \min_{i\in\overline{F}} x_i = 0$, which means $\forall F \in \mathscr{F}_\mu$, $\exists i \notin F$ such that $x_i = 0$. So there must be at least one violated property outside each focal set of μ.

5.3 *Example for the Cube of Choquet Integral*

Let us consider the menu of a traditional restaurant in Lyon.[1] We leave it in French (due to the lack of precise equivalent terms in English for most dishes):

Starter

Saladier lyonnais: museau, pieds de veau, cervelas, lentilles, pommes de terre, saucisson pistaché, frisée, oreilles de cochon

[1]This example is specially dedicated to Ron Yager in remembrance of a dinner in Lyon in a traditional restaurant, which took place at the occasion of the CNRS Round Table on Fuzzy Sets organized by Robert Féron [31] in Lyon on June 23–25, 1980 [32]. This Round Table was an important meeting for the development of fuzzy set research, because most of the active researchers of the field were there. Interestingly enough, Robert Féron had the remarkable intuition to invite Gustave Choquet in the steering committee, at a time where no fuzzy set researcher was mentioning Choquet integrals! This meeting also included, as usual, some nice moments of relaxation and good humor. In particular, at the above-mentioned dinner, to which quite a number of people took part (including two of the authors of this paper), Ron enjoyed very much a pigs feet dish. He was visibly very happy with his choice, so Lotfi Zadeh told him, “Ron, you should have been a pig in another life”, to which Ron replied “no, Lotfi, it is in this life”, while continuing to suck pigs’ bones with the greatest pleasure.

Oeuf meurette: oeuf poché, crotons, champignons, sauce vin rouge et lardons
Harengs pommes de terre à l'huile

Main course

Gratin d'andouillettes, sauce moutarde
Rognons de veau au Porto et moutarde
Quenelles de brochet, sauce Nantua et riz pilaf

Dessert

Gnafron: sorbet cassis et marc de Bourgogne
Baba au rhum et chantilly
Crème caramel

A tourist wants to eat some typical dishes of Lyon. His preferred dishes are "saladier lyonnais" (which offers a great sampling of meats from the Lyon region) and "gratin d'andouillettes" since he wants to eat some gourmet delicatessen products. The evaluation scale is the real interval [0, 1], so the "saladier lyonnais" and "gratin d'andouillettes" get the maximal rating 1. The other dishes receive a smaller rating. The set of criteria is $\mathscr{C} = \{s, c, d\}$, where s, c, d refer to starter, main course, and dessert respectively. We consider the Möbius transform: $m : 2^{\mathscr{C}} \to [0, 1]$ defined by $m(s) = m(s, c) = 0.5$ and 0 otherwise. Such a weighting clearly stresses the importance of the starter, and acknowledges the fact that the main course is only of interest with a starter, while dessert is not an issue for this tourist. A chosen menu is represented by a vector (x_s, x_c, x_d) where x_i is the rating corresponding to the chosen dish for the criterion i. The Choquet integral of x with respect to the capacity μ associated to m is:

$$C_\mu(x) = 0.5x_s + 0.5 \cdot \min(x_s, x_c).$$

$\overline{m}$ is the set function defined by $\overline{m}(d) = \overline{m}(c, d) = 0.5$ and 0 otherwise. It is easy to check that $\overline{m}$ is a Möbius transform. The Choquet integral of x with respect to $\overline{\mu}$, the capacity defined with $\overline{m}$ is:

$$C_{\overline{\mu}}(x) = 0.5x_d + 0.5 \cdot \min(x_c, x_d).$$

Let us look at the choices that get a perfect global evaluation on the cube of Choquet integrals:

- **A**: $C_\mu(x) = 1$ iff $x_s = x_c = 1$: a menu with a maximal evaluation contains the "saladier lyonnais" and the "gratin d'andouillette."
- **I**: $C_\mu(1 - x) = 0$ iff $x_s = 1$ or $x_s = x_c = 1$: a menu with a maximal evaluation contains the "saladier lyonnais" and may contain the "gratin d'andouillette."
- **a**: $C_{\overline{\mu}}(1 - x) = 1$ iff $x_c = x_d = 0$: a menu with a maximal evaluation contains neither the "gratin d'andouillette", nor the best dessert.
- **i**: $C_{\overline{\mu}}(x) = 0$ iff $x_d = 0$ ou $x_c = x_d = 0$: a menu with a maximal evaluation does not contain the best dessert, but may contain the "gratin d'andouillette".

