Chapter 2
Maximum and Minimum Cost Flow Finding in Networks in Fuzzy Conditions

The problems of the maximum and the minimum cost flow finding with zero and nonzero lower flow bounds are relevant, since they allow solving the problems of economic planning, logistics, transportation management, etc. In the area of transportation networks flow tasks enable to find the cargo transportation of the maximum volume between given points taking into account restrictions on the arc capacities of the cargo transmission paths, choose the routes of the optimal cost with the set lower flow bounds, which can be found after the profitability analysis of the cargo transportation along the particular road section. In considering these tasks it is necessary to take into account the inherent uncertainty of the network parameters, since environmental factors, measurement errors, repair work on the roads, the specifics of the constantly changing structure of the network influence the upper and lower flow bounds and transportation costs.

2.1 Maximum Flow Finding in a Network with Fuzzy Arc Capacities

The complex nature of environment factors, in particular, the inherent uncertainty is the basis for considering the maximum flow tasks in fuzzy dynamic network (Definition 2.1).

Definition 2.1 *Fuzzy transportation network is a fuzzy directed graph* [1, 2] $\widetilde{G} = (X, \widetilde{A})$, where $X = \{x_1, x_2, \ldots, x_n\}$ is the set of nodes, $\widetilde{A} = \{\mu_{\tilde{A}}\langle x_i, x_j\rangle / \langle x_i, x_j\rangle\}$, $\langle x_i, x_j\rangle \in X^2$, $\mu_{\tilde{A}}\langle x_i, x_j\rangle$—the fuzzy set of the arcs, where $\mu_{\tilde{A}}(x_i, x_j)$ is a grade of membership of the directed arc $\langle x_i, x_j\rangle$ to the fuzzy set of the directed arcs $\widetilde{A}$. Fuzzy arc capacity is applied as $\mu_{\tilde{A}}(x_i, x_j)$.

The key notion of the maximum flow task is the fuzzy residual (incremental) network, since we will search the maximum flow each time in the fuzzy residual

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network that changes its structure based on the flow values passing along its arcs. Present the Definition 2.2 of the fuzzy residual network [3], modified for using in a fuzzy way.

Definition 2.2 *A fuzzy residual (incremental) transportation network* $\widetilde{G}^{\mu}$ is a fuzzy network defined as a fuzzy directed graph that sets two arcs: $\left(x_i^{\mu}, x_j^{\mu}\right)$ with the residual arc capacity $\tilde{u}_{ij}^{\mu} = \tilde{u}_{ij} - \tilde{\xi}_{ij}$ and $\left(x_j^{\mu}, x_i^{\mu}\right)$ with the residual arc capacity $\tilde{u}_{ji}^{\mu} = \tilde{\xi}_{ij}$ for each arc $\left(x_i, x_j\right)$ of the original graph $\widetilde{G}$. Thus, a fuzzy residual network contains only arcs with the positive arc capacities.

Then the model of the maximum flow problem in a fuzzy transportation network [4] is defined as:

$$\tilde{v} = \sum_{x_j \in \Gamma(s)} \tilde{\xi}_{sj} = \sum_{x_k \in \Gamma^{-1}(t)} \tilde{\xi}_{kt} \to \max, \tag{2.1}$$

$$\sum_{x_j \in \Gamma(x_i)} \tilde{\xi}_{ij} = \sum_{x_k \in \Gamma^{-1}(x_i)} \tilde{\xi}_{ki} = \begin{cases} \tilde{v}, & x_i = s, \\ -\tilde{v}, & x_i = t, \\ \tilde{0}, & x_i \neq s, t, \end{cases} \tag{2.2}$$

$$\tilde{\xi}_{ij} \leq \tilde{u}_{ij}, \quad \forall \left(x_i, x_j\right) \in \widetilde{A}. \tag{2.3}$$

In the model (2.1)–(2.3) $\tilde{v}$—the maximum flow value in the fuzzy network; $\tilde{\xi}_{ij}$—fuzzy flow value passing along the arc $\left(x_i, x_j\right)$; s—the initial node of the graph (the source); t—the terminal node of the graph (sink); $\Gamma(x_i)$—the set of nodes, arcs from the node $x_i \in X$ go to; $\Gamma^{-1}(x_i)$—the set of nodes, arcs from the node $x_i \in X$ go from; u_{ij}—the maximum amount of flow that can pass through the arc $\left(x_i, x_j\right)$ (arc capacity).

It is necessary to perform the algorithm of solving the present task in fuzzy conditions. Let us introduce the method of the augmenting path finding in the network using breadth-first search [5] modified for using in fuzzy environment and for finding the shortest fuzzy path.

Method of the augmenting path finding by the breadth-first search in fuzzy conditions:

1. Form a queue consisting of the nodes Q. Initially Q contains only the source-node s.
2. Mark the node s as visited but without predecessor. Mark other nodes as unvisited.
3. Check the nodes in the queue:

   3.1 If the queue is empty, then stop and exit the algorithm, since there is no way.

   3.2 If the queue is not empty and the first node in the queue is $x_i = t$, go to the step 6.

3.3 If the queue is not empty and the first node in the queue is $x_i \neq t$, delete the first node in the queue x_i.

4. Check the arcs $\{x_i, x_j\} \in \widetilde{A}$, such that the node x_j has not visited yet.

4.1 If there are no such arcs $\{x_i, x_j\} \in \widetilde{A}$ that the node x_j has not visited yet, go to the step 3.

4.2 If there are arcs $\{x_i, x_j\} \in \widetilde{A}$ that the node x_j has not visited yet, mark x_j as visited with the predecessor x_i.

5. Insert the node x_j at the end of the queue and go to the step 3.
6. Look through the nodes of the queue in the reverse order of t to s, each time passing to the predecessor. Return the path in the reverse order.

An algorithm for the maximum flow task solving in the network with fuzzy capacities

Consider the algorithm for the maximum flow finding in the network with fuzzy capacities.

Step 1. Construct fuzzy residual network $\widetilde{G}^{\mu} = \left(X^{\mu}, \widetilde{A}^{\mu}\right)$, where X^{μ}—the set of nodes of the fuzzy residual network coincides with the set of nodes of the network $\widetilde{G}$ and $\widetilde{A}^{\mu} = \{\langle \tilde{u}_{ij}^{\mu}/(x_i,\, x_j)\rangle\}$—fuzzy set of the arcs of the network $\widetilde{G}^{\mu}$ determined depending on the flow values coming along the arcs of the graph $\widetilde{G}$. Fuzzy arc capacities are used as the membership functions of the arcs. If $\tilde{\xi}_{ij} < \tilde{u}_{ij}$, then $\tilde{u}_{ij}^{\mu} = \tilde{u}_{ij}^{\mu} - \tilde{\xi}_{ij}$. If $\tilde{\xi}_{ij} > \tilde{0}$, then $\tilde{u}_{ji}^{\mu} = \tilde{\xi}_{ij}$. Initially the residual network coincides with the initial network (due to the equality of the arc flow to $\tilde{0}$).

Step 2. Search the shortest path $\widetilde{P}^{\mu}$ according to the criterion of the number of arcs from the source to the sink in the constructed fuzzy residual network, starting from zero flows. The selection is made using the breadth-first search in fuzzy conditions.

2.1. If the path $\widetilde{P}^{\mu}$ is found, go to the **step 3**.

2.2. If the path is failed to find, then the maximum flow $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu} = \tilde{v}$ is obtained in the initial graph $\widetilde{G}$, the **exit**.

Step 3. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}_{ij}^{\mu}\right]$, $\left(x_i, x_j\right) \in \widetilde{P}^{\mu}$ flow units along the path.

Step 4. Update the flow values in $\widetilde{G}$: replace the flow $\tilde{\xi}_{ji}$ along the corresponding arcs $\left(x_j, x_i\right)$ from $\widetilde{G}$ by $\tilde{\xi}_{ji} - \tilde{\delta}^{\mu}$ from $\tilde{\xi}_{ji}$ for arcs $\left(x_i^{\mu}, x_j^{\mu}\right) \notin \widetilde{A}$, $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}^{\mu}$ in $\widetilde{G}^{\mu}$. Replace the flow $\tilde{\xi}_{ij}$ along the corresponding arcs $\left(x_i, x_j\right)$ from $\widetilde{G}$ by $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}$ from $\tilde{\xi}_{ij}$ for arcs $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}$, $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}^{\mu}$ in $\widetilde{G}^{\mu}$. Replace the flow value in the graph $\widetilde{G}$: $\tilde{\xi}_{ij} \rightarrow \tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ and turn to the **step 1** starting with updated flow value along the arcs.

2.2 Method of Fuzzy Calculations with Fuzzy Numbers

It is necessary to perform sum, subtraction and comparing operations solving the maximum flow tasks and other tasks in the book in networks with fuzzy arc capacities. Conventional operations of the sum, subtraction are presented earlier in this chapter. Present methods can be used for solving flow tasks in fuzzy conditions, but they have some disadvantages.

The disadvantage of the conventional method of operating fuzzy numbers is strong "blurring" of borders of the resulting number and, as a result, loss of information content with such numbers. Standard subtraction operation of two equal fuzzy numbers does not lead to zero fuzzy number, which is unacceptable for flow tasks, where at least one arc must become saturated each iteration. It creates problems of the correct display of uncertainty of the proper system. In this case, we propose to use nonstandard subtraction operation [6], which does not lead to the strong "blurring" of the borders of the resulting number and produces a fuzzy number is equal to zero by subtracting equal fuzzy numbers. While specifying the fuzzy numbers the fact that the degree of borders blurring depends on the size of the center is not usually taken into account. Therefore, the more the center, the more "blurred" the borders should be (while measuring 1 kg of material, we are talking "about 1 kg", implying the number "from 900 to 1100 g", but while measuring 1 t. of material, imply that "about 1 t." is the number "from 990 to 1110 kg").

Otherwise, if absolute values are small, and deviation borders are large, the effectiveness of the fuzzy logic application decreases. Subtraction of fuzzy numbers in the case where it is impossible to uniquely identify the largest can lead to negative values, which is unacceptable for flows.

Let us represent two fuzzy triangular numbers: $\widetilde{A}_1 = (10, 2, 3)$ and $\widetilde{A}_2 = (9, 2, 7)$ of the form $\widetilde{A}_1 = (a_1, \gamma_1, \delta_1)$ and $\widetilde{A}_2 = (a_2, \gamma_2, \delta_2)$, where a_1 and a_2—the centers of triangular numbers, γ_1 and γ_2—the left deviations, δ_1 and δ_2—the right deviations. Assume, it is necessary to determine the subtraction of such numbers. It is necessary to determine the largest before subtraction of fuzzy numbers. We can't uniquely identify the largest number comparing fuzzy numbers $\widetilde{A}_1$ и $\widetilde{A}_2$, as $a_1 > a_2$, $a_1 - \gamma_1 > a_2 - \gamma_2$, $a_1 + \delta_1 < a_2 + \delta_2$, therefore, comparing them by the center of gravity, we find that the center of gravity of the second number is larger, therefore, it is estimated higher. Hence, the subtraction operation can be expressed as follows (9,2,7)–(10,2,3) = (−1,5,9), i.e., we get fuzzy triangular number with a negative center and, accordingly, the negative left border, which contradicts the condition of non-negative flows values.

Therefore, a different approach to handling fuzzy numbers in the study of flows is required, which is in operating the central values of fuzzy numbers during the algorithm, and the operation of non-standard deduction will be used during the subtraction; "blurring" of the resulting number will be provided once at the end of the algorithm based on the basic values set by experts. It is considered that the numbers with the large centers have large uncertainty bounds.

Consequently, the following method [7] is proposed to use when operating triangular fuzzy numbers. Suppose there are the values of lower, upper flow bounds or transmission costs in a form of fuzzy triangular numbers on the number axis set by the expert according to his experience, knowledge and information about the particular road or its section. Then when adding two original triangular fuzzy numbers their centers will be added: $a_1 + a_2$, subtracted: $a_1 - a_2$, and $a_1 \geq a_2$. The fact that the number with the larger center will have the larger deviation borders is taken into account. To calculate the deviations it is necessary to define required value by adjacent values. Let the fuzzy parameter (for example, arc capacity) "near $.\tilde{a}'$." is between two adjacent values "near $\tilde{a}_1$" and "near $\tilde{a}_2$", $(\tilde{a}_1 \leq \tilde{a}' \leq \tilde{a}_2)$ which membership functions $\mu_{\tilde{a}_1}(a_1)$ and $\mu_{\tilde{a}_2}(a_2)$ have a triangular form, thus, the borders of membership function of fuzzy arc capacity $\mu_{\tilde{a}\prime}(a)$ of the fuzzy parameter "near $\tilde{x}'$" one can set by the linear combination of left and right borders of adjacent values according to the method described in [8] and presented as:

$$
\begin{aligned}
l^L &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^L + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^L, \\
l^R &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^R + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^R.
\end{aligned} \tag{2.4}
$$

In (2.4) l^L—the left deviation border of the fuzzy triangular number with the center a'; l^R—the right deviation border of the fuzzy triangular number with the center a'. It is shown in Fig. 2.1.

In the case when the central value of triangular number resulting by adding (subtracting) repeats the already marked value on the number axis, its deviation borders coincide with the deviation borders of the number marked on the number axis. If required central value is not between two numbers, but precedes the first marked value on the number axis, its deviation borders coincide with those of the first marked on the axis. The same applies to the case when the required central value follows the last marked value on the axis.

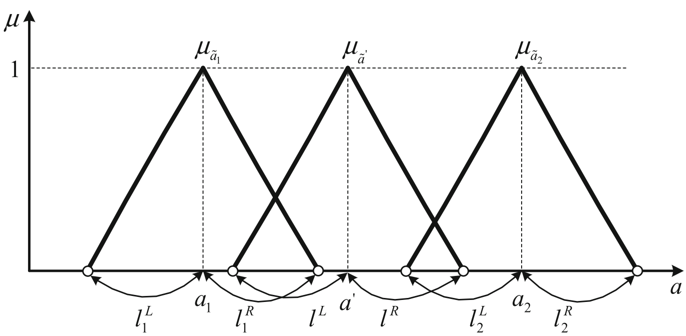


Fig. 2.1 Defining of the membership function $\mu_{\tilde{a}\prime}(a)$

The same method is used for calculations of the deviation borders of the trapezoidal numbers. Let us introduce the following equation for the left l^L and the right l^R trapezoidal deviation borders:

$$\begin{aligned} l^L &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_l^L + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^L, \\ l^R &= \frac{(b_2 - b')}{(b_2 - b_1)} \times l_l^R + \left(1 - \frac{(b_2 - b')}{(b_2 - b_1)}\right) \times l_2^R. \end{aligned} \tag{2.5}$$

In the Eq. (2.5) l^L—the left deviation border of the fuzzy trapezoidal number with the center $[a', b']$; l^R—the right deviation border of the fuzzy trapezoidal number with the center $[a', b']$. It is represented in Fig. 2.2.

The present method allow us to avoid receiving of the negative fuzzy numbers, which are unacceptable for the flows, simplifies the calculations and leads to the strong "blurring" of the borders of fuzzy numbers.

Let us consider the numerical example that implements the algorithm for the maximum flow finding in the network with fuzzy arc capacities.

Numerical example 1

Let that network is presented in the form of the fuzzy directed graph, given in Fig. 2.3.

The arc capacities in the form of the fuzzy intervals set by the experts are assigned to the arcs of the graph. It is necessary to find the maximum flow in the graph and perform it in the form of the fuzzy trapezoidal number, if the basic values of arc capacities in the form of the fuzzy trapezoidal numbers are given, as shown in Fig. 2.4.

Step 1. Fuzzy residual network coincides with the initial network $\widetilde{G}$, shown in Fig. 2.3 by the equality of the arc flows to $[\tilde{0}, \tilde{0}]$.

Step 2. Search the shortest path according to the number of arcs from s to t in fuzzy residual network. Use the breadth-first-search for it:

The queue Q consists of the vertices s. Note s as visited without predecessors.

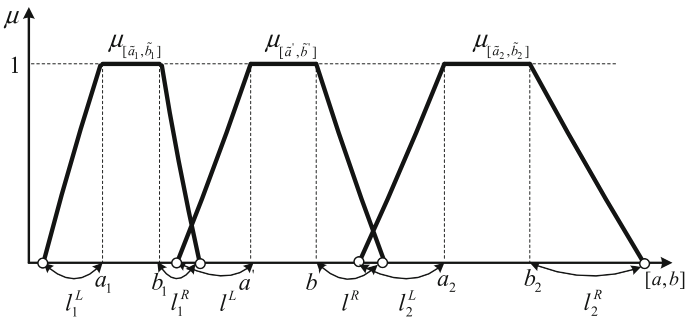


Fig. 2.2 Defining of the membership function $\mu_{\tilde{a}'}(a)$

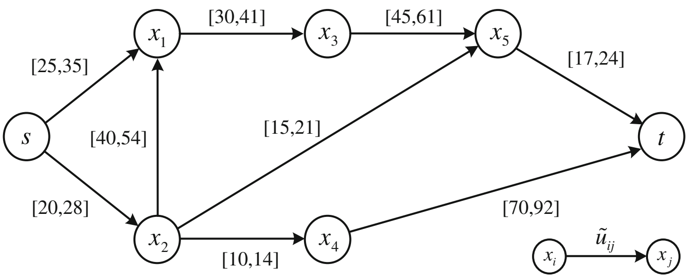


Fig. 2.3 The initial network $\widetilde{G}$

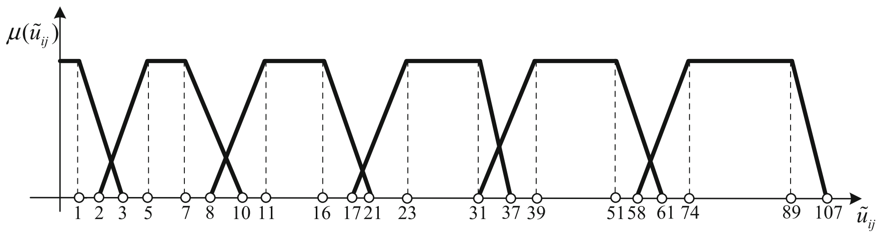


Fig. 2.4 Membership functions of the basic values of arc capacities of the network $\widetilde{G}$

Delete the first node in queue s. Form the set of nodes arcs from s go to and that have not yet visited: $\{x_1, x_2\}$. The queue consists of the nodes $\{x_1, x_2\}$. The nodes s, x_1, x_2 have been visited. The nodes x_1, x_2 have the predecessor s.

Delete the first node in queue x_1. Form the set of nodes arcs from x_1 go to and that have not yet visited: $\{x_3\}$. The queue consists of the nodes $\{x_2, x_3\}$. The nodes s, x_1, x_2, x_3 have been visited. The nodes x_1, x_2 have the predecessor s, the node x_3 has the predecessor x_1.

Delete the first node in queue x_2. Form the set of nodes arcs from x_2 go to and that have not yet visited: $\{x_4, x_5\}$. The queue consists of the nodes $\{x_3, x_4, x_5\}$. The nodes $s, x_1, x_2, x_3, x_4, x_5$ have been visited. The nodes x_1, x_2 have the predecessor s, the node x_3 has the predecessor x_1, the nodes x_4, x_5 have the predecessor x_2.

Delete the first node in queue x_3. Form the set of nodes arcs from x_3 go to and that have not yet visited: $\{\emptyset\}$. The node x_3 is delete from the queue. The queue consists of the nodes $\{x_4, x_5\}$. The nodes $s, x_1, x_2, x_3, x_4, x_5$ have been visited. The nodes x_1, x_2 have the predecessor s, the node x_3 has the predecessor x_1, the nodes x_4, x_5 have the predecessor x_2.

Delete the first node in the queue x_4. Form the set of nodes arcs from x_4 go to and that have not yet visited: $\{t\}$. The arc (x_4, t) is found and the algorithm terminates. The queue consists of the nodes $\{x_5, t\}$. All nodes have been visited. The nodes x_1, x_2 have the predecessor s, the node x_3 has the predecessor x_1, the nodes x_4, x_5 have the predecessor x_2, the node t has the predecessor x_4. Go backwards by predecessors and receive the path $s \to x_2 \to x_4 \to t$.

Step 3. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}^{\mu}_{ij}\right]$, i.e. $([2\tilde{0}, 2\tilde{8}], [1\tilde{0}, 1\tilde{4}], [7\tilde{0}, 9\tilde{2}]) = [1\tilde{0}, 1\tilde{4}]$ flow units along the path $s \to x_2 \to x_4 \to t$.

Step 4. Update the flow values in the graph $\widetilde{G}$.

The flow $\tilde{\xi}_{ij}\left[\tilde{0}, \tilde{0}\right]$ goes to $[1\tilde{0}, 1\tilde{4}]$. Construct the graph with the new flow value, as shown in Fig. 2.5 and turn to the **step 1**.

Step 1. Construct fuzzy residual network $\widetilde{G}^{\mu}$ for the graph in Fig. 2.5 according to the flow values passing along the arcs of the graph, as shown in Fig. 2.6.

