Chapter 2
Compensators of Random Times

Given a random time τ, we study the compensator of the default (indicator) process $A := 1\!\!1_{[\![\tau,\infty[\![}$ and the associated compensated martingale M. As a financial application, we establish properties of the intensity rate for a single default, and give pricing formulae for defaultable derivatives.

2.1 Compensator of a Default Process in Its Natural Filtration

Let τ be a random time on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We work with the default indicator process $A := 1\!\!1_{[\![\tau,\infty[\![}$ and its natural filtration $\mathbb{A} := (\mathscr{A}_t, t \geq 0)$, both introduced in Definition 1.44. Equivalently, the filtration $\mathbb{A}$ is the smallest filtration satisfying the usual conditions, which renders τ a stopping time. It is important (and obvious) to note that $\int_0^t g(u)dA_u := \int_{(0,t]} g(u)dA_u = (A_t - A_0)g(\tau)$ where g is a Borel function. We denote by F the right-continuous cumulative distribution function of τ, defined as $F(t) = \mathbb{P}(\tau \leq t)$ for any $t \in \mathbb{R}$ and $F(\infty) = \lim_{t\to\infty} F(t) = 1 - \mathbb{P}(\tau = \infty)$. Note that, unless τ is $\mathbb{P}$-a.s. finite, $F(\infty)$ is not equal to one. As $\Delta F(t) = F(t) - F(t-) = \mathbb{P}(\tau = t)$, τ is an $\mathbb{A}$-totally inaccessible stopping time if and only if F is continuous (see [77, Chap. IV]). Remark as well that the càg process A_- is $\mathbb{A}$-predictable.

Lemma 2.1 *The filtration $\mathbb{A}$ satisfies $\mathscr{A}_t = \sigma(\{\tau \leq s\} : s \leq t) \vee \mathscr{N}^{\mathbb{P}}$, where $\mathscr{N}^{\mathbb{P}}$ is the set of $\mathbb{P}$-null sets.*

Proof Let us introduce an auxiliary filtration $\mathbb{A}^0 := (\mathscr{A}_t^0 : t \geq 0)$ defined by

$$\mathscr{A}_t^0 := \sigma(\{\tau \leq s\} : s \leq t) \vee \mathscr{N}^{\mathbb{P}}.$$

It is then enough to show that $\mathbb{A}^0$ is right-continuous. Since $\mathscr{A}_\infty^0 = \sigma(\tau)$ any element of $\mathscr{A}_t^0$ can be written as $\{\tau \in B\}$ for $B \in \mathscr{B}(\mathbb{R}^+)$ such that $B = B \cap [0, t]$. Since for

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any $\{\tau \in B\} \in \mathscr{A}^0_{t+}$, the Borel set B satisfies

$$B = \bigcap_{r>t} B \cap [0, r] = B \cap [0, t]$$

we deduce that $\mathscr{A}^0_{t+} = \mathscr{A}^0_t$. □

We present an important lemma, which explains the structure of $\mathscr{A}_t$-measurable random variables.

Lemma 2.2 *A random variable Y is $\mathscr{A}_t$-measurable if and only if it is of the form $Y = h 1\!\!1_{\{t<\tau\}} + g(\tau) 1\!\!1_{\{\tau \le t\}}$ a.s. where h is a constant and g is a Borel function.*

In particular, if $A \in \mathscr{A}_t$, then $A \cap \{t < \tau\} = \widehat{A} \cap \{t < \tau\}$ for some $\widehat{A} \in \mathscr{A}_0$ (note that $\mathbb{P}(\widehat{A}) = 0$ or $\mathbb{P}(\widehat{A}) = 1$).

Proof Let $\mathscr{V}$ be the vector space of random variables defined as follows

$$\mathscr{V} := \{Y = h 1\!\!1_{\{t<\tau\}} + g(\tau) 1\!\!1_{\{\tau \le t\}} : h \in \mathbb{R}, \quad g : \overline{\mathbb{R}}^+ \to \mathbb{R} \text{ is a Borel function}\}$$

Any random variable $Y \in \mathscr{V}$ is $\mathscr{A}_t$-measurable since, for any $B \in \mathscr{B}(\mathbb{R})$, we have

$$Y^{-1}(B) = \{\omega : h \in B,\ t < \tau(\omega)\} \cup \{\omega : g(\tau(\omega) \wedge t) \in B, \tau(\omega) \le t\} \in \mathscr{A}_t$$

as $\tau \wedge t$ is an $\mathscr{A}_t$-measurable random variable. On the other hand, since $\mathscr{V}$ satisfies

1. $1 \in \mathscr{V}$,
2. if $Y^n := h^n 1\!\!1_{\{t<\tau\}} + g^n(\tau) 1\!\!1_{\{\tau \le t\}}$ where, for each n, h^n is a constant and g^n is a Borel function, $Y^n \nearrow Y$ and Y is finite, then $Y = h 1\!\!1_{\{t<\tau\}} + g(\tau) 1\!\!1_{\{\tau \le t\}}$, where $h := \limsup h^n$ and $g := \limsup g^n$ is a Borel function,
3. for each $s \le t$ and $N \in \mathscr{N}^{\mathbb{P}}$, $1\!\!1_N 1\!\!1_{\{\tau \le s\}} \in \mathscr{V}$,

by the monotone class theorem [112, Theorems 1.2, 1.4], we conclude that $\mathscr{V}$ contains all $\sigma(\{\tau \le s\} : s \le t) \vee \mathscr{N}^{\mathbb{P}}$-measurable random variables.

Since $\mathscr{A}_t = \sigma(\{\tau \le s\} : s \le t) \vee \mathscr{N}^{\mathbb{P}}$ the proof is completed. □

2.1.1 A Key Lemma

The following lemma provides an essential tool to compute the conditional expectation of an integrable random variable given $\mathscr{A}_t$.

Lemma 2.3 *Let X be an integrable, $\mathscr{F}$-measurable r.v. Then,*[1]

$$\mathbb{E}[X|\mathscr{A}_t] = 1\!\!1_{\{\tau>t\}} \frac{\mathbb{E}[X 1\!\!1_{\{\tau>t\}}]}{\mathbb{P}(\tau > t)} + \mathbb{E}[X|\sigma(\tau)] 1\!\!1_{\{\tau \le t\}}\,. \tag{2.1}$$

[1]Here, and in the rest of the book, we write $\frac{1}{b} 1\!\!1_{\{b>0\}}$ for the quantity equal to $\frac{1}{b}$ if $b > 0$ and equal to 0 if $b = 0$.

Proof Let $t^* := \inf\{s : F(s) = 1\}$. Then τ is bounded a.s. by t^* and $\mathbb{P}(\tau > t)$ is positive for $t^* > t$. Let $A \in \mathscr{A}_t$, then, by Lemma 2.2 there exists $\widehat{A} \in \mathscr{A}_0$ such that $\{\tau > t\} \cap A = \{\tau > t\} \cap \widehat{A}$, and we have

$$\mathbb{E}\left[X\mathbb{P}(\tau > t)\mathbb{1}_{\{\tau>t\}}\mathbb{1}_A\right] = \mathbb{E}\left[\mathbb{P}(\tau > t)\mathbb{1}_{\widehat{A}}\mathbb{E}\left[X\mathbb{1}_{\{\tau>t\}}\right]\right] = \mathbb{E}\left[\mathbb{E}[X\mathbb{1}_{\{\tau>t\}}]\mathbb{1}_{\{\tau>t\}}\mathbb{1}_A\right]$$

which leads to

$$\mathbb{E}\left[X\mathbb{1}_{\{\tau>t\}}|\mathscr{A}_t\right] = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}[X\mathbb{1}_{\{\tau>t\}}]}{\mathbb{P}(\tau > t)}.$$

Lemma 2.2 implies also that

$$\mathbb{E}\left[X\mathbb{1}_{\{\tau\le t\}}|\mathscr{A}_t\right] = \mathbb{E}\left[\mathbb{E}[X|\sigma(\tau)]\mathbb{1}_{\{\tau\le t\}}|\mathscr{A}_t\right] = \mathbb{E}[X|\sigma(\tau)]\mathbb{1}_{\{\tau\le t\}}.$$

Thus, (2.1) is proven. □

2.1.2 *Martingales and Predictable Representation Property*

The Doob–Meyer decomposition of the increasing process A is derived in the next proposition. Particular attention is given to its martingale part M, which is a fundamental martingale enjoying the predictable representation property (PRP) in the filtration $\mathbb{A}$, as proven in Proposition 2.7.

Proposition 2.4 *The process M given by*

$$M_t := A_t - \int_0^{\tau\wedge t} \frac{dF(s)}{1 - F(s-)} \tag{2.2}$$

is an $\mathbb{A}$-martingale.

If F is absolutely continuous w.r.t. the Lebesgue measure with density f, the process

$$M_t := A_t - \int_0^{\tau\wedge t} \lambda(s)ds = A_t - \int_0^t \lambda(s)(1 - A_s)ds$$

is an $\mathbb{A}$-martingale, where $\lambda(s) = \dfrac{f(s)}{1 - F(s)}\mathbb{1}_{\{F(s)<1\}}$ is a deterministic non-negative function, called **the intensity rate of τ**.