Without surprise, the Choquet integral in **A** is maximal if the menu includes both the "saladier lyonnais" and the "gratin d'andouillette," while **I** is maximal as soon as the menu includes at least the "saladier lyonnais". The maximality conditions in **a** (and in **i**) are less straightforward to understand. Here we should remember that already in cube of Fig. 2, **a** entails **I** provided that x is normalized (i.e., $\exists i, x_i = 1$), which ensures that the expression attached to **a** is smaller or equal to the one associated with **I**. The same condition is enough for having

$C_{\overline{\mu}}(1-x) = \sum_{F \subseteq \mathscr{C}} m_\mu(F) \cdot \min_{i \in \overline{F}} 1 - x_i \leq$ 1- $C_\mu(1-x) = \sum_{F \subseteq \mathscr{C}} m_\mu(F) \cdot \max_{i \in F} x_i$ provided that $\sum_{F \subseteq \mathscr{C}} m_\mu(F) = 1$. Indeed, let $x_{i^*} = 1$, then for all $F \subseteq \mathscr{C}$, either $x_{i^*} \in F$ or $x_{i^*} \in \overline{F}$. Thus, either $\min_{i \in \overline{F}} 1 - x_i = 0$, or $\max_{i \in F} x_i = 1$, which ensures the inequality.

Thus going back to the example, since the evaluation in **a** is maximal for $x_c = x_d = 0$, the normalization forces $x_s = 1$, which means that the menu includes the "saladier lyonnais". Note also that $x_s = 1, x_c = 0, x_d = 0$ is a minimal normalized evaluation vector x, for which the desintegral associated with **a** is maximal. Considering the evaluation in **i** the normalization entails that $x_s = 1$ or $x_c = 1$ so the menu includes the "saladier lyonnais" or the "gratin d'andouillette".

5.4 The Cube of OWA Operators

Ordered Weighted Averages (OWA) [8–10] and their weighted extension [33] have been found useful in many applications. Since OWAs are a particular case of Choquet integrals [34], one may wonder about a square, and then a cube of opposition associated to OWAs as a particular case of the cube of Fig. 7. Let us first recall what is an OWA.

An OWA_w is a real mapping on $\mathscr{C}$ associated to a collection of weights $w = (w_1, \ldots, w_n)$ such that $w_i \in [0, 1]$ for all $i \in \{1, \ldots, n\}$, $\sum_{i=1}^n w_i = 1$, and defined by:

$$OWA_w(x) = \sum_{i=1}^{n} w_i \cdot x_{(i)}$$

where $x_{(1)} \leq \cdots \leq x_{(n)}$.

This includes noticeable particular cases:

- $w = (1, 0, \ldots, 0) \Rightarrow OWA_w(x) = \min_{i=1}^n x_i$,
- $w = (0, \ldots, 0, 1) \Rightarrow OWA_w(x) = \max_{i=1}^n x_i$,
- $w = (\frac{1}{n}, \ldots, \frac{1}{n}) \Rightarrow OWA_w(x) = \frac{\sum_{i=1}^n x_i}{n}$.

In [8], Yager also defines measures of orness and andness:

$$orness(OWA_w) = \frac{1}{n-1} \sum_{i=1}^{n} (n-i) \cdot w_i; \;\; andness(OWA_w) = 1 - orness(OWA_w).$$

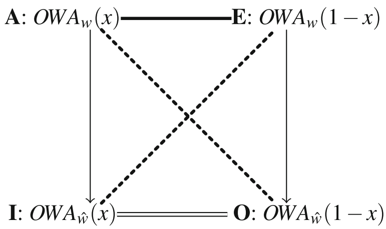


Fig. 8 Square of opposition of OWA

Note that $orness(OWA_w), andness(OWA_w) \in [0, 1]$. The closer the OWA_w is to an *or* (resp. *and*), the closer $orness(OWA_w)$ is to 1 (resp. 0).

In the same article, Yager also defines the measure of dispersion (or entropy) of an OWA associated to w by

$$disp(OWA_w) = -\sum_{i=1}^{n} w_i \ln w_i.$$

The measure of dispersion estimates the degree to which we use all the aggregates equally.

The dual of OWA_w (see, e.g., [35]) is $OWA_{\hat{w}}$ with the weight $\hat{w} = (w_n, \dots, w_1)$. More precisely we have $\hat{w}_i = w_{n-i+1}$. It is easy check that $disp(OWA_{\hat{w}}) = disp(OWA_w)$ and $orness(OWA_{\hat{w}}) = 1 - orness(OWA_w) = andness(OWA_w)$.

The following duality relation holds

$$\begin{aligned} OWA_w(1-x) &= \textstyle\sum_{i=1}^{n} w_i(1 - x_{(n-i+1)}) = 1 - \sum_{i=1}^{n} w_{n-i+1} x_{(i)} \\ &= 1 - OWA_{\hat{w}}(x) \end{aligned}$$

In particular, it changes min into max and conversely.