Step 2. Search the shortest path according to number of arcs from s to t in the fuzzy residual network $\widetilde{G}^{\mu}$. Use the breadth-first-search and find the path: $s \to x_2 \to x_5 \to t$.

Step 3. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}^{\mu}_{ij}\right]$, i.e. min $([1\tilde{0}, 1\tilde{4}], [1\tilde{5}, 2\tilde{1}], [1\tilde{7}, 2\tilde{4}]) = [1\tilde{0}, 1\tilde{4}]$ flow units along the path: $s \to x_2 \to x_5 \to t$.

Step 4. Update the flow values in the graph $\widetilde{G}$.

The flow $\tilde{\xi}_{ij} = [1\tilde{0}, 1\tilde{4}]$ turns to $[2\tilde{0}, 2\tilde{8}]$. Construct the graph with the new flow value, as shown in Fig. 2.7 and turn to the **step 1**.

Step 1. Construct fuzzy residual network $\widetilde{G}^{\mu}$ for the graph in Fig. 2.7 by the flow values passing along the arcs of the graph, as shown in Fig. 2.8.

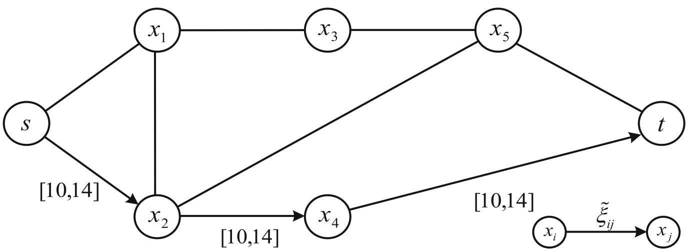


Fig. 2.5 Graph $\widetilde{G}$ with the new flow value of [10,14] units

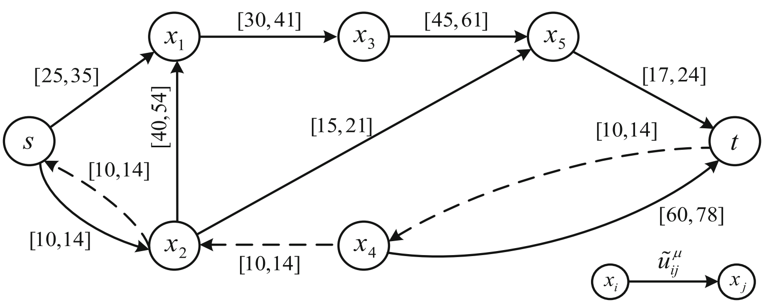


Fig. 2.6 Fuzzy residual network $\widetilde{G}^{\mu}$ for the graph in Fig. 2.5

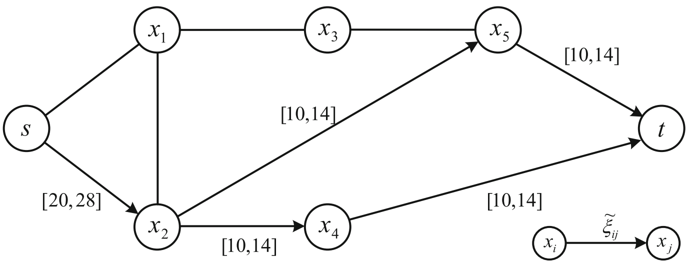


Fig. 2.7 $\widetilde{G}$ with the new flow value of $[\tilde{20}, \tilde{28}]$ units

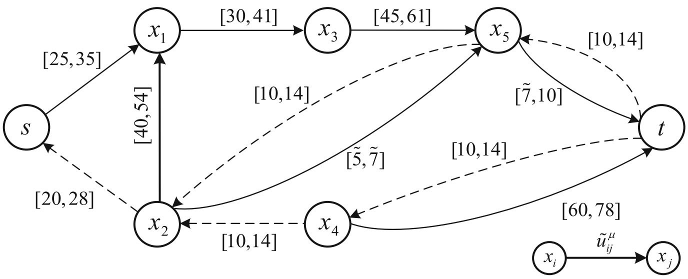


Fig. 2.8 Fuzzy residual network $\widetilde{G}^{\mu}$ for the graph in Fig. 2.7

Step 2. Search the shortest path according to number of arcs from s to t in the fuzzy residual network $\widetilde{G}^{\mu}$ in Fig. 2.8. Use the breadth-first-search and find the path: $s \to x_1 \to x_3 \to x_5 \to t$.

Step 3. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}_{ij}^{\mu}\right]$, i.e. min $([\tilde{25}, \tilde{35}], [\tilde{30}, \tilde{41}], [\tilde{45}, \tilde{61}], [\tilde{7}, \tilde{10}]) = [\tilde{7}, \tilde{10}]$ flow units along the path $s \to x_1 \to x_3 \to x_5 \to t$.

Step 4. Update the flow values in the graph $\widetilde{G}$.

The flow $\tilde{\xi}_{ij} = [\tilde{20}, \tilde{28}]$ turns to $[\tilde{20}, \tilde{28}] + [\tilde{7}, \tilde{10}] = [\tilde{27}, \tilde{38}]$. Construct the graph with the new flow value, as shown in Fig. 2.9 and turn to the **step 1**.

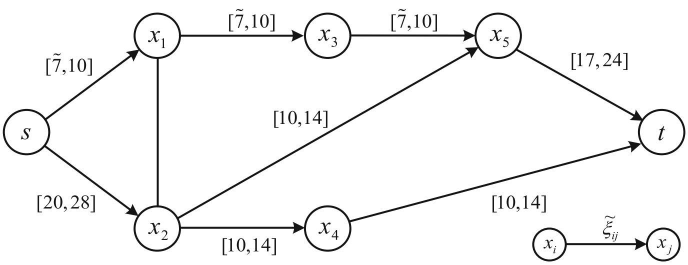


Fig. 2.9 Graph $\widetilde{G}$ with the new flow value of [27,38] units

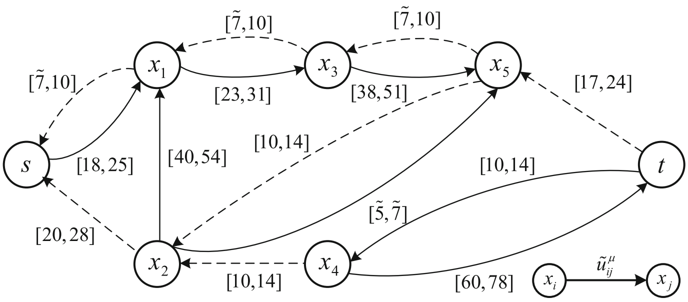


Fig. 2.10 Fuzzy residual network $\widetilde{G}^{\mu}$ for the graph in Fig. 2.9

Step 1. Build fuzzy residual network $\widetilde{G}^{\mu}$ for the graph in Fig. 2.9 by the flow values passing along the arcs of the graph, as show in Fig. 2.10.

Step 2. Search the shortest path according to the number of arcs from s to t in the fuzzy residual network $\widetilde{G}^{\mu}$ in Fig. 2.10. Use the breadth-first-search and find the path: the path doesn't exist.

The maximum flow $\tilde{\tilde{\xi}}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu} = \tilde{v}$ is obtained in the initial graph $\widetilde{G}$ of the value of $[2\tilde{7}, 3\tilde{8}]$ units. Network with the maximum flow is represented in Fig. 2.9.

Let us define deviation borders of the obtained fuzzy interval $[2\tilde{7}, 3\tilde{8}]$ corresponded to the maximum flow in the graph $\widetilde{G}$. The detected result is between two adjacent basic values of the arc capacities: $[2\tilde{3}, 3\tilde{1}]$ with the left deviation $l_1^L = 6$, the right deviation—$l_1^R = 6$ and $[3\tilde{9}, 5\tilde{1}]$ with the left deviation $l_2^L = 8$, the right deviation $-l_2^R = 10$. According to (2.5) we obtain:

$$\begin{aligned} l^L &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^L + \left(1 - \frac{(a_2 - a\prime)}{(a_2 - a_1)}\right) \times l_2^L = \frac{(39-27)}{(39-23)} \times 6 + \left(1 - \frac{(39-27)}{(39-23)}\right) \times 8 \\ &= 6.5 \approx 7, \\ l^R &= \frac{(b_2 - b')}{(b_2 - b_1)} \times l_1^R + \left(1 - \frac{(b_2 - b\prime)}{(b_2 - b_1)}\right) \times l_2^R = \frac{(51-37)}{(51-31)} \times 6 + \left(1 - \frac{(51-37)}{(51-31)}\right) \times 10 \\ &= 7.2 \approx 7. \end{aligned}$$

Therefore, the maximum flow in the fuzzy network in Fig. 2.3 is obtained, which can be represented as a fuzzy trapezoidal number of (27,38,7,7) units, as shown in Fig. 2.11.

According to the Fig. 2.11 the maximum flow with the degree of confidence equal to 0.7 is within the interval [25,40], but anyway the maximum flow is no less than 20 units and no more than 45 units.

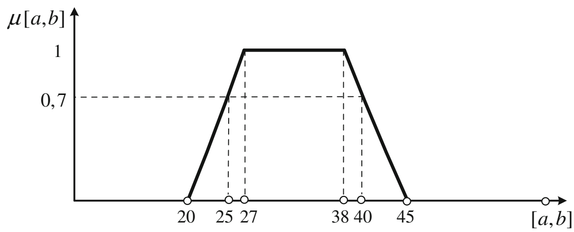


Fig. 2.11 The maximum flow in the form of the fuzzy trapezoidal number of (27,38,7,7) units

2.3 Maximum Flow Finding in a Network with Fuzzy Nonzero Lower and Upper Flow Bounds

Let the network considered in the previous section contains the lower flow bounds besides arc capacities. It occurs in the case, when one deals with transportation profitability. For example, a transport plane will carry out the flight at the lowest feasible amount of load of 1 ton; passenger aircraft will fly if at least twenty tickets will be sold. These figures are considered as lower flow bounds. Thus, nonzero lower flow bounds must be considered in some flow problems. The capacity in such problems is the upper flow bound. Thus, the flow must satisfy the constraint on upper and lower flow bounds, i.e. expression (1.13).

The complexity of the problem statement of the maximum flow with lower bounds specified on the arcs (1.11)–(1.13) lies in the fact that the feasible flow cannot exist, in contrast to the conventional problem of the maximum flow finding (zero flow is valid). Therefore it is necessary to solve two problems: on the admissibility of the flow, and then—to find the maximum flow in the case of the existence of the feasible flow. Considering the fuzzy nature of the upper and lower flow bounds, we come to the maximum flow problem formulation in the network, taking into account the nonzero lower and upper flow bounds, presented in the fuzzy form [9]:

$$\tilde{v} = \sum_{x_j \in \Gamma(s)} \tilde{\xi}_{sj} = \sum_{x_k \in \Gamma^{-1}(t)} \tilde{\xi}_{kt} \to \max, \tag{2.6}$$

$$\sum_{x_j \in \Gamma(x_i)} \tilde{\xi}_{ij} = \sum_{x_k \in \Gamma^{-1}(x_i)} \tilde{\xi}_{ki} = \begin{cases} \tilde{v}, & x_i = s, \\ -\tilde{v}, & x_i = t, \\ \tilde{0}, & x_i \neq s, t, \end{cases} \tag{2.7}$$

$$\tilde{l}_{ij} \leq \tilde{\xi}_{ij} \leq \tilde{u}_{ij}, \quad \forall \left(x_i, x_j\right) \in \tilde{A}. \tag{2.8}$$

In the model (2.6)–(2.8) $\tilde{l}_{ij}$—fuzzy lower flow bound for the arc (x_i, x_j).

To solve this problem one must turn to the fuzzy graph without lower bounds flows [10], that is reflected in the introduced *rule 2.1* of turning to the corresponding fuzzy graph without lower flow bounds from the fuzzy graph with nonzero lower flow bounds, modified for the using in the fuzzy form:

Rule 2.1 of turning to the corresponding fuzzy graph without lower flow bounds from fuzzy graph with nonzero lower bounds flows for solving the maximum flow problem with fuzzy nonzero lower flow bounds

Turn to the fuzzy graph $\widetilde{G}^* = (X^*, \widetilde{A}^*)$ without lower flow bounds from the given fuzzy graph $\widetilde{G} = (X, \widetilde{A})$ with nonzero lower flow bounds. Introduce artificial nodes s^* and t^*, the arc (t, s) with $\tilde{u}^*_{ts} = \infty$, $\tilde{l}^*_{ts} = \tilde{0}$ in the new graph $\widetilde{G}$. For each node (x_i, x_j) in $\widetilde{G}$ with $\tilde{l}_{ij} \neq \tilde{0}$: (1) decrease $\tilde{u}_{ij}$ to $\tilde{u}^*_{ij} = \tilde{u}_{ij} - \tilde{l}_{ij}$, $\tilde{l}_{ij}$ to $\tilde{0}$. (2) Introduce arcs (s^*, x_j) and (x_i, t^*) with flow bounds equal to $\tilde{u}^*_{s^*x_j} = \tilde{u}^*_{x_i t^*} = \tilde{l}_{ij}$, $\tilde{l}^*_{s^*x_j} = \tilde{l}^*_{x_i t^*} = \tilde{0}$. Arcs without lower flow bounds are the same for $\widetilde{G}^*$: for any arc (x_i, x_j) with $\tilde{l}_{ij} = \tilde{0}$ is $\tilde{u}^*_{ij} = \tilde{u}_{ij}$.

Thus, artificial arcs have capacities equal to the lower flow bounds in the generated graph without lower flow bounds and initial arcs with nonzero lower bounds are introduced with a capacity equals the difference between the initial upper and lower flow bound. Artificial arc from *t* and *s* defines the feasible flow in the original graph.

Start to search for the maximum flow in the fuzzy residual network corresponding to the graph without lower flow bounds [10] and constructed according to the Definition 2.3, modified for use in a fuzzy way after the transition to the corresponding graph without lower flow bounds from the fuzzy graph. The criterion of the feasible flow existing in the initial fuzzy graph is finding the maximum flow equals the sum of the lower flow bounds in the transformed graph without lower bounds.

Definition 2.3 *Fuzzy residual network for the graph* $\widetilde{G}^* = \left(X^*, \widetilde{A}^*\right)$ *without lower flow bounds for solving the maximum flow problem with fuzzy nonzero lower flow bounds—the network* $\widetilde{G}^{*\mu} = \left(X^{*\mu}, \widetilde{A}^{*\mu}\right)$, where $X^{*\mu} = X^*$—the set of the nodes of the fuzzy residual network with artificial nodes, $\widetilde{A}^{*\mu} = \left\{\left\langle \tilde{u}^{*\mu}_{ij} / \left(x^{*\mu}_i, x^{*\mu}_j\right)\right\rangle\right\}$—fuzzy set of the arcs of the network $\widetilde{G}^{*\mu}$, constructed according the following rules: for all arcs, if $\tilde{\xi}^*_{ij} < \tilde{u}^*_{ij}$, then include corresponding arc in $\widetilde{G}^{*\mu}$ with arc capacity $\tilde{u}^{*\mu}_{ij} = \tilde{u}^*_{ij} - \tilde{\xi}^*_{ij}$. For all arcs, if $\tilde{\xi}^*_{ij} > \tilde{0}$, then include corresponding arc in $\widetilde{G}^{*\mu}$ with arc capacity $\tilde{u}^{*\mu}_{ji} = \tilde{\xi}^*_{ij}$.

Reverse transition to the initial graph with the feasible flow from the modified graph with the maximum flow [11] is according to the introduced *rule 2.2*, modified for the use in fuzzy terms.

The rule 2.2 of transition to the graph with the feasible flow from the graph without lower flow bounds with the maximum flow for solving the maximum flow problem with fuzzy nonzero lower flow bounds

Turn to the graph $\widetilde{G}$ from the graph $\widetilde{G}^*$ as following: reject artificial nodes and arcs, connecting them with other nodes. The feasible flow vector $\tilde{\xi} = \left(\tilde{\xi}_{ij}\right)$ of the value $\tilde{\sigma}$ is defined as: $\tilde{\xi}_{ij} = \tilde{\xi}^*_{ij} + \tilde{l}_{ij}$, where $\tilde{\xi}^*_{ij}$—the flows, going along the arcs of the graph $\widetilde{G}^*$ after deleting all artificial nodes and connecting arcs.

If the feasible flow is found in the initial fuzzy graph, transform it by finding the maximum [10], according to the *rule 2.3*, presented in the fuzzy form. If the maximum flow equals the sum of the lower flow bounds *is not found in the modified graph, therefore, there is no feasible flow in the fuzzy initial graph and the task has no solution*

The rule 2.3 of the fuzzy residual network constructing with the feasible flow vector for solving the maximum flow problem with fuzzy nonzero lower flow bounds

For all arc, if $\tilde{\xi}_{ij} < \tilde{u}_{ij}$, then include the corresponding arc in $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ with arc capacity $\tilde{u}^{\mu}_{ij} = \tilde{u}_{ij} - \tilde{\xi}_{ij}$. For all arc, if $\tilde{\xi}_{ij} > \tilde{l}_{ij}$, then include the corresponding arc in $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ with arc capacity $\tilde{u}^{\mu}_{ji} = \tilde{\xi}_{ij} - \tilde{l}_{ij}$.

Let us represent the formal algorithm for solving the proposed problem based on the defined rules and definitions.

Algorithm of the maximum flow finding in the network with nonzero lower flow bounds in fuzzy conditions

Let us represent the formal algorithm for solving the proposed problem based on the defined rules and definitions [9].

Step 1. Let us define if the initial graph $\widetilde{G} = \left(X, \widetilde{A}\right)$ has the feasible flow. Turn to the graph $\widetilde{G}^* = \left(X^*, \widetilde{A}^*\right)$ without lower flow bounds according to the *rule 2.1.*

Step 2. Find maximum flow in $\widetilde{G}^*$ between artificial nodes. Build a fuzzy residual network starting with zero flows according to the Definition 2.3.

Step 3. Search the shortest path $\widetilde{P}^{*\mu}$ in terms of the number of arcs from the artificial source s^* to the artificial sink t^* in the constructed fuzzy residual network starting with zero flow values. The choice of the shortest path is according to the breadth-first search.

3.1. If the $\widetilde{P}^{*\mu}$ is found, go to the **step 4**.

3.2. The flow value $\tilde{\phi} < \sum_{\tilde{l}_{ij} \neq 0} \tilde{l}_{ij}$ is obtained, which is the maximum flow in $\widetilde{G}^*$, if the path is failed to find. It means that it is impossible to pass any unit of flow, but not all the artificial arcs are saturated. Therefore, initial graph $\widetilde{G}$ has no feasible flow and the task has no solution. Exit.

Step 4. Pass the minimum from the arc capacities $\tilde{\delta}^{*\mu} = \min\left[\tilde{u}\left(\widetilde{P}^{*\mu}\right)\right]$, $\tilde{u}\left(\widetilde{P}^{*\mu}\right) = \min\left[\tilde{u}^{*\mu}_{ij}\right]$, $\left(x^{*\mu}_i, x^{*\mu}_j\right) \in \widetilde{P}^{*\mu}$ along the path $\widetilde{P}^{*\mu}_p$.

Step 5. Update the fuzzy flow values in the graph $\widetilde{G}^*$: change the fuzzy flow $\tilde{\xi}^*_{ji}$ along the corresponding arcs $\left(x^*_j, x^*_i\right)$ from $\widetilde{G}^*$ by $\tilde{\xi}^*_{ji} - \tilde{\delta}^{*\mu}$ for arcs $\left(x^{*\mu}_i, x^{*\mu}_j\right) \notin \widetilde{A}^*$, $\left(x^{*\mu}_i, x^{*\mu}_j\right) \in \widetilde{A}^{*\mu}$ in $\widetilde{G}^{*\mu}$. Change the fuzzy flow $\tilde{\xi}^*_{ij}$ along the arcs

$\left(x_i^*, x_j^*\right)$ from $\widetilde{G}^*$ by $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}$ for arcs $\left(x_i^{*\mu}, x_j^{*\mu}\right) \in \widetilde{A}^*$, $\left(x_i^{*\mu}, x_j^{\mu *}\right) \in \widetilde{A}^{\mu *}$ in $\widetilde{G}^{*\mu}$. Replace $\tilde{\xi}_{ij}^*$ by $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}$.

Step 6. Provide a comparison of the flow vector $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu} \times P^{*\mu}$ of the value $\tilde{\sigma}^*$ and the sum of the lower flow bounds $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ of the original graph:

6.1. If the flow vector $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu} \times \widetilde{P}^{*\mu}$ of the value $\tilde{\sigma}^*$ is less than $\sum_{\tilde{l}_{ij} \neq 0} \tilde{l}_{ij}$, i.e. not all artificial arcs become saturated, go to the **step 2**, i.e. to constructing of the new incremental graph with the flow passing along the arcs until $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}$ becomes equal to $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$.

