Chapter 2
Integral Volterra Equations

2.1 Overview

Any functional equation in which the unknown function appears under the sign of integration is called an integral equation. Integral equations arise in a great many branches of science; for example, in potential theory, acoustics, elasticity, fluid mechanics, irradiative transfer, theory of population, etc. In many instances the integral equation originates from the conversion of a boundary value problem or an initial value problem associated with a partial or an ordinary differential equation, but many problems lead directly to integral equations and cannot be formulated in terms of differential equations. Integral equations are of many types; here, we attempt to indicate some of the main distinguishing features with particular regard to the use and construction of algorithms.

In the classical theory of integral equations one distinguishes between Fredholm equations and Volterra equations. In a Fredholm equation the region of integration is fixed, whereas in a Volterra equation the region is variable. Thus, the equation

$$\varphi(x) - \lambda \int_a^b K(x,s)\varphi(s)\mathrm{d}s = \mathrm{cf}(x); \quad a \le x \le b \tag{2.1}$$

is an example of Fredholm equation, and the equation

$$\varphi(x) - \lambda \int_a^x K(x,s)\varphi(s)\mathrm{d}s = \mathrm{cf}(x); \quad a \le x \tag{2.2}$$

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is an example of a Volterra equation. Here, the forcing function f(x) and the kernel function.

$K(x, s, \varphi(s))$ are prescribed, while $\varphi(x)$ is the unknown function to be determined. (More generally the integration and the domain of definition of the functions may extend to more than one dimension.) The parameter λ is often omitted; it is, however, of importance in certain theoretical investigations (e.g., stability) and in the eigenvalue problem discussed below. If in (1.1) or (1.2), $c = 0$, the integral equation is said to be of the first kind. If $c = 1$, the equation is said to be of the second kind. Equations (1.1) and (1.2) are linear if the kernel $K(x, s, \varphi(s)) = k(x, s)\ \varphi(s)$, otherwise they are nonlinear. Note: in a linear integral equation, $k(x, s)$ is usually referred to as the kernel. We adopt this convention throughout. These two types of equations are broadly analogous to problems of initial- and boundary value type for an ordinary differential equation (ODE); thus the Volterra equation, characterized by a variable upper limit of integration, is amenable to solution by methods of marching type whilst most methods for treating Fredholm equations lead ultimately to the solution of an approximating system of simultaneous algebraic equations. For comprehensive discussion of numerical methods, see [1, 2]. In what follows, the term "integral equation" is used in its general sense, and the type is distinguished when appropriate.

2.1.1 Structure of Kernel

When considering numerical methods for integral equations, particular attention should be paid to the character of the kernel, which is usually the main factor governing the choice of an appropriate quadrature formula or system of approximating functions. Various commonly occurring types of singularity call for individual treatment. Likewise, provision can be made for cases of symmetry, periodicity or other special structure, where the solution may have special properties and/or economies may be affected in the solution process. We note in particular the following cases to which we shall often have occasion to refer in the description of individual algorithms.

(a) A linear integral equation with a kernel $k(x, s) = k(s, x)$ is said to be symmetric. This property plays a key role in the theory of Fredholm integral equations.
(b) If $k(x, s) = k(a + b - x, a + b - s)$ in a linear integral equation, the kernel is called Centro-symmetric.
(c) If in Eqs. (1.1) or (1.2) the kernel has the form $K(x, s), \varphi(s) = k(x - s)\ g(s, \varphi(s))$, the equation is called a convolution integral equation; in the linear case $g(s, \varphi(s)) = \varphi(s)$.
(d) If the kernel in (1.1) has the form

$$K(x,\, s,\, \varphi(s)) = K1(x,\, s,\, \varphi(s)), \quad a \le s \le x,$$

$$K(x,\, s,\, \varphi(s)) = K2(x,\, s,\, \varphi(s)), \quad x \le s \le b,$$

where the functions $K1$ and $K2$ are well behaved, whilst K or its s-derivative is possibly discontinuous, may be described as discontinuous or of '*split*' type; in the linear case $K(x, s, \varphi(s)) = k(x, s)\varphi(s)$ and consequently $K1 = k1(\varphi(s))$ and $K2 = k2$ $(\varphi(s))$. Examples are the commonly occurring kernels of the type $k(x - s)$ and the Green's functions (influence functions) which arise in the conversion of ODE boundary value problems to integral equations. It is also of interest to note that the Volterra Eq. (1.2) may be conceived as a Fredholm equation with kernel of split type, with $K2(x, s, \varphi(s)) \equiv 0$; consequently methods designed for the solution of Fredholm equations with split kernels are also applicable to Volterra equations.

2.1.2 Singular and Weakly Singular Equations

An integral equation may be called singular if either

(a) its kernel contains a singularity, or
(b) the range of integration is infinite,

and it is said to be weakly singular if the kernel becomes infinite at $s = x$. Sometimes a solution can be affected by a simple adaptation of a method applicable to a nonsingular equation: for example, an infinite range may be truncated at a suitably chosen point. In other cases, however, theoretical considerations will dictate the need for special methods and algorithms. Examples are:

(i) Integral equations with singular kernels of Cauchy type;
(ii) Equations of Wiener–Hopf type;
(iii) Various dual integral equations arising in the solution of boundary value problems of mathematical physics;
(iv) The well-known Abel integral equation, an equation of Volterra type, whose kernel contains an inverse square-root singularity at $s = x$.

2.1.3 Eigenvalue Problem

Closely connected with the linear Fredholm integral equation of the second kind is the eigenvalue problem represented by the homogeneous equation

$$\varphi(x) - \lambda \int_a^b K(x, s)\varphi(s)\mathrm{d}s = 0; \quad a \le x \le b \tag{2.3}$$

If λ is chosen arbitrarily this equation in general possesses only the trivial solution $\varphi(x) = 0$. However, for a certain critical set of values of λ, the characteristic values or eigenvalues (the latter term is sometimes reserved for the reciprocals $\mu = 1/\lambda$), there exist nontrivial solutions φ(x), termed characteristic functions or eigenfunctions, which are of fundamental importance in many investigations. The analogy with the Eigen problem of linear algebra is readily apparent, and indeed most methods of solution of Eq. (1.3) entail reduction to an approximately equivalent algebraic problem.

2.1.4 Nonlinear Fredholm Equation of the Second Kind

$$\varphi(x) - \boldsymbol{\lambda} \int_a^b K(x, s, \varphi(s))\mathrm{d}s = f(x); \quad a \le x \le b \tag{2.4}$$

The numerical solution of Eq. (1.4) is usually accomplished either by simple iteration or by a more sophisticated iterative scheme based on Newton's method; in the latter case it is necessary to solve a sequence of linear integral equations. Convergence may be demonstrated subject to suitable conditions of Lipchitz continuity of the function K with respect to the argument φ. Examples of Fredholm type (for which the provision of algorithms is contemplated) are:

(a) the Uryson equation

$$\varphi(x) - \int_0^1 K(x, s, \varphi(s))\mathrm{d}s = 0; \quad 0 \le x \le 1 \tag{2.5}$$

(b) the Hammerstein equation a nonlinear integral equation of the type

$$\varphi(x) - \int_a^b F(x, s), g(s, \varphi(s))\mathrm{d}s = 0; \quad a \le x \le b \tag{2.6}$$

Where $K(x, s)$ and $g(x, \varphi(s))$ are given functions, while $\varphi(s)$ is the unknown function. Named after Hammerstein [3], who considered the case where $K(x, s)$ is a

symmetric and positive Fredholm kernel, i.e., all its eigenvalues are positive. If, in addition, the function $g(x, \varphi(s))$ is continuous and satisfies the condition $|g(x, \varphi(s))| \leq C_1|s| + C_2$, where C_1 and C_2 are positive constants and C_1 is smaller than the first eigenvalue of the kernel $K(x, s)$, the Hammerstein equation has at least one continuous solution. If, on the other hand, $g(x, \varphi(s))$ happens to be a non-decreasing function of s for any fixed x from the interval $[a, b]$, Hammerstein's equation cannot have more than one solution. This property holds also if $g(x, \varphi(s))$ satisfies the condition.

$|G(x, \varphi(s_1)) - g(x, \varphi(s_2))| \leq C|s_1 - s_2|$, where the positive constant C is smaller than the first eigenvalue of the kernel $K(x, s)$. A solution of the Hammerstein equation may be constructed by the method of successive approximation.

2.1.5 *Integral Equations Volterra*

Volterra equations may be regarded as a special case of Fredholm equations, with the kernel $K(x, s)$ defined on the square $a < x < b$; $a < s < b$, and vanishing in the triangle $a < x < s < b$. A Volterra equation of the second kind without free term is called a homogeneous Volterra equation. The expression $\int_a^x K(x, s)\varphi(s)\mathrm{d}s$ defines an integral operator acting in L_2; it is known as the Volterra operator. The principal result of the theory of Volterra equations of the second kind may be described as follows. For each complex $\lambda \neq \infty$ there exists a square-integrable solution of the Volterra equation of the second kind which is, moreover, unique. This solution may be obtained by successive approximation, i.e., as the limit of a mean-square-convergent sequence.

Consider the Volterra integral equation of the first kind

$$\int_a^x K(x, s)\varphi(s)\mathrm{d}s = f(x); \quad a \leq x \tag{2.7}$$

Clearly it is necessary that $f(a) = 0$; otherwise no solution to (1.7) can exist. The following types of Volterra integral equations of the first kind occur in real life problems: equations with unbounded kernel at $s = x$, equations with sufficiently smooth kernel.