Proof Note that

$$\int_0^{\tau\wedge t} \frac{dF(s)}{1 - F(s-)} = \int_0^t (1 - A_{s-})\frac{dF(s)}{1 - F(s-)}.$$

Let $t \le u$. Then, on the one hand, we obtain

$$\mathbb{E}[A_u - A_t | \mathscr{A}_t] = \mathbb{E}\left[1\!\!1_{\{t<\tau\le u\}} | \mathscr{A}_t\right] = 1\!\!1_{\{t<\tau\}} \frac{F(u) - F(t)}{1 - F(t)}, \tag{2.3}$$

where the second equality follows from Lemma 2.3 applied to $1\!\!1_{\{u \ge \tau\}}$. On the other hand, applying once again Lemma 2.3, we obtain

$$\mathbb{E}\left[\int_{(t\wedge\tau, u\wedge\tau]} \frac{dF(s)}{1 - F(s-)} \,\Big|\, \mathscr{A}_t\right] = 1\!\!1_{\{t<\tau\}} \frac{1}{\mathbb{P}(t < \tau)} \mathbb{E}\left[\int_{(t,u]} 1\!\!1_{\{s\le\tau\}} \frac{dF(s)}{1 - F(s-)}\right]$$
$$= 1\!\!1_{\{t<\tau\}} \frac{1}{\mathbb{P}(t < \tau)} \int_{(t,u]} \mathbb{P}(s \le \tau) \frac{dF(s)}{1 - F(s-)} = 1\!\!1_{\{t<\tau\}} \frac{F(u) - F(t)}{1 - F(t)}.$$

In view of (2.3), this proves the result. □

Example 2.5 Let N be an inhomogeneous Poisson process with deterministic intensity function λ and let τ be the first jump time of N. Then the default indicator process of τ is $A_t = N_{t\wedge\tau}$. Since $N_t - \int_0^t \lambda(s)ds$ is a martingale, it is also a martingale when stopped at time τ, i.e., $A_t - \int_0^{t\wedge\tau} \lambda(s)ds$, is a martingale. Therefore λ is the intensity rate of τ.

Corollary 2.6 *For any bounded Borel function $h : [0,\infty] \to \mathbb{R}$, the process M^h given by*

$$M_t^h := 1\!\!1_{\{\tau\le t\}} h(\tau) - \int_0^{t\wedge\tau} h(u) \frac{dF(u)}{1 - F(u-)}$$

satisfies $dM_t^h = h(t)dM_t$ where M is defined in (2.2) and is an $\mathbb{A}$-martingale.

Proof This is indeed true since

$$M_t^h = \int_0^t h(u)dA_u - \int_0^t (1 - A_{u-})h(u) \frac{dF(u)}{1 - F(u-)} = \int_0^t h(u)dM_u .$$

The martingale property follows from the fact that h is $\mathbb{A}$-predictable and bounded. □

We will generalize the above results in Sect. 2.2.3.

Proposition 2.7 *Any $\mathbb{A}$-local martingale can be written as a stochastic integral w.r.t. the martingale M defined in (2.2); in other terms, M has the PRP in the filtration $\mathbb{A}$.*

Proof Using the fact that $\mathscr{A}_\infty = \sigma(\tau)$, any bounded $\mathbb{A}$-martingale Y can be written as $Y_t = \mathbb{E}[h(\tau)|\mathscr{A}_t]$, where h is a bounded Borel function $h : [0,\infty] \to \mathbb{R}$. Then, by Lemma 2.3, we have

$$Y_t = \mathbb{E}[h(\tau)|\mathscr{A}_t] = 1\!\!1_{\{\tau\le t\}} h(\tau) + \frac{1\!\!1_{\{t<\tau\}}}{1 - F(t)} \left(\int_t^\infty h(u)\, dF(u) + h(\infty)(1 - F(\infty))\right).$$

Recall that $\tau \le t^*$ a.s. for $t^* := \inf\{t : F(t) = 1\}$. Therefore $Y_t = h(\tau)$ for $t \ge t^*$ and the previous formula implies that $Y_{t^*} = \lim_{t \nearrow t^*} Y_t$ since $\lim_{t \nearrow t^*} Y_t = h(\tau)$. Then, it is enough to show the integral representation of Y up to $t < t^*$. Note that, for $t < t^*$, one has $d(1 - F(t))^{-1} = ((1 - F(t))(1 - F(t-)))^{-1} dF(t)$. By integration by parts (see Proposition 1.16 (b)), for $t < t^*$, we firstly deduce that

$$d\frac{1 - A_t}{1 - F(t)} = \frac{1 - A_{t-}}{(1 - F(t))(1 - F(t-))} dF(t) - \frac{dA_t}{1 - F(t)}$$

and secondly that

$$d\left(\frac{1 - A_t}{1 - F(t)} \int_t^\infty h(u)\, dF(u)\right) = -\frac{1 - A_{t-}}{1 - F(t-)} h(t) dF(t)$$
$$+ \left(\int_t^\infty h(u)\, dF(u)\right) \frac{1 - A_{t-}}{(1 - F(t))(1 - F(t-))} dF(t) - \left(\int_t^\infty h(u)\, dF(u)\right) \frac{dA_t}{1 - F(t)}.$$

Finally, denoting

$$K(t) := \frac{1}{1 - F(t)} \mathbb{1}_{\{F(t) < 1\}} \left(\int_t^\infty h(u)\, dF(u) + h(\infty)(1 - F(\infty))\right),$$

we conclude that $dY_t = (h(t) - K(t)) dM_t$ for each $t \ge 0$. It is then standard, by localizing argument that the PRP is valid for any local martingale. $\square$

2.2 Compensator of the Default Process in a General Setting

In this section, a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} := (\mathscr{F}_t, t \ge 0)$ satisfying the usual conditions of $\mathbb{P}$-completeness, right-continuity and such that $\mathscr{F}_\infty \subset \mathscr{F}$, is given. We define two filtrations $\mathbb{G}^0 := (\mathscr{G}_t^0)_{t \ge 0}$ and $\mathbb{G} := (\mathscr{G}_t)_{t \ge 0}$ by

$$\mathscr{G}_t^0 := \mathscr{F}_t \vee \mathscr{A}_t \quad \text{and} \quad \mathscr{G}_t := \cap_{\varepsilon > 0} \mathscr{G}_{t+\varepsilon}^0, \tag{2.4}$$

or, equivalently $\mathbb{G}^0 := \mathbb{F} \vee \mathbb{A}$ and $\mathbb{G} := \mathbb{F} \triangledown \mathbb{A}$ (see notation on p. 1). In other terms, $\mathbb{G}$ is the progressively enlarged filtration which is the smallest right-continuous filtration containing $\mathbb{F}$ and making τ a stopping time.

From Proposition 1.46, the Doob–Meyer decomposition of the $\mathbb{F}$-Azéma supermartingale $Z := {}^{o,\mathbb{F}}(1 - A)$, or equivalently $Z_t = \mathbb{P}(\tau > t | \mathscr{F}_t)$, is $Z = n - A^p$ where n is an $\mathbb{F}$-martingale and A^p is the $\mathbb{F}$-dual predictable projection of the increasing process $A := \mathbb{1}_{[\![\tau, \infty[\![}$.

2.2.1 A Key Lemma

Lemma 2.9 provides an essential tool to compute the conditional expectation of an integrable random variable given $\mathscr{G}_t$ in terms of the conditional expectations given $\mathscr{F}_t$. Before we proceed to that key lemma we establish the following proposition which gives the equivalent form of $\mathscr{G}_t^0$-measurable r.v.'s.

Proposition 2.8 *For a fixed t, a random variable Y is $\mathscr{G}_t^0$-measurable if and only if it is of the form*

$$Y = y 1\!\!1_{\{t<\tau\}} + \widehat{y}(\tau) 1\!\!1_{\{\tau\le t\}} \tag{2.5}$$

for an $\mathscr{F}_t$-measurable random variable y and an $\mathscr{F}_t \otimes \mathscr{B}(\overline{\mathbb{R}}^+)$-measurable mapping $(\omega, u) \to \widehat{y}(\omega, u)$.

Proof Let $\mathscr{V}$ be the vector space of random variables defined as follows

$$\mathscr{V} := \{Y : Y = y 1\!\!1_{\{t<\tau\}} + \widehat{y}(\tau) 1\!\!1_{\{\tau\le t\}} \text{ for } \mathscr{F}_t \text{ -measurable r.v. } y \\ \text{and } \mathscr{F}_t \otimes \mathscr{B}(\overline{\mathbb{R}}^+)\text{-measurable mapping } \widehat{y}\}.$$

First note that any r.v. $Y \in \mathscr{V}$ is $\mathscr{G}_t^0$-measurable since, for any $B \in \mathscr{B}(\mathbb{R})$, we have

$$Y^{-1}(B) = \{\omega : y \in B,\ t < \tau(\omega)\} \cup \{\omega : \widehat{y}(\tau(\omega) \wedge t) \in B, \tau(\omega) \le t\} \in \mathscr{G}_t^0$$

as $\tau \wedge t$ is a $\mathscr{G}_t^0$-measurable random variable. On the other hand, $\mathscr{V}$ satisfies that (i) $1 \in \mathscr{V}$; (ii) if $Y^n := y^n 1\!\!1_{\{t<\tau\}} + \widehat{y}^n(\tau) 1\!\!1_{\{\tau\le t\}}$ where y^n and $\widehat{y}^n$ are respectively $\mathscr{F}_t$-measurable r.v. and $\mathscr{F}_t \otimes \mathscr{B}(\overline{\mathbb{R}}^+)$-measurable mapping, $Y^n \nearrow Y$ and Y is finite, then $Y = y 1\!\!1_{\{t<\tau\}} + \widehat{y}(\tau) 1\!\!1_{\{\tau\le t\}}$, where $y := \limsup y^n$ and $\widehat{y} := \limsup \widehat{y}^n$ are measurable accordingly; and (iii) for each $s \le t$ and $F \in \mathscr{F}_t$, $1\!\!1_F 1\!\!1_{\{\tau\le s\}} \in \mathscr{V}$. By the monotone class theorem, we conclude that $\mathscr{V}$ contains all $\sigma(\{\tau \le s\} : s \le t) \vee \mathscr{F}_t$-measurable random variables which completes the proof. □

Lemma 2.9 *Let X be an $\mathscr{F}$-measurable integrable r.v. Then, for any $t \ge 0$,*

$$\mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{G}_t\right] = 1\!\!1_{\{\tau>t\}} \frac{1}{Z_t} \mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{F}_t\right].$$