6.2. If the flow value $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu} \times P^{*\mu}$ is equal to $\sum_{\tilde{l}_{ij} \neq 0} \tilde{l}_{ij}$, i.e. all arcs from the artificial source to the artificial sink become saturated, then the value $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu} \times P^{*\mu}$ is required value of maximum flow $\tilde{\sigma}^*$ in $\widetilde{G}^*$ In this case the flow $\tilde{\xi}_{ts}^*$ passing along the artificial arc (t, s) in $\widetilde{G}^*$ determines the feasible flow in the initial graph $\widetilde{G}$ of the value $\tilde{\sigma} = \tilde{\xi}_{ts}^*$. Turn to the graph $\widetilde{G}$ from the graph $\widetilde{G}^*$ according to the *rule 2.2*. The network $G\left(\tilde{\xi}\right)$ is obtained. Go to the **step 7**.

Step 7. Construct the residual network $G\left(\tilde{\xi}^\mu\right)$ taking into account the feasible flow vector $\tilde{\xi} = \left(\tilde{\xi}_{ij}\right)$ in the graph $\widetilde{G}$ according to the *rule 2.3*.

Step 8. Define the shortest path $\widetilde{P}^\mu$ according to the number of arcs from the artificial source to the artificial sink in the constructed residual network $G\left(\tilde{\xi}^\mu\right)$. The choice of the shortest path is according to the breadth-first search.

8.1. Go to the **step 9** if the augmenting path $\widetilde{P}^\mu$ is found.

8.2. The maximum flow $\tilde{\xi}_{ij} + \tilde{\delta}^\mu \times \widetilde{P}^\mu = \tilde{v}$ in $\widetilde{G}$ is found if the path is failed to find, then stop.

Step 9. Pass $\tilde{\delta}^\mu = \min\left[\tilde{u}\left(\widetilde{P}^\mu\right)\right]$, $\tilde{u}\left(\widetilde{P}^\mu\right) = \min\left[\tilde{u}_{ij}^\mu\right]$, $\left(x_i^\mu, x_j^\mu\right) \in \widetilde{P}^\mu$ along the found path.

Step 10. Update the fuzzy flow values in the graph $\widetilde{G}$: replace the fuzzy flow $\tilde{\xi}_{ji}$ along the corresponding arcs $\left(x_j, x_i\right)$ from $\widetilde{G}$ by $\tilde{\xi}_{ji} - \tilde{\delta}^\mu$ for arcs $\left(x_i^\mu, x_j^\mu\right) \notin \widetilde{A}$, $\left(x_i^\mu, x_j^\mu\right) \in \widetilde{A}^\mu$ in $\widetilde{G}^\mu\left(\tilde{\xi}\right)$ and change the fuzzy flow $\tilde{\xi}_{ij}$ along the arcs $\left(x_i, x_j\right)$ from $\widetilde{G}$ by $\tilde{\xi}_{ij} + \tilde{\delta}^\mu$ for arcs $\left(x_i^\mu, x_j^\mu\right) \in \widetilde{A}$, $\left(x_i^\mu, x_j^\mu\right) \in \widetilde{A}^\mu$, in $\widetilde{G}^\mu\left(\tilde{\xi}\right)$ and replace the flow value in $\widetilde{G}$: $\tilde{\xi}_{ij} \to \tilde{\xi}_{ij} + \tilde{\delta}^\mu \times \widetilde{P}^\mu$ and turn to the **step 7** starting from the new flow value along the arcs.

Thus the described algorithm allows to find the maximum flow in networks with lower flow bounds in fuzzy conditions or show that the feasible flow doesn't exist.

Let us consider the proof of the main provisions of the proposed algorithm.

Let us show that if there is the maximum flow equals the sum of the lower flow bounds in the graph $\widetilde{G}^*$, there is the feasible flow the original graph. It is necessary to introduce the Theorem 2.1.

Theorem 2.1 *If the maximum flow in the graph $\widetilde{G}^*$ is equal to the sum of the lower flow bounds $\tilde{\sigma}^* = \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$, therefore, there is the feasible flow of the value $\tilde{\sigma} = \tilde{\xi}^*_{ts}$ in the original graph $\widetilde{G}$.*

Proof Let us assume that it is false, i.e. if the maximum flow in the graph $\widetilde{G}^*$ is not equal to the sum of the lower flow bounds $\tilde{\sigma}^* \neq \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$, therefore, there is the feasible flow of the value $\tilde{\sigma} = \tilde{\xi}^*_{ts}$ in the graph $\widetilde{G}$. Graph $\widetilde{G}^*$ is obtained by addition of the parameter $\tilde{l}_{ij}$ to artificial arcs and subtraction of the parameter $\tilde{l}_{ij}$ from $\tilde{u}_{ij}$ from the arcs with nonzero lower flow bounds. The flow passing along the arc (t, s) defines the feasible one. Applying the inverse transformation assume that the flow is not equal to the sum of the lower flow bounds in $\widetilde{G}^*$ gives the feasible one in $\widetilde{G}$. Since the flow leaving the artificial arcs of the graph $\widetilde{G}^*$ can not be than $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$, consider the case when the flow $\tilde{\sigma}^* < \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ Then, by subtracting $\tilde{l}_{ij}$ from the flow $\tilde{\xi}^*_{ij}$ along the artificial arcs of the graph $\widetilde{G}^*$, obtain negative flow values (because the flow less than $\tilde{l}_{ij}$ is passing the artificial arcs), which contradicts the non-negativity flow condition as well as the fact that during equivalent transformations we had to get zero values of the flow and liquidate artificial arcs. Therefore, our hypothesis is not true, and the theorem is proved.

Let us show that if the feasible flow in $\widetilde{G}^*$ is equal to the sum of the lower flow bounds in the original graph, and it has a value of $\tilde{\sigma} = \tilde{\xi}^*_{ts}$ and defined as $\tilde{\xi}_{ij} = \tilde{\xi}^*_{ij} + \tilde{l}_{ij}$. It is necessary to introduce the Corollary 2.1.

Corollary 2.1 *If $\tilde{\sigma}^* = \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$, than the feasible flow vector $\tilde{\xi} = \left(\tilde{\xi}_{ij}\right)$ of the value $\tilde{\sigma}$ and defining as $\tilde{\xi}_{ij} = \tilde{\xi}^*_{ij} + \tilde{l}_{ij}$, is the feasible flow in $\widetilde{G}$ of the value $\tilde{\sigma} = \tilde{\xi}^*_{ts}$.*

Proof The proof follows from the Theorem 2.1 and the rules of transition to the graph without lower flow bounds $\widetilde{G}^*$. Since the lower flow bounds are deducted from the arc capacities during transition from the graph $\widetilde{G}$ to the graph $\widetilde{G}^*$, then artificial arcs are rejected and lower flow bounds are added to the flow value when reverse transition to the graph $\widetilde{G}$. Thus, we subtract and then add the same amount. The following flow is the feasible one because the addition of lower flow bounds ensures that the flow greater or equal to the lower flow bound of this arc will pass along any arc of the graph, i.e. conditions on the flow restrictions are satisfied for any arc of the graph $\widetilde{G}$. The corollary is proved.

Let us show that the flow defined in a fuzzy form and received on the **step 8** of the maximum flow algorithm with nonzero lower flow bounds is the fuzzy maximum flow in the original graph $\widetilde{G}$ based on the Theorem 2.2.

Theorem 2.2 *Fuzzy flow* $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu} \times \widetilde{P}^{\mu} = \tilde{v}$ *obtained at the* ***step 8*** *of the maximum flow algorithm with nonzero lower flow bounds is a fuzzy maximum flow in the original graph* $\widetilde{G}$.

Proof To show that the fuzzy flow $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu} \times \widetilde{P}^{\mu} = \tilde{v}$ is the maximum flow in the graph $\widetilde{G}$, assume that it is not true. Then the incremental fuzzy path from the source to the sink must exit in the graph $\widetilde{G}$. However, the algorithm terminates when the path doesn't exist, therefore, we have a contradiction. Thus, the obtained flow $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu} \times \widetilde{P}^{\mu} = \tilde{v}$ is the maximum flow in a graph. The theorem is proved.

Let us show, if the maximum flow in the graph $\widetilde{G}^{*}$ is less than the sum of the lower flow bounds of the initial graph, then the feasible flow doesn't exist in $\widetilde{G}$, that is reflected in the Corollary 2.2.

Corollary 2.2 *If the maximum flow* $\tilde{\sigma}^{*}$ *in* $\widetilde{G}^{*}$ *is less than the sum of the lower flow bounds of the initial graph, i.e.* $\tilde{\sigma}^{*} < \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$, *then the feasible flow doesn't exist in* $\widetilde{G}$.

Proof At the reverse transition to the initial graph from the transformed graph without lower flow bounds the flow value $\tilde{\sigma}^{*} < \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ means that the maximum flow in the original graph is less than the sum of the lower flow bounds, i.e. for any arc which has nonzero lower flow bound the flow greater than or equal to it can not be transferred. Hence, the feasible flow doesn't exist. The corollary is proved.

Consider the numerical example, which implements operations of the maximum flow finding algorithm in fuzzy network with nonzero lower flow bounds.

Numerical example 2

Let the network is represented in the form of the fuzzy directed graph given in Fig. 2.12. Values of the upper and lower flow bounds in the form of fuzzy intervals are assigned to the arcs of the graph. It is necessary to find the maximum flow in the graph and represent it in the form of the fuzzy triangular number, if the basic values of arc capacities in the form of trapezoidal fuzzy numbers are known, as shown in Fig. 2.4.

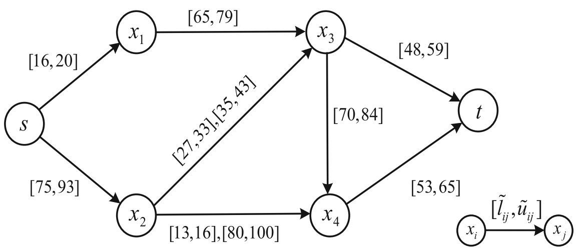


Fig. 2.12 Initial network $\widetilde{G}$

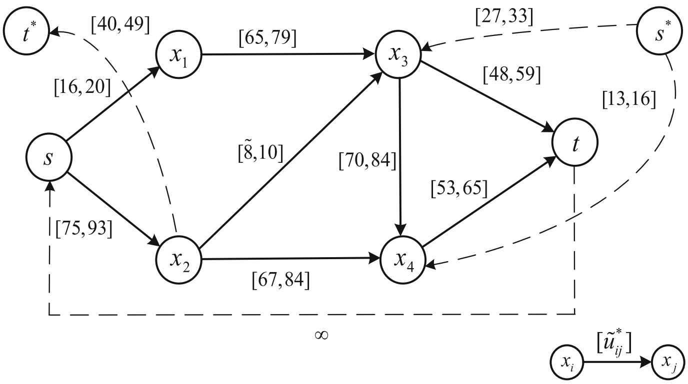


Fig. 2.13 Graph $\widetilde{G}^*$ without lower flow bounds

Step 1. Turn to the graph without lower flow bounds according to the *rule 2.1*, as shown in Fig. 2.13.

Step 2. Fuzzy residual network $\widetilde{G}^{*\mu}$ at the **step 2** coincides with the graph $\widetilde{G}^*$ without lower flow bounds, presented in Fig. 2.13 by the equality of the arc flows to $\left[\tilde{0}, \tilde{0}\right]$.

Step 3. Find the shortest path according to the number of arcs from s* to t* in the fuzzy residual network $\widetilde{G}^{*\mu}$ in Fig. 2.13 by the breadth-first-search. Obtain the path $s^* \to x_3 \to t \to s \to x_2 \to t^*$.

Step 4. Pass $\tilde{\delta}^{*\mu} = \min\left[\tilde{u}_{ij}^{\mu}\right]$, i.e. min $([27{,}33], [48{,}59], \infty, [75{,}93], [40{,}49]) =$ $[27{,}33]$ flow units along the path $s^* \to x_3 \to t \to s \to x_2 \to t^*$.

Step 5. Update the values of flows in $\widetilde{G}^*$.

The flow $\tilde{\xi}_{ij}^* = \left[\tilde{0}, \tilde{0}\right]$ turns to $\tilde{\xi}_{ij}^* = \left[\tilde{0}, \tilde{0}\right] + [27{,}33] = [27{,}33]$.

Construct a graph with the new flow value, as shown in Fig. 2.14.

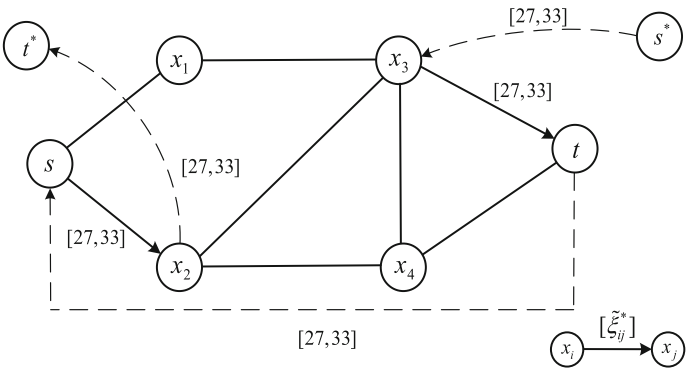


Fig. 2.14 Graph $\widetilde{G}^*$ with the new flow value of [27,33] units

Step 6.1. Turn to the **step 2**, i.e. to constructing the fuzzy residual network $\widetilde{G}^{*\mu}$ taking into account the flow in Fig. 2.14, as the obtained flow is less than the sum of the lower flow bounds $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ in $\widetilde{G}^{*}$ ($[27{,}33] < [40{,}49]$).

Step 2. Define arc capacities of the fuzzy residual network $\widetilde{G}^{*\mu}$ according to the flow values going along the arcs in Fig. 2.14.

Construct fuzzy residual network, as shown in Fig. 2.15.

Step 3. Find the shortest path according to the number of arcs from s^* to t^* in the fuzzy residual network. Use the breadth-first-search and obtain the path $s^* \rightarrow x_4 \rightarrow t \rightarrow s \rightarrow x_2 \rightarrow t^*$.

Step 4. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}_{ij}^{\mu}\right]$, i.e. min $([13{,}16], [53{,}65], \infty, [48{,}60], [27{,}33]) = [13{,}16]$ flow units along the path $s^* \rightarrow x_4 \rightarrow t \rightarrow s \rightarrow x_2 \rightarrow t^*$

Step 5. Update the flow values in the graph $\widetilde{G}^{*}$.

The flow $\tilde{\xi}_{ij}^{*} = [27{,}33]$ turns to $\tilde{\xi}_{ij}^{*} = [27{,}33] + [13{,}16] = [40{,}49]$. Build the graph with the new flow value, as shown in Fig. 2.16.

Step 6.2. As the obtained flow equals the sum of the lower flow bounds $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ in $\widetilde{G}^{*}$ ([40,49] = [40,49]), the maximum flow in $\widetilde{G}^{*}$ and, therefore, the feasible flow in $\widetilde{G}$, defined by the flow passing along the artificial reverse arc (t, s) are obtained.. Thus, the feasible flow in $\widetilde{G}$ is [40,49] units. Graph $\widetilde{G}\left(\tilde{\xi}\right)$ with the feasible flow, defined according to the *rule 2.2*, is presented in Fig. 2.17.

Step 7. Determine arc capacities of the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ according to the rule *2.3* for the graph in Fig. 2.17 by the flow values, passing along the arcs of the graph.

Build fuzzy residual network, as shown in Fig. 2.18.

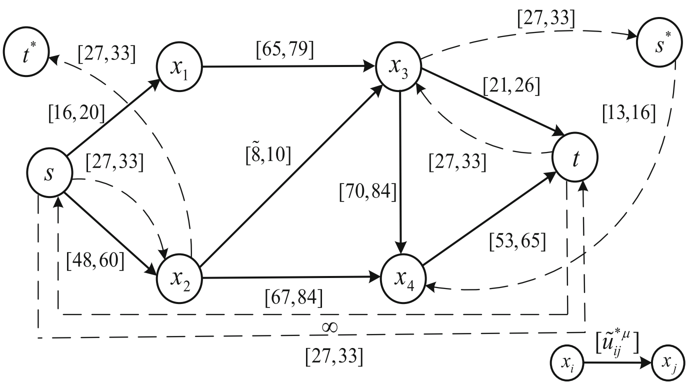


Fig. 2.15 Fuzzy residual network $\widetilde{G}^{*\mu}$ for the graph in Fig. 2.14

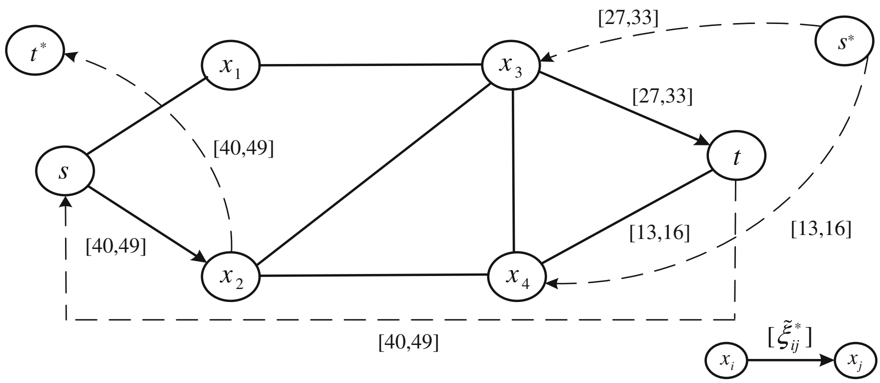


Fig. 2.16 Graph $\widetilde{G}^*$ with the new flow value of [40,49] units

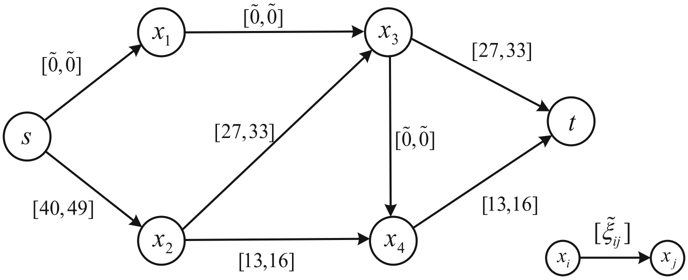


Fig. 2.17 Graph $\widetilde{G}(\tilde{\xi})$ with the feasible flow of [40,49] units

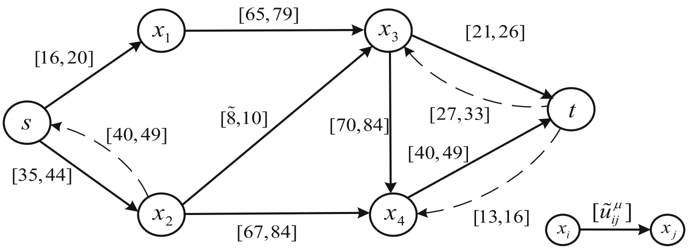


Fig. 2.18 Fuzzy residual network $\widetilde{G}^{\mu}(\tilde{\xi})$ for the graph in Fig. 2.17

Step 8. Search the shortest path according to the number of arcs from s^* to t^* in the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ in Fig. 2.18. Use the breadth-first-search and obtain the path $s \to x_1 \to x_3 \to t$.

Step 9. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}^{\mu}_{ij}\right]$, i.e. min ([16,20], [65,79], [21,26]) = [16,20] flow units along the path $s \to x_1 \to x_3 \to t$

Step 10. Update the flow values in $\widetilde{G}$.

The flow $\tilde{\xi}_{ij}$ = [40,49] turns to [40,49] + [16,20] = [56,69]. Construct the graph with the new flow value, as represented in Fig. 2.19, and turn to the **step 7**.

Step 7. Determine arc capacities of the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ according to the *rule 2.3* for the graph in Fig. 2.19 by the flow values, passing along the arcs of the graph.

Build fuzzy residual network, as shown in Fig. 2.20.

Step 8. Search the shortest path according to the number of arcs from s^* to t^* in the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ in Fig. 2.20. Use the breadth-first-search and obtain the path $s \to x_2 \to x_3 \to t$.

Step 9. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}^{\mu}_{ij}\right]$, i.e. min $\left([35,44],[\tilde{8},10],[\tilde{5},\tilde{6}]\right) = [\tilde{5},\tilde{6}]$ flow units along the path $s \to x_2 \to x_3 \to t$.

Step 10. Update the flow values in the graph $\widetilde{G}$.

The flow $\tilde{\xi}_{ij} = [56{,}69]$ turns to $[56{,}69] + [\tilde{5},\tilde{6}] = [61{,}75]$. Construct the graph with the new flow value, as shown in Fig. 2.21 and turn to the **step 7**.

Step 7. Define arc capacities of the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ according to the *rule 2.3* in Fig. 2.21 by the flow values passing along the arcs of the graph.

Build a fuzzy residual network, as shown in Fig. 2.22.