In general, a nonsingular Volterra equation of the first kind presents less computational difficulty than the Fredholm Eq. (1.4) with a smooth kernel. A Volterra equation of the first kind may, under suitable conditions, be converted by differentiation to one of the second kind or by integration by parts to an equation of the second kind for the integral of the wanted function.

A very general Volterra equation of the second kind is given by

$$\varphi(x) - \int_a^x K(x, s, \varphi(s))\mathrm{d}s = f(x); \quad a \le x \tag{2.8}$$

The resemblance of Volterra equations to ODEs suggests that the underlying methods for ODE problems can be applied to Volterra equations. Indeed this turns out to be the case. The main advantages of implementing these methods are their well-developed theoretical background, i.e., convergence and stability. Many Volterra integral equations arising in real-life problems have a convolution kernel [4].

However, a subclass of these equations which have kernels of the form $K(x - s) = \sum_{j=0}^{N} \boldsymbol{\lambda}_j (x - s)^j$ where λ_j are real numbers can be converted into a system of linear or nonlinear ODEs. Equations of type (1.2) were first systematically studied by Volterra [5, 6]. The principal result of the theory of Volterra equations of the second kind may be described as follows. For each complex $\lambda \neq \infty$ there is a square-integrable solution of the Volterra equation of the second kind which is, moreover, unique. This solution may be obtained by successive approximation (cf. Sequential approximation, method below), i.e., as the limit of a mean-square-convergent sequence:

The contracting-mapping principle implies that if $|K(x, y)| \le M$ and $|\boldsymbol{\lambda}| < \frac{1}{M(b-a)}$ or $|\boldsymbol{\lambda}| < \left(\int_a^b \int_a^b K^2(x, s)\varphi_n(s))\mathrm{d}x\mathrm{d}s \right)^{-1/2}$, then the considered integral equation has a unique solution in the space L_2[a, b] which can be constructed by the method of successive approximation (1.3). Function φ_0 is an arbitrary square-integrable function. In the case of a continuous kernel $K(x, s)$ and $f \in C[a, b]$, this sequence converges uniformly on $[a, b]$ to a unique continuous solution. In practical applications of Volterra equations of the second kind, it is very important that its solution be found at least approximately, e.g., by the method of successive approximation. However, other methods are usually more convenient, and some of them such will now be described. Let f and K be continuous functions. The interval $[a, b]$ is subdivided into N equal parts with the aid of partitioning points x_i, and $x_0 = a$, $x_N = b$. To find the approximate value of $\varphi(x_i)$, the integral over the interval is replaced by a quadrature sum:

$$\int_a^b F(x)\mathrm{d}x = \sum_{k=1}^{N} A_k F(x_k) + \boldsymbol{\rho} \tag{2.9}$$

Where: A_k and x_k are constants for a given interval $[a, b]$ and the quadrature sum formula. For example:

1. *Using the rectangle formula with nodes* $x_0,\ldots, x_{i-1}$:

$$\int_{x_0}^{x_i} K(x_i, s)\varphi(s)\mathrm{ds} \approx \frac{b-a}{N}\sum_{j=0}^{i-1} K(x_i, x_j)\varphi(x_j) \tag{2.10}$$

$$x_1 = a;\ x_2 = a + (b-a)/n; \ldots x_N = a + (N-1)\,(b-a)/n;\ A_k = (b-a)/N$$

The approximate value of $\varphi(x_i)$ is then obtained using collocation:

$$\tilde{\varphi}(x_i) = \boldsymbol{\lambda}\frac{b-a}{N}\sum_{j=0}^{i-1} K(x_i, x_j)\tilde{\varphi}(x_j) + f(x_i);\quad \tilde{\varphi}(x_0) = f(a)$$

The values of the approximate solution at the points on $[a, b]$ situated between the partitioning points may be found, for example, from the relation

$$\tilde{\varphi}(x) \simeq \boldsymbol{\lambda}\frac{b-a}{N}\sum_{j=0}^{i-1} K(x, x_j)\tilde{\varphi}(x_j) + f(x);\quad x_{j-1} < x \leq x_j$$

For $N \to \infty$ this approximate solution converges uniformly to the exact solution of the Volterra equation of the second kind.

2. *Using the trapezoidal formula*:

$$x_1 = a;\ x_2 = a + (b-a)/(N-1); x_N = b;\ A_1 = A_N = (b-a)/2(N-1);$$
$$A_2 = \ldots = A_{N-1} = (b-a)/(N-1);$$

3. *Using tangent formula*:

$$x_1 = a + (b-a)/2N;\ x_2 = a + 3(b-a)/2N; \ldots x_N = a + (2N-1)\,(b-a)/2N;\ A_k = (b-a)/N;$$

4. *Using Chebyshev formula*:

$$x_k = (b-a)/2 + x_k^{(N)}(b-a)/2;\ A_k = (b-a)/N;$$

Chebyshev nodes in the interval (−1, 1) are:

$$x_k^{(N)} = \cos[(2k-1)\pi/2\mathrm{N}];\ k = 1, 2, \ldots, N$$

5. *Using Gauss–Legendre quadrature formula* [7]:

$$x_k = a + (b-a)x_k^{(N)};\ A_k = (b-a)A_k^{(N)};$$

$x_k^{(N)}$—root of P_N—Legendre polynomials,

For the simplest integration problem stated above, the associated polynomials are Legendre polynomials, $P_N(x)$, and the method is usually known as Gauss–Legendre quadrature. With the N-th polynomial normalized to give $P_N(1) = 1$, the k-th Gauss node, x_k, is the k-th root of P_N are listed in Table 2.1.

Change of interval

An integral over $[a, b]$ must be changed into an integral over $[-1, 1]$ before applying the Gaussian quadrature rule. This change of interval can be done in the following way:

$$\int_a^b f(x)\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}z + \frac{a+b}{2}\right)\mathrm{d}z$$

Table 2.1 Roots and weights of Legendre polynomials

Number of points (N)	Points x_k	Weights $A_k^{(N)}$
1	0	2
2	$\pm\sqrt{\frac{1}{3}}$	1
3	0	8/9
	$\pm\sqrt{\frac{3}{5}}$	5/9
4	$\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$
5	0	128/225
	$\pm\frac{1}{3}\sqrt{5 - 2\sqrt{\frac{10}{7}}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5 + 2\sqrt{\frac{10}{7}}}$	$\frac{322-13\sqrt{70}}{900}$

Applying the Gaussian quadrature rule then results in the following approximation:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{2}\sum_{k=1}^{N} A_k f\left(\frac{b-a}{2}z_k + \frac{a+b}{2}\right) \tag{2.11}$$

After applying the formula (1.11) to the integral in the left-hand side of Eq. (1.2) we obtain the equality

$$\tilde{\varphi}(x) - \boldsymbol{\lambda}\sum_{k=1}^{N} A_k K(x, x_k)\tilde{\varphi}(x_k) = f(x) + \lambda\rho(x) \tag{2.12}$$

The approximate value of $\varphi(x_i)$ is then obtained using collocation

$$\tilde{\varphi}(x_i) - \boldsymbol{\lambda}\sum_{k=1}^{N} A_k K(x_i, x_k)\tilde{\varphi}(x_k) = f(x_i) + \lambda\rho_i; \quad i = 1, 2, \ldots, N \tag{2.13}$$

Discarding the right-hand side of Eq. (1.12) small value $\lambda\rho_i$, we obtain the following system of equations with N unknowns, which can be written as follows in the expanded form:

$$\begin{cases} \tilde{\varphi}(x_1)[1 - \lambda A_1 K(x_1, x_1)] - \lambda\tilde{\varphi}(x_2)A_2 K(x_1, x_2) - \ldots - \lambda\tilde{\varphi}(x_N)A_N K(x_1, x_N) = f(x_1) \\ -\lambda\tilde{\varphi}(x_1)A_1 K(x_2, x_1) + \tilde{\varphi}(x_2)[1 - \lambda A_2 K(x_2, x_2)] - \ldots - \lambda\tilde{\varphi}(x_N)A_N K(x_2, x_N) = f(x_2) \\ \ldots \\ -\lambda\tilde{\varphi}(x_1)A_1 K(x_n, x_1) - \lambda\tilde{\varphi}(x_2)A_2 K(x_n, x_2) - \ldots + \tilde{\varphi}(x_N)[1 - \lambda A_N K(x_N, x_N)] = f(x_N) \end{cases} \tag{2.14}$$

Solving this system of Eq. (2.14), we find approximations for the values of $\tilde{\varphi}(x_1); \tilde{\varphi}(x_2); \ldots \tilde{\varphi}(x_N)$ unknown function $\varphi(x_1); \varphi(x_2); \ldots \varphi(x_N)$. Obviously, the accuracy of the result obtained by replacing the integral Eq. (1.2) with the systems of linear Eq. (1.14), will be higher than a smaller error we make by replacing the integral by the sum. Accurate error estimation is presented in [8, 9].