This corresponds to the expected relation for the diagonals of the square of opposition of Fig. 8 for OWAs. Then the entailment relations of the vertical sides require to have

$$\sum_{i=1}^{n} w_i \cdot x_{(i)} \leq \sum_{i=1}^{n} w_i \cdot x_{(n-i+1)}$$

This can be rewritten as

$$\begin{aligned} 0 &\leq w_1 \cdot (x_{(n)} - x_{(1)}) + w_2 \cdot (x_{(n-1)} - x_{(2)}) + \cdots + w_n \cdot (x_{(1)} - x_{(n)}) \\ &= (w_1 - w_n) \cdot (x_{(n)} - x_{(1)}) + (w_2 - w_{n-1}) \cdot (x_{(n-1)} - x_{(2)}) + \cdots \end{aligned}$$

In order to guarantee that the above sum adds positive terms only, it is enough to enforce the following condition for the weights:

$$w_1 \geq w_2 \geq \cdots \geq w_n,$$

which expresses a demanding aggregation. We are not surprised to observe that the w associated to max violates the above condition. The situation is similar to the one already encountered with Sugeno integrals where we had to display integrals based on pessimistic or optimistic fuzzy measures depending on the vertices of the square and similar to the situation of belief functions, which are pessimistic, which ensures a regular square of opposition without any further condition.

Besides, in [34, 36–38] it is proved that a capacity μ depends only on the cardinality of subsets if and only if there exists $w \in [0,1]^n$ such that $C_\mu(x) = OWA_w(x)$. Moreover we have the following relations. The fuzzy measure μ associated to OWA_w is given by: $\mu(T) = \sum_{i=n-t+1}^{n} w_i$ where t denotes the cardinality of T. It is worth noticing that the Moebius transform is $m(T) = \sum_{j=0}^{t-1} \binom{t-1}{j} (-1)^{t-1-j} w_{n-j}$, so m depends only on the cardinality of the subsets. It is worth noticing that while the particular cases min and average are associated with simple positive mass functions ($m(\mathscr{C}) = 1$, and $m(\{i\}) = 1/n$ respectively), max is associated with a mass function that has negative weights (remember that plausibility measures do not have a positive Moebius transform).

Conversely we have $w_{n-t} = \mu(T \cup i) - \mu(T) = \sum_{K \subseteq T} m(K \cup i)$ $i \in \mathscr{C}$ $T \subseteq \mathscr{C} \backslash i$. So if μ depends only on the cardinality of the subsets, $\overline{\mu}$, the capacity associated to $\overline{m}$, depends only on the cardinality of subsets (since the Moebius transform depends only on the cardinality of subsets). The weight of the OWA associated to $\overline{\mu}$: $\overline{w}_{n-t} = \overline{\mu}(T \cup i) - \overline{\mu}(T)$. Moreover, note that $m(T)$ involves weights from w_{n-t+1} to w_n, while $\overline{m}(T) = \sum_{j=0}^{n-t-1} \binom{n-t-1}{j} (-1)^{n-t-1-j} w_{n-j}$ involves weights from w_{t+1} to w_n, and $\hat{m}(T) = \sum_{j=0}^{t-1} \binom{t-1}{j} (-1)^{t-1-j} \hat{w}_{n-j}$ involves weights from w_0 to w_t, since $\hat{w}_{n-j} = w_{j+1}$. This indicates that these mass functions are different.

Hence we obtain the cube associated to the OWA's presented on Fig. 9.

A deeper investigation of this cube in relation with conditions ensuring entailments from top facet to bottom facet, and the positivity of associated mass functions is left for further research.

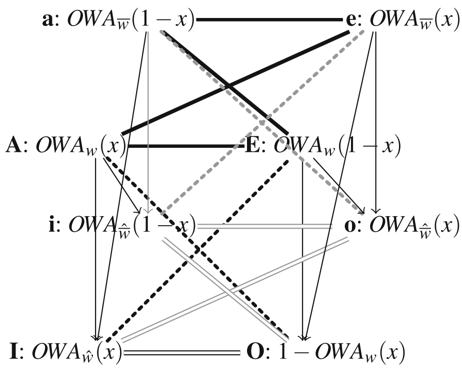


Fig. 9 The cube of opposition for OWA operators

6 Concluding Remarks

This paper has first shown how the structure of the cube of opposition extends from ordinary sets to weighted min- and max-based aggregations and more generally to Sugeno integrals, which constitute a very important family of qualitative aggregation operators. Then, a similar construct has been exhibited for Choquet integrals and OWA operators. The cube exhausts all the possible aggregation attitudes. Moreover, as mentioned in Sect. 2, it is compatible with a bipolar view where we distinguish between desirable properties and rejected properties. It thus provides a rich theoretical basis for multiple criteria aggregation. Still further research is needed for a better understanding of the interplay of the vertices in the different cubes.

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