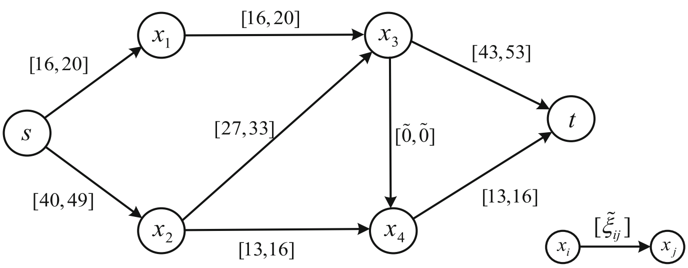


Fig. 2.19 Graph $\widetilde{G}(\tilde{\xi})$ with the feasible flow of [56,69] units

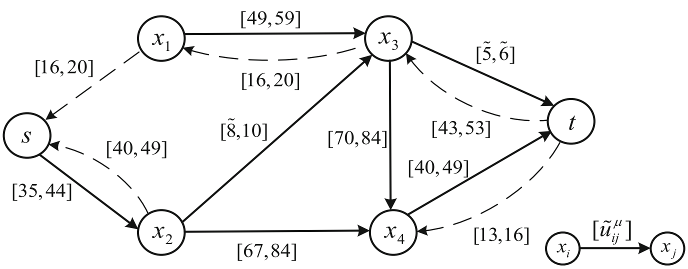


Fig. 2.20 Fuzzy residual network $\widetilde{G}^{\mu}(\xi)$ for the graph in Fig. 2.19

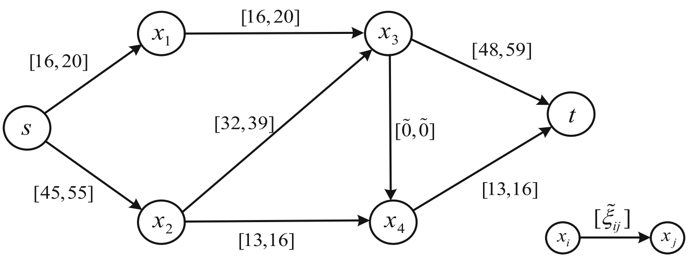


Fig. 2.21 Graph $\widetilde{G}(\tilde{\xi})$ with the feasible flow of [61,75] units

Step 8. Search the shortest path according to the number of arcs from s^* to t^* in the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ in Fig. 2.22. Use the breadth-first-search and obtain the path $s \to x_2 \to x_4 \to t$.

Step 9. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}_{ij}^{\mu}\right]$, i.e. min [30,38], [67,84], [40,49] = [30,38] flow units along the path $s \to x_2 \to x_4 \to t$.

Step 10. Update the flow values in the graph $\widetilde{G}\left(\tilde{\xi}\right)$.

Construct the graph with the new flow value, as shown in Fig. 2.23 and turn to the **step 7**.

Step 7. Define arc capacities of the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ according to the *rule 2.3* for the graph in Fig. 2.23 by the flow values, passing along the arcs of the graph.

Construct fuzzy residual network, as represented in Fig. 2.24.

Step 8. Search the shortest path according to the number of arcs from s^* to t^* in the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ in Fig. 2.24. Use the breadth-first-search and find that there is no such a path. Therefore, the maximum flow $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu} = \tilde{v}$ of the value [91,113] flow units is obtained in the initial graph. Network with the maximum flow is represented in Fig. 2.23.

Define the borders of uncertainty of the resulting fuzzy interval [91,113] corresponding to the maximum flow in a graph $\widetilde{G}$. The obtained result follows the

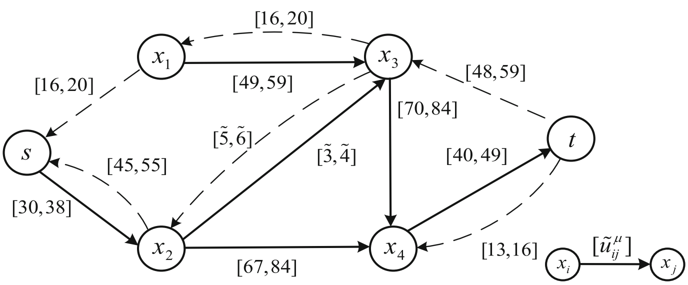


Fig. 2.22 Fuzzy residual network $\widetilde{G}^{\mu}(\xi)$ for the graph in Fig. 2.21

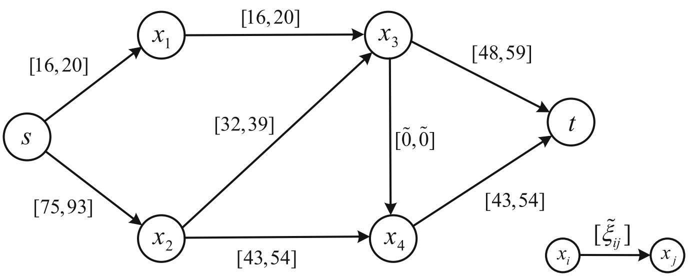


Fig. 2.23 Graph $\widetilde{G}(\tilde{\xi})$ with the feasible flow of [91,113] units

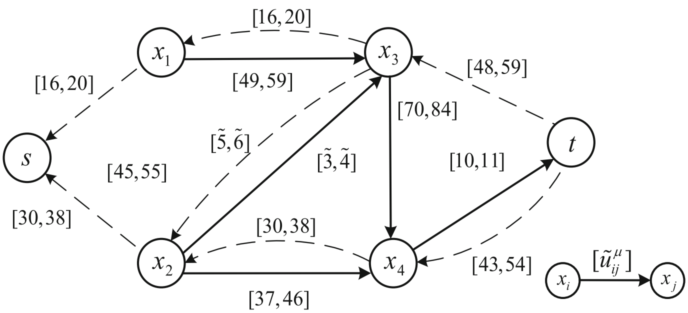


Fig. 2.24 Fuzzy residual network $\widetilde{G}^{\mu}(\xi)$ for the graph in Fig. 2.23

basic value of fuzzy arc capacity: [74,89] with the left deviation $l_1^L = 16$ and the right deviation $l_1^R = 18$ in the form of the fuzzy trapezoidal number (74,89,16,18). Therefore, the deviation borders of the fuzzy interval [91,113] coincide with the deviation borders of the previous number.

Therefore, we can present the value of the maximum flow in the form of the trapezoidal fuzzy number (91,113,16,18), as shown in Fig. 2.25.

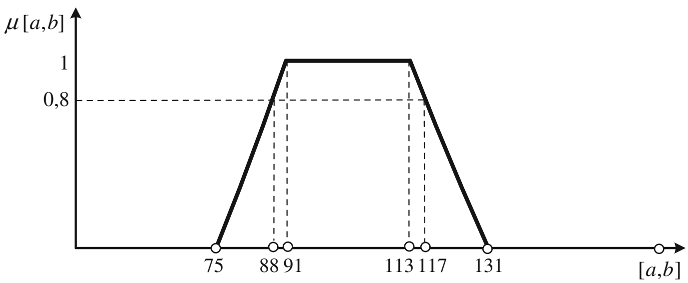


Fig. 2.25 The maximum flow in the form of the fuzzy trapezoidal number (91,113,16,18) of the flow units

The maximum flow with the degree of confidence equals 0,8 is in the interval [88,117], but anyway the maximum flow will be no less than 75 and no more than 131 units.

2.4 Minimum Cost Flow Finding in a Network with Fuzzy Arc Capacities and Transmission Costs

Suppose it is necessary to determine the minimum transportation cost of a certain quantity of cargo from a given point to the terminal one, given the restrictions on arc capacities of the road sections. Then we come to the problem of the minimum cost flow determining in the transportation network. Considering the fuzzy nature of arc capacities and transmission cost, we obtain the problem of determining the minimum cost flow in the transportation network in fuzzy conditions. Thus, the problem statement of the minimum cost flow finding in the transportation network with fuzzy arc capacities and costs can be represented as follows [4]:

$$\sum_{(x_i,x_j)\in\widetilde{A}} \tilde{c}_{ij}\tilde{\xi}_{ij} \to \min, \tag{2.9}$$

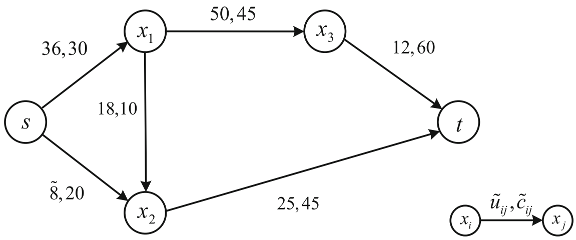
$$\sum_{x_j\in\Gamma(x_i)} \tilde{\xi}_{ij} = \sum_{x_k\in\Gamma^{-1}(x_i)} \tilde{\xi}_{ki} = \begin{cases} \tilde{\rho}, & x_i = s, \\ -\tilde{\rho}, & x_i = t, \\ \tilde{0}, & x_i \neq s,t, \end{cases} \tag{2.10}$$

$$\tilde{\xi}_{ij} \le \tilde{u}_{ij}, \quad \forall\,(x_i,x_j)\in\widetilde{A}. \tag{2.11}$$

In the model (2.9)–(2.11) $\tilde{c}_{ij}$—transmission cost one the one flow unit along the arc (x_i,x_j); $\tilde{\rho}$—given flow value, which transmission cost should be minimized.

Basaker and Gowan's algorithms [12] of the sequential search of the shortest paths and M. Klein's algorithm of the negative weight cycles detection and removal are used for solving this problem in clear terms, as was described in the first chapter

Fig. 2.26 Initial network $\widetilde{G}$



of the following monograph. Since the area of research is transportation networks, the original flow transmission costs are non-negative, then, the original graph will contain no cycles of the negative cost, therefore, it is inappropriate to apply M. Klein's at the first step of the minimal cost flow algorithm. It is also worth to notice that Floyd and Warshall's algorithm [13] operating routes from each node to any node is traditionally applied for detection of the negative cost cycles, which is not effective in this case because the it is sufficient to find the distance from the source to the sink. Therefore, P. Basaker, R. Gowan's algorithm and its modifications should be considered to find the minimum cost flow.

The search of the minimum cost path and maximum flow transmission along it, taking into account capacities of the arcs included in this path is carried out in the P. Basaker, R. Gowan's minimum cost flow algorithm. As the search of the minimum cost path is equivalent to finding the shortest path in the graph (in this case, the cost is equivalent to the length of the path), there is a question of choosing the optimal algorithm of the shortest path searching in the graph.

There are many algorithms for finding the shortest path, such as algorithms by Dijkstra [14], Bellman and Ford [15, 16], Floyd and Warshall [13], Levit [17], Johnson [18]. Traditionally, Dijkstra's algorithm [14] proposed in 1959 is used for finding the shortest paths (paths of the minimum cost), it is in searching for the shortest path from a given node to all the other vertices of the graph.

There are three sets that are supported during the algorithm: nodes distance to which has already been calculated (but perhaps not entirely); nodes, distance to which is calculated; nodes distance to which has not been not yet calculated. Time complexity of E. Dijkstra's algorithm depends on the data structure used to implement the queues with priority and representation of the input graph. In general, its running-time is $O(X^2)$ [19]. Algorithm of Levit [17] is also looking for the shortest path from a given node to all other vertices of the graph. Its running-time is exponential in the worst case, but in practice usually indicates time [19]. There are the same three sets that are supported during the algorithm. The nodes included in the first set are divided into two ordered sets—basic and priority queue. Each vertex is associated with a non-negative value of the length of the shortest of the currently known ways in it from the initial node. The disadvantage of the B. Levit's method is the necessity of the reprocessing of the nodes, while the advantage is the best running-time in graphs based on the real networks. But both methods are not applied to the graphs with the negative arcs lengths (transmission costs). In general, algorithm of B. Levit can be applied to such a graphs, but necessity to take into account such arcs makes the method more difficult.

Method of Bellman and Ford [15, 16] searches the shortest path from the node to other for the time $O(XA)$ and can be used for the graphs with the negative arcs lengths (transmission costs). The peculiarity of the algorithm is that it detects either the presence of the negative weight cycles that can exist if the original graph has arcs of the negative length (cost).

Algorithm of the authors Floyd and Warshall [13] looks for the shortest paths between all pairs of vertices in time $O(X^3)$. The method can be applied to graphs with negative path length (costs). It is used either to identify the cycles of the negative cost (for example, in the search algorithm of the minimum cost flow of M. Klein). This algorithm allows finding the shortest distance between the nodes without saving paths.

Algorithm of Johnson [18] looks for the shortest path between all pairs of nodes in time $O(X^2 \log X + XA)$ in general. It can be used for graphs with negative path lengths (costs). The algorithm is in the use of E. Dijkstra's algorithm for finding the shortest path previously changing weights of the edges, escaping from the negative lengths (costs). The new weights are introduced using the method of R. Bellman and L. Ford

We look for the shortest path in the fuzzy residual network in the algorithm of the minimum cost flow finding. Thus, arcs of the negative costs appear as in the case of the passing the flow along the arc $\left(x_i, x_j\right) \in \widetilde{A}$ with the transmission cost $\tilde{c}_{ij}$ the reverse arc $\left(x_j^{\mu}, x_i^{\mu}\right) \in \widetilde{A}^{\mu}$ appears in the fuzzy residual network with the cost $-\tilde{c}_{ij}$. Therefore, we cannot apply an effective E. Dijkstra's algorithm for finding the minimum cost path. Therefore, to find the minimum cost flow it is appropriate to apply the algorithm proposed by R. Busacker and P. Gowen, where L. Ford's algorithm is applied at the stage of finding the shortest chain, which allows operating negative parameters of costs, or algorithm, which reduces the negative numbers to non-negative.

The method which allows turning from negative numbers to non-negative, is based on the introduction of the potentials for each node of the graph. More effective algorithms for searching the given minimum cost flow are algorithms of the authors Edmonds and Karp [20] and Tomizawa [21], transforming the negative cost of transportations via node potentials to non-negative. The main idea of the method is that at each step of the fuzzy residual network construction transmission costs are recalculated based on the values of the potentials assigned to nodes of the graph such a way that they are non-negative. In turn, the node potentials are defined by finding the lengths of the shortest paths from the source to the specific node, where the length of the path is its transmission cost.

Thus, let us consider basic definitions of the minimum cost flow finding in a network with fuzzy arc capacities and costs.

2.4.1 Potential Method for the Minimum Cost Flow Finding in a Network with Fuzzy Arc Capacities and Transmission

Let us consider basic concepts used in the method of the minimum cost flow finding of Tomizawa's [21] based on the algorithm of the potentials and modified costs introduction described in [22] and modified for the case of application of fuzzy numbers.

The key concept of the method is the concept of fuzzy modified (reduced) costs (Definition 2.4) which is a modification of the definition, taken from [22].

Definition 2.4 Let $\tilde{\pi}(x_i)$, $x_i \in X, i = 1, \ldots, n$—some specified node weights (node potentials). We define the so-called fuzzy "modified" or reduced costs $\tilde{c}_{ij}^{\pi}$ associated with the arcs of the residual network, such as:

$$\tilde{c}_{ij}^{\pi} = \tilde{c}_{ij}^{\mu} - \tilde{\pi}(x_i) + \tilde{\pi}\left(x_j\right). \tag{2.12}$$

In (2.12) $\tilde{c}_{ij}^{\mu}$ is the transmission cost $\left(x_i, x_j\right)$ in the fuzzy residual network $\widetilde{G}^{\mu}$, i.e.

$$\tilde{c}_{ij}^{\mu} = \begin{cases} \tilde{c}_{ij}, & \mathit{if} \ \ (x_i, x_j) \in \widetilde{A}^{\mu}, (x_i, x_j) \in \widetilde{A}, \\ -\tilde{c}_{ji}, & \mathit{if} \ \ (x_i, x_j) \in \widetilde{A}^{\mu}, (x_i, x_j) \notin \widetilde{A}. \end{cases}$$

For some cycle $\widetilde{H}^{\mu}$ in the residual network:

$$\sum_{\left(x_i^{\mu}, x_i^{\mu}\right) \in \widetilde{H}^{\mu}} \tilde{c}_{ij}^{\pi} = \sum_{\left(x_i^{\mu}, x_i^{\mu}\right) \in \widetilde{H}^{\mu}} \tilde{c}_{ij}^{\mu}.$$

According to the condition from [22], let us consider the Definition 2.5.

Definition 2.5 The following equality is true for any path from s to t :

$$\sum_{\left(x_i^{\mu}, x_i^{\mu}\right) \in \widetilde{P}^{\mu}} \tilde{c}_{ij}^{\pi} = \tilde{\pi}(t) - \tilde{\pi}(s) + \sum_{\left(x_i^{\mu}, x_i^{\mu}\right) \in \widetilde{P}^{\mu}} \tilde{c}_{ij}^{\mu}. \tag{2.13}$$

According to condition (2.13) node potentials can't change the path of the minimum cost between any pair of nodes.

To prove the optimality of the flow, obtained after the introduction of the potentials, let us present Theorem 2.3, which is a modification of the theorem presented in [22].

Theorem 2.3 *The flow* $\tilde{\xi}^{*}$ *is optimal when and only when there is a vector potential, such as* $\tilde{c}_{ij}^{\pi} \geq \tilde{0}$ *for all* $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}^{\mu}$.

Proof Let us assume that there are such values $\tilde{\pi}$, that $\tilde{c}_{ij}^{\pi} \geq \tilde{0}$ for $\forall \left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}^{\mu}$. Let $\widetilde{H}^{\mu}$ be a cycle in $\widetilde{G}^{\mu}$, then:

$$\tilde{c}\left(\widetilde{H}^{\mu}\right) = \sum_{\left(x_i^{\mu}, x_i^{\mu}\right) \in \widetilde{H}^{\mu}} \tilde{c}_{ij}^{\mu} = \sum_{\left(x_i^{\mu}, x_i^{\mu}\right) \in \widetilde{H}^{\mu}} \tilde{c}_{ij}^{\pi} \geq \tilde{0}.$$

Consequently, a negative cost cycle in $\widetilde{G}^{\mu}$ does not exist and, consequently, the flow is optimal. The theorem is proved.

Let us show that if there is no cycle of the negative cost in the residual network $\widetilde{G}^{\mu}$, then there is a flow vector with non-negative reduced costs.

Assume all nodes in the residual network $\widetilde{G}^{\mu}$ can be reachable from s. Let $\tilde{\gamma}^{\mu}(s, x_j)$ determines the length of the shortest path from s to x_j^{μ} in $\widetilde{G}^{\mu}$, assuming $\tilde{c}_{ij}^{\mu}$ are path lengths. If $\widetilde{G}^{\mu}$ has no cycle of the negative cost, then $\tilde{\gamma}^{\mu}(s, x_j) \leq \tilde{\gamma}^{\mu}(s, x_i) + \tilde{c}_{ij}^{\mu}$. Therefore, setting $\tilde{\pi}(x_j) = -\tilde{\gamma}^{\mu}(s, x_j)$ we obtain:

$$\tilde{c}_{ij}^{\pi} = \tilde{c}_{ij}^{\mu} - \tilde{\pi}(x_i) + \tilde{\pi}(x_j) = \tilde{c}_{ij}^{\mu} + \tilde{\gamma}^{\mu}(s, x_i) - \tilde{\gamma}^{\mu}(s, x_j) \geq \tilde{0}.$$

For flow $\tilde{\xi}$ and node $x_j \in X$ the node condition x_j is set as:

$$\tilde{e}(x_j) = \tilde{\rho}_j + \sum_{(x_i, x_j) \in \widetilde{A}} \tilde{\xi}_{ij} - \sum_{(x_j, x_i) \in \widetilde{A}} \tilde{\xi}_{ji}. \tag{2.14}$$

According to (2.14):

If $\tilde{e}(x_j) > \tilde{0}$, then the condition in the node is excess.

If $\tilde{e}(x_j) > \tilde{0}$, then the condition in the node is deficit.

If $\tilde{e}(x_j) = \tilde{0}$, then the condition in the node is balance.

$\tilde{\rho}_j$—the given flow value for the node x_j, called the balance of the node.

If $\tilde{\rho}_j > \tilde{0}$, then node x_j is called the source, $\tilde{\rho}_j$—supply of the node x_i.

If $\tilde{\rho}_j = \tilde{0}$, the node x_j is intermediate node.

If $\tilde{\rho}_j < \tilde{0}$, the node x_j is called a sink, $\tilde{\rho}_j$—the demand of the node x_i.

For further proof of the equality $\tilde{\pi}(x_j) = -\tilde{\gamma}^{\mu}(s, x_j)$ give Lemma 2.1, which is modification of the Lemma given in [22] for using in fuzzy conditions.