Finally, it should be noted that the method of replacing the integral equation to a system of algebraic also suitable for the approximate solution of the problem of finding the eigenvalues (or characteristic values) and eigenfunctions (or characteristic functions). To solve this problem, instead of the system of Eq. (2.13) is necessary to consider the corresponding homogeneous system.

$$\tilde{\varphi}(x_i) - \boldsymbol{\lambda}\sum_{k=1}^{N} A_k K(x_i, x_k)\tilde{\varphi}(x_k) = 0; \quad i = 1, 2, \ldots, N \tag{2.15}$$

Determinant of this system will be

$$\Delta(\lambda) = \begin{vmatrix} [1 - \lambda A_1 K(x_1, x_1)] - \lambda A_2 K(x_1, x_2) - \ldots - \lambda A_N K(x_1, x_N) \\ -\lambda A_1 K(x_2, x_1) + [1 - \lambda A_2 K(x_2, x_2)] - \ldots - \lambda A_N K(x_2, x_N) \\ \ldots \\ -\lambda A_1 K(x_n, x_1) - \lambda A_2 K(x_n, x_2) - \ldots + [1 - \lambda A_N K(x_N, x_N)] \end{vmatrix} \quad (2.16)$$

Equating to zero the determinant (2.16) and solving the corresponding equation $\Delta(\lambda) = 0$, we find the values of λ at which the system of Eq. (2.16) has a solution $\{\tilde{\varphi}(x_i)\}$ that is not identically zero. These values are in fact the approximate eigenvalues values of λ. Further, taking these values λ equal to the roots of the Eq. (2.16), and obtaining for these values the corresponding independent solution of (2.15), we can obtain approximate expressions for the eigenfunctions of the equation.

Everything said so far also applies to Volterra equations whose kernel $K(x, s)$ is a matrix of dimension r × r, and where φ and f are r-dimensional vector-functions. The name Volterra equation or generalized Volterra equation is also given to a more general integral equation (GIE), of the form:

$$\varphi(P) - \boldsymbol{\lambda} \int_{D(P)} K(P, Q)\varphi(Q)\mathrm{d}Q = f(P) \quad (2.17)$$

Assuming that the successive approximations such are in some sense convergent (e.g., uniformly or on the average) on the domain of definition of the functions φ and f for all $\lambda \neq \infty$. Here, P and Q are points of the n-dimensional Euclidean space; $D(P)$ is the domain of integration, which usually depends on the point P, and $D(P) \subseteq D$ for any P. The following equation may serve as an example:

$$\varphi(x, y) - \boldsymbol{\lambda} \int_a^x \int_a^b K(x, y, \boldsymbol{\xi}, \boldsymbol{\eta})\varphi(\boldsymbol{\xi}, \boldsymbol{\eta})\mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{\eta} = f(x, y) \quad (2.18)$$

If the function $K(x, y, \xi, \eta)$ is square-integrable for $a \leq x \leq b$, $a \leq y \leq b$, $a \leq \xi \leq b$, $a \leq \eta \leq b$, while $f(x, y)$ is square-integrable for $a \leq x \leq b$, $a \leq y \leq b$, the sequence (1.3) is mean-square-convergent for $\lambda \neq \infty$. Generalized Volterra equations of the first kind usually cannot be reduced to Volterra equations of the second kind, though this may be possible in special cases. GIE (or mixed Volterra–Fredholm integral equations and Fredholm integral equations on the unbounded interval $[0, \infty)$ occur in mechanics and many related fields of engineering and mathematical physics [10]. Recently, many papers have been devoted to the existence of solutions of nonlinear functional integral equations [11].

Equations of type (2.18) and their nonlinear counterparts arise in the theory of parabolic boundary value problems. Actually few numerical methods for (2.18) are

known. In [11] a method for numerical treatment of (2.18) is given and also shown existence and unicity of the solution of (2.18) independently of the value chosen for λ. In Chap. 4, we give a numerical solution obtained by continuous time collocation and time discretization method.

Nonlinear Volterra equations is the name sometimes given to Volterra equations in which the product $K(x, s)\ \varphi(s)$ has been replaced by some function $K(x, s, \varphi(s))$ which is nonlinear with respect to $\varphi(s)$. Equations of this type are frequently encountered in theoretical and in applied studies. Thus, the Cauchy problem for an ordinary differential equation may be readily reduced to the problem of solving a nonlinear Volterra equation. The application of potential theory to boundary value problems for equations of parabolic type reduces such problems to a generalized Volterra equation. In the case of nonlinear Volterra equations it may be shown, if certain assumptions are made with respect to $K(x, s, \varphi(s))$, that successive approximations of type (2.3) converge on an interval $[a, a + \Delta a]$, where Δa is sufficiently small. Approximate solutions of nonlinear Volterra equations are found by using the recurrence relation (2.5); it is sufficient to replace $K(x, s, \varphi(s))$ by $K(x, s)\ \varphi(s)$. If $K(x, s, \varphi(s))$ is independent of x, for example, $K(x, s, \varphi(s)) = a(x)\ b(s)\ f[\varphi(s)]$.

2.2 Reduction of ODEs to the Volterra Integral Equation (IE)

In mathematics, an *integral equation* is an equation in which an unknown function appears under an integral sign. There is a close connection between differential and integral equations, and some problems may be formulated either way. Consider an ordinary differential equation (ODE) of the first order $\frac{dy}{dx} = f(x; y)$, where $f(x; y)$ is defined and continuous in a two-dimensional domain G which contains the point $(x0; y0)$. Integrating the ODE subject to the initial condition $y(x0) = y0$, we obtain

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\mathrm{d}t \tag{2.19}$$

This is the *Volterra integral equation of the second kind* with respect to the unknown function $\varphi(t)$. This equation is equivalent to the initial value problem of ODE. Note that this is generally a *nonlinear* integral equation with respect to $\varphi(t)$.

Consider now a linear ODE of the second order with variable coefficients

$$y'' + A(x)y' + B(x)y = g(x) \tag{2.20}$$

The initial condition

$$y(x_0) = y_0; y'(x_0) = y_1 \tag{2.21}$$

$A(x)$ and $B(x)$ are given functions continuous in an interval G which contains the point x_0. Integrating y'' in (2.20) we obtain

$$y'(x) = -\int_{x_0}^{x} A(t)y'(x)\mathrm{d}x - \int_{x_0}^{x} B(x)y(x)\mathrm{d}x + \int_{x_0}^{x} g(x)\mathrm{d}x + y_1 \tag{2.22}$$

Integrating the first integral on the right-hand side in (1.22) by parts yields

$$y'(x) = -A(x)y(x) - \int_{x_0}^{x} [B(x) - A'(x)]y(x)\mathrm{d}x + \int_{x_0}^{x} g(x)\mathrm{d}x + A(x_0)y_0 + y_1$$

Integrating a second time, we obtain

$$\begin{aligned} y(x) = &-\int_{x_0}^{x} A(x)y(x)\mathrm{d}x - \int_{x_0}^{x}\int_{x_0}^{x} [B(t) - A'(t)]y(t)\mathrm{d}x\mathrm{d}t + \int_{x_0}^{x}\int_{x_0}^{x} g(t)\mathrm{d}x\mathrm{d}t \\ &+ [A(x_0)y_0 + y_1](x - x_0) + y_0 \end{aligned} \tag{2.23}$$

Using the relationship $\int_{x_0}^{x}\int_{x_0}^{x} g(t)\mathrm{d}x\mathrm{d}t = \int_{x_0}^{x} (x - t)g(t)\mathrm{d}t$ we transform (2.23) to obtain

$$\begin{aligned} y(x) = &-\int_{x_0}^{x} \{A(t) + (x - t)[B(t) - A'(t)]\}y(t)\mathrm{d}t \\ &+ \int_{x_0}^{x} (x - t)g(t)\mathrm{d}t + [A(x_0)y_0 + y_1](x - x_0) + y_0 \end{aligned} \tag{2.24}$$

Separate the known functions in (2.24) and introduce the notation for the *kernel function*

$$\begin{aligned} K(x,t) &= -A(t) + (t - x)[B(t) - A'(t)], \\ f(x) &= \int_{x_0}^{x} (x - t)g(t)\mathrm{d}t + [A(x_0)y_0 + y_1](x - x_0) + y_0 \end{aligned} \tag{2.25}$$

Then (2.25) becomes

$$y(x) = f(x) + \int_{x_0}^{x} K(x;t)y(t))\mathrm{d}t \tag{2.26}$$

This is the Volterra IE of the second kind with respect to the unknown function $y(t)$. This equation is equivalent to the initial value problem (2.22) and (2.23). Note that here we obtain a *linear* integral equation with respect to $y(x)$.

Example 2.1 Consider a homogeneous linear ODE of the second order with constant coefficients

$$y'' + \omega^2 y = 0 \tag{2.27}$$

Initial conditions (at $x_0 = 0$)

$$Y(0) = 0;\ y'(0) = \omega \tag{2.28}$$

We see that here: $A(x) \equiv 0; B(x) \equiv \boldsymbol{\omega}^2; g(x) \equiv 0; y_0 = 0; y_0' = \boldsymbol{\omega}$. We use the same notation as in (2.25). Substituting into (2.24) and calculating $K(x;\ t) = \omega^2 (t - x)$; $f(x) = \boldsymbol{\omega} x$; we find that the IE (2.26), equivalent to the initial value problem (2.27) and (2.28), takes the form (2.29).

$$y(x) = \boldsymbol{\omega} x - \boldsymbol{\omega}^2 \int_{0}^{x} (x - t)y(t)\mathrm{d}t \tag{2.29}$$

The solution of the differential Eqs. (2.27) and (2.28) is: $y(x) = \sin(\omega x)$. Therefore, the solution of IE (2.11) is the same.