Proof The right-hand side of the above identity is well-defined since for each $t \ge 0$ one has $0 = \mathbb{E}[1\!\!1_{\{Z_t=0\}} Z_t] = \mathbb{E}[1\!\!1_{\{Z_t=0\}} 1\!\!1_{\{\tau>t\}}]$ which implies $\{\tau > t\} \subset \{Z_t > 0\}$. By Proposition 2.8, for any $G \in \mathscr{G}_t^0$ there exists $F \in \mathscr{F}_t$ s.t. $G \cap \{\tau > t\} = F \cap \{\tau > t\}$. Hence:

$$\begin{aligned}
\mathbb{E}\left[X 1\!\!1_{\{\tau>t\}\cap G} Z_t\right] &= \mathbb{E}\left[X 1\!\!1_{\{\tau>t\}\cap F} Z_t\right] = \mathbb{E}\left[\mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{F}_t\right] 1\!\!1_F Z_t\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{F}_t\right] 1\!\!1_F \mathbb{E}\left[1\!\!1_{\{\tau>t\}} | \mathscr{F}_t\right]\right] = \mathbb{E}\left[\mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{F}_t\right] 1\!\!1_{F\cap\{\tau>t\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{F}_t\right] 1\!\!1_{G\cap\{\tau>t\}}\right]
\end{aligned}$$

which shows that

$$\mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{G}_t^0\right] = 1\!\!1_{\{\tau>t\}} \frac{1}{Z_t} \mathbb{E}[X 1\!\!1_{\{\tau>t\}} | \mathscr{F}_t] \,.$$

By Theorem 1.10 the right-hand side in the above equality has a right-continuous modification (as a process) and, by Theorem 1.11, the following equalities hold:

$$\mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{G}_t\right] = \mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{G}_{t+}^0\right] = \lim_{u \downarrow t} \mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{G}_u^0\right] = \mathbb{E}\left[X 1\!\!1_{\{\tau>t\}} | \mathscr{G}_t^0\right].$$

That completes the proof. □

Combining Lemma 2.9 and the definition of dual projections provides us with the following result.

Corollary 2.10 *(a) Let h be a bounded $\mathbb{F}$-optional process. Then*

$$\mathbb{E}[h_\tau | \mathscr{G}_t] = h_\tau 1\!\!1_{\{\tau \le t\}} + 1\!\!1_{\{\tau>t\}} \frac{1}{Z_t} \mathbb{E}\left[\int_t^\infty h_u dA_u^o + h_\infty Z_\infty \middle| \mathscr{F}_t\right]. \tag{2.6}$$

(b) Let h be a bounded $\mathbb{F}$-predictable process. Then

$$\mathbb{E}[h_\tau | \mathscr{G}_t] = h_\tau 1\!\!1_{\{\tau \le t\}} + 1\!\!1_{\{\tau>t\}} \frac{1}{Z_t} \mathbb{E}\left[\int_t^\infty h_u dA_u^p + h_\infty Z_\infty \middle| \mathscr{F}_t\right]. \tag{2.7}$$

Proof By Lemma 2.9 one has

$$\mathbb{E}[h_\tau | \mathscr{G}_t] = h_\tau 1\!\!1_{\{\tau \le t\}} + 1\!\!1_{\{\tau>t\}} \frac{1}{Z_t} \mathbb{E}[h_\tau 1\!\!1_{\{t<\tau\le\infty\}} | \mathscr{F}_t] \,.$$

We note that $h_\tau 1\!\!1_{\{t<\tau\}} = \int_t^\infty h_u dA_u + h_\infty 1\!\!1_{\{\tau=\infty\}}$. Then the assertion (a) follows by Definition 1.39 (a) of the $\mathbb{F}$-dual optional projection of τ.

The proof of assertion (b) follows by an analogous argument, using the $\mathbb{F}$-dual predictable projection of τ (see Definition 1.39 (b)). □

2.2.2 $\mathbb{G}$-*Measurability Versus* $\mathbb{F}$-*Measurability*

Proposition 2.11 *(a) For a $\mathbb{G}$-optional process Y, the $\mathbb{F}$-optional process y defined as $y := {}^{o,\mathbb{F}}(Y 1\!\!1_{[\![0,\tau[\![}) 1\!\!1_{\{Z>0\}} \frac{1}{Z}$ satisfies*

$$1\!\!1_{[\![0,\tau[\![} Y = 1\!\!1_{[\![0,\tau[\![} \, y \,. \tag{2.8}$$

The process y is called the $(\mathbb{F}, \tau)$-optional reduction of Y.

(b) A process Y is $\mathbb{G}$-predictable if and only if it is of the form

$$Y = \mathbb{1}_{[\![0,\tau]\!]} y + \mathbb{1}_{]\!]\tau,\infty[\![} \; \widehat{y}(\tau) \tag{2.9}$$

where y is $\mathbb{F}$-predictable and $(\omega, t, u) \mapsto \widehat{y}_t(\omega, u)$ is a $\mathscr{P}(\mathbb{F}) \otimes \mathscr{B}(\overline{\mathbb{R}}^+)$-measurable function. Moreover one can choose $y := {}^{p,\mathbb{F}}(Y \mathbb{1}_{[\![0,\tau]\!]}) \mathbb{1}_{\{Z_- > 0\}} \frac{1}{Z_-}$ which is called the $(\mathbb{F}, \tau)$-predictable reduction of Y.

Proof (a) Stochastic processes of the form $f(t)X(\omega)$ where f is a Borel function and X is an $\mathscr{F}$-measurable r.v. generate all stochastic processes by the monotone class theorem (see Definition 1.1 where stochastic processes are required to be measurable). Hence the processes of the form $f(t)X_t(\omega)$ where f is a Borel function and X is a càdlàg $\mathbb{G}$-martingale generate all $\mathbb{G}$-optional processes. It is therefore enough to establish (2.8) for any càdlàg $\mathbb{G}$-martingale X. Lemma 2.9 implies that for any $t \geq 0$ and integrable r.v. X_∞,

$$X_t \mathbb{1}_{\{t<\tau\}} = \mathbb{E}[X_\infty | \mathscr{G}_t] \mathbb{1}_{\{t<\tau\}} = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}\left[X_\infty \mathbb{1}_{\{\tau>t\}} | \mathscr{F}_t\right]}{Z_t}.$$

Note that, by Theorem 1.10, the process $\mathbb{E}\left[X_\infty \mathbb{1}_{\{\tau>t\}} | \mathscr{F}_t\right] Z_t^{-1} \mathbb{1}_{\{Z_t>0\}}$ has a càdlàg modification x. Hence, since x is $\mathbb{F}$-adapted, x is also an $\mathbb{F}$-optional process satisfying $\mathbb{1}_{[\![0,\tau[\![} X = \mathbb{1}_{[\![0,\tau[\![} x$. Therefore, for any $\mathbb{G}$-optional process Y there exists an $\mathbb{F}$-optional process $\widetilde{y}$ such that $\mathbb{1}_{[\![0,\tau[\![} Y = \mathbb{1}_{[\![0,\tau[\![} \widetilde{y}$ and taking $\mathbb{F}$-optional projections:

$${}^{o}(\mathbb{1}_{[\![0,\tau[\![} Y) = {}^{o}(\mathbb{1}_{[\![0,\tau[\![} \widetilde{y}) = {}^{o}(\mathbb{1}_{[\![0,\tau[\![}) \, \widetilde{y} = Z \widetilde{y}.$$

Hence $\widetilde{y}$ is only uniquely determined on $\{Z > 0\}$ and satisfies

$$\mathbb{1}_{\{Z>0\}} \widetilde{y} = \frac{{}^{o}(\mathbb{1}_{[\![0,\tau[\![} Y)}{Z} \mathbb{1}_{\{Z>0\}}.$$

The $(\mathbb{F}, \tau)$-reduction y consists of choosing the process satisfying $y \mathbb{1}_{\{Z=0\}} = 0$.
(b) A process given by the right-hand side of (2.9) where $\widehat{y}_t(\omega, u) = \widetilde{y}_t(\omega) f(u)$, with a Borel function $f : \overline{\mathbb{R}}^+ \to \mathbb{R}$ and an $\mathbb{F}$-predictable process $\widetilde{y}$, is clearly $\mathbb{G}$-predictable. The statement for a general $\widehat{y}$ follows by the usual argument based on the monotone class theorem.

To show that any $\mathbb{G}$-predictable process is of the given form, also by the monotone class theorem, it is enough to consider the elements from the generator of $\mathbb{G}$-predictable processes. Since, by Theorem 3.21 in [112], $\mathbb{G}$-predictable processes are generated by the processes of the form $Y_t(\omega) = f(s \wedge \tau(\omega)) F_s(\omega) \mathbb{1}_{\{t \geq s\}}$, where s is a fixed real number, f is a Borel function and F_s is an $\mathscr{F}_s$-measurable r.v., we conclude that (2.9) holds.

The second part of the statement about the form of $(\mathbb{F}, \tau)$-predictable reduction of Y follows by an argument analogous to the one used in the proof of (b). □

Corollary 2.12 *For any $\mathbb{G}$-stopping time S there exists an $\mathbb{F}$-stopping time T such that $S \wedge \tau = T \wedge \tau$. For any $\mathbb{G}$-predictable stopping time S there exists an $\mathbb{F}$-predictable stopping time T such that $S \wedge \tau = T \wedge \tau$.*

Proof In the first case it is enough to apply Proposition 2.11 (b) to the process $1\!\!1_{[\![0,S]\!]}$, and in the second case to $1\!\!1_{[\![0,S[\![}$. □

Remark 2.13 A characterization result analogous to (2.9) for optional processes does *not* hold in general. We refer to Barlow's counterexample presented in Example 5.13 and Corollary 5.12 which concerns the case of an honest time.