Lemma 2.1 *Let us assume that the flow $\tilde{\xi}$ and potentials, assigned to the nodes, satisfy the optimality criteria of reduced costs: $\tilde{c}_{ij}^{\pi} = \tilde{c}_{ij}^{\mu} - \tilde{\pi}(x_i) + \tilde{\pi}(x_j) \geq \tilde{0}$ for $\forall (x_i^{\mu}, x_j^{\mu}) \in \widetilde{A}^{\mu}$. For each node $x_j^{\mu} \in X^{\mu}$ let $\tilde{\gamma}^{\mu}(s, x_j)$ define the length of shortest path from s to x_j^{μ} in $\widetilde{G}^{\mu}$, assuming $\tilde{c}_{ij}^{\pi}$ as arc lengths.*

The flow $\tilde{\xi}$ satisfies the optimality criteria of the reduced costs relative to the potentials, i.e. the following condition is satisfied:

$$\tilde{\pi}'(x_j) = \tilde{\pi}(x_j) - \tilde{\gamma}^{\mu}(s, x_j) \quad \text{for } \forall\, x_j \in X. \tag{2.15}$$

Proof As $\tilde{\gamma}^{\mu}(s, x_j)$ are lengths of the shortest paths from s to x_j^{μ} in $\widetilde{G}^{\mu}$ in (2.15) for $\forall\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}$, then the following condition is satisfied:

$$\tilde{\gamma}^{\mu}(s, x_j) \leq \tilde{\gamma}^{\mu}(s, x_i) + \tilde{c}_{ij}^{\pi}. \tag{2.16}$$

Therefore, based on (2.16):

$$\begin{aligned}\tilde{c}_{ij}^{\pi'} &= \tilde{c}_{ij}^{\mu} - \tilde{\pi}'(x_i) + \tilde{\pi}'(x_j) = \tilde{c}_{ij}^{\mu} - (\tilde{\pi}(x_i) - \tilde{\gamma}^{\mu}(s, x_i)) + (\tilde{\pi}(x_j) - \tilde{\gamma}^{\mu}(s, x_j)) \\ &= \tilde{c}_{ij}^{\pi} + \tilde{\gamma}^{\mu}(s, x_i) - \tilde{\gamma}^{\mu}(s, x_j) \geq \tilde{0}.\end{aligned}$$

Let us synthesize algorithm, which implements finding the minimum cost flow in the network with fuzzy arc capacities and costs, as well as the rule of the fuzzy residual network construction for finding the minimum cost flow, presented in [22] and modified for fuzzy conditions.

Rule 2.4 of construction of the fuzzy residual network for minimum cost flow finding in the network with fuzzy arc capacities and costs

Build fuzzy residual network $\widetilde{G}^{\mu} = (X^{\mu}, A^{\mu})$, where $X^{\mu} = X$—the set of nodes of the residual network equals to the set of nodes X of the network $\widetilde{G}$, a $\widetilde{A}^{\mu} = \left\{\left\langle \tilde{u}_{ij}^{\mu} / \left(x_i^{\mu}, x_j^{\mu}\right)\right\rangle\right\}$—fuzzy arcs set of the network $\widetilde{G}^{\mu}$, according to the following rules: for all arcs: if $\tilde{\xi}_{ij} < \tilde{u}_{ij}$, then include the corresponding arc in $\widetilde{G}^{\mu}$ with arc capacity $\tilde{u}_{ij}^{\mu} = \tilde{u}_{ij} - \tilde{\xi}_{ij}$, reduced cost $\tilde{c}_{ij}^{\pi} = \tilde{c}_{ij}^{\mu} - \tilde{\pi}_i + \tilde{\pi}_j$, where $\tilde{c}_{ij}^{\pi} = \tilde{c}_{ij}$. For all arcs, if $\tilde{\xi}_{ij} > \tilde{0}$, then include the corresponding arc in $\widetilde{G}^{\mu}$ with arc capacity $\tilde{u}_{ji}^{\mu} = \tilde{\xi}_{ij}$ and modified cost $\tilde{c}_{ji}^{\pi} = \tilde{c}_{ji}^{\mu} + \tilde{\pi}_i - \tilde{\pi}_j$, where $\tilde{c}_{ji}^{\mu} = -\tilde{c}_{ij}$.

Using the properties and lemmas presented below, we introduce algorithm for the minimum cost flow finding in the network with fuzzy arc capacities and transmission costs.

Algorithm of the minimum cost flow finding in the network with fuzzy arc capacities and transmission costs

Step 1. Introduce initial values: $\tilde{\xi}_{ij} = \tilde{0}, \tilde{\pi}(x_i) = \tilde{0}, \tilde{e}(x_i)\tilde{\rho}_i$ for $\forall\, x_i \in X$.

Step 2. Check the existence of the exceed in the node s.

2.1. If there is the exceed in the node s, i.e. $\tilde{e}(s) \geq \tilde{0}$, go to the **step 3**.

2.2. If there is no exceed in the node s, therefore, given flow value is found. To find its minimum cost turn to the given costs from the reduced ones, **exit**.

Step 3. Build fuzzy residual network $\widetilde{G}^{\mu}$ according to the *rule 2.4*. In the first step fuzzy residual network coincides with the original one due to the equality $\tilde{\xi}_{ij} = \tilde{0}$, and reduced costs coincide with initial ones, since $\tilde{\pi} = \tilde{0}$.

Step 4. Determine the minimum cost path $\widetilde{P}^{\mu}$ from s to t using E. Dijkstra's algorithm in the residual network, based on the reduced costs $\tilde{c}_{ij}^{\pi}$.

4.1. If the path exists, i.e. the permanent label is assigned to the node t, stop and go to the **step 5**.

4.2. If the path does not exist, i.e. vertex **t** is not reachable, then the task has no solution, **exit**.

Step 5. Pass $\tilde{\delta}^{\mu} = \min\left\{\tilde{u}\left(\widetilde{P}^{\mu}\right), \tilde{e}(s), -\tilde{e}(t)\right\}$ flow units along the found path $\widetilde{P}^{\mu}$, where $\tilde{u}\left(\widetilde{P}^{\mu}\right)$—arc capacity of the path $\widetilde{P}^{\mu}$, defined by the minimum from the arc capacities of this path, i.e. $\tilde{u}\left(\widetilde{P}^{\mu}\right) = \min\left[\tilde{u}_{ij}^{\mu}\right]$, $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{P}^{\mu}$.

Step 6. Define new values of the node potentials, as:

$$\tilde{\pi}(x_i) = \begin{cases} \tilde{\pi}(x_i) - \tilde{\gamma}^{\mu}(s, x_i), & \textit{if the node } x_i^{\mu} \textit{ has permanent label}, \\ \tilde{\pi}(x_i) - \tilde{\gamma}^{\mu}(s, t), & \textit{in other case}. \end{cases} \tag{2.17}$$

Step 7. Update flow values in the graph $\widetilde{G}$: for arcs $\left(x_i^{\mu}, x_j^{\mu}\right) \notin \widetilde{A}$, $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}^{\mu}$ in $\widetilde{G}^{\mu}$ change the flow $\tilde{\xi}_{ji}$ along the corresponding arcs $\left(x_j, x_i\right)$ of $\widetilde{G}$ from $\tilde{\xi}_{ji}$ to $\tilde{\xi}_{ij} - \tilde{\delta}^{\mu}$. For the arcs $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}$, $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}^{\mu}$ in $\widetilde{G}^{\mu}$ change the flow $\tilde{\xi}_{ij}$ along the corresponding arcs $\left(x_i, x_j\right)$ of $\widetilde{G}$ from $\tilde{\xi}_{ij}$ to $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}$, replace the flow value in the graph $\widetilde{G}$: $\tilde{\xi}_{ij} \rightarrow \tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ and turn to the **step 2**.

Since using the algorithm of E. Dijkstra we are interested in the way of the minimum cost from the source to the sink, there is no need to find the minimum cost path from the source to all vertices (as it is traditionally applied in the algorithm of E. Dijkstra). Consequently, the termination criterion of the minimum cost path algorithm of E. Dijkstra is not attributing the permanent marks to all vertices, and the attribution of the permanent mark to the sink node. Then updating the node potential, as it is presented in the **step 6** of the minimum cost flow algorithm will occur according to (2.17).

Let us show that this statement is true.

Thus, the new potential of the node x_i is replaced by the difference $\tilde{\pi}(x_i) - \tilde{\gamma}^{\mu}(s, x_i)$ in the shortest path from s to x_i^{μ}, if the node is assigned a permanent mark, which was shown in Lemma 2.1. If some vertices x_i have temporary labels, assign value $\tilde{\pi}(x_i) - \tilde{\gamma}^{\mu}(s, t)$ to the new values of the node potentials, i.e., the difference between the old value of potential and the shortest path length (path of the minimum cost) from s to **t**. This is true, because $\tilde{\gamma}^{\mu}(s, t)$ it is the greatest from the permanent marks, thus, these nodes, except t are not involved in the choice of the minimum cost path. That is, when the sink is reached, there is no need to consider nodes that haven't received permanent marks, so assign them the maximum of the obtained marks, i.e. $\tilde{\gamma}^{\mu}(s, t)$.

If it is necessary to find the maximum flow having a minimal cost, you can apply conventional algorithm of the minimum cost flow finding, but this algorithm requires preliminary assignment of supply and demand nodes. Since, the maximum required flow value is used as supply for the source and demand for the sink it is necessary to solve preliminary the maximum flow task in the fuzzy network for the supply and demand nodes assigning. Considering the computational complexity of J. Edmonds and R. Karp's algorithm for the maximum flow finding, it is advisable to consider the so-called algorithm of the subsequent shortest chain, described in [12] and the chain is found by Ford's algorithm [16].

Consider the numerical example that implements the algorithm of the minimum cost flow finding in the network with fuzzy arc capacities and costs.

Numerical example 3

Let the network is represented in the form of the fuzzy directed graph in Fig. 2.26. Values of arc capacities and transmission costs of the one flow unit are assigned to the arcs of the graph. It is necessary to define the minimum transportation cost of the fuzzy flow "near 18" units and represent the result in the form of the fuzzy triangular number, if there are basic values of arc capacities and transmission costs, represented in Figs. 2.27, 2.28, 2.29 and 2.30.

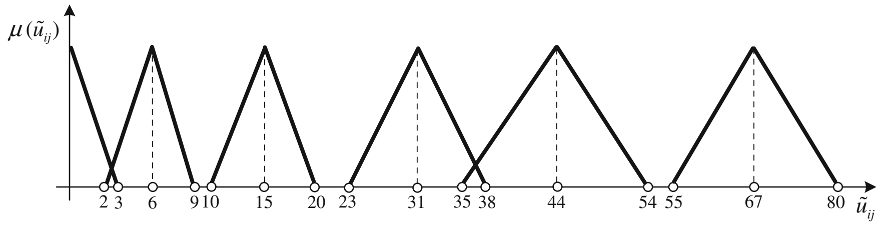


Fig. 2.27 Basic values of arc capacities

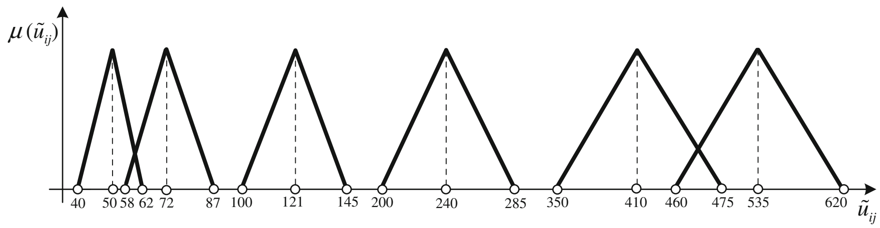


Fig. 2.28 Basic values of transmission costs

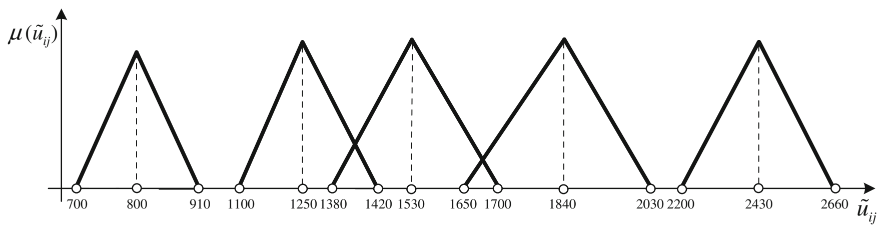


Fig. 2.29 Basic values of transmission costs

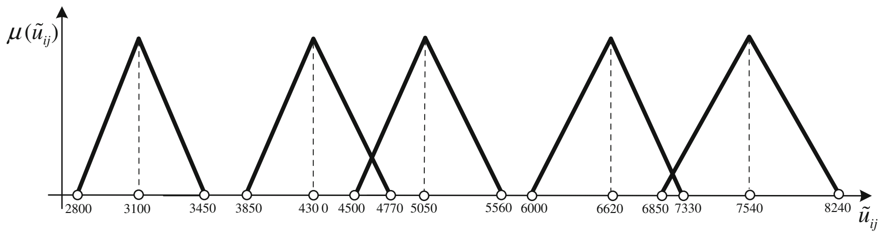


Fig. 2.30 Basic values of transmission costs

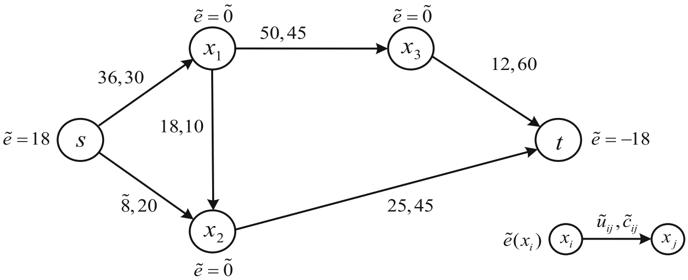


Fig. 2.31 Graph $\widetilde{G}$ with given balances of the nodes

Step 1. Define initial values of flows and node potentials equal to $\tilde{0}$ i.e. $\tilde{\xi}_{ij} = \tilde{0}, \tilde{\pi}(x_i) = \tilde{0}$. Assign the corresponding value of the exceed to each node $\tilde{e}\left(x_j\right) = \tilde{\rho}_j + \sum_{\left(x_i,x_j\right)\in\tilde{A}} \tilde{\xi}_{ij} - \sum_{\left(x_j,x_i\right)\in\tilde{A}} \tilde{\xi}_{ji}$, as shown in Fig. 2.31.

Step 2.1. Define the condition of the node s: $\tilde{e}(s) = 18 + \tilde{0} - \tilde{0} > \tilde{0}$, i.e. the node has exceed, therefore, turn to the **step 3**.

Step 3. Fuzzy residual network $\widetilde{G}^{\mu} = \left(X^{\mu}, \widetilde{A}^{\mu}\right)$ coincides with the initial one, presented in Fig. 2.31, as the arc flows equal to $\tilde{0}$.

Step 4. Define the minimum cost path from s to t using E. Dijkstra's algorithm in the residual network, represented in Fig. 2.31.

Let $\tilde{l}(x_i) = \tilde{0}$—the label of the node x_i.

Step 1. Let $\tilde{l}(s) = \tilde{0}$ and consider this label as constant one. Let $\tilde{l}(x_i) = \infty$, $\forall x_i \neq s$ and consider these labels as temporary. Let $p = s$.

First iteration

Step 2. Change the labels for all nodes with temporary labels according to the following equality:

$$\tilde{l}(x_i) = \min\left[\tilde{l}(x_i), \tilde{l}(p) + \tilde{c}(p, x_i)\right]. \tag{2.18}$$

$\Gamma(p) = \Gamma(s) = \{x_1, x_2\}$—all labels are temporary. Consider x_1. We obtain the following equation from (2.18):

$$\tilde{l}(x_1) = \min\left[\infty, \tilde{0}^+ + 30\right] = 30,$$
$$\tilde{l}(x_2) = \min\left[\infty, \tilde{0}^+ + 20\right] = 20.$$

Step 3. Define the minimum value for nodes $x_1, x_2, x_3, t:$ $\min[30{,}20, \infty, \infty] = 20$, that corresponds to x_2.

Step 4. Consider the label of the node x_2 as constant $\tilde{l}(x_2) = 20$, $p = x_2$.

Step 5. $p \neq t$, turn to the *step 2*.

Second iteration

Step 2. Change the labels for all nodes with temporary labels.

$\Gamma(p) = \Gamma(x_2) = t$—all labels are temporary. We obtain the following equation from (2.18):

$$\tilde{l}(t) = \min\,[\infty, 20 + 45] = 65.$$

Step 3. Define the minimum value for nodes $x_1, x_3, t:$ $\min[30, \infty, 65] = 30$, that corresponds to x_1.

Step 4. Consider the label of the node x_1 as constant, $\tilde{l}(x_1) = 30^+$, $p = x_1$.

Step 5. $p \neq t$, turn to the *step 2*.

Third iteration

Step 2. Change the labels for all nodes with temporary labels.

$\Gamma(p) = \Gamma(x_1) = \{x_2,\, x_3\}$—only x_3 has temporary label. We obtain the following equation from (2.18):

$$\tilde{l}(x_3) = \min[\infty, 30 + 45] = 75.$$

Step 3. Define the minimum value for nodes $x_3, t:$ $\min[75{,}65] = 65$, that corresponds to t.

Step 4. Consider the label of the node t as constant, $\tilde{l}(t) = 65^+$, $p = t$.

Step 5. $p = t$, therefore, we obtain the path of the minimum cost $\tilde{l}(t) = 65$. Find this path according to the following equation:

$$\tilde{l}(x_i') + \tilde{c}(x_i', x_i) = \tilde{l}(x_i). \qquad (2.19)$$

In (2.19) x_i'—the node, proceeding to the node x_i in the path of the minimum cost from s to x_i.

Define the node, proceeding to the node t in the path of the minimum cost from s to t.

$$\tilde{l}(x_3) + \tilde{c}(x_3, t) \neq \tilde{l}(t), \quad \text{since } 75 + 60 \neq 65,$$
$$\tilde{l}(x_2) + \tilde{c}(x_2, t) = \tilde{l}(t), \quad \text{since } 20 + 45 = 65.$$

Obtain the path $x_2 \to t$. Define the node, proceeding to the node x_2 in the path of the minimum cost from s to t.

$$\tilde{l}(x_1) + \tilde{c}(x_1, x_2) \neq \tilde{l}(x_2), \quad \text{since } 30 + 10 \neq 20,$$
$$\tilde{l}(s) + \tilde{c}(s, x_2) = \tilde{l}(x_2), \quad \text{since } \tilde{0} + 20 = 20.$$

Therefore, the path $s \to x_2 \to t$ of the cost 65 units is obtained

Step 5. Pass $\tilde{\delta}^{\mu} = \min\{\tilde{u}(\widetilde{P}^{\mu}), \tilde{e}(s), -\tilde{e}(t)\}$, i.e. $\tilde{\delta}^{\mu} = \min\{\tilde{8}, 25, 18, 18\} = \tilde{8}$ flow units along the found path.

Step 6. Define new values of the node potentials:

$$\tilde{\pi}(s) = \tilde{0} - \tilde{0} = \tilde{0},$$
$$\tilde{\pi}(x_1) = \tilde{0} - 30 = -30,$$
$$\tilde{\pi}(x_2) = \tilde{0} - 20 = -20,$$
$$\tilde{\pi}(x_3) = \tilde{0} - 65 = -65,$$
$$\tilde{\pi}(t) = \tilde{0} - 65 = -65.$$

Step 7. Update the flow values in $\widetilde{G}$.

The flow $\tilde{\xi}_{ij} = \tilde{0}$ turns to $\tilde{8}$ units. Build the graph with the new flow value, as shown in Fig. 2.32 and turn to the **step 2**.

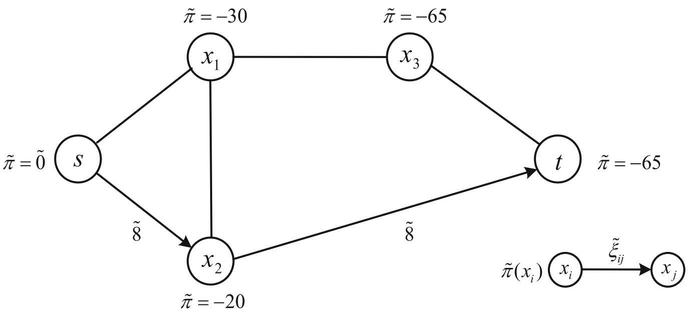


Fig. 2.32 Graph $\widetilde{G}$ with the flow of $\tilde{8}$ flow units and node potentials

Step 2. Check, if there is exceed in the node s. Since $\tilde{e}(s) = 18 + \tilde{0} - \tilde{8} > \tilde{0}$, i.e. the node has exceed, therefore, turn to the **step 3**.