The following theorems apply to Volterra equations of the first kind. If $f(s)$ and $K(x,\ s)$ are differentiable, $K(x,x) \neq 0\, x \in [a,b]$, and if $K(x,\ x)$ and $K'_x(x,\ x)$ are square-summable on $[a,\ b]$ and on $a < x < b$; $a < s < b$, respectively, a Volterra equation of the first kind is equivalent to the Volterra equation of the second kind obtained by differentiation of the Volterra equation of the first kind and having the form:

$$\varphi(x) + \int_a^x \frac{K'_x(x,s)}{K(x,x)}\varphi(s)\mathrm{d}s = \frac{f'(x)}{K(x,x)}f(x)$$

If $K(x,\ x) = 0$ at least at one point, the solution of the Volterra equation of the first kind must be more thoroughly investigated. If, on the other hand, $K(x,x) \equiv 0$, then the differentiation operation may be repeated under certain conditions. If the differentiation is impossible or does not result in a Volterra equation of the second kind, this equation of the first kind may be solved, for example, using a regularization algorithm.

In practical applications of Volterra equations of the second kind it is very important that its solution be found at least approximately, e.g., by the method of successive approximation. However, other methods are usually more convenient, and one such method will now be described. Let f and K be continuous functions. The interval $[a, b]$ is subdivided into N equal parts with the aid of partitioning points x_i, and $x_0 = a$, $x_N = b$. To find the approximate value of $\varphi(x_i)$, the integral over the interval is replaced by a quadrature sum, for example, using the rectangle formula with nodes $x_0 \ldots x_{i-1}$:

2.3 Sequential Approximation Method (*Method of Successive Approximation*)

This is a method to construct an approximating equation for approximate (and numerical) solutions of certain kinds of linear and nonlinear integral equations. However, the main type of integral equations suitable for solving by this method is linear one-dimensional integral Fredholm equations of the second kind. The method as applied to such equations consists of an approximation which replaces the kernel $K(x, s)$ of the integral Eq. (1.2) by a degenerate kernel of the type $K_N(x, s) = \sum_{n=1}^{N} a_n(x) b_{n(s)}$ followed by the solution of the Fredholm degenerate integral equation

$$\tilde{\varphi}(x) - \boldsymbol{\lambda} \int_a^b K_N(x, s)\tilde{\varphi}(s)\mathrm{d}s = f(x); \quad a \le x \le b; \quad a \le s \le b \tag{2.30}$$

Solving (2.30) is reduced to solving a system of linear algebraic equations. The degenerate kernel $K_N(x, s)$ may be found from the kernel $K(x, s)$ in several ways, e.g., by expanding the kernel into a Taylor series or a Fourier series (for other methods see Strip method (integral equations)—see below). The method of degenerate kernels may be applied to systems of integral equations of the type (2.2), to multi-dimensional equations with relatively simple domains of integration and to certain nonlinear equations of Hammerstein type (cf. Hammerstein equation—see above). In many cases, the good convergence properties of the approximations constructed by this method allow one to apply it to practical computations.

We seek a solution of the integral Eq. (2.2) in the form of a series arranged in powers of parameter λ.

$$\varphi(x) = \tilde{\varphi}_0(x) + \lambda\tilde{\varphi}_1(x) + \lambda^2\tilde{\varphi}_2(x) + \ldots \tag{2.31}$$

Substituting this series in Eq. (2.2), we find

$$\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \ldots = f(x) + \lambda \int_a^b K(x,y)[\varphi_0(y) + \lambda\varphi_1(y) + \lambda^2\varphi_2(y) + \ldots]\mathrm{d}y$$

Equating the coefficients of equal powers of λ, we obtain

$$\begin{aligned} \varphi_0(x) &= f(x) \\ \varphi_1(x) &= \int_a^b K(x,y)[\varphi_0(y)]\mathrm{d}y \\ \varphi_2(x) &= \int_a^b K(x,y)[\varphi_1(y)]\mathrm{d}y \\ &\ldots \end{aligned} \tag{2.32}$$

From these equations, we can determine sequentially all the functions $\varphi_1(x)$, $\varphi_2(x)$, i.e., if we introduce the so-called *iterated kernel*, we can write the following expression for the functions $\varphi_n(x)$

$$\begin{aligned} \varphi_n(x) &= f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,y) f(y)\mathrm{d}y \\ K_1(x,y) &= K(x,y); \quad K_m(x,y) = \int_a^b K_{m-1}(y,t)K(x,t)\mathrm{d}t \end{aligned} \tag{2.33}$$

$K_m(x; y)$ is called *the m-th iterated kernel*. One can easily prove that the iterated kernels satisfy a more general relationship

$$K_m(x,y) = \int_a^b K_r(x,t)K_{m-r}(y,t)\mathrm{d}t; \quad r = 1, 2, \ldots, m-1 \quad (m = 2, 3\ldots) \tag{2.34}$$

Assume that there exists a constant C_1 such that

$$\sqrt{\int_a^b |K(x,y)|^2\mathrm{d}y} \le C_1 \quad \forall x \in [a,b] \quad \lambda \le \frac{1}{C_1} \tag{2.35}$$

Then the sequence φ_n of successive approximations (2.31) converges uniformly for all λ, satisfying (2.35). The limit of the sequence φ_n is the unique solution to IE (2.2).

Let us introduce now the notation for *the resolvent*

$$\Gamma(x, y, \lambda) = \sum_{m=1}^{n} \lambda^{m-1} \int_a^b K_m(x, y)\mathrm{d}y \tag{2.36}$$

Changing the order of summation and integration in (2.36), we obtain the solution in the compact form

$$\varphi(x) = f(x) + \boldsymbol{\lambda} \int_a^b \Gamma(x, y, \boldsymbol{\lambda}) f(y)\mathrm{d}y \tag{2.37}$$

One can show that the resolvent satisfies the IE

$$\Gamma(x, y, \boldsymbol{\lambda}) = K(x, y) + \boldsymbol{\lambda} \int_a^b \Gamma(t, y, \boldsymbol{\lambda}) K(x, t)\mathrm{d}t \tag{2.38}$$

To study the convergence properties of the sequence (2.36) and to prove the existence of a solution to (1.2), the contracting-mapping principle formulated below is widely used. This principle implies that if inequality (2.35) holds and the kernel $K(x, y)$ is limited ($|K(x, y)| \leq M$), then the sequences (2.36) converge and the considered integral equation has a unique solution in the space $L_2[a, b]$ which can be constructed by the method of successive approximation.

If λ satisfies (2.35), then the series (2.36) can be successfully used to approximate solution of integral Eq. (2.2). Often required quadrature (2.32) can be performed accurately, then the resulting series (2.31) converges at least as a geometric progression with ratio $|\lambda|M(b - a)$. The error, which will occur, if we restrict ourselves only to "n" members of the series (2.31) can be estimated easily. That is, assuming $f(x) \leq N$ is easy to get sequentially that $|\varphi_n(x)| < NM^n(b - a)^n$, and so the remainder of the series (1.31) after "n" members.

$$|\lambda^n \varphi_n(x) + \lambda^{n+1} \varphi_{n+1}(x) + \ldots| < \frac{NM^n(b - a)^n |\lambda|^n}{1 - M(b - a)^{|\lambda|}} \tag{2.39}$$

Practically, estimating of the error of the solution can be done differently. First of all, with the help of some interpolation formula (for example the method of least squares), one can, using the numbers found $\varphi(x_i)$, obtain an approximate expression for $\varphi(x)$ in the whole interval $[a, b]$. To test how well the function $\varphi(x)$ satisfies the integral Eq. (2.8), it can be substituted into the integral equation, and then this

integral should be replaced by a sum using the other quadrature formula, and check what the error for several values of x_i.

Power series (2.39) is an expansion in the λ near the point $\lambda = 0$ and will therefore converge to the first eigenvalue of the kernel $K(x, y)$, i.e., $|\lambda| < |\lambda_1|$. Consequently, the power series (1) cannot be used if $|\lambda| > |\lambda_1|$, since it diverges, and difficult to use, if λ is close to λ_1, since then the series converges slowly. In these cases, it is often convenient (when we know the approximate location of the eigenvalues of the integral equation), apply the so-called method of analytic continuation for the approximate solution of Eq. (1.8). We shall assume for convenience that $\lambda_1 = -1$, since this can be achieved by replacing λ with $\bar{\lambda} = -\frac{\lambda}{\lambda_1}$. Solution with $|\lambda| < 1$ is given by a convergent series.

$$\varphi = \varphi_0 + \lambda\varphi_1 + \lambda^2\varphi_2 + \ldots$$

With this series, we can calculate the value of the function φ (x, λ) and its derivatives with respect to λ (for example, when $\lambda = 0.5$). In turn, this will enable the expansion of the solution of Eq. (1) in powers of ($\lambda - 1/2$). This expansion can be obtained directly by substituting in the power series λ on ($\lambda' + 1/2$) (referring to $\lambda' = \lambda - 1/2$), and expanding power series in powers of λ'. The distance between point $\lambda = 1/2$ and eigenvalue $\lambda = -1$ is equal to 1.5 and, therefore, the resulting series in powers of λ' will have a radius of convergence of 1.5 and will allow us to, for example, to compute the solution for $\lambda = 1$, which was not possible with the original series. Rebuilt series will converge at $\lambda = 1$, i.e. $\lambda' = 0.5$, unless this value itself is not an eigenvalue. This method is called the method of *analytic continuation*. Rebuilt power series is as follows.

$$\varphi = \varphi_0 + (\lambda' + \frac{1}{2})\varphi_1 + (\lambda' + \frac{1}{2})^2\varphi_2 + \ldots = [\varphi_0 + (\frac{1}{2})\varphi_1 + (\frac{1}{2})^2\varphi_2 + (\frac{1}{2})^3\varphi_3 + \ldots] + [\varphi_1 + \varphi_2 + \frac{3}{4}\varphi_3 + \ldots]\lambda' + [\varphi_2 + \frac{3}{2}\varphi_3 + \ldots](\lambda')^2 + \ldots \quad (2.40)$$

2.4 Linear Volterra IE of the Second Kind

2.4.1 *Basic Steps of the Method of Moments (MoM) Galerkin Method*

We restrict our attention here to the MoM applications to solving integral equations, where the unknowns are stress functions. The basic idea of the MoM is as follows. The unknown quantity (φ) is expanded in terms of a set of linearly independent known functions, φ_n (referred to as basis or expansion functions), i.e., it is approximated by the following finite series $\varphi \approx \sum_{n=1}^{N} a_n\varphi_n$, where a_n are unknown

coefficients yet to be determined. The expansion functions should be chosen, usually based on experience, so that reasonable approximation of φ is obtained with a small number of terms, N.