Lemma 2.14 *The process Z is positive on the stochastic interval $[\![0, \tau[\![$ and the processes Z_- and $\widetilde{Z}$ are positive on the stochastic interval $[\![0, \tau]\!]$.*

Proof We shall use the optional section theorem to prove that the $\mathbb{G}$-optional set $\{Z = 0\} \cap [\![0, \tau[\![$ is evanescent. Corollary 2.12 implies that for any $\mathbb{G}$-stopping time S there exists an $\mathbb{F}$-stopping time T such that $S \wedge \tau = T \wedge \tau$. For any $\mathbb{F}$-stopping time T, it holds $0 = \mathbb{E}[1\!\!1_{\{Z_T=0\}} Z_T] = \mathbb{E}[1\!\!1_{\{Z_T=0\}} 1\!\!1_{\{\tau>T\}}]$. Therefore the supermartingale Z is positive on the interval $[\![0, \tau[\![$.

Using a similar argument based on the predictable section theorem and the fact that $0 = \mathbb{E}[1\!\!1_{\{Z_{T-}=0\}} Z_{T-}] = \mathbb{E}[1\!\!1_{\{Z_{T-}=0\}} 1\!\!1_{\{\tau \geq T\}}]$ for any $\mathbb{F}$-predictable stopping time T, we conclude that the $\mathbb{G}$-predictable set $\{Z_- = 0\} \cap [\![0, \tau]\!]$ is evanescent. Therefore the left-limit Z_- is positive on $[\![0, \tau]\!]$. The proof for $\widetilde{Z}$ is analogous. □

2.2.3 Martingales

The generalisation of Proposition 2.4 to the case of a non-trivial reference filtration $\mathbb{F}$ is given below.

Proposition 2.15 *The process M given by*

$$M_t := A_t - \int_0^{t\wedge\tau} \frac{1}{Z_{u-}} dA_u^p \tag{2.10}$$

is a u.i. $\mathbb{G}$-martingale. In other words, the process $(1 - A_-)\frac{1}{Z_-} \cdot A^p$ is the $\mathbb{G}$-compensator of τ, and its $(\mathbb{F}, \tau)$-predictable reduction equals $\frac{1}{Z_-} 1\!\!1_{\{Z_->0\}} \cdot A^p$.

Proof Note that, by Lemma 2.14, the process $(1 - A_-)\frac{1}{Z_-}$ is well-defined. The assertion is a consequence of Proposition 1.36 (a) as soon as we show the desired form of the $\mathbb{G}$-dual predictable projection of A (or $\mathbb{G}$-compensator of A, since A is $\mathbb{G}$-adapted). It is equivalent to prove that, for any $\mathbb{G}$-predictable bounded process H, one has

$$\mathbb{E}\left[\int_{[0,\infty)} H_s dA_s\right] = \mathbb{E}\left[\int_{[0,\infty)} H_s(1 - A_{s-})\frac{dA_s^p}{Z_{s-}}\right].$$

Let h be the $(\mathbb{F}, \tau)$-predictable reduction of H (see Proposition 2.11 (b)). By the definition of the dual predictable projection,

$$\mathbb{E}\left[\int_{[0,\infty)} H_s dA_s\right] = \mathbb{E}\left[\int_{[0,\infty)} h_s 1\!\!1_{\{Z_{s-}>0\}} dA_s\right] = \mathbb{E}\left[\int_{[0,\infty)} h_s 1\!\!1_{\{Z_{s-}>0\}} dA_s^p\right].$$

Identities (1.9) and (1.18) imply that the last term equals

$$\mathbb{E}\left[\int_{[0,\infty)} \frac{h_s}{Z_{s-}} {}^p(1-A_-)_s 1\!\!1_{\{Z_{s-}>0\}} dA_s^p\right] = \mathbb{E}\left[\int_{[0,\infty)} \frac{h_s}{Z_{s-}}(1-A_{s-}) dA_s^p\right]$$

and the result follows by recalling that h is the $(\mathbb{F}, \tau)$-predictable reduction of H. □

Similarly to Corollary 2.6, the following result is in force.

Corollary 2.16 *For a bounded $\mathbb{G}$-predictable process H, the process M^h given by*

$$M_t^h := H_\tau \, A_t - \int_0^{t\wedge\tau} \frac{H_u}{Z_{u-}} dA_u^p$$

satisfies $dM_t^h = H_t \, dM_t$ where M is defined in (2.10) and is a u.i. $\mathbb{G}$-martingale.

In the next proposition the distribution of the $\mathbb{G}$-compensator of τ sampled at time τ is studied. We also refer to the connected result given in Sect. 3.3.2.

Proposition 2.17 *Assume that τ is a finite random time such that A^p is continuous. Then the random variable $\Lambda_\tau := \int_0^\tau \frac{1}{Z_{u-}} dA_u^p$ has a unit exponential law.*

Proof Denote the $\mathbb{G}$-compensator of A by Λ, i.e., $\Lambda_t = \int_0^{\tau\wedge t} \frac{1}{Z_{u-}} dA_u^p$ (see Proposition 2.15) and note that the definition of r.v. Λ_τ is consistent with this notation. Let φ be a bounded Borel function, $\Phi(t) = \int_0^t \varphi(s)ds$ and consider the martingale

$$\int_0^t \varphi(\Lambda_s) dM_s = \varphi(\Lambda_\tau) 1\!\!1_{\{\tau\le t\}} - \int_0^t \varphi(\Lambda_s) d\Lambda_s = \varphi(\Lambda_\tau) 1\!\!1_{\{\tau\le t\}} - \Phi(\Lambda_t).$$

Then, with $t\to\infty$, using the fact that $\Lambda_\infty = \Lambda_\tau$, one has $\mathbb{E}[\varphi(\Lambda_\tau)] = \mathbb{E}[\Phi(\Lambda_\tau)]$. Taking $\varphi(t) = -ae^{-at}$, one gets $\mathbb{E}[\exp(-a\Lambda_\tau)] = (1+a)^{-1}$ which characterizes the unit exponential law. □

Proposition 2.18 *Define the process Υ by $\Upsilon := (1-A)Z^{-1}$.*
(a) Let X be a non-negative $\mathbb{F}$-supermartingale. Then, $X\Upsilon$ is a $\mathbb{G}$-supermartingale.
(b) Assume that Z is positive and X is an $\mathbb{F}$-martingale. Then $X\Upsilon$ is a $\mathbb{G}$-martingale.
(c) If the process Z is positive, decreasing and continuous, then the $\mathbb{G}$-compensator of τ equals $\Lambda^\tau := (\Lambda_{t\wedge\tau}, t \ge 0)$ where $\Lambda := -\ln Z$. In other words, the $\mathbb{G}$-martingale M, introduced in (2.10), is $M = A - \Lambda^\tau$. Moreover $\Upsilon = \Upsilon_0 \mathscr{E}(-M)$.

Proof Note that Υ is well-defined since the random sets $\{Z = 0\}$ and $[\![0, \tau[\![$ are disjoint as shown in Lemma 2.14.
(a) For $s \leq t$, one has

$$\mathbb{E}\left[\Upsilon_t X_t | \mathscr{G}_s\right] = \mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} \frac{1}{Z_t} X_t \Big| \mathscr{G}_s\right] = \mathbb{1}_{\{\tau>s\}} \frac{1}{Z_s} \mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} \frac{1}{Z_t} X_t \Big| \mathscr{F}_s\right]$$
$$= \mathbb{1}_{\{\tau>s\}} \frac{1}{Z_s} \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} | \mathscr{F}_t\right] \mathbb{1}_{\{Z_t>0\}} \frac{1}{Z_t} X_t \Big| \mathscr{F}_s\right] = \Upsilon_s \mathbb{E}\left[X_t \mathbb{1}_{\{Z_t>0\}} | \mathscr{F}_s\right] \leq \Upsilon_s X_s\,,$$

where the second equality holds by Lemma 2.9.
(b) This is a consequence of similar computations as (a).
(c) Under the hypotheses on Z, Proposition 2.15 implies that the $\mathbb{G}$-compensator of A is $-(1 - A_-)\frac{1}{Z} \bullet Z = \Lambda^\tau$, hence the first assertion holds true. It remains to show that $d\Upsilon_t = -\Upsilon_{t-} dM_t$ which follows from integration by parts and $A^p = 1 - Z$. □

Remark 2.19 (a) We illustrate the difference between (a) and (b) in the above proposition with a trivial example. Let τ be an $\mathbb{F}$-stopping time, then $Z = \mathbb{1}_{[\![0,\tau[\![}$ and it vanishes at time τ. The process $\Upsilon = (1 - A)$ is indeed a $\mathbb{G}$-supermartingale but is not a $\mathbb{G}$-martingale.
(b) Assertion (b) in the above proposition seems to be related to a change of probability. It is important to note that here, one changes the filtration, not the probability measure. Moreover, setting $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathscr{G}_t} = \Upsilon_t$ does not define a probability $\mathbb{Q}$ *equivalent* to $\mathbb{P}$, since the non-negative martingale Υ vanishes. The probability $\mathbb{Q}$ would be only absolutely continuous w.r.t. $\mathbb{P}$. See Collin-Dufresne et al. [64].
(c) Assume that Z is continuous and positive. We recall that the Doob–Meyer decomposition of Z is $Z = n - A^p$. By integration by parts one obtains

$$d\Upsilon_t = -(1 - A_{t-})\frac{1}{Z_t^2}\left(dn_t - dA_t^p\right) + \frac{1 - A_{t-}}{Z_t^3} d\langle n\rangle_t^{\mathbb{F}} - \frac{1}{Z_t} dA_t$$

and it follows that

$$d\Upsilon_t + \frac{1}{Z_t} dM_t = -(1 - A_{t-})\frac{1}{Z_t^2}\left(dn_t - \frac{1}{Z_t} d\langle n\rangle_t^{\mathbb{F}}\right).$$

Due to the $\mathbb{G}$-martingale property of Υ and M, the quantity $(1 - A_-)\frac{1}{Z^2} \bullet \left(n - \frac{1}{Z} \bullet \langle n\rangle^{\mathbb{F}}\right)$ must be a $\mathbb{G}$-local martingale (stopped at τ). We shall see, in Chap. 5, that the martingale property of $n - \frac{1}{Z} \bullet \langle n\rangle^{\mathbb{F}}$ stopped at τ is a consequence of the general Jeulin's formula (5.1).