Step 3. Calculate the new values of the reduced costs according to the paths lengths from s to t:

$$\begin{aligned}
\tilde{c}^{\pi}_{sx_1} &= \tilde{c}^{\mu}_{sx_1} - \tilde{\pi}_s + \tilde{\pi}_{x_1} = 30 - \tilde{0} - 30 - \tilde{0},\\
\tilde{c}^{\pi}_{x_2 s} &= \tilde{c}^{\mu}_{x_2 s} - \tilde{\pi}_{x_2} + \tilde{\pi}_s = 20 + 20 + \tilde{0} = \tilde{0},\\
\tilde{c}^{\pi}_{x_1 x_2} &= \tilde{c}^{\mu}_{x_1 x_2} - \tilde{\pi}_{x_1} + \tilde{\pi}_{x_2} = 10 + 30 - 20 = 20,\\
\tilde{c}^{\pi}_{x_1 x_3} &= \tilde{c}^{\mu}_{x_1 x_3} - \tilde{\pi}_{x_1} + \tilde{\pi}_{x_3} = 45 + 30 - 65 = 10,\\
\tilde{c}^{\pi}_{x_2 t} &= \tilde{c}^{\mu}_{x_2 t} - \tilde{\pi}_{x_2} + \tilde{\pi}_t = 45 + 20 - 65 = \tilde{0},\\
\tilde{c}^{\pi}_{t x_2} &= \tilde{c}^{\mu}_{t x_2} - \tilde{\pi}_t + \tilde{\pi}_{x_2} = -45 + 65 - 20 = \tilde{0},\\
\tilde{c}^{\pi}_{x_3 t} &= \tilde{c}^{\mu}_{x_3 t} - \tilde{\pi}_{x_3} + \tilde{\pi}_t = -60 + 65 - 65 = 60.
\end{aligned}$$

Build fuzzy residual network $\widetilde{G}^{\mu}$ according to the flow values passing along the arcs of the graph in Fig. 2.32 and calculated reduced costs, as shown in Fig. 2.33.

Step 4. Define the minimum cost path from s to t according to the E. Dijkstra's algorithm in the residual network, represented in Fig. 2.33: $s \to x_1 \to x_2 \to t$.

Step 5. Pass $\tilde{\delta}^{\mu} = \min\{\tilde{u}(\widetilde{P}^{\mu}), \tilde{e}(s), -\tilde{e}(t)\}$, i.e. $\tilde{\delta}^{\mu} = \min\{36, 18, 17, (18 + \tilde{0} - \tilde{8}), -(-18 + \tilde{8} - \tilde{0})\} = 10$ flow units along the found path.

Step 6. Define new values of the node potentials:

$$\begin{aligned}
\tilde{\pi}(s) &= \tilde{0} - \tilde{0} = \tilde{0},\\
\tilde{\pi}(x_1) &= -30 - \tilde{0} = -30,\\
\tilde{\pi}(x_2) &= -20 - 20 = -40,\\
\tilde{\pi}(x_3) &= -65 - 10 = -75,\\
\tilde{\pi}(t) &= -65 - 20 = -85.
\end{aligned}$$

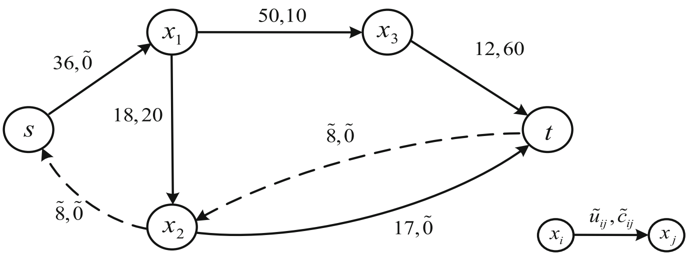


Fig. 2.33 Fuzzy residual network $\widetilde{G}^{\mu}$ for the graph in Fig. 2.32

Step 7. Update the flow values in $\widetilde{G}$.

The flow $\tilde{\xi}_{ij} = \tilde{8}$ turns to 18 units. Build the graph with the new flow value, as shown in Fig. 2.34 and turn to the **step 2**.

Step 2. Check, if there is exceed in the node s. Since $\tilde{e}(s) = 18 + \tilde{0} - 18 = \tilde{0}$, then the node s is balanced. Therefore, given flow value of the minimum cost is found.

Define the cost of the flow, which is equal to 18 flow units, as: $10 \cdot 30 + \tilde{8} \cdot 20 + 10 \cdot 10 + 18 \cdot 45 = 1370$ conventional units.

Let us define the uncertainty borders of the obtained fuzzy number 18, corresponded to the maximum flow in the graph $\widetilde{G}$. Found result is between two basic neighboring values of arc capacities: 15 with the left deviation $l_1^L = 5$, the right deviation—$l_1^R = 5$ and 31 with the left deviation $l_2^L = 8$, the right deviation—$l_2^R = 7$, presented in the form of fuzzy triangular numbers. According to (2.4) we obtain:

$$
\begin{aligned}
l^L &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^L + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^L = \frac{(31 - 18)}{(31 - 15)} \times 5 + \left(1 - \frac{(31 - 18)}{(31 - 15)}\right) \times 8 \\
&= 5.5625 \approx 6, \\
l^R &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^R + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^R = \frac{(31 - 18)}{(31 - 15)} \times 5 + \left(1 - \frac{(31 - 18)}{(31 - 15)}\right) \times 7 \\
&= 5.375 \approx 5.
\end{aligned}
$$

Therefore, we can represent the value of the maximum flow in the form of the fuzzy triangular number (18,6,5), as shown in Fig. 2.35.

According to the Fig. 2.35 the flow (18,6,5) of the minimum cost with the degree of confidence equals to 0,7 should be in the interval [16,20], but anyway the flow of the minimum cost will be no less, than 12 units and no more than 23 units.

Let us represent the minimum transportation cost of (18,6,5) flow units equals to 1370 conventional units in the form of the fuzzy triangular number.

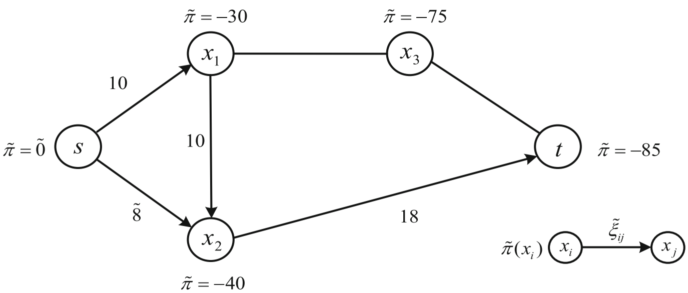


Fig. 2.34 Graph $\widetilde{G}$ with the flow of 18 units and node potentials

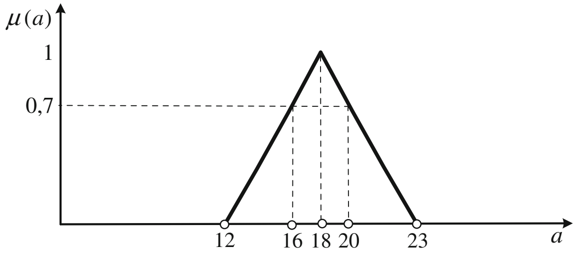


Fig. 2.35 Given flow in the form of the fuzzy triangular number of (18,6,5) units

Let us define the uncertainty borders of the obtained fuzzy number 1370 conventional units. Found result is between two basic neighboring values of arc capacities: 1250 with the left deviation $l_1^L = 150$, right deviation—$l_1^R = 170$ and 1530 with the left deviation $l_1^L = 150$, right deviation—$l_1^R = 170$. According to (2.4) we obtain:

$$l^L = \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^L + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^L = \frac{(1530 - 1370)}{(1530 - 1250)} \times 150 + \left(1 - \frac{160}{280}\right) \times 150$$
$$= 150.2 \approx 150,$$
$$l^R = \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^R + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^R = \frac{(1530 - 1370)}{(1530 - 1250)} \times 170 + \left(1 - \frac{160}{280}\right) \times 170$$
$$= 170.1 \approx 170.$$

Therefore, we can represent the minimum flow transportation cost in the form of the fuzzy triangular number (1370,150,170), as shown in Fig. 2.36.

According to the Fig. 2.36 the minimum transportation cost of (18,6,5) flow units with the degree of confidence equals to 0,8 should be in the interval [1340,1404] conventional units, but anyway the minimum transportation cost of (18,6,5) units should be no less than 1220 and no more than 1540 conventional units.

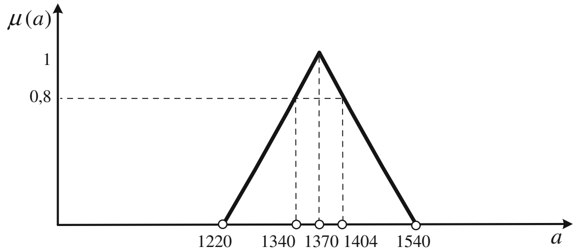


Fig. 2.36 The minimum transportation cost in the form of the fuzzy triangular number of (1370,150,170) conventional units

2.5 Minimum Cost Flow Finding in a Network with Fuzzy Nonzero Lower, Upper Flow Bounds and Transmission Costs

Fundamental problem, arising in the flow researches is the minimum cost flow finding in the network with nonzero lower flow bounds. This problem statement [23] is valid, when it is necessary to define the transportation routes of the minimum cost, taking into account transportation profitability set as the minimum flow value, that should be passed along the arcs.

In the real life it is impossible to consider this problem excluding changes in the environment and human activity, as such parameters of the network, as lower and upper flow bounds and transmission costs cannot be accurately measured. Therefore, these parameters should be represented in the fuzzy form. Thus, we obtain the problem statement of the minimum cost flow finding in the network with fuzzy nonzero lower, upper flow bounds and costs. This task will have a solution if the restriction on arc capacities will be satisfied for all arcs of the graph (2.22), and if the given flow value will not exceed the maximum flow in the graph. The problem statement can be represented as follows:

$$\sum_{(x_i,x_j)\in\widetilde{A}} \tilde{c}_{ij}\tilde{\xi}_{ij} \to \min, \tag{2.20}$$

$$\sum_{x_j\in\Gamma(x_i)} \tilde{\xi}_{ij} = \sum_{x_k\in\Gamma^{-1}(x_i)} \tilde{\xi}_{ki} = \begin{cases} \tilde{\rho}, & x_i = s, \\ -\tilde{\rho}, & x_i = t, \\ \tilde{0}, & x_i \neq s, t, \end{cases} \tag{2.21}$$

$$\tilde{l}_{ij} \le \tilde{\xi}_{ij} \le \tilde{u}_{ij}, \quad \forall\ (x_i, x_j) \in \widetilde{A}. \tag{2.22}$$

The flow task of the minimum cost flow finding in the network with fuzzy nonzero lower flow bounds, presented by the model (2.20)–(2.22) was considered in [10]. The methods of its solution in fuzzy conditions were not considered in the literature. Therefore, it is necessary to develop method, realized this task in fuzzy conditions.

To solve this problem it is necessary to introduce the rules of turning to the fuzzy graph without lower bounds flows (*the rule 2.5)* and search the maximum flow in it. The minimum cost flow is defined in the fuzzy residual network without lower flow bounds, constructed according to the Definition 2.6. After transition to the graph with the feasible flow, search given flow value of the minimum cost according to the construction of the fuzzy residual network with the feasible flow (*rule 2.6).* Present rules and definitions obtained by the synthesis of basic properties of the algorithm of the minimum cost flow finding with nonzero lower flow bounds, described in [10].

Rule 2.5 of turning to the corresponding fuzzy graph without lower flow bounds from fuzzy graph with nonzero lower bounds flows for solving the minimum cost flow problem with fuzzy nonzero lower flow bounds

Turn to the fuzzy graph $\widetilde{G}^* = (X^*, \widetilde{A}^*)$ without lower flow bounds from the given fuzzy graph $\widetilde{G} = (X, \widetilde{A})$ with nonzero lower flow bounds. Introduce artificial nodes s^* and t^*, the arc (t, s) with $\tilde{u}^*_{ts} = \infty$, $\tilde{l}^*_{ts} = \tilde{0}$, $\tilde{\xi}_{ij} + \tilde{\delta}^\mu \widetilde{P}^\mu$ in the new graph $\widetilde{G}$. For each node (x_i, x_j) in $\widetilde{G}$ with $\tilde{l}_{ij} \neq \tilde{0}$: (1) decrease $\tilde{u}_{ij}$ to $\tilde{u}^*_{ij} = \tilde{u}_{ij} - \tilde{l}_{ij}$, $\tilde{l}_{ij}$ to $\tilde{0}$, $\tilde{c}(\tilde{\xi}_{ij} + \tilde{\delta}^\mu \widetilde{P}^\mu)$. (2) Introduce arcs (s^*, x_j) and (x_i, t^*) with flow bounds equal to $\tilde{u}^*_{s^* x_j} = \tilde{u}^*_{x_i t^*} = \tilde{l}_{ij}$, $\tilde{l}^*_{s^* x_j} = \tilde{l}^*_{x_i t^*} = \tilde{0}$, costs $\tilde{c}^*_{s^* x_j} = \tilde{c}^*_{x_i t^*} = \tilde{0}$. Arcs without lower flow bounds are the same for $\widetilde{G}^*$: for any arc (x_i, x_j) with $\tilde{l}_{ij} = \tilde{0}$ is $\tilde{u}^*_{ij} = \tilde{u}_{ij}$, $\tilde{c}^*_{ij} = \tilde{c}_{ij}$.

Definition 2.6 *Fuzzy residual network for the graph* $\widetilde{G}^* = (X^*, \widetilde{A}^*)$ *without lower flow bounds for solving the minimum cost flow problem with fuzzy nonzero lower flow bounds—the network* $\widetilde{G}^{*\mu} = (X^{*\mu}, \widetilde{A}^{*\mu})$, where $X^{*\mu} = X^*$—the set of the nodes of the fuzzy residual network with artificial nodes, $\widetilde{A}^{*\mu} = \left\{ < \tilde{u}^{*\mu}_{ij} / (x^{*\mu}_i, x^{*\mu}_j) > \right\}$—fuzzy set of the arcs of the network $\widetilde{G}^{*\mu}$, constructed according the following rules: for all arcs, if $\tilde{\xi}^*_{ij} < \tilde{u}^*_{ij}$, then include corresponding arc in $\widetilde{G}^{*\mu}$ with arc capacity $\tilde{u}^{*\mu}_{ij} = \tilde{u}^*_{ij} - \tilde{\xi}^*_{ij}$, modified cost $\tilde{c}^{*\mu}_{ij} = \tilde{c}^*_{ij}$. For all arcs, if $\tilde{\xi}^*_{ij} > \tilde{0}$, then include corresponding arc in $\widetilde{G}^{*\mu}$ with arc capacity $\tilde{u}^{*\mu}_{ji} = \tilde{\xi}^*_{ij}$, modified cost $\min[28, \tilde{8}] = \tilde{8}$.

The rule 2.6 of the fuzzy residual network constructing with the feasible flow vector for solving the minimum cost flow problem with fuzzy nonzero lower flow bounds

For all arc, if $\tilde{\xi}^*_{s^* x_3} = \tilde{0}$, then include the corresponding arc in $\widetilde{G}^\mu(\tilde{\xi})$ with arc capacity $\tilde{u}^\mu_{ij} = \tilde{u}_{ij} - \tilde{\xi}_{ij}$, modified cost $\tilde{c}^\mu_{ij} = \tilde{c}_{ij}$. For all arc, if $\tilde{\xi}_{ij} > \tilde{l}_{ij}$, then include the corresponding arc in $\widetilde{G}^\mu(\tilde{\xi})$ with arc capacity $\tilde{u}^\mu_{ji} = \tilde{\xi}_{ij} - \tilde{l}_{ij}$, modified cost $\tilde{c}^\mu_{ji} = -\tilde{c}_{ij}$.

Based on the proposed rule represent algorithm for solving this task.

The minimum cost flow finding algorithm in the network with fuzzy nonzero lower, upper flow bounds and costs

Strep 1. Determine, if the task has a solution

1.1. If $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij} \leq \tilde{\rho}$, turn to the **step 2**.

1.2. If $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij} > \tilde{\rho}$, the task has no solution, **exit**.

Step 2. Let us define if the initial graph $\widetilde{G} = (X, \widetilde{A})$ has the feasible flow. Turn to the graph $\widetilde{G}^* = (X^*, \widetilde{A}^*)$ without lower flow bounds according to the *rule 2.5*.

Step 3. Build a fuzzy residual network starting with zero flows according to the Definition 2.6.

Step 4. Search the shortest path $\widetilde{P}^{*\mu}$ in terms of the number of arcs from the artificial source s^* to the artificial sink t^* in the constructed fuzzy residual network starting with zero flow values. The choice of the shortest path is according to the breadth-first search.

4.1. If the $\widetilde{P}^{*\mu}$ is found, go to the **step 5**.

4.2. The flow value $\tilde{\phi} < \sum_{\tilde{l}_{ij} \neq 0} \tilde{l}_{ij}$ is obtained, which is the maximum flow in $\widetilde{G}^*$, if the path is failed to find. It means that it is impossible to pass any unit of flow, but not all the artificial arcs are saturated. Therefore, initial graph $\widetilde{G}$ has no feasible flow and the task has no solution. Exit.

Step 5. Pass the minimum from the arc capacities $\tilde{\delta}^{*\mu} = \min[\tilde{u}(\widetilde{P}^{*\mu})]$, $\tilde{u}(\widetilde{P}^{*\mu}) = \min[\tilde{u}_{ij}^{*\mu}]$, $(x_i^{*\mu}, x_j^{*\mu}) \in \widetilde{P}^{*\mu}$ along the path $\widetilde{P}^{*\mu}$.

Step 6. Update the fuzzy flow values in the graph $\widetilde{G}^*$: change the fuzzy flow $\tilde{\xi}_{ji}^*$ along the corresponding arcs (x_j^*, x_i^*) from $\widetilde{G}^*$ by $\tilde{\xi}_{ji}^* - \tilde{\delta}^{*\mu}$ for arcs $(x_i^{*\mu}, x_j^{*\mu}) \notin \widetilde{A}^*$, $(x_i^{*\mu}, x_j^{*\mu}) \in \widetilde{A}^{*\mu}$ in $\widetilde{G}^{*\mu}$. Change the fuzzy flow $\tilde{\xi}_{ij}^*$ along the arcs (x_i^*, x_j^*) from $\widetilde{G}^*$ by $\tilde{\xi}_{ij}^* - \tilde{\delta}^{*\mu}$ for arcs $(x_i^{*\mu}, x_j^{*\mu}) \in \widetilde{A}^*$, $(x_i^{*\mu}, x_j^{*\mu}) \in \widetilde{A}^{*\mu}$ in $\widetilde{G}^{*\mu}$. Replace $\tilde{\xi}_{ij}^*$ by $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}$.

Step 7. Provide a comparison of the $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}$ и $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$:

7.1. If the flow value $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}$ of the minimum cost $\tilde{c}(\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu})$ of the value $\tilde{\sigma}^*$ is less than $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ and less than given flow value $\tilde{\rho}$, i.e. $\tilde{\sigma}^* < \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij} \leq \tilde{\rho}$, i.e. not all artificial arcs become saturated, go to the **step 3**, i.e. to constructing of the new incremental graph with the flow passing along the arcs until $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}$ becomes equal to $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$.

7.2. If the flow value $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}$ of the minimum cost $\tilde{c}\left(\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}\right)$ of the value $\tilde{\sigma}^*$ is equal to $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ and no more than $\tilde{\rho}$, $\tilde{\sigma}^* = \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij} \leq \tilde{\rho}$, i.e. all artificial arcs become saturated, then the value $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}$ is required value of the maximum flow $\tilde{\sigma}^*$ of the minimum cost $\tilde{c}\left(\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\widetilde{P}^{*\mu}\right)$. In this case, all arcs leaving the artificial source and entering artificial sink become saturated, and the flow along the artificial arc (t, s) in $\widetilde{G}^*$ determines the feasible flow in the initial graph $\widetilde{G}$ of the value $\tilde{\sigma} = \tilde{\xi}_{ts}^*$. Turn to the graph $\widetilde{G}$ from the graph $\widetilde{G}^*$ according to the *rule 2.2*. The network $\widetilde{G}(\tilde{\xi})$ with the feasible is obtained. Determine, if the flow is optimal.

7.2.1. If the flow in $\widetilde{G}(\tilde{\xi})$ is equal to $\tilde{\rho}$ of the cost $\sum_{(x_i, x_j) \in \tilde{A}} \tilde{c}_{ij}\tilde{\xi}_{ij}$, we find the minimum cost flow, **the end**.

7.2.2. If the flow in $\widetilde{G}\left(\tilde{\xi}\right)$ is less than $\tilde{\rho}$ of the cost $\sum_{(x_i, x_j) \in \tilde{A}} \tilde{c}_{ij}\tilde{\xi}_{ij}$, we obtain the network $\widetilde{G}\left(\tilde{\xi}\right)$ with the feasible flow. Turn to the **step 8**.

Step 8. Construct the residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ taking into account the feasible flow vector $\tilde{\xi} = \left(\tilde{\xi}_{ij}\right)$ in the graph $\widetilde{G}$ according to the *rule 2.6*.

Step 9. Define the shortest path $\widetilde{P}^{\mu}$ from the source to sink in the constructed residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ according to L. Ford's algorithm.