When function φ is substituted into (2.18), one obtains the approximate equation

$$L\left(\sum_{n=1}^{N} a_n \varphi_n\right) \approx f \tag{2.41}$$

Note that Eq. (2.41) cannot be exactly satisfied at all points, as we have a finite number of terms in the series. Exceptions are rare examples that do have analytical solutions, but which are not of our interest here. The unknown coefficients (a_n) should now be determined such that Eq. (2.41) is satisfied in a sense. Hence, a measure is needed describing the degree of accuracy to which the left side and the right side of Eq. (2.41) match.

In the MoM, this measure is obtained in the following way. Both sides of Eq. (2.41) are multiplied by a known, properly selected function, referred to as the weighting function, w_n, and the results integrated over a spatial region. This integration is a special, but very frequent case of an inner product of two functions, f and φ_n, which is denoted by $<f\ \varphi>$. Generally, the inner product of elements f and φ of a given space is a scalar, which satisfies the following conditions:

$$\langle \varphi, f\rangle = \langle f, \varphi\rangle; \langle \alpha\varphi + \beta f, h\rangle = \alpha\langle \varphi, h\rangle + \beta\langle f, h\rangle; \langle \varphi, \varphi^*\rangle > 0$$
$$\text{if } \varphi \neq 0, \text{and} \langle \varphi, \varphi^*\rangle = 0 \quad \text{if } \varphi = 0,$$

where α and β are arbitrary scalars, and h is another element of the same space.

Equation (1.41) represents a system of N ordinary linear equations in N unknowns, and it can be solved using various techniques. To prepare a computer code that uses the MoM to solve a complex high temperature creep problem, usually requires a lot of work and experience. Often, codes are specialized for certain classes of problems. In most cases, a useful measure of accuracy of the solution obtained does not exist. In spite of this deficiency, the MoM is the most powerful tool available nowadays for analysis of fairly general creep deformation field problems that involve linear integral equations Volterra of second type.

The expansion and testing functions can be arbitrary. However, to provide an efficient solution, the expansion functions should be selected such that the solution can be well approximated by a relatively small number of functions. Similarly, the weighting functions should provide a reliable measure of discrepancy between the two sides of equation. On the other hand, all these functions should be selected bearing in mind complexity and speed of computations, and flexibility to accommodate to a wide range of problems [12].

Expansion and testing functions may coincide, i.e., we can take $\varphi_n = w_n$; $n = 1, 2, \ldots, N$. In this case, we have a Galerkin solution, which is equivalent to the Rayleigh-Ritz variational method, often used in the finite-element approach.

Both expansion and testing functions can be divided into two categories. The first category is sub-domain functions. The domain, where the unknown function (φ) is defined, is divided into a number of small sub-domains.

Each basis function is defined only on one sub-domain (i.e., it is assumed zero elsewhere), and it is a very simple function. Such a choice simplifies evaluation of integral equation, and it can relatively easily accommodate an arbitrary memory function. However, it may result in instabilities as the approximation of the unknown function is discontinuous or has discontinuous derivatives, and it may require a large number of basis functions for an accurate solution.

The piecewise-constant approximation is discontinuous. A better approximation is the piecewise approximation, which is continuous, but has a discontinuous first derivative. Analytically, this approximation can be constructed in two ways. For simplicity, we consider a one-dimensional expansion. The first way is assuming a linear function on a sub-domain, and then matching the approximations on adjacent sub-domains to obtain continuity. Alternatively, a sub-domain memory function (the kernel) can be approximated by the piecewise function, which is continuous, but has a discontinuous first derivative.

More sophisticated functions can be designed using more complicated sub-domain functions and introducing additional constraints. An expansion function that closely resembles creep function may expedite the numerical solution. Approximation by expansion functions involved in the MoM means not only an approximation of the unknown function, but also of the material property parameters (MPP). The approximation of the MPP means a modification of their spectra of the sub-domains where the unknown function is defined.

In practice, however, the entire domain is divided into a small number of relatively large sub-domains. For example, in case of high temperature creep analysis the dimensionless temperature θ ranges from 0 to 10 (for metallic materials) and it is divided into ten segments.

The expansion and testing functions are then defined on these large sub-domains. This procedure is referred to as the almost-entire-domain approximation.

This kind of functions may well accommodate complex temperature creep analysis problems and yield good results with a smaller number of unknowns.

The more complicated the basis functions are, the more analytical preparation is usually required before starting to write the computer code. A set of basis functions is usually suitable for a certain class of problems, but not for a general structure. Hence, a code customized for a class of problems is usually more efficient than a general code.

2.4.2 *Linear Integral Equations (IEs) with Degenerate Kernels*

The importance of degenerate integral equations in the general theory of Fredholm equations is based on the fact that the solution of any Fredholm equation of the

second kind can be approximated by solutions of degenerate integral equations in the mean-square (and certain other) metrics to any degree of accuracy. Their degenerate kernels approximate the kernel of the initial equation in one sense or another. An IE (1.8) with *a degenerate kernel* $K(x,y) = \sum_{i=1}^{n} a_i(x)b_i(x)$ can be represented, by changing the order of summation and integration, in the form:

$$L(\varphi) = \varphi(x) - \lambda \sum_{i=1}^{n} a_i(x) \int_a^x b_i(y)\varphi(y)\mathrm{d}y = f(x) \tag{2.42}$$

Here, one may assume that functions $a_i(x)$ (and $b_i(y)$) are linearly independent (otherwise, the number of terms in (2.42) can be reduced). It is easy to solve such IEs with degenerate and separable kernels. Denote

$$\tilde{\varphi}(x) = f(x) + \lambda \sum_{i=1}^{n} A_i a_i(x) \tag{2.43}$$

A_i is unknown constants. Equation (1.42) becomes:

The problem reduces to the determination of unknowns A_i. To this end, substitute (2.43) into Eq. (2.42) to obtain, after simple algebra,

$$f(x) + \lambda \sum_{i=1}^{n} A_i a_i(x) - \lambda \sum_{i=1}^{n} a_i(x) \int_a^x \left[b_j(y)\{f(y) + \lambda \sum_{j=1}^{n} A_j a_j(y)\} \right] \mathrm{d}y = f(x) \tag{2.44}$$

Denoting $\psi_j(x) = \int_0^x b_j(y)a_j(y)\mathrm{d}y$; $\vartheta_i(x) = \int_a^x b_i(y)f(y)\mathrm{d}y$ and substituting into (1.44) we have

$$\sum_{i=1}^{n} a_i(x) \left[A_i - \sum_{i=1}^{n} \{\vartheta_i(x) + \lambda \sum_{j=1}^{n} A_j \psi_j(x)\} \right] = 0 \tag{2.45}$$

Since functions $a_i(x)$ are linearly independent, equality (2.45) yields

$$A_i - \sum_{i=1}^{n} \{\mathfrak{I}_i(x) + \boldsymbol{\lambda} \sum_{j=1}^{n} A_j \boldsymbol{\psi}_j(x)\} = 0 \tag{2.46}$$

For a function $\tilde{\varphi}(x)$ to represent the exact solution of the Eq. (2.42), it is necessary that the operator $L(\varphi)$ is equals identically zero, and this is equivalent to the fact that it (the integral equation) must be orthogonal to all the functions a_i that is:

$$\int_a^b \left[\sum_{i=1}^{n}\{A_i - \vartheta_i(x) - \lambda\sum_{j=1}^{n} A_j\psi_j(x)\}\right] a_i(x)\mathrm{d}x = 0 \tag{2.47}$$

Denoting $a_{ij} = \int_a^b \psi_j(x)a_i(x)\mathrm{d}x; f_i = \int_a^b \vartheta_i(x)a_i(x)\mathrm{d}x$ and changing the order of summation and integration in (2.47) we obtain the solution in the compact form

$$A_i - \lambda\sum_{j=1}^{n} a_{ij}A_j = f_i \quad i = 1, 2\ldots n \tag{2.48}$$