Definition 2.20 If there exists a $\mathbb{G}$-predictable (resp. $\mathbb{F}$-predictable) process $\lambda^{\mathbb{G}}$ (resp. $\lambda^{\mathbb{F}}$) such that

$$A_t - \int_0^t \lambda_s^{\mathbb{G}} ds = A_t - \int_0^{\tau\wedge t} \lambda_s^{\mathbb{F}} ds$$

is a $\mathbb{G}$-martingale, then $\lambda^{\mathbb{G}}$ (resp. $\lambda^{\mathbb{F}}$) is called the $\mathbb{G}$**-intensity rate of** τ (resp. the $\mathbb{F}$**-intensity rate of** τ).

Remark 2.21 Assume that A^p is absolutely continuous w.r.t. the Lebesgue measure, i.e., there exists a measurable process $\widetilde{a}$ such that $A^p = \int_0^\cdot \widetilde{a}_s ds$. Note that ${}^p\widetilde{a}$ exists since it is non-negative. Since $(A^p)^p = A^p$, identity (1.9) implies that $A^p = \int_0^\cdot a_s ds$ where $a = {}^p\widetilde{a}$ is $\mathbb{F}$-predictable. It was proven in Proposition 2.15 that the process

$$A_t - \int_0^{t\wedge\tau} \lambda_u du = A_t - \int_0^t (1 - A_{u-})\lambda_u du$$

where $\lambda_u = \frac{a_u}{Z_{u-}} \mathbb{1}_{\{Z_{u-}>0\}}$ is a $\mathbb{G}$-martingale. Therefore λ is the $\mathbb{F}$-intensity rate of τ and $\lambda \mathbb{1}_{[\![0,\tau]\!]}$ is the $\mathbb{G}$-intensity rate of τ.

The Ethier–Kurtz Criterion establishes that, if there exists $K < \infty$ such that for any $s < t$, $\mathbb{E}[A_t - A_s|\mathscr{G}_s] \le K(t-s)$, then A^p is absolutely continuous (see [93, 126]).

Lemma 2.22 *The $\mathbb{F}$-intensity rate process λ, if it exists, satisfies*

$$\lambda_t = \lim_{h\to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t+h|\mathscr{F}_t)}{\mathbb{P}(t < \tau|\mathscr{F}_t)} .$$

Proof The martingale property of M implies that

$$\mathbb{E}[\mathbb{1}_{\{t<\tau\le t+h\}}|\mathscr{G}_t] = \int_t^{t+h} \mathbb{E}[(1-A_s)\lambda_s|\mathscr{G}_t]ds .$$

It follows that, on $\{t < \tau\}$

$$\lambda_t = \lim_{h\to 0} \frac{1}{h}\mathbb{P}(t < \tau < t+h|\mathscr{G}_t) = \lim_{h\to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t+h|\mathscr{F}_t)}{\mathbb{P}(t < \tau|\mathscr{F}_t)} .$$

□

Proposition 2.23 *Suppose that the filtration $\widehat{\mathbb{F}}$ satisfies $\widehat{\mathbb{F}} \subset \mathbb{F}$. Denote by $\widehat{\mathbb{G}}$ the filtration $\widehat{\mathbb{G}} = \widehat{\mathbb{F}} \triangledown \mathbb{A}$, by $\widehat{Z}$ the $\widehat{\mathbb{F}}$-Azéma supermartingale of τ and by $A^{p,\widehat{\mathbb{F}}}$ the $\widehat{\mathbb{F}}$-dual predictable projection of A. Suppose that the $\mathbb{F}$-dual predictable projection of A is absolutely continuous w.r.t. the Lebesgue measure, i.e., $A^p = \int_0^\cdot a_s ds$ for an $\mathbb{F}$-predictable process a. Then, the $\widehat{\mathbb{F}}$-intensity rate of τ equals*

$${}^{p,\widehat{\mathbb{F}}}a_s \frac{1}{\widehat{Z}_{s-}} \mathbb{1}_{\{\widehat{Z}_{s-}>0\}}.$$

Proof By virtue of the Proposition 2.15, Definition 2.20 and Remark 2.21, it is enough to check that $A^{p,\widehat{\mathbb{F}}} = \int_0^\cdot {}^{p,\widehat{\mathbb{F}}}a_s ds$ which follows from (1.9) and $A^{p,\widehat{\mathbb{F}}} = (A^p)^{p,\widehat{\mathbb{F}}}$. □

2.3 Cox Model and Extensions

This section is devoted to a particular construction of random times. It is a fundamental construction of a default time in finance. In a credit risk setting, the random variable τ represents the time when a default occurs. In the literature, models for default times are often based on a threshold: the default occurs when some driving process X passes a given barrier. Based on this observation, we consider a random time in a general threshold model. Let X be a stochastic process and Θ be a barrier which shall be made precise later. Define a random time as the first passage time at the level Θ:

$$\tau := \inf\{t \,:\, X_t \geq \Theta\}\,.$$

In classical structural models, a reference filtration $\mathbb{F}$ is given, the process X is $\mathbb{F}$-adapted and represents the value of a firm and the barrier Θ is a constant. So, τ is an $\mathbb{F}$-stopping time. If τ is an $\mathbb{F}$-predictable stopping time (e.g. if $\mathbb{F}$ is a Brownian filtration), the $\mathbb{F}$-compensator of $A := 1\!\!1_{[\![\tau,\infty[\![}$ is A. The goal is then to compute the conditional law of the default $\mathbb{P}(\tau > \theta|\mathscr{F}_t)$, for $\theta > t$.

In a reduced form approach (say, if τ is not the first time where an observable process passes a constant barrier) there are two sources of information: information arriving from market prices modelled via a reference filtration $\mathbb{F}$ and the information about the default time, i.e., the knowledge whether the default occurred or not, modelled via filtration $\mathbb{A}$.

At the intuitive level, $\mathbb{F}$ is generated by prices of some assets, or by other economic factors (e.g. interest rates). The case where $\mathbb{F}$ is the trivial filtration is studied in Sect. 2.1. Though in typical examples $\mathbb{F}$ is chosen to be the Brownian filtration, most theoretical results do not rely on such a specification of the filtration $\mathbb{F}$.

2.3.1 Construction of a Cox Model with a Given Intensity

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Assume that λ is a non-negative $\mathbb{F}$-adapted process and that there exists a random variable Θ, independent of $\mathscr{F}_\infty$, with a unit exponential law. In the **Cox model**, the default time τ is defined as the first time when the increasing process $\Lambda_t := \int_0^t \lambda_s\, ds$ crosses the random level Θ, i.e.,

$$\tau := \inf\{t \geq 0 \,:\, \Lambda_t \geq \Theta\}.$$

In particular, using the increasing property of Λ, one gets $\{\tau > s\} = \{\Lambda_s < \Theta\}$. We assume that $\Lambda_t < \infty$, for all t and $\Lambda_\infty = \infty$, in particular τ is a finite random time.

Remark 2.24 (a) If a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is given, in order to construct a r.v. Θ independent of $\mathscr{F}$, one may need to enlarge the probability space as follows. Let $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{\mathbb{P}})$ be an auxiliary probability space with a r.v. Θ with exponential law. We

introduce the product probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}}) = (\Omega \times \widehat{\Omega}, \mathscr{F} \otimes \widehat{\mathscr{F}}, \mathbb{P} \otimes \widehat{\mathbb{P}})$.
(b) Some authors define the default time as

$$\tau = \inf\{t \geq 0 \ : \ X_t \geq \Theta\} = \inf \left\{ t \geq 0 \ : \ \sup_{s \leq t} X_s \geq \Theta \right\}$$

where X is a given $\mathbb{F}$-semimartingale.
(c) One can define the time of default as $\tau = \inf\{t \ : \ \Lambda_t \geq \Sigma\}$ where Σ is a non-negative r.v. independent of $\mathscr{F}_\infty$. Assume that the cumulative distribution function of Σ, denoted by Φ, is continuous and increasing. This model then reduces to the previous one: the r.v. $\Phi(\Sigma)$ has a uniform distribution and

$$\tau = \inf\{t \ : \ \Phi(\Lambda_t) \geq \Phi(\Sigma)\} = \inf \left\{ t \ : \ \Psi^{-1} \circ \Phi(\Lambda_t) \geq \Theta \right\}$$

where Ψ is the cumulative distribution function of the unit exponential law.

2.3.2 Conditional Expectations and Immersion

Lemma 2.25 *The conditional distribution of τ given $\mathscr{F}_t$ for $t \in [0, \infty]$ equals*

$$\mathbb{P}(\tau > \theta | \mathscr{F}_t) = \mathbb{E}\left[\exp\left(-\Lambda_\theta\right) | \mathscr{F}_t\right] \quad \textit{for} \quad \theta \in \mathbb{R}^+.$$

Equivalently, the conditional density function of τ given $\mathscr{F}_t$ for $t \in [0, \infty]$ equals

$$\mathbb{P}(\tau \in d\theta | \mathscr{F}_t) = \mathbb{E}\left[\lambda_\theta \exp(-\Lambda_\theta) | \mathscr{F}_t\right] d\theta \quad \textit{for} \quad \theta \in \mathbb{R}^+.$$

In particular, the Azéma supermartingale $Z = \exp(-\Lambda)$ is decreasing, positive and continuous and satisfies $Z_t = \mathbb{P}(\tau > t | \mathscr{F}_\infty)$ and the dual predictable projection of τ equals $A^p = 1 - \exp(-\Lambda)$.