9.1. Go to the **step 10** if the augmenting path $\widetilde{P}^{\mu}$ is found.

9.2. Given flow value exceeds the maximum flow in the graph $\widetilde{G}$ if the path is failed to find, i.e. $\tilde{\rho} > \tilde{v}$ and the task has no solution, **the end**.

Step 10. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}\left(\widetilde{P}^{\mu}\right)\right]$, $\tilde{u}\left(\widetilde{P}^{\mu}\right) = \min\left[\tilde{u}_{ij}^{\mu}\right]$, $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{P}^{\mu}$ along the found path.

Step 11. Update the fuzzy flow values in the graph $\widetilde{G}$: replace the fuzzy flow $\tilde{\xi}_{ji}$ along the corresponding arcs (x_j, x_i) from $\widetilde{G}$ by $\tilde{\xi}_{ji} - \tilde{\delta}^{\mu}$ for arcs $\left(x_i^{\mu}, x_j^{\mu} \notin \widetilde{A}\right)$, $\left(x_i^{\mu}, x_j^{\mu} \in \widetilde{A}^{\mu}\right)$ in $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ and change the fuzzy flow $\tilde{\xi}_{ij}$ along the arcs (x_i, x_j) from $\widetilde{G}$ by $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}$ for arcs $\left(x_i^{\mu}, x_j^{\mu} \in \widetilde{A}\right)$, $\left(x_i^{\mu}, x_j^{\mu}\right) \in \widetilde{A}^{\mu}$ in $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ and replace the flow value in $\widetilde{G}$: $\tilde{\xi}_{ij} \to \tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ and turn to the **step 12**.

Step 12. Compare the flow value $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ and $\tilde{\rho}$:

12.1. If the flow value $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ of the minimum cost $\tilde{c}\left(\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}\right)$ is less than $\tilde{\rho}$, then replace $\tilde{\xi}_{ij}$ by $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ and turn to the **step 8**.

12.2. If the flow value $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ of the minimum cost $\tilde{c}\left(\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}\right)$ is equal to $\tilde{\rho}$, therefore, the given flow value of the minimum cost is found, **the end**.

12.3. If the flow value $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu} = \tilde{h}$ of the minimum cost $\tilde{c}\left(\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}\right)$ is more than $\tilde{\rho}$, then required flow is $\tilde{\xi}_{ij} + \left(\tilde{\delta}^{\mu} - \tilde{h} + \widetilde{P}\right)\widetilde{P}^{\mu}$ of the minimum cost $\tilde{c}\left(\tilde{\xi}_{ij} + \left(\tilde{\delta}^{\mu} - \tilde{h} + \widetilde{P}\right)\widetilde{P}^{\mu}\right)$, **the end**.

Let us consider **step 9** of the algorithm in details. As the lengths of the shortest paths from s to any other node are found (they are the final label values), then paths can be obtained via recursive procedure using the following relation:

$$l(x_i) + \tilde{c}(x_i, t) = l(t). \tag{2.23}$$

As the node x_i precedes the node t in the shortest path from s to t, then for any node x_i' the corresponding preceding arc x_i can be found as one of the remaining arcs, for which (2.23) is true. Due to the specific of flow tasks and rules of construction the incremental graphs (the reverse arc $\left(x_j^{\mu}, x_i^{\mu}\right)$ is introduced with arc capacity $\tilde{u}_{ji}^{\mu} = \tilde{\xi}_{ij}$, when the flow $\tilde{\xi}_{ij} > \tilde{0}$ is passing along the arc and artificial arcs of zero cost exist) finding the shortest path via recursive procedure, cycles can appear. If the node enters the cycle, the path is deleted and the alternative path of the minimum cost is selected (the alternative path cost and the path with cycles cost is equal).

Minimum cost maximum flow finding in the network with fuzzy nonzero lower, upper flow bounds and costs $\tilde{c}_{ij}^{\mu} \leq \tilde{0}$

Let it is necessary to find the minimum transmission cost of the maximum flow in the network taking into account restrictions on upper and lower flow bounds. This task isn't considered in the literature in fuzzy conditions, as lower, upper flow bounds and transmission costs can not be precisely defined. The range of factors, described in 1.2.1 influence parameters. Therefore, we come to the minimum cost maximum flow finding task with nonzero lower flow bounds in fuzzy conditions. The problem statement of this task is as follows:

$$\sum_{(x_i,x_j)\in\tilde{A}} \tilde{c}_{ij}\tilde{\xi}_{ij} \to \min, \tag{2.24}$$

$$\sum_{x_j\in\Gamma(x_i)} \tilde{\xi}_{ij} = \sum_{x_k\in\Gamma^{-1}(x_i)} \tilde{\xi}_{ki} = \begin{cases} \tilde{v}, & x_i = s, \\ -\tilde{v}, & x_i = t, \\ \tilde{0}, & x_i \neq s, t, \end{cases} \tag{2.25}$$

$$\tilde{l}_{ij} \leq \tilde{\xi}_{ij} \leq \tilde{u}_{ij}, \quad \forall \left(x_i, x_j\right) \in \widetilde{A} \tag{2.26}$$

In the model (2.24)–(2.26) $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$—the maximum fuzzy flow value, which transmission cost should be minimized.

Let us consider algorithm of the minimum cost maximum flow finding in the network with fuzzy nonzero upper, lower flow bounds and costs [24].

Step 1. Let us define if the initial graph $\widetilde{G} = (X, \widetilde{A})$ has the feasible flow. Turn to the graph $\widetilde{G} = \left(X^*, \widetilde{A}^*\right)$ without lower flow bounds according to the *rule 2.5.*

Step 2. Starting with zero flows, build a fuzzy residual network $\widetilde{G}^{*\mu}$ starting with zero flows according to the Definition 2.6.

Step 3. Search the shortest path $\widetilde{P}^{*\mu}$ in terms of the number of arcs from the artificial source s^* to the artificial sink t^* in the constructed fuzzy residual network starting with zero flow values. The choice of the shortest path is according to L. Ford's algorithm.

3.1. If the $\widetilde{P}^{*\mu}$ is found, go to the **step 4**.

3.2. The flow value $\tilde{\varphi}^* < \sum_{\tilde{l}_{ij}\neq\tilde{0}} \tilde{l}_{ij}$ is obtained, which is the maximum flow in $\widetilde{G}^*$, if the path is failed to find. It means that it is impossible to pass any unit of flow, but not all the artificial arcs are saturated. Therefore, initial graph $\widetilde{G}$ has no feasible flow and the task has no solution. Exit.

Step 4. Pass the minimum from the arc capacities $\tilde{\delta}^{*\mu} = \min[\tilde{u}(\widetilde{P}^{*\mu})]$, $\tilde{u}(\widetilde{P}^{*\mu}) = \min[\tilde{u}_{ij}^{*\mu}]$, $(x_i^{*\mu}, x_j^{*\mu}) \in \widetilde{P}^{*\mu}$ along the path $\widetilde{P}^{*\mu}$.

Step 5. Update the fuzzy flow values in the graph $\widetilde{G}^*$: change the fuzzy flow $\tilde{\xi}_{ji}^*$ along the corresponding arcs (x_j^*, x_i^*) from $\widetilde{G}^*$ by $\tilde{\xi}_{ji}^*$ - $\tilde{\delta}^{*\mu}$ for arcs $(x_i^{*\mu}, x_j^{*\mu}) \notin \widetilde{A}^*$,

$(x_i^{*\mu}, x_j^{*\mu}) \in \tilde{A}^{*\mu}$ in $\tilde{G}^{*\mu}$. Change the fuzzy flow $\tilde{\xi}_{ij}^*$ along the arcs (x_i^*, x_j^*) from $\tilde{G}^*$ by $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}$ for arcs $(x_i^{*\mu}, x_j^{*\mu}) \in \tilde{A}^*$, $(x_i^{*\mu}, x_j^{*\mu}) \in \tilde{A}^{*\mu}$ in $\tilde{G}^{*\mu}$. Replace $\tilde{\xi}_{ij}^*$ by $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\tilde{P}^{*\mu}$.

Step 6. Provide a comparison of the flow vector $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\tilde{P}^{*\mu}$ and $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$:

6.1. If the flow vector $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\tilde{P}^{*\mu}$ of the minimum cost $\tilde{c}(\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\tilde{P}^{*\mu})$ of the value $\tilde{\sigma}^*$ is less than $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$, i.e. not all artificial arcs become saturated, go to the **step 2**, i.e. to constructing of the new incremental graph with the flow passing along the arcs until $\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\tilde{P}^{*\mu}$ becomes equal to $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$.

6.2. If the flow value $\tilde{\xi}_{x_1x_3}^* = \tilde{0} + \tilde{0} = \tilde{0}$ of the minimum cost $\tilde{\xi}_{x_1t^*}^* = \tilde{8} + \tilde{0} = \tilde{8}$ is equal to $\tilde{\xi}_{x_2x_3}^* = \tilde{0} + \tilde{0} = \tilde{0}$, i.e. all arcs from the artificial source to the artificial sink become saturated, then the value $\tilde{\xi}_{x_2t}^* = \tilde{0} + \tilde{0} = \tilde{0}$ is required value of the maximum flow $\tilde{\xi}_{x_3t}^* = \tilde{0} + \tilde{8} = \tilde{8}$ of the minimum cost $\tilde{c}(\tilde{\xi}_{ij}^* + \tilde{\delta}^{*\mu}\tilde{P}^{*\mu})$. In this case the flow passing along the artificial arc $\tilde{\xi}_{ij}^* = \tilde{8}$ in $\tilde{G}^*$ determines the feasible flow in the initial graph $\tilde{G}^*$ of the value $\tilde{\sigma} = \tilde{\xi}_{ts}^*$. Turn to the graph $\tilde{G}$ from the graph $\tilde{G}^*$ according to the *rule 2.2*. The network is obtained. Go to the **step 7**.

Step 7. Construct the residual network $\tilde{G}^{\mu}(\tilde{\xi})$ taking into account the feasible flow vector $\tilde{\xi} = (\tilde{\xi}_{ij})$ in the graph $\tilde{G}$ according to the *rule 2.6*.

Step 8. Define the shortest path $\tilde{P}^{\mu}$ according to L. Ford's algorithm in the constructed residual network $\tilde{G}^{\mu}(\tilde{\xi})$.

8.1. Go to the **step 9** if the augmenting path $\tilde{P}^{\mu}$ is found.

8.2. The maximum flow $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\tilde{P}^{\mu} = \tilde{v}$ in $\tilde{G}$ is found if the path is failed to find, then **stop**.

Step 9. Pass $\tilde{\delta}^{\mu} = \min[\tilde{u}(\tilde{P}^{\mu})]$, $\tilde{u}(\tilde{P}^{\mu}) = \min[\tilde{u}_{ij}^{\mu}]$, $(x_i^{\mu}, x_j^{\mu}) \in \tilde{P}^{\mu}$ along the found path.

Step 10. Update the fuzzy flow values in the graph $\tilde{\xi}_{sx_2}^* < \tilde{u}_{sx_2}^*$: replace the fuzzy flow $\tilde{\xi}_{ji}$ along the corresponding arcs (x_j, x_i) from $\tilde{G}$ by $\tilde{\xi}_{ji} - \tilde{\delta}^{\mu}$ for arcs $(x_i^{\mu}, x_j^{\mu}) \notin \tilde{A}$, $(x_i^{\mu}, x_j^{\mu}) \in \tilde{A}^{\mu}$ in $\tilde{G}^{\mu}(\tilde{\xi})$ and change the fuzzy flow $\tilde{\xi}_{ij}$ along the arcs (x_i, x_j) from $\tilde{G}$ by $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}$ for arcs $(x_i^{\mu}, x_j^{\mu}) \in \tilde{A}$, $(x_i^{\mu}, x_j^{\mu}) \in \tilde{A}^{\mu}$ in $\tilde{G}^{\mu}(\tilde{\xi})$ and replace the flow value in $\tilde{G}$: $\tilde{\xi}_{ij} \to \tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\tilde{P}^{\mu}$ and turn to the **step 7** starting from the new flow value along the arcs.

Give the proof of the main items of the algorithm. Introduce Theorem 2.4, that shows if the flow equals the sum of the lower flow bounds in $\tilde{G}$ is found in $\tilde{G}^*$, then original graph $\tilde{G}$ has feasible flow.

Theorem 2.4 *If the maximum flow of the minimum cost in $\tilde{G}^*$ is equal to the sum of the lower flow bounds $\tilde{\sigma}^* = \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$, then the flow of the value $\tilde{\sigma} = \tilde{\xi}_{ts}^*$ exists in $\tilde{G}$.*

Proof Proof that if the maximum flow in $\widetilde{G}^{*}$ is equal to the sum of the lower flow bounds $\tilde{\sigma}^{*} = \sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$, then the flow of the value $\tilde{\sigma} = \tilde{\xi}^{*}_{ts}$ exists in $\widetilde{G}$ is presented in the Theorem 2.1. The required flow is the flow of the minimum cost according to the correctness of L. Ford's operations and equivalence of the minimum cost flow finding in $\widetilde{G}$ and $\widetilde{G}^{*}$. Thus, it is necessary to pass flow value equals the sum of the lower flow bounds along the paths of the minimum cost in $\tilde{8} < 36$, where the flow along the artificial arcs is used instead of lower flow bounds. If the feasible flow in $\widetilde{G}$ exists, then all arcs with nonzero lower flow bounds in $\widetilde{G}^{*\mu}$ should be saturated, i.e. the flow more, than lower flow bounds should pass along them. The costs of artificial arcs are equal to $\tilde{\xi}^{*}_{s^{*}x_1} < \tilde{u}^{*}_{s^{*}x_1}$, therefore, the choice of these arcs and passing the flow along them in $\tilde{u}^{*\mu}_{s^{*}x_1} = 28 - \tilde{8} = 20$ doesn't influence the minimum cost search in $\tilde{\xi}^{*}_{s^{*}x_1} > \tilde{0}$. Hence, we should pass the flow along the arcs with nonzero lower flow bounds, arcs from artificial source and arcs to artificial sink according to L. Ford's algorithm in $\tilde{u}^{*\mu}_{x_1 s^{*}} = \tilde{0} + \tilde{8} = \tilde{8}$ and obtain the minimum cost flow, that is equivalent to the search of the minimum cost flow in the initial graph and doesn't influence the choice of the minimum cost paths. Theorem is proved.

Let us consider numerical example, implementing the minimum cost flow finding algorithm with fuzzy nonzero lower, upper flow bounds and costs.

Numerical example 4

Let the network is represented in the form of the fuzzy directed graph in Fig. 2.37. Values of upper and lower flow bounds and transmission costs of one flow unit are assigned to arcs of the graph. It is necessary to determine the minimum transmission cost 36 flow units for the given graph with lower flow bounds and present results in the form of fuzzy triangular numbers or present, that this flow doesn't exist.

Step 1. Define, if the task has a solution.

Step 1.1. As $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij} \leq \tilde{\rho}$, i.e. $36 = 36$, then turn to the **step 2**.

Step 2. Turn to the construction of the graph without lower flow bounds according to the *rule 2.5*, as shown in Fig. 2.38.

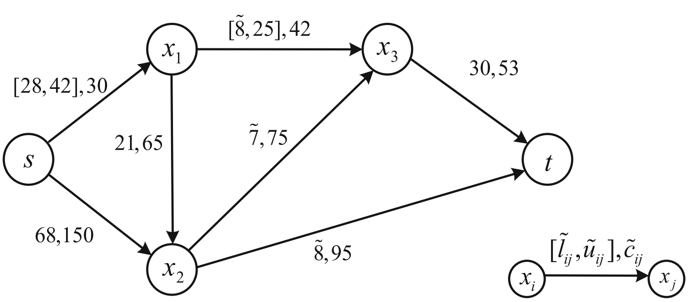


Fig. 2.37 Initial network $\widetilde{G}$

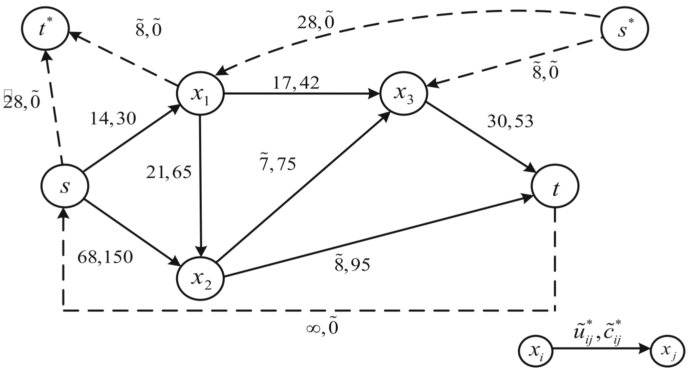


Fig. 2.38 Graph $\widetilde{G}^*$ without lower flow bounds

Step 3. Fuzzy residual network $\widetilde{G}^{*\mu}$ coincides with the graph $\widetilde{G}^*$ without lower flow bounds, presented in Fig. 2.38, as arc flows equal $\tilde{0}$.

Step 4. Find the shortest path according to the number of arcs from s^* to t^* in fuzzy residual network $\widetilde{G}^{*\mu}: s^* \to x_1 \to t^*$ of the cost $\tilde{0}$ conventional units.

Step 5. Pass $\tilde{\delta}^{*\mu} = \min\left[\tilde{u}^{*\mu}_{ij}\right]$, i.e. $\min[28, \tilde{8}] = \tilde{8}$ flow un its along the path $s^* \to x_1 \to t^*$

Step 6. Update the flow values in $\widetilde{G}^*$.

The flow $\tilde{\xi}^*_{ij} = \tilde{0}$ turns to $\tilde{\xi}^*_{ij} = \tilde{0} + \tilde{8} = \tilde{8}$.

Build a graph with the new flow value. As shown in Fig. 2.39.

Step 7.1. As the obtained flow is less than the sum of the lower flow bounds $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ in $\widetilde{G}^* \left(\tilde{8} < 36\right)$, turn to the **step 3**, i.e. to construction of the fuzzy residual network $\widetilde{G}^{*\mu}$ with the flow, presented in Fig. 2.39.

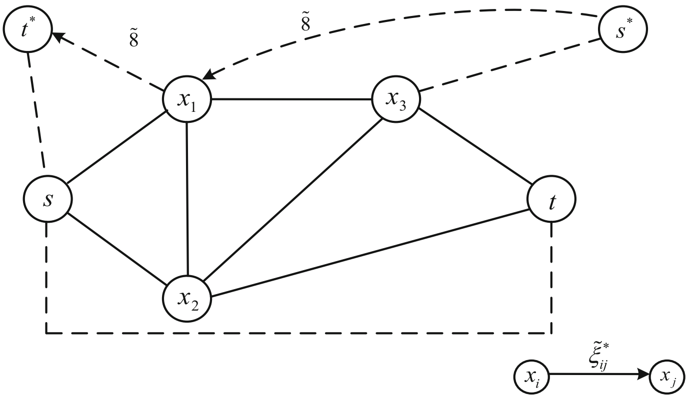


Fig. 2.39 Graph $\widetilde{G}^*$ with the new flow value of $\tilde{8}$ units

Step 3. Define arc capacities of the fuzzy residual network $\widetilde{G}^{*\mu}$ due to the flow values, passing along the arcs of the graph in Fig. 2.39.

Build fuzzy residual network, as shown in Fig. 2.40.

Step 4. Find the shortest path according to the number of arcs from s^* to t^* in $\widetilde{G}^{*\mu}$ due to the L. Ford's algorithm: $s^* \to x_3 \to t \to s \to t^*$ of the cost 53 conventional units.

Step 5. Pass $\tilde{\delta}^{*\mu} = \min\left[\tilde{u}_{ij}^{*\mu}\right]$, i.e. $\min\left[\tilde{8}, 30, \infty, 28\right] = \tilde{8}$ flow units along the path $s^* \to x_3 \to t \to s \to t^*$.

Step 6. Update the flow values in $\widetilde{G}^*$.

The flow $\tilde{\xi}_{ij}^* = \tilde{8}$ turns to $\tilde{\xi}_{ij}^* = \tilde{8} + \tilde{8} = 16$.

Build graph with the new flow value, as shown in Fig. 2.41.

Step 7.1. Since the obtained flow is less than the sum of the lower flow bounds $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ in $\widetilde{G}^*(16 < 36)$, turn to the **step 3**.

Step 3. Determine arc capacities of the network $\widetilde{G}^{*\mu}$ due to the flow values, passing along the arcs of the graph in Fig. 2.41.

Build fuzzy residual network, as shown in Fig. 2.42.

Step 4. Find the shortest path according to the number of arcs from s^* to t^* in $\widetilde{G}^{*\mu}$ due to the L. Ford's algorithm: $s^* \to x_1 \to x_3 \to t \to s \to t^*$ of the cost of 95 units.

Step 5. Pass $\tilde{\delta}^{*\mu} = \min\left[\tilde{u}_{ij}^{*\mu}\right]$, i.e. $\min[20, 17, 22, \infty, 20] = 17$ flow units along the path $s^* \to x_1 \to x_3 \to t \to s \to t^*$.