Assume now that functions $\varphi(x)$; $f(x)$; $a(x)$ and $b(x)$ in Eq. (2.41) are continuously differentiable functions, and λ is a parameter. Differentiating the Eq. (2.42) we obtain

$$L'(\varphi) = \varphi'(x) - \lambda\sum_{i=1}^{n} a_i(x)b_i(y)\varphi(y) - \lambda\sum_{i=1}^{n} a_i'(x)\int_a^x b_i(y)\varphi(y)\mathrm{d}y = f'(x) \tag{2.49}$$

Denoting $Z_i = \int_a^x b_j(y)\varphi(y)\mathrm{d}y$ or : $Z_i' = b_j(y)\varphi(y)$ and substituting in (1.47) we have now $(n + 1)$ ordinary differential equations (ODE) of first order with zero initial conditions.

$$\begin{aligned}
&\frac{\mathrm{d}\varphi}{\mathrm{d}x} = f'(x) + \lambda\sum_{i=1}^{n} a_i(x)b_i(x)\varphi(x) + \lambda\sum_{i=1}^{n} a_i'(x)Z_i(x)\\
&\frac{\mathrm{d}Z_i}{\mathrm{d}x} = b_i(x)\varphi(x);\ \varphi(\mathrm{x} = \mathrm{a}) = 0;\ Z(x = a) = 0; \quad i = 1, 2, \ldots, n
\end{aligned} \tag{2.50}$$

It should be noted that the above inequality (2.35) must be satisfied. Otherwise, the interval $[a, b]$ should be divided into a finite number of smaller intervals, and the solution of the integral Eq. (1.8) is found by applying the so-called strip method.

2.5 Special Types of Integral Equations

2.5.1 Strip Method (Integral Equations)

If the interval $[a, b]$ is large, then in many cases the inequality (1.35) is not satisfied and therefore the contracting-mapping principle is not applicable. This method is based on replacing the kernel in a special way by a degenerate kernel, evaluating the resolvent of the degenerate equation and then improving the approximate solution through the use of a rapidly convergent iterative algorithm. To construct the degenerate kernel, divide the square $[a \le x \le b, a \le s \le b]$ into N strips

$\{\frac{b-a}{N} i \le x \le \frac{b-a}{N}(i+1), a \le s \le b\} i = 0, 1, \ldots N-1$. In each strip, say the i-th, the function $K(x, s)$ is approximated in the mean square, or uniformly, by functions $K_i(x, s) = P_i(x)Q_i(s)$. In this case, $K_i(x, s) = K(\xi_i, s)\xi_i \in \{\frac{b-a}{N} i \le x \le \frac{b-a}{N}(i+1)\}$, $a \le s \le b$.

The function $K_i(x, s)$ is now used to construct a degenerate kernel:

$$\begin{aligned} K_N(x,s) &= \sum_{i=0}^{N-1} P_i(x)Q_i(s) \\ P_i(x) &= \begin{cases} P_i(x), & x \in \left[\frac{b-a}{N} i \le x \le \frac{b-a}{N}(i+1)\right] \\ 0, & x \notin \left[\frac{b-a}{N} i \le x \le \frac{b-a}{N}(i+1)\right] \end{cases} \\ Q_i(s) &= \begin{cases} Q_i(s), & s \in \left[\frac{b-a}{N} i \le s \le \frac{b-a}{N}(i+1)\right] \\ 0, & s \notin \left[\frac{b-a}{N} i \le s \le \frac{b-a}{N}(i+1)\right] \end{cases} \end{aligned} \tag{2.51}$$

The solution of the equation with the degenerate kernel (2.51) approximates the solution of Eq. (2.2), generally, the larger the number N of strips and the function $K_N(x, s)$ the better the approximation of $K(x, s)$ in each strip [13]. The approximate solution $\varphi_N(x)$ can be further improved by using the iterative algorithm that insures the continuation of the final solution $\varphi(x)$ in the whole interval $[a, b]$ at the same time. The contracting-mapping principle is of special interest for nonlinear equations; see e.g., [14].

2.5.2 Power Series Solution for Integral Equations

In many cases if the Kernel of the integral equation is of the form $K(xs)$ and the Mellin transform of $K(t)$ exists we can find the solution of the integral equation

$$g(s) = s \int_0^\infty K(st) f(t) \mathrm{d}t \tag{2.52}$$

In a form of a power series

$$\begin{aligned} f(t) &= \sum_0^\infty \frac{a_n}{M(n+1)} t^n \\ g(s) &= \sum_{n=0}^\infty a_n s^{-n}; \quad M(n+1) = \int_0^\infty t^n K(t) \mathrm{d}t \end{aligned} \tag{2.53}$$

Where $g(s)$ is the Z-transform of the function $g(s)$ and $M(n + 1)$ is the Mellin transform of the Kernel [15].

2.5.3 Hammerstein Integral Equation

A nonlinear integral equation of the type

$$\varphi(x) + \int_a^b K(x,s) f[s, \varphi(s)] \mathrm{d}s = 0; \quad a \le x \le b \tag{2.54}$$

Where $K(x, s)$ and $f(x, s)$ are given functions, while $\varphi(x)$ is the unknown function. Named after Hammerstein [16], who considered the case where $K(x, s)$ is a symmetric and positive Fredholm kernel, i.e., all its eigenvalues are positive. If, in addition, the function $f(x, s)$ is continuous and satisfies the condition $|f(x,s)| \le C_1|s| + C_2$, where C1 and C2 are positive constants and C_1 is smaller than the first eigenvalue of the kernel $K(x, s)$, the Hammerstein equation has at least one continuous solution. If, on the other hand, $f(x, s)$ happens to be a non-decreasing function of s for any fixed x from the interval $[a, b]$, Hammerstein's equation cannot have more than one solution. This property holds also if f-(x, s) satisfies the condition $|f(x,s_1) - f(x,s_2)| \le C|s_1 - s_2|$, where the positive constant C is smaller than the first eigenvalue of the kernel $K(x, s)$. A solution of the Hammerstein equation may be constructed by the method of successive approximation.

Below are examples of solutions of Volterra and Fredholm integral equations of the second kind (and the mixed type Fredholm–Volterra equation), which are intended to acquaint the reader with the approximate numerical methods for solving these equations used in the subsequent chapters of this book, as well as give an engineering assessment of convergence of these solutions by comparison with known exact solutions. Of the large number of examples available in mathematical physics are selected only those that is close to the phenomenological theory of creep of materials and its practical application to the structural analysis and design. Some examples are solved by applying different numerical methods (more than one) in order to compare them and to obtain upper and lower bounds with respect to the exact solution of the given integral equation.

2.6 Examples

Example 2.2 (Method #1) Data: Consider a Volterra IE of the 2nd kind

$$f(x) = \varphi(x) - \lambda \int_0^x \mathrm{e}^{x-y} \varphi(y) \mathrm{d}y \tag{2.55}$$

The kernel $K(x, y) = e^{x-y}$, $f(x)$ is a given continuously differentiable function, and $\lambda\lambda\psi$ is a parameter.

Solution: method #1.

Differentiating (1.54) we obtain

$$f(x) = \varphi(x) - \lambda \int_0^x e^{x-y}\varphi(y)dy$$

$$f'(x) = \varphi'(x) - \lambda[\varphi(x) + \int_0^x e^{x-y}\varphi(y)dy]$$

Subtracting term wise we obtain an ordinary differential equation (ODE) of the first order

$$\varphi'(x) - (\lambda + 1)\varphi(x) = f'(x) - f(x) = F(x) \tag{2.56}$$

with respect to the (same) unknown function $\varphi(x)$. Setting $x = 0$ in (2.54) we obtain the initial condition for the unknown function $\varphi(x)$: $\varphi(0) = f(0)$. Thus, we have obtained the initial value problem for $\varphi(x)$. Integrating (2.56) subject to the initial condition we obtain the Volterra IE (1.54), which is therefore equivalent to the initial value problem (2.56).

Later (see Chap. 4) equation similar to (2.55) will represent the simplest case of creep deformations.

$$f(x) = \varphi(x) - \lambda \int_0^x e^{-\alpha(x-y)}\varphi(y)dy \tag{2.57}$$

where α − material property parameter (MPP) and $\lambda = -1$

The corresponding differential Eq. (1.57) is:

$$\begin{aligned} &\varphi'(x) - (\lambda - \alpha)\varphi(x) = f'(x) + \alpha f(x) = F(x, \alpha) \\ &\text{where } f(x) = xe^{-0.15x}; f'(x) = e^{-0.15x}(1 - 0.15x); \alpha = 0.33 \\ &\text{initial condition: } \varphi(0) = 0 \end{aligned} \tag{2.58}$$

Solution of Eq. (2.58) in this case is as follows (using POLYMATH software):

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0	0	1.0	1.0
2	y	0	0	0.5595	0.5595

Differential equation

$$1 \quad \mathrm{d}(y)/\mathrm{d}(t) = -1.33 * y + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + t * \exp(-0.15 * t)) * 0.33$$

See Fig. 2.1.

The Stress–Temperature function can be approximated by quadratic parabola:

Model: $y = a0 + a1 * t + a2 * t\text{^}2 = 0.0083 + 0.892t - 0.347t^2$

Variable	Value
$a0$	0.00883
$a1$	0.892
$a2$	−0.347

See Fig. 2.2.