Proof The proof follows from the equality $\{\tau > \theta\} = \{\Lambda_\theta < \Theta\}$ for $\theta \in \mathbb{R}^+$ and the independence of Θ and $\mathscr{F}_\infty$. The $\mathscr{F}_t$-measurability of Λ_θ for $\theta \leq t$, implies

$$\mathbb{P}(\tau > \theta | \mathscr{F}_t) = \mathbb{P}\left(\Lambda_\theta < \Theta \mid \mathscr{F}_t\right) = \exp\left(-\Lambda_\theta\right).$$

The $\mathscr{F}_\theta$-measurability of Λ_θ for $\theta > t$, implies

$$\mathbb{P}(\tau > \theta | \mathscr{F}_t) = \mathbb{E}\left[\mathbb{P}(\tau > \theta | \mathscr{F}_\theta) | \mathscr{F}_t\right] = \mathbb{E}\left[\exp\left(-\Lambda_\theta\right) | \mathscr{F}_t\right]$$

and the result follows. In particular, we conclude that

$$Z_t = \mathbb{P}(\tau > t | \mathscr{F}_t) = \mathbb{P}(\tau > t | \mathscr{F}_\infty). \tag{2.11}$$

Hence the Azéma supermartingale is decreasing, positive and continuous. □

Corollary 2.26 *The process M defined as $M_t := A_t - \Lambda_{t\wedge\tau}$ is a $\mathbb{G}$-martingale.*

Proof This is an immediate application of Proposition 2.15 and the above lemma since, in a Cox model,

$$\int_0^t (1 - A_{s-}) \frac{1}{Z_{s-}} dA_s^p = \int_0^{t\wedge\tau} \exp(\Lambda_s) d(1 - \exp(-\Lambda_s)) = \Lambda_{t\wedge\tau}.$$

□

In a Cox model, Lemma 2.9 and Corollary 2.10 can be written as follow for r.v.'s of a particular form.

Lemma 2.27 *(a) Let X be an integrable $\mathscr{F}_T$-measurable r.v. Then, for $t \le T$*

$$\mathbb{E}\left[X \mathbb{1}_{\{T<\tau\}} | \mathscr{G}_t\right] = \mathbb{1}_{\{\tau>t\}} \mathbb{E}\left[X e^{-(\Lambda_T - \Lambda_t)} \Big| \mathscr{F}_t\right]. \tag{2.12}$$

(b) Let h be a bounded $\mathbb{F}$-predictable process. Then, for $t \le T$

$$\mathbb{E}\left[h_\tau \mathbb{1}_{\{\tau<T\}} | \mathscr{G}_t\right] = h_\tau \mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{\tau>t\}} e^{\Lambda_t} \mathbb{E}\left[\int_t^T h_u \lambda_u e^{-\Lambda_u} du \Big| \mathscr{F}_t\right]. \tag{2.13}$$

The immersion between $\mathbb{F}$ and $\mathbb{G}$ (see Definition 1.20) holds in a Cox model.

Lemma 2.28 *Let τ be a default time in a Cox model. Then $\mathbb{F}$ is immersed in $\mathbb{G}$, i.e., each $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.*

Proof Since Θ is independent of $\mathscr{F}_\infty$, by Proposition 1.21, we obtain that any $\mathbb{F}$-martingale Y is an $\mathbb{F}^{\sigma(\Theta)} := \mathbb{F} \vee \sigma(\Theta)$-martingale. Since $\mathbb{F} \subset \mathbb{G} \subset \mathbb{F}^{\sigma(\Theta)}$, it follows from Proposition 1.24 that Y is a $\mathbb{G}$-martingale. □

Remark 2.29 (a) We shall see in Sect. 3.2, Lemma 3.8, that the immersion in a Cox model also follows by (2.11).
(b) Immersion has important implications regarding the PRP stability under filtration enlargement. See Sect. 3.2.3 for progressive enlargement of filtration setting and also Proposition 1.23.

2.3.3 Generalisation of Cox Model

Instead of absolutely continuous process Λ we consider an increasing $\mathbb{F}$-adapted process Γ. We emphasize that not only do we not assume that Γ is absolutely continuous, we are even interested in the case where Γ fails to be continuous. Similarly to before, the default time τ is defined as the first time when an increasing process Γ is above the random level Θ, i.e., $\tau := \inf\{t \ge 0 \,:\, \Gamma_t \ge \Theta\}$. An analogous argument to previously yields $Z = e^{-\Gamma}$. However, since Γ can fail to be predictable, the $\mathbb{G}$-compensator of A is no longer equal to Γ, as we now demonstrate in an example.

Example 2.30 Let Γ be a compound Poisson process with positive jumps given by $\Gamma_t = \sum_{n=1}^{N_t} Y_n$ where N is a Poisson process with intensity λ and $(Y_n)_{n\geq 1}$ are positive random variables, i.i.d. and independent from N, and let $\mathbb{F}$ be the natural filtration of Γ. Assume that τ is constructed as above.

For $\psi := \int_0^\infty (1 - e^{-y}) F(dy)$ where F is the cumulative distribution function of Y_1, the process $(\mu_t := e^{-\Gamma_t + t\lambda\psi}, t \geq 0)$ is an $\mathbb{F}$-martingale. Then, from $Z_t = \mu_t e^{-t\lambda\psi}$, using integration by parts, one deduces that

$$dZ_t = e^{-t\lambda\psi} d\mu_t - e^{-t\lambda\psi}\mu_t \lambda\psi dt$$

which provides the Doob–Meyer decomposition of Z. It follows, from Proposition 2.15, that

$$M_t := 1\!\!1_{\{\tau \leq t\}} - (t \wedge \tau)\lambda\psi$$

is an $\mathbb{F}$-martingale, in particular the $\mathbb{F}$-intensity rate of τ is $\lambda\psi$.

2.4 Compensators in a Two Defaults Setting

In this section, in order to underline the role of the filtration in the computations of the compensator, the simplest model with two random times τ_1 and τ_2 is presented. The similar methodology, with more complex computations, can be developed for several default times.

Denote by $\mathbb{A}^i$ the natural filtration of the process $A^i := 1\!\!1_{[\![\tau_i, \infty[\![}$ for $i = 1, 2$ and by $\mathbb{G}$ the filtration $\mathbb{G} := \mathbb{A}^1 \vee \mathbb{A}^2$. Assume that the pair (τ_1, τ_2) has non-atomic law and its survival probability function $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, defined as $G(t, s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)$, is positive and continuously differentiable in both variables. Note that $G(t, 0) = \mathbb{P}(\tau_1 > t)$ is the survival probability of τ_1. We denote by $\partial_i G$ the first partial derivative w.r.t. the ith variable and $\partial_{ij} G$ the second partial derivative w.r.t. the ith and jth variables for $i, j \in \{1, 2\}$.

Proposition 2.31 *(a) The $\mathbb{A}^1$-compensator of τ_1 equals $-\ln G(t \wedge \tau, 0)$, i.e., the process $A^1_t + \ln G(t \wedge \tau, 0)$ is a $\mathbb{A}^1$-martingale. Equivalently, $\frac{-\partial_1 G(t,0)}{G(t,0)}$ is the intensity rate of τ_1.*
(b) The $\mathbb{G}$-compensator of τ_1 equals $\int_0^t (1 - A^1_s)\widehat{\lambda}_s ds$, where

$$\widehat{\lambda}_t = -1\!\!1_{\{t \leq \tau_2\}} \frac{\partial_1 G(t, t)}{G(t, t)} - 1\!\!1_{\{\tau_2 \leq t\}} \frac{\partial_{12} G(t, \tau_2)}{\partial_2 G(t, \tau_2)}$$

is $\mathbb{A}^2$-intensity rate of τ_1. Equivalently the process $A^1_t - \int_0^t (1 - A_s)\widehat{\lambda}_s ds$ is a $\mathbb{G}$-martingale.

Proof Assertion (a) is a straightforward consequence of Proposition 2.4.

(b) In order to apply Proposition 2.15 let us first compute the Doob–Meyer decomposition of the supermartingale $Z := {}^{o,\mathbb{A}^2}(1\!\!1_{[\![0,\tau_1[\![})$. By Lemma 2.9, one has that

$$Z_t = A_t^2 \mathbb{P}(\tau_1 > t|\sigma(\tau_2)) + (1 - A_t^2)\frac{\mathbb{P}(\tau_1 > t, \tau_2 > t)}{\mathbb{P}(\tau_2 > t)} = A_t^2 h(t, \tau_2) + (1 - A_t^2)\psi(t)$$

where $h(t, v) = \frac{\partial_2 G(t,v)}{\partial_2 G(0,v)}$ and $\psi(t) = \frac{G(t,t)}{G(0,t)}$. Using the integration by parts formula, one obtains

$$dZ_t = (h(t,t) - \psi(t))\, dA_t^2 + \big(A_t^2 \partial_1 h(t, \tau_2) + (1 - A_t^2)\psi'(t)\big)\, dt.$$

By the assertion (a), the process $M^2 = A^2 + \int_0^{\cdot}(1 - A_t^2)\frac{\partial_2 G(0,t)}{G(0,t)}dt$ is an $\mathbb{A}^2$-martingale and

$$\psi'(t) = (h(t,t) - \psi(t))\,\frac{\partial_2 G(0,t)}{G(0,t)} + \frac{\partial_1 G(t,t)}{G(0,t)}.$$

It follows that $dZ_t = (h(t,t) - \psi(t))dM_t^2 - dA_t^p$ where

$$dA_t^p = -A_t^2 \partial_1 h(t, \tau_2)dt - (1 - A_t^2)\frac{\partial_1 G(t,t)}{G(0,t)}dt.$$

Finally the form of the $\mathbb{G}$-compensator of τ_1 can be concluded from Proposition 2.4 and the identity $\frac{\partial_1 h(t,\tau_2)}{h(t,\tau_2)} = \frac{\partial_{12} G(t,\tau_2)}{\partial_2 G(t,\tau_2)}$. □

We refer to [194] for related computations.