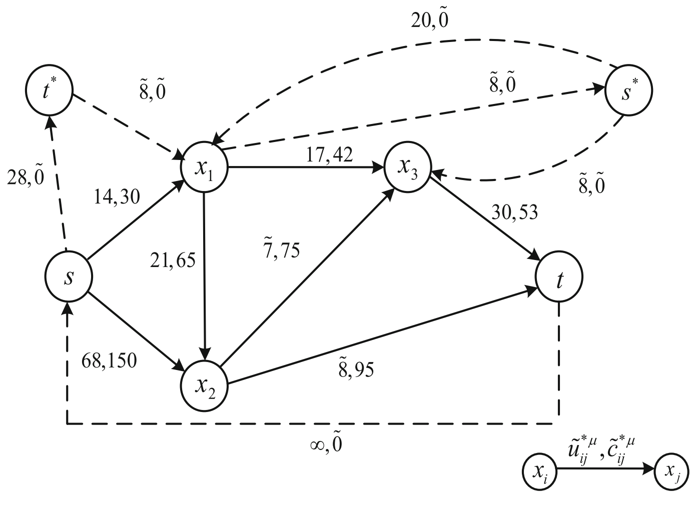


Fig. 2.40 Fuzzy residual network $\widetilde{G}^{*\mu}$ for graph in Fig. 2.39

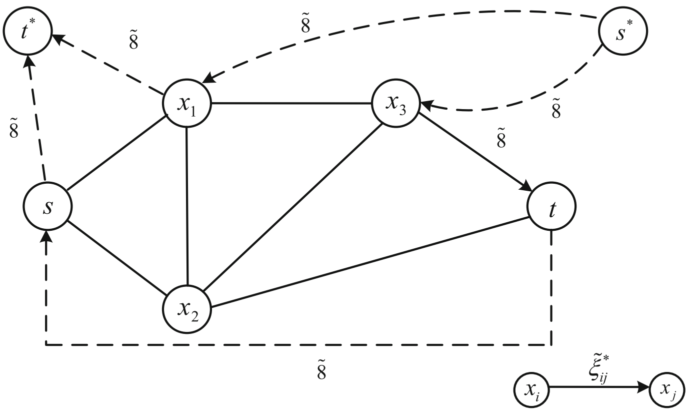


Fig. 2.41 Graph $\widetilde{G}^*$ with the new flow value of 16 units

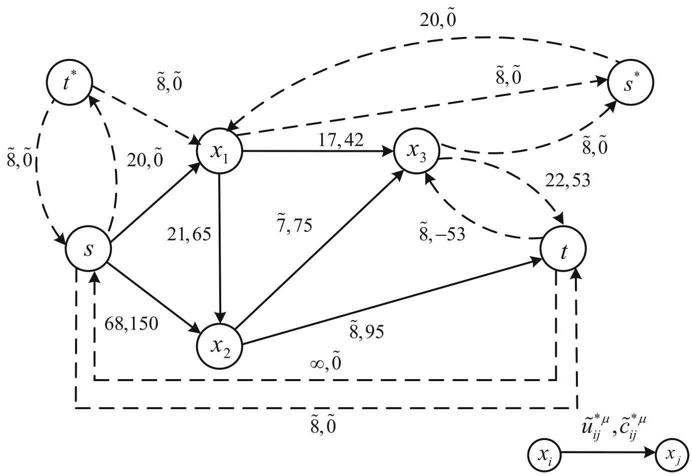


Fig. 2.42 Fuzzy residual network $\widetilde{G}^{*\mu}$ for graph in Fig. 2.41

Step 6. Update flow values in $\widetilde{G}^*$.

The flow $\tilde{\xi}^*_{ij} = 16$ turns to $\tilde{\xi}^*_{ij} = 16 + 17 = 33$.

Build graph with the new flow value, as shown in Fig. 2.43.

Step 7.1. Since the obtained flow is less than the sum of the lower flow bounds $\sum_{\tilde{l}_{ij} \neq \tilde{0}} \tilde{l}_{ij}$ in $\widetilde{G}^*(33 < 36)$, turn to the **step 3**.

Step 3. Define arc capacities of the fuzzy residual network $\widetilde{G}^{*\mu}$ due to the flow values, passing along the arcs of the graph in Fig. 2.43.

Build fuzzy residual network, as shown in Fig. 2.44.

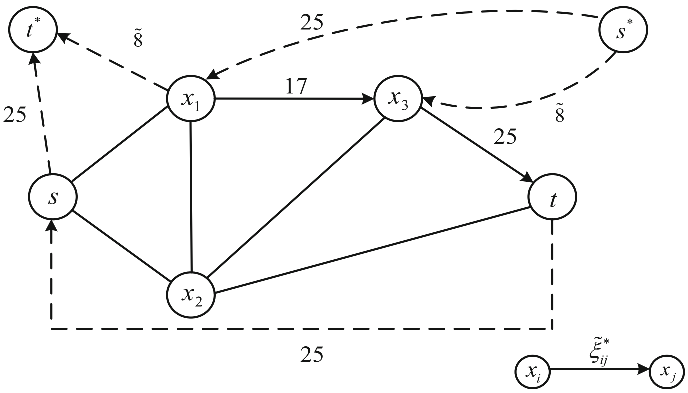


Fig. 2.43 Graph $\widetilde{G}^*$ with the new flow value of 33 units

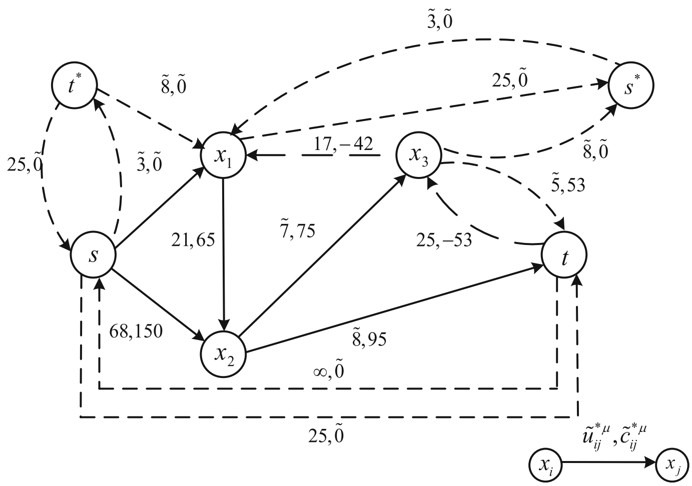


Fig. 2.44 Fuzzy residual network $\widetilde{G}^{*\mu}$ for the graph in Fig. 2.43

Step 4. Find the shortest path according to the number of arcs from s^* to t^* in $\widetilde{G}^{*\mu}$ due to the L. Ford's algorithm: $s^* \to x_1 \to x_2 \to t \to s \to t^*$ of the cost и 160 conventional units.

Step 5. Pass $\tilde{\delta}^{*\mu} = \min\left[\tilde{u}^{*\mu}_{ij}\right]$, i.e. $\min[3,\ 21, \tilde{8},\ \infty,\ \tilde{3}] = \tilde{3}$ flow units along the path $s^* \to x_1 \to x_2 \to t \to s \to t^*$.

Step 6. Update flow values in $\widetilde{G}^*$.

The flow $\tilde{\xi}^*_{ij} = 33$ turns to $\tilde{\xi}^*_{ij} = 33 + \tilde{3} = 36$. Build graph with the new flow value, as shown in Fig. 2.45.

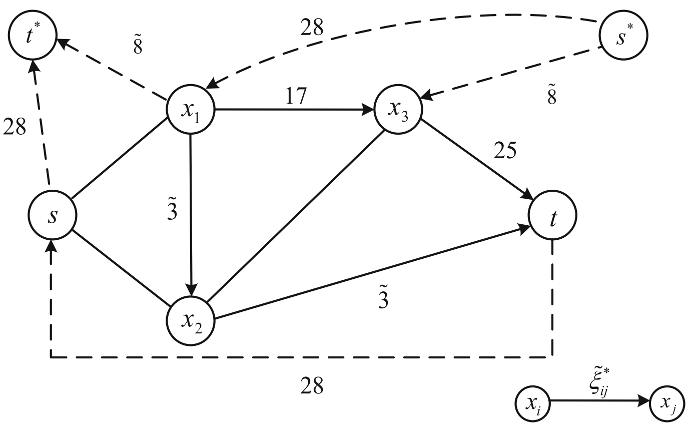


Fig. 2.45 Graph $\widetilde{G}^*$ with the new flow value of 36 units

Step 7.2.2. Since the obtained flow is equal to the sum of the lower flow bounds $\sum_{\tilde{l}_{ij}\neq\tilde{0}} \tilde{l}_{ij}$ and given flow $\tilde{\rho}$ $(36 = 36 = 36)$ in $\widetilde{G}$, the maximum flow is found in $\widetilde{G}^*$, therefore, there is a feasible flow in the initial $\widetilde{G}$, which is equal to the flow, passing along the artificial arc (t, s). Therefore, the feasible flow in $\widetilde{G}$, defined by the *rule 2.2,* is presented in Fig. 2.46 and is equal to 28 units, that is less, than 36 units, thus, turn to the **step 8**.

Step 8. Define arc capacities of the network $\widetilde{G}^\mu\left(\tilde{\xi}\right)$ according to the *rule 2.6* for the graph in Fig. 2.46 according to the flow values, passing along the arcs of the graph.

Build fuzzy residual network, as shown in Fig. 2.47.

Step 9. Find the shortest path according to the number of arcs from s to t in $\widetilde{G}^\mu$ due to the L. Ford's algorithm: $s \to x_1 \to x_2 \to t$ of the cost 202 conventional units.

Step 10. Pass $\tilde{\delta}^\mu = \min\left[\tilde{u}^\mu_{ij}\right]$, i.e. $\min\left[14,18,\tilde{5}\right] = \tilde{5}$ flow units along the path $s \to x_1 \to x_2 \to t$.

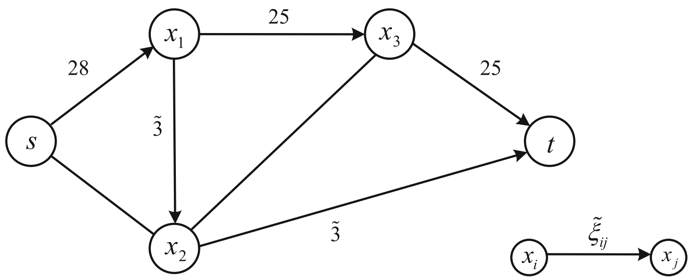


Fig. 2.46 Graph $\widetilde{G}(\tilde{\xi})$ with the feasible flow of units 28

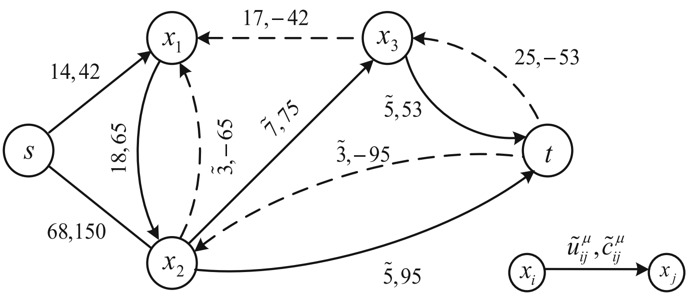


Fig. 2.47 Fuzzy residual network $\widetilde{G}^{\mu}(\tilde{\xi})$ for the graph in Fig. 2.46

Step 11. Update the flow values in $\widetilde{G}\left(\tilde{\xi}\right)$.

Step 12.1. Compare the flow value $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ and $\tilde{\rho}$. As $33 < 36$ change $\tilde{\xi}_{ij}$ by $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$, as shown in Fig. 2.48.

The flow $\tilde{\xi}_{ij} = 28$ turns to $\tilde{\xi}_{ij} = 28 + \tilde{5} = 33$, go to the **step 9**.

Step 8. Define arc capacities of the fuzzy residual network $\widetilde{G}^{\mu}\left(\tilde{\xi}\right)$ due to the *rule 2.7* for the graph in Fig. 2.48 according to the flow values, passing along the arcs of the graph.

Build fuzzy residual network, as shown in Fig. 2.49.

Step 9. Find the shortest path according to the number of arcs from s to t in $\widetilde{G}^{\mu}$ due to the L. Ford's algorithm: $s \to x_1 \to x_2 \to x_3 \to t$ of the cost 223 conventional units.

Step 10. Pass $\tilde{\delta}^{\mu} = \min\left[\tilde{u}_{ij}^{\mu}\right]$, i.e. $\min\left[\tilde{9}, 13, \tilde{7}, \tilde{5} = \tilde{5}\right]$ flow units along the path $s \to x_1 \to x_2 \to x_3 \to t$.

Step 11. Update flow values in $\widetilde{G}\left(\tilde{\xi}\right)$.

The flow $\tilde{\xi}_{ij} = 33$ turns to $\tilde{\xi}_{ij} = 33 + \tilde{5} = 38$. Build a graph with the new flow value, as shown in Fig. 2.50 and turn to the **step 12**.

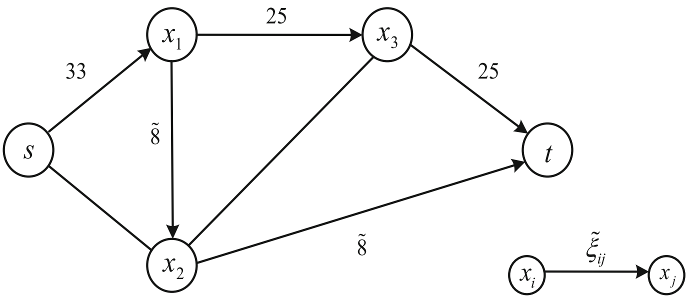


Fig. 2.48 Graph $\widetilde{G}(\tilde{\xi})$ with the feasible flow of 33 units

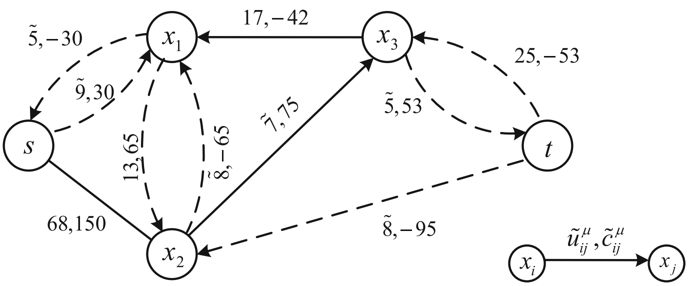


Fig. 2.49 Fuzzy residual network $\widetilde{G}^{\mu}(\tilde{\xi})$ for the graph in Fig. 2.48

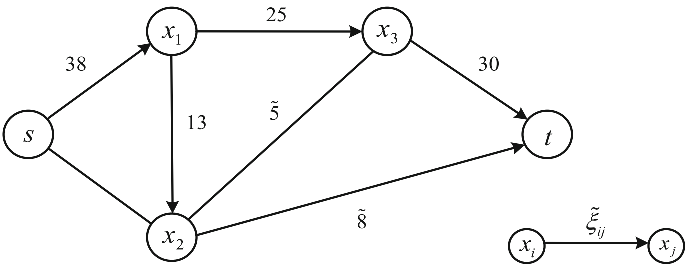


Fig. 2.50 Graph $\widetilde{G}(\tilde{\xi})$ with the feasible flow of 38 units

Step 12.3. Compare flow value $\tilde{\xi}_{ij} + \tilde{\delta}^{\mu}\widetilde{P}^{\mu}$ and $\tilde{\rho}$. Since 38 > 36, then required flow value will be $33 + (\tilde{5} - 38 + 36) \times \widetilde{P}^{\mu}$, i.e. $33 + \tilde{3} \times \widetilde{P}^{\mu}$, where $\widetilde{P}^{\mu} = s \to x_1 \to x_2 \to x_3 \to t$, that presented in Fig. 2.51.

Minimum transmission cost of 36 flow units is $36 \cdot 30 + 11 \cdot 65 + 25 \cdot 42 + \tilde{3} \cdot 75 + \tilde{8} \cdot 95 + 28 \cdot 53 = 5314$ conventional units.

Let us define deviation borders of obtained fuzzy number of 36 units, that corresponds to the given flow value in $\widetilde{G}$. The found result is between two basic neighboring values of fuzzy arc capacities: 31 with the left deviation $l_1^L = 8$, right

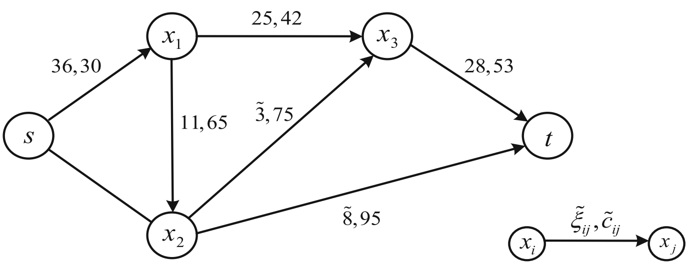


Fig. 2.51 Graph $\widetilde{G}(\tilde{\xi})$ with the flow 36 units

deviation—$l_1^R = 7$ and 44 with the left deviation $l_2^L = 9$, right deviation—$l_2^R = 10$. According to (2.4) it is obtained:

$$\begin{aligned} l^L &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^L + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^L = \frac{(44 - 36)}{(44 - 31)} \times 8 + (1 - \frac{(44 - 36)}{(44 - 31)}) \times 9 \\ &= 8.34 \approx 8, \\ l^R &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^R + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^R = \frac{(44 - 36)}{(44 - 31)} \times 7 + (1 - \frac{(44 - 36)}{(44 - 31)}) \times 10 \\ &= 8.11 \approx 8. \end{aligned}$$

Therefore, now we can represent the value of the given flow in the form of the fuzzy triangular number (36,8,8), as shown in Fig. 2.52.

Due to the Fig. 2.52 the minimum cost flow with the degree of confidence equals 0,6 will be within the interval [33,39] , but anyway it will no less than 28 units and no more than 44 units.

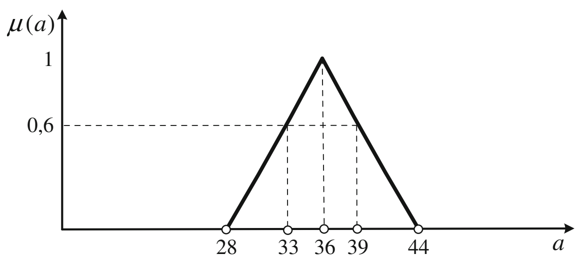
Let us represent the minimum transmission of (36,8,8) flow units equals 5314 conventional units in the form of the fuzzy triangular number.

Define uncertainty borders of the given fuzzy flow of 5314 conventional units. The obtained result is between two basic neighboring values of fuzzy transmission costs: 5050 with the left deviation $l_1^L = 550$, right deviation—$l_1^R = 510$ and 6620 with the left deviation $l_2^L = 620$, right deviation—$l_2^R = 710$. According to (2.4) it is obtained:

$$\begin{aligned} l^L &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^L + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^L = \frac{(6620 - 5314)}{(6620 - 5050)} \times 550 + \left(1 - \frac{1306}{1570}\right) \times 620 \\ &= 562.9 \approx 563, \\ l^R &= \frac{(a_2 - a')}{(a_2 - a_1)} \times l_1^R + \left(1 - \frac{(a_2 - a')}{(a_2 - a_1)}\right) \times l_2^R = \frac{(6620 - 5314)}{(6620 - 5050)} \times 510 + \left(1 - \frac{1306}{1570}\right) \times 710 \\ &= 544.9 \approx 545. \end{aligned}$$

Therefore, it is to represent the minimum transportation cost in the form of the fuzzy triangular number (5314,563,545), as shown in Fig. 2.53.

Fig. 2.52 Given flow value in the form of the fuzzy triangular number of (36,8,8) units



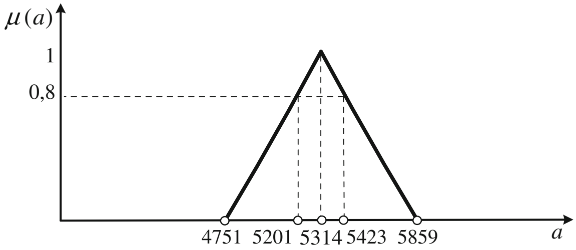


Fig. 2.53 Minimum transmission cost in the form of the fuzzy triangular number of (5314,563,545) conventional units

Due to the Fig. 2.53 the minimum transmission cost of (36,8,8) flow units with the degree of confidence 0,8 will be within the interval of [5201,5423] conventional units, but anyway the minimum transmission cost of (36,8,8) flow units will be no less, than 4751 and no more, than 5859 conventional units.

2.6 Summary

This chapter deals with main concepts of fuzzy logic. The basic flow tasks in fuzzy networks are described. The factors leading to the problem statements in fuzzy conditions are given. The conclusion is in necessity of flow tasks considering in fuzzy conditions.

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