Method #2 Equation (1.55) is as follows:

$$\varphi'(x) = f'(x) - \lambda\varphi(x) + \lambda\alpha \int_0^x \mathrm{e}^{-\alpha(x-y)}\varphi(y)\mathrm{d}y$$

$$\text{or}: \begin{cases} \varphi'(x) = f'(x) - \lambda\varphi(x) + \lambda\alpha \mathrm{e}^{-\alpha x} Z \\ Z' = \mathrm{e}^{\alpha y}\varphi(y) \end{cases} \quad \varphi(0) = 0;\ Z(0) = 0 \tag{2.59}$$

Consider now the larger interval $[a, b] = [0, 10]$. If inequality (1.35) does not hold then the strip method will be applied: the whole interval is divided in 10 equal subintervals.

Partitioning points are as follows: $x_0 = 0$; $x_1 = 1$; $x_{10} = 10$ Let us find the solution of Eq. (1.59) using POLYMATH software for (initial condition for subinterval "i" is the end point of the solution of subinterval "$i - 1$"):

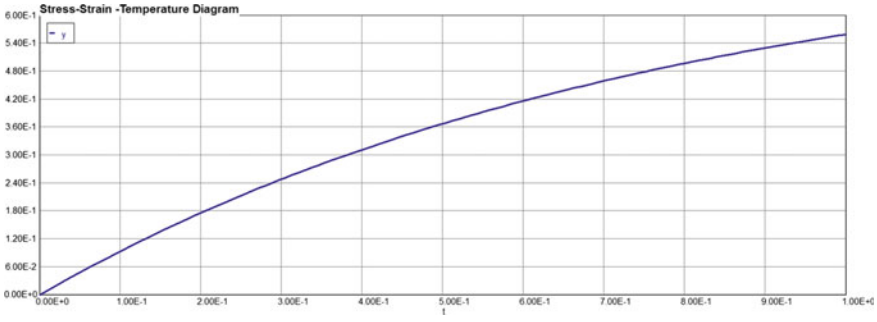

Fig. 2.1 Stress–Temperature diagram

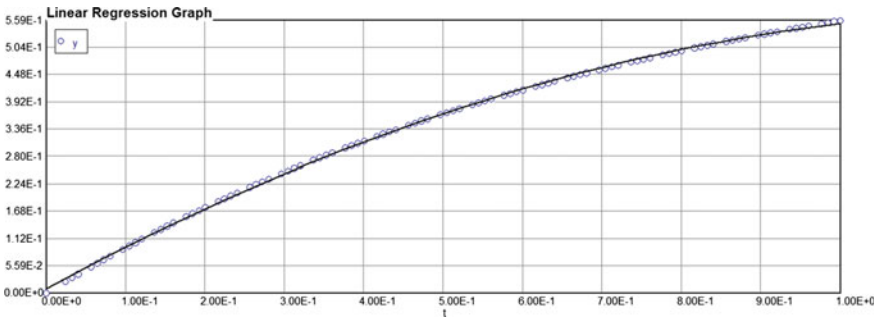

Fig. 2.2 Stress–Temperature (Strain) function

Subinterval [0, 1]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0	0	1.0	1.0
3	$y1$	0	0	0.5594981	0.5594981
4	z	0	0	0.4189734	0.4189734

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1 + 0$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t)) + 0$$

Note: One can see that methods 1 and 2 obviously provide identical results for function φ.

Subinterval [1, 2]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	1.0	1.0	2.0	2.0
3	$y1$	0.56	0.56	0.7081798	0.7081798
4	z	0.42	0.42	1.498877	1.498877

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Subinterval [2, 3]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	2.0	2.0	3.0	3.0
3	$y1$	0.71	0.71	0.7376851	0.7376851
4	z	1.5	1.5	3.172535	3.172535

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Subinterval [3, 4]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	3.0	3.0	4.0	4.0
3	$y1$	0.74	0.7293919	0.7401333	0.7293919
4	z	3.2	3.2	5.547809	5.547809

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Subinterval [4, 5]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	4.0	4.0	5.0	5.0
3	$y1$	0.74	0.7049854	0.74	0.7049854
4	z	5.5	5.5	8.700056	8.700056

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Subinterval [5, 6]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	5.0	5.0	6.0	6.0
3	$y1$	0.71	0.6703445	0.71	0.6703445
4	z	8.7	8.7	12.95057	12.95057

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Subinterval [6, 7]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	6.0	6.0	7.0	7.0
3	$y1$	0.67	0.6311913	0.67	0.6311913
4	z	13.0	13.0	18.57578	18.57578

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Subinterval [7, 8]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	7.0	7.0	8.0	8.0
3	$y1$	0.63	0.5900709	0.63	0.5900709
4	z	18.6	18.6	25.87087	25.87087

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Subinterval [8, 9]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	8.0	8.0	9.0	9.0
3	$y1$	0.6	0.5534172	0.6	0.5534172
4	z	25.9	25.9	35.44505	35.44505

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Subinterval [9, 10]

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	9.0	9.0	10.0	10.0
3	$y1$	0.55	0.5087269	0.55	0.5087269
4	z	35.4	35.4	47.59758	47.59758

Differential equations

$$2 \quad \mathrm{d}(y1)/\mathrm{d}(t) = -1.0 * y1 + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + z * (\exp(-0.33 * t)) * 0.33 * 1$$

$$3 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

Let us check if inequality (1.35) holds for the whole interval [0, 10]:

$$|\lambda| = 1 < \frac{1}{|\mathrm{e}^{-0.33x}| \sqrt{\int_0^{10} [0.33(\mathrm{e}^{-(10-x)})]^2 \mathrm{d}x}} = \frac{1}{\sqrt{00.1648}} = 2.46$$

Table 2.2 Stress–Temperature data

θ	0	1	2	3	4	5	6	7	8	9	10
φ	0	0.56	0.71	0.74	0.73	0.71	0.67	0.63	0.59	0.55	0.51
y	0	0.56	0.71	0.74	0.73	0.70	0.66	0.63	0.58	0.54	0.50

Therefore, the solution can be found in this case ($\alpha = 0.33$) without subdividing the whole interval in ten subintervals. However, on the other hand it proves that this form of strip method application works properly. The comparison results are provided in Table 2.2.

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0	0	10.0	10.0
2	y	0	0	0.7364601	0.5006154

Differential equations

$$1 \quad \mathrm{d}(y)/\mathrm{d}(t) = -1.33 * y + (\exp(-0.15 * t)) * 1.0 * (1 - 0.15 * t) + t * (\exp(-0.15 * t)) * 0.33$$

$$2 \quad \mathrm{d}(z)/\mathrm{d}(t) = y1 * (\exp(0.33 * t))$$

See Fig. 2.3.

If Eq. (1.55) has a form:$\varphi'(x) = f'(x) - \lambda n\varphi(x) + \lambda \sum_{i=1}^{n} \alpha_i \int_0^x \mathrm{e}^{-\alpha_i(x-y)}\varphi(y)\mathrm{d}y$ then

$$\begin{cases} \varphi'(x) = f'(x) - \lambda n\varphi(x) + \lambda \sum\limits_{i=1}^{n} \alpha_i Z_i \\ Z_i' = \mathrm{e}^{\alpha_i x}\varphi(x) \end{cases} \qquad (2.60)$$
$$\varphi(0) = 0; Z_i(0) = 0;$$

Example 2.3 Data: Consider a Volterra IE of the second kind

$$l\varphi(x) = x - \int_0^x (x - t)\varphi(t)\mathrm{d}t 0 \le x \le 1$$

$$\text{Exact solution} \rightarrow \varphi(x) = \sin x$$

The exact solution is easy to obtain by differentiating the integral equation above twice. We have now: $\varphi''(x) + \varphi(x) = 0$. Initial conditions: $\varphi(0) = 0; \varphi'(0) = 1$. Solution $\varphi(x) = \sin x$.

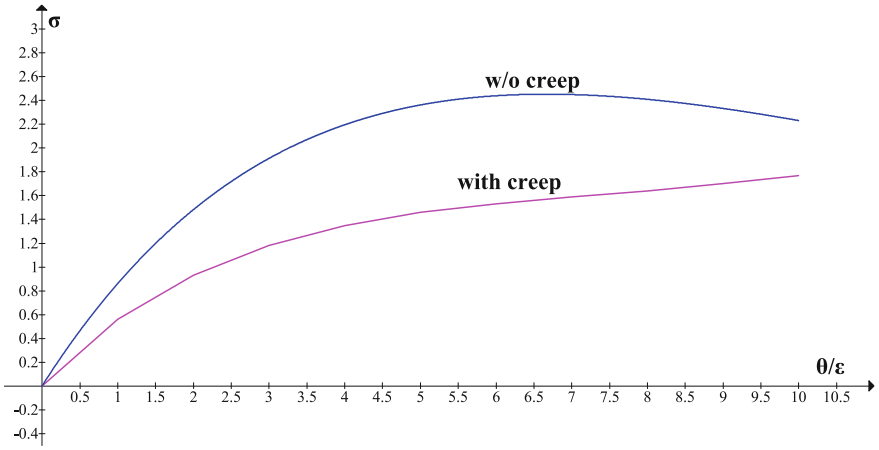

Fig. 2.3 Stress–Temperature (Strain) diagram

Method #1 *Gauss quadrature sum: n = 2*

K(x, t) = x-t f(x) = x; λ = -1; x1 = 0.2113; x_2 = 0.7887; $K_{1,1}$ = 1; $K_{1,2}$ = 0; $K_{2,1}$ = 0.2887; $K_{2,2}$ = 1; f_1 = 0.2113; f_2 = 0.7887;

$$\varphi(x_1) = \frac{\begin{vmatrix} 00.2113 & 0 \\ 0.7887 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0.2887 & 1 \end{vmatrix}} = 0.2113; \quad \varphi(x_2) = \frac{\begin{vmatrix} 1 & 0.2113 \\ 0.2887 & 0.7887 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0.2887 & 1 \end{vmatrix}} = 0.7277$$

$$\varphi(x) = x - \frac{1}{2}[(x - 0.2113)0.2113 + (x - 0.7887)0.7277] = 0.5x + 0.309$$

$$\varphi(x = 1) = 0.809; \quad \sin(1) = 0.841 \quad (\text{error: } 30.96\%)$$

See Fig. 2.4.