2.5 Construction of Random Time with Given Intensity

One of the approaches in credit risk is based on the intensity rate, i.e., the knowledge of a process λ such that $A - \int_0^{\cdot}(1 - A_s)\lambda_s ds$ is a martingale. In this formulation, no reference filtration is given, which leads to the following questions. Given a non-negative process λ, is it possible to construct τ such that the previous martingale property holds? The most popular answer is the Cox model (see Sect. 2.3). One can give other constructions as soon as a random time τ, such that the multiplicative decomposition of the Azéma supermartingale is $Ne^{-\Lambda}$, can be constructed. In this section one such a construction is presented; we refer the reader to [133] for a setting where there are infinitely many constructions with the property that the Azéma supermartingale of τ is $Ne^{-\Lambda}$, even in the case $N = 1$. The problem is solved by constructing a random time τ and a probability measure $\mathbb{Q}$ on the product space $\Omega \times \mathbb{R}^+$ such that the Azéma supermartingale of τ under $\mathbb{Q}$ equals $Ne^{-\Lambda}$ and the restriction to $\mathbb{Q}$ to $\mathscr{F}_\infty$ is $\mathbb{P}$. The random variable τ is defined by $\tau((\omega, x)) = x$, and $\mathbb{Q}$ is constructed so that $Y_t(u) := \mathbb{Q}(\tau \le u|\mathscr{F}_t)$ is a martingale for a fixed u, increasing in u for a fixed t, and $Y_t(t) = \mathbb{Q}(\tau \le t|\mathscr{F}_t) = 1 - N_t e^{-\Lambda_t} = 1 - Z_t$.

Proposition 2.32 *We assume that N and Λ are continuous and N is positive. Let $0 < u < \infty$ be fixed and consider the process $Y(u)$ defined for $t \in [u, \infty)$ by*

$$Y_t(u) := (1 - Z_t) \exp\left\{-\int_u^t \frac{Z_s}{1 - Z_s} d\Lambda_s\right\}.$$

Then, for any u, the process $Y(u) = (Y_t(u), u \leq t \leq \infty)$ is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale and, for fixed t, the family $(Y_t(u), u \leq t)$ is increasing in u.

Proof Applying the integration by parts formula for $t \geq u$ one gets

$$dY_t(u) = -\exp\left\{-\int_u^t \frac{Z_s}{1 - Z_s} d\Lambda_s\right\} e^{-\Lambda_t} dN_t.$$

Therefore, $Y(u)$ is an $(\mathbb{F}, \mathbb{P})$-local martingale on $[u, \infty)$. Being clearly positive and bounded by 1, it is a uniformly integrable martingale on $[u, \infty)$. □

2.6 Dynamics of Prices

The goal of this section is to give the dynamics of prices of some contingent claims. We assume that the probability measure $\mathbb{P}$ is the pricing measure, i.e., discounted prices are $(\mathbb{G}, \mathbb{P})$-local martingales. We assume that Z is positive, decreasing and continuous and condition **(A)** holds. Denote by Λ the process $\Lambda := -\ln Z$. Then, by Proposition 2.18 (c), the process $\Upsilon := (1 - A)\exp(\Lambda)$ is a $\mathbb{G}$-martingale which satisfies $\Upsilon = 1 - \Upsilon_- \bullet M$. For discounting, the $\mathbb{F}$-adapted interest rate process r and $R_t := \int_0^t r_s ds$ are used.

A **defaultable zero-coupon bond** of maturity T pays one monetary unit at time T, if the default has not occurred before T, and its price at time t is denoted $D_t(T)$.

Proposition 2.33 *Let m^Λ be the $\mathbb{F}$-martingale $m_t^\Lambda := \mathbb{E}_\mathbb{P}\left[\exp(-(R_T + \Lambda_T))|\mathscr{F}_t\right]$. Then the price of the defaultable zero-coupon bond has the following dynamics*

$$dD_t(T) = -D_{t-}(T)dM_t + \Upsilon_{t-}\exp(R_t)dm_t^\Lambda + D_t(T)r_t dt$$

Proof By Lemma 2.9 we obtain that the price $D_t(T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D_t(T) &= \mathbb{E}_\mathbb{P}\left[\mathbb{1}_{\{T<\tau\}} \exp\left(-(R_T - R_t)\right) \,\Big|\, \mathscr{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}_\mathbb{P}\left[\exp\left(-(R_T + \Lambda_T - R_t - \Lambda_t)\right) \,\Big|\, \mathscr{F}_t\right] = m_t^\Lambda \Upsilon_t \exp(R_t). \end{aligned}$$

Note that condition **(A)** implies that the process $[m^K, \Upsilon]$ is constant, since Υ is purely discontinuous martingale with a single jump at τ. Then, by integration by parts,

$$\begin{aligned}dD_t(T) &= -m_{t-}^{\Lambda}\Upsilon_{t-}\exp(R_t)dM_t + \Upsilon_{t-}\exp(R_t)dm_t^{\Lambda} + m_t^{\Lambda}\Upsilon_t\exp(R_t)dR_t \\ &= -D_{t-}(T)dM_t + \Upsilon_{t-}\exp(R_t)dm_t^{\Lambda} + D_t(T)r_t dt\,.\end{aligned}$$

□

Corollary 2.34 *Moreover if Λ is absolutely continuous, i.e., $\Lambda_t = \int_0^t \lambda_s ds$, and both λ and r are deterministic processes, we have*

$$dD_t(T) = -D_{t-}(T)dM_t + D_t(T)r_t dt\,.$$

Proof In the particular case where λ and r are deterministic, $dm_t^{\Lambda} = 0$ since m^{Λ} is a constant process, namely $m^{\Lambda} = \exp(-(R_T + \Lambda_T))$ and the result follows. □

Let K be a given $\mathbb{F}$-predictable process. A claim with **recovery payment at maturity** is a contract which pays K_τ at date T if $\tau \leq T$ and there is no payment in the case $\tau > T$. Its price at time t, for $t \leq T$ is denoted V_t. Instead, a claim with **recovery payment at default time** is a contract which pays K_τ at time τ if $\tau \leq T$ and there is no payment otherwise. Its price at time t, for $t \leq \tau$ is denoted U_t.

Proposition 2.35 *Let m^K be the $\mathbb{F}$-martingale $m_t^K := \mathbb{E}_{\mathbb{P}}\left[\int_0^T K_u dA_u^p | \mathscr{F}_t\right]$ for a given $\mathbb{F}$-predictable process K. Assume that $r = 0$. Then:*
(a) The process V has dynamics

$$V = V_0 + (K - V_-) \cdot M + \Upsilon_- \cdot m^K.$$

(b) The process U has dynamics

$$U = U_0 - U_- \cdot M + \Upsilon_- \cdot m^K - \Upsilon_- K \cdot A^p.$$

Proof (a) An immediate application of Corollary 2.10 (b) shows that the price at time t of a claim with recovery payment at maturity is

$$\begin{aligned}V_t &:= \mathbb{E}_{\mathbb{P}}\left[K_\tau \mathbb{1}_{\{\tau\leq T\}}|\mathscr{G}_t\right] = K_\tau \mathbb{1}_{\{\tau\leq t\}} + \mathbb{1}_{\{t<\tau\}}\frac{1}{Z_t}\mathbb{E}_{\mathbb{P}}\left[\int_t^T K_u dA_u^p \middle| \mathscr{F}_t\right] \\ &= \int_0^t K_u dA_u + \Upsilon_t\left(-\int_0^t K_u dA_u^p + m_t^K\right).\end{aligned}$$

By integration by parts we obtain

$$\Upsilon m^K = \Upsilon_- \cdot m^K + m_t^K \cdot \Upsilon + [m^K, \Upsilon].$$

As in the proof of Proposition 2.33, the process $[m^K, \Upsilon]$ is constant. Therefore, applying integration by parts again, we deduce that

$$V = V_0 + (K - V_-) \cdot M + \Upsilon_- \cdot m^K.$$

(b) Similarly as for V we derive that

$$U_t = 1\!\!1_{\{t<\tau\}}\mathbb{E}_{\mathbb{P}}\left[K_\tau 1\!\!1_{\{t<\tau\le T\}}|\mathscr{G}_t\right] = 1\!\!1_{\{t<\tau\}}\frac{1}{Z_t}\mathbb{E}_{\mathbb{P}}\left[\int_t^T K_u dA_u^p\middle|\mathscr{F}_t\right].$$

By applying integration by parts, we conclude that the dynamics of U are

$$U = U_0 - U_- \bullet M + \Upsilon_- \bullet m^K - \Upsilon_- K \bullet A^p.$$

□

Remark 2.36 (a) Note that, from the definition, the process V is a $\mathbb{G}$-martingale. Therefore Proposition 2.35 (a) implies that m^K stopped at τ is a $\mathbb{G}$-martingale. We refer the reader to Sect. 5.5 where random times with this property are studied.
(b) Note that $U + \Upsilon_- K \bullet A = V$ is a $\mathbb{G}$-martingale. Assume that the $\mathbb{F}$-intensity rate of τ exists and is denoted by λ. Therefore the process $U_t + \int_0^{t\wedge\tau} K_s\lambda_s ds$ is a $\mathbb{G}$-martingale. The quantity $K_t\lambda_t$ which appears can be interpreted as a dividend K_t paid at rate λ_t, or with probability $\lambda_t dt = \mathbb{P}(t < \tau < t + dt|\mathscr{F}_t)/\mathbb{P}(t < \tau|\mathscr{F}_t)$.

The next proposition concerns pricing and hedging a **defaultable call option** in a Cox model.

Proposition 2.37 *Assume that $\mathbb{F}$ is the natural filtration of a Brownian motion B and τ is a default time in a Cox model such that the process λ is deterministic. Let $r = 0$ and S satisfy $dS_t = S_t\sigma dB_t$, where σ is a constant. Then a defaultable call option, with payoff $1\!\!1_{\{T<\tau\}}(S_T - K)^+$, can be perfectly hedged by investing an amount equal to the price of this option in the defaultable bond.*

Proof By Lemma 2.9, the price of a defaultable call option is

$$C_t = \mathbb{E}_{\mathbb{P}}\left[1\!\!1_{\{T<\tau\}}(S_T - K)^+|\mathscr{G}_t\right] = 1\!\!1_{\{t<\tau\}}\frac{1}{Z_t}\mathbb{E}_{\mathbb{P}}\left[Z_T(S_T - K)^+|\mathscr{F}_t\right] = \Upsilon_t m_t^S$$

where $m_t^S := \mathbb{E}_{\mathbb{P}}\left[Z_T(S_T - K)^+|\mathscr{F}_t\right]$. In the particular case where λ is deterministic,

$$m_t^S = e^{-\Lambda_T}\mathbb{E}_{\mathbb{P}}\left[(S_T - K)^+|\mathscr{F}_t\right] = e^{-\Lambda_T}C_t^S$$

where C^S the price of a call in the Black and Scholes model, is of the form $C_t^S = C^S(t, S_t)$, hence

$$C_t = \Upsilon_t e^{-\Lambda_T}C_t^S = D_t(T)C_t^S.$$

Using the continuity of C_t^S and the fact that $dC_t^S = \Delta_t dS_t$ where Δ_t is the Delta-hedge ($\Delta_t = \partial_y C^S(t, S_t)$), we deduce that

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(\Upsilon_t dC_t^S + C_t^S d\Upsilon_t) = e^{-\Lambda_T}(\Upsilon_t\Delta_t dS_t - C_t^S\Upsilon_{t-}dM_t) \\ &= e^{-\Lambda_T}(\Upsilon_t\Delta_t dS_t - C_t^S\Upsilon_{t-}dM_t). \end{aligned}$$

Therefore, since by Corollary 2.34 $dD_t(T) = -e^{-\Lambda_T}\Upsilon_{t-}dM_t$, we obtain

$$dC_t = e^{-\Lambda_T}\Upsilon_t\Delta_t dS_t - C_t^S dD_t(T) = e^{-\Lambda_T}\Upsilon_t\Delta_t dS_t + \frac{C_{t-}}{D_{t-}(T)}dD_t(T)\,,$$

hence a hedging strategy consists of holding $\frac{C_{t-}}{D_{t-}(T)}$ defaultable zero-coupon bonds and $e^{-\Lambda_T}\Upsilon_{t-}\Delta_t$ risky asset S. Note that, on $\{t < \tau\}$, one has $C_{t-} = C_t$. □

The result obtained in Proposition 2.37 can be generalized to the case of stochastic intensity. See the so-called balance condition in [35].

2.7 Bibliographic Notes

The basic case studied in Sect. 2.1 is presented in Brémaud [44] and Dellacherie [71, 72]. Dellacherie and Meyer [76, Chap. IV, paragraph 107] consider also the filtration $\mathbb{A}^* = (\mathscr{A}_t^*, t \geq 0)$ where the σ-field $\mathscr{A}_t^*$ is generated by $\tau \wedge t$ and contains the set $\{\tau \geq t\}$ which cannot be split into two non-null sets from $\mathscr{A}_t^*$ (thus the set $\{\tau \geq t\}$ is an atom of $\mathscr{A}_t^*$). The filtration $\mathbb{A}^*$ is not right-continuous: $\mathscr{A}_{t+}^*$ is obtained by splitting the atom $\{\tau \geq t\}$ into $\{\tau = t\}$ and $\{\tau > t\}$. Setting $\mathscr{A}_t = \mathscr{A}_{t+}^*$, the random time τ is an $\mathbb{A}$-stopping time, but is not an $\mathbb{A}^*$-stopping time. Note that indeed $\mathbb{A}$ is the natural filtration of the process $\mathbb{1}_{[\![\tau,\infty[\![}$. It is proved that any $\mathbb{A}^*$-stopping time is predictable and that, if the law of τ is atomic and not degenerated, then τ is $\mathbb{A}$-accessible and not $\mathbb{A}$-predictable.

The predictable representation property is important in a financial framework, and some results (in particular the case presented in Sect. 2.1.2) can be found in Chou and Meyer [56].

The paper of Herdegen and Herrman [113] contains a systematic study of the case presented in the first section of this chapter, in a very general setting (no specific assumption on the regularity of the law of τ).

A deep study on extension of Proposition 2.11 (b) to optional processes can be found in Song [201].

A chapter in Protter [189] is devoted to the study of compensators. Janson et al. [126] and Zeng [220] present conditions which ensure that the compensator of A is absolutely continuous w.r.t. the Lebesgue measure. Their approach is based on the Ethier–Kurtz criterion. See Janson et al. [126] for an extension of Proposition 2.23 to absolutely continuous compensators.

Compensators can be computed using the *Laplacien approché* methodology (see Dellacherie and Meyer [79, VII, 22] and Dellacherie [74]), also called Aven's lemma in credit risk [23] (see more details in Zeng [220]). As we shall see in Chap. 5, compensators appear in many places for progressive enlargement framework. See also Coculescu [60] for some examples of compensators of default times and Last and Brandt [166] for the more general case of marked point processes.

The intensity based model was introduced in Jarrow and Turnbull [128], and Jarrow et al. [127] (see also Bielecki and Rutkowski [39]). The problem which appear in Sect. 2.5 is studied in Gapeev et al. [103], Jeanblanc and Song [133, 134] and in Li and Rutkowski [168]. It is proven in Song [200] that, for a given supermartingale valued in [0, 1] with multiplicative decomposition $Ne^{-\Lambda}$, there exist various ways to construct τ so that $Z = Ne^{-\Lambda}$, which implies that Λ^τ is the compensator of A. Some constructions lead to immersion, the others do not.

A reduced form approach is presented in Bielecki et al. [36, 37], Elliott et al. [89] and Kusuoka [163] among others. More information on pricing defaultable claims can be found in Bielecki et al. [38]. General presentations of modeling credit risk are done by Giesecke [105] and Bélanger et al. [32]. In particular, Cox models are used in a great number of studies (see, e.g., Lando [164]).

Structural models are models for default time, in which the default time is defined as a stopping time in the given filtration. These models do not involve enlargement of filtration and are not presented here. Guo and Zheng [111] present some explicit computation of compensators for hitting times and Okhrati et al. [187] study compensators of processes of the form $g(t, X_t)\mathbb{1}_{\{\tau>t\}}$ where τ is a hitting time for a Lévy process X. The structural approach faced some difficulties. One of them is that, if the filtration taken into account satisfies condition **(C)**, the random time τ is predictable, hence prices of defaultable zero coupon must go to 0 before τ, and this fact is not observed in the data. Another difficulty is to compute the conditional law of τ, especially in the case of multidimensional default times (see Blanchet and Patras [42]).

2.8 Exercises

Exercise 2.1 Let B be a Brownian motion and $\tau = \inf\{t : B_t = a\}$. Find the $\mathbb{F}^B$-compensator of τ and the $\mathbb{A}$-compensator of τ, where $\mathbb{A}$ is the natural filtration of the process A.

Exercise 2.2 We are in the setting of Sect. 2.1 and we assume that F is differentiable, $F < 1$ and $F(\infty) = 1$. Set $\Lambda(t) = \int_0^t \frac{dF(s)}{1-F(s)}$.
(a) Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a (bounded) Borel measurable function. Prove that the process

$$Y_t := \exp\left(\mathbb{1}_{\{\tau \le t\}} h(\tau)\right) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\, d\Lambda(u)$$

is an $\mathbb{A}$-martingale. Find an $\mathbb{A}$-predictable process φ such that $dY_t = \varphi_t dM_t$.
(b) Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative Borel measurable function such that the random variable $h(\tau)$ is integrable. Prove that the process

$$Y_t := (1 + \mathbb{1}_{\{\tau \le t\}} h(\tau)) \exp\left(-\int_0^{t\wedge\tau} h(u)\, d\Lambda(u)\right).$$

is an $\mathbb{A}$-martingale. Find an $\mathbb{A}$-predictable process φ such that $dY_t = \varphi_t dM_t$. Give a condition on h so that Y is positive. In that case, find an $\mathbb{A}$-predictable process ψ such that $dY_t = Y_{t-}\psi_t dM_t$.

Exercise 2.3 Assume that

$$dS_t = S_t(rdt + \sigma dB_t), \quad S_0 = 1$$

where B is a Brownian motion and let $\tau = \inf\{t : S_t \leq \alpha\}$, with $\alpha < 1$. Define $\mathbb{H} = (\mathscr{H}_t, t \geq 0)$ as the filtration generated by the observations of S at given times $t_1, \ldots, t_n$, i.e., for $t \in [t_n, t_{n+1})$, that is, $\mathscr{H}_t = \sigma(S_s, s \leq t_n)$ for $t_n \leq t < t_{n+1}$. Compute the $\mathbb{H}$-intensity rate of τ.

Exercise 2.4 (a) Prove that, in a Cox model, τ is independent of $\mathscr{F}_\infty$ if and only if λ is a deterministic function.
(b) Prove that, in general, in a Cox model, $\mathbb{A}$ is not immersed in $\mathbb{G}$. Prove that, if λ is deterministic, $\mathbb{A}$ is immersed in $\mathbb{G}$.

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