Method #2 *Galerkin Method*

$\varphi(x) = Ax + B$

$$\varphi_1 = x; \varphi_2 = 1$$

$$\begin{cases} \int_0^1 x(Ax+B)\mathrm{d}x + \int_0^1 \int_0^x t(At+B)(x-t)\mathrm{d}t\mathrm{d}x = \int_0^1 x^2\mathrm{d}x \\ \int_0^1 (Ax+B)\mathrm{d}x + \int_0^1 \int_0^x (At+B)(x-t)\mathrm{d}t\mathrm{d}x = \int_0^1 x\mathrm{d}x \end{cases}$$

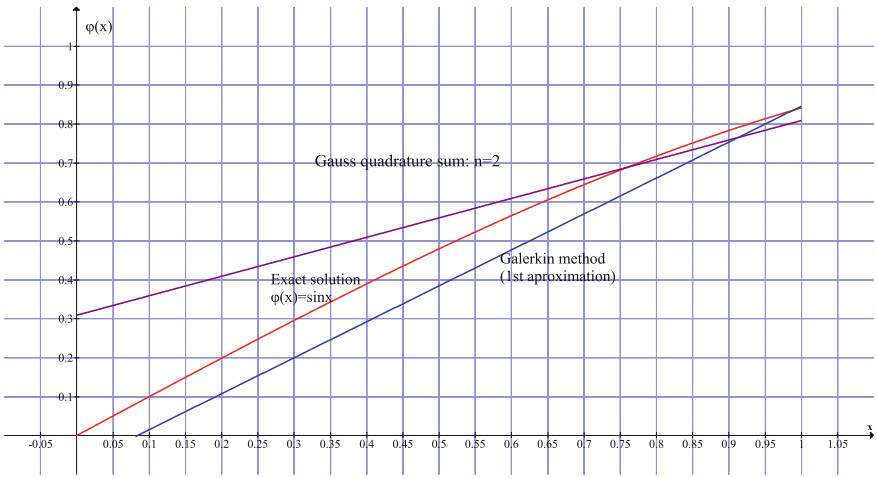

Fig. 2.4 Numerical methods comparison (Gauss and Galerkin)

$5A + 8B = 4$
$4A + 9B = 3$
$A = 12/13$; $B = -1/13$
$\varphi(x = 1) = 0.846$ $\sin(1) = 0.841$ (error: 0.6 %)

Method #3 *Successive approximation (Sequential approximation method)*

1. $\varphi_0 = x$; $\lambda = -1$

$$\varphi_1 = \int_0^x (x - t)t\mathrm{d}t = \frac{x^3}{3!}$$

$$\varphi_2 = \int_0^x (x - t)\frac{t^3}{3!}\mathrm{d}t = \frac{x^5}{5!}$$

$$\ldots$$

$$\varphi(x) = \varphi_0 + \lambda\varphi_1 + \lambda^2\varphi_2 + \ldots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sin x$$

Example 2.4 Data: Consider a Volterra IE of the second kind

$$\varphi(x) = \mathrm{e}^x + \int_0^x \varphi(t)\mathrm{d}t \quad 0 \le x \le 1$$

$$\text{Exact solution} \rightarrow \varphi(x) = \mathrm{e}^x(1 + x)$$

Successive approximation (Sequential approximation method)

1. $\varphi_0 = e^x$; $\lambda = 1$

$$\varphi_1 = \int_0^x e^t dt = e^x - 1$$

$$\varphi_2 = \int_0^x (e^t - 1)dt = e^x - 1 - x$$

$$\varphi_3 = \int_0^x (e^t - 1 - x)dt = e^x - 1 - x - \frac{x^2}{2}$$

...

$$\varphi_n(x) = \varphi_0 + \lambda\varphi_1 + \lambda^2\varphi_2 + \ldots = e^x + e^x - n - (n-1)x - \sum_{k=3}^{n} \frac{x^{k-1}}{k-1}$$

$$\varphi_3(x = 1) = 4e - 3 - 2 - \frac{1}{2} = 5.373$$

$$\varphi(x = 1) = e^x(1 + x) = 5.436 \approx 5.373(1.2\%)$$

Example 2.5 Consider the following nonlinear Volterra integral equation (with the exact solution $x(t) = \sin t$.)

$$x(t) - \frac{1}{2}\int_0^t x^2(s)ds = \sin t + \frac{1}{8}\sin 2t - \frac{1}{4}t; \quad 0 < t < 10$$

Inequality (1.35) is $|-1/2| < 1$. Therefore: $0 < t < \infty$.
Solution:
Differentiating the integral equation above, we obtain

$$x'(t) = (\cos t) + \frac{1}{4}(\cos 2t) - \frac{1}{4} + \frac{1}{2}x^2(t); \quad x(0) = 0$$

Differential equations

$d(u)/d(t) = \cos(t) + 0.25 * \cos(2 * t) - 0.25 + 0.5 * u^2$

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0	0	10.0	10.0
2	u	0	−0.9977534	0.9989641	−0.5440211

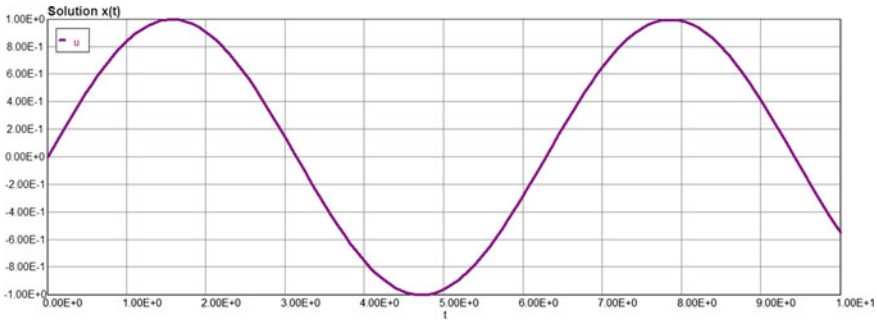

Fig. 2.5 Solution of nonlinear Volterra integral equation

See Fig. 2.5.

Example 2.6 Data: Consider the linear Volterra integral equation. The analytical solution of the above problem is given by $u(x) = \exp(-x)$

$$u(x) = \cos x - \sin x + 2\int_0^x [\sin(x-t)]u(t)\mathrm{d}t; \quad 0 < x < 1$$

Solution:

Differentiating the integral equation above we obtain

$$u'(x) = -\sin x - \cos x + 2(\cos x)Z_1 + 2(\sin x)Z_2$$
$$\frac{\mathrm{d}Z_1}{\mathrm{d}x} = (\cos x)u; \frac{\mathrm{d}Z_2}{\mathrm{d}x} = (\sin x)u; \quad u(0) = 1; Z_1(0) = Z_2(0) = 0$$

Using POLYMATH software for ODE system above, we have

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0	0	10.0	10.0
2	u	1.0	6.737E-05	1.0	6.737E-05

Differential equations

1 $\mathrm{d}(u)/\mathrm{d}(t) = -\sin(t) - \cos(t) + 2 * (\cos(t)) * z3 + 2 * (\sin(t)) * z4$
2 $\mathrm{d}(z1)/\mathrm{d}(t) = (\cos(t)) * u$
3 $\mathrm{d}(z2)/\mathrm{d}(t) = (\sin(t)) * u$

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0	0	1.0	1.0
2	u	1.0	0.3678794	1.0	0.3678794

The analytical solution $u(x = 1) = e^{-1} = 0.36788$

Example 2.7 Data: Consider the nonlinear Fredholm integral equation.

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 \mathrm{xtu}^2(t)\mathrm{d}t$$

The analytical solution of the above integral equation is given by $u(x) = x$.

$$\tilde{u}(x) = \frac{7}{8}x + Ax; \quad a(x) = x; \quad b(t) = t;$$

$$Ax = \frac{x}{2}\int_0^1 t\left[\frac{7}{8}t + At\right]^2 \mathrm{d}t; \quad A = \frac{1}{2}\int_0^1 t^3\left[\frac{7}{8} + A\right]^2 \mathrm{d}t = \frac{\left[\frac{7}{8} + A\right]^2}{8}$$

$$A^2 - 6.25A + 0.765 = 0; \quad A = 0.125; \quad \tilde{u} = 1.0x = 1.0x$$

Example 2.8 Data: Consider the nonlinear Fredholm integral equation.

$$\varphi(x) - \frac{1}{2}\int_0^1 e^{xs}\varphi(s)\mathrm{d}s = 1 - \frac{1}{2x}(e^x - 1)$$

The analytical solution of the problem is given by $\varphi(x) = 1$. Let us apply the Gauss quadrature formula for the approximate solution of integral equation above. Replacing the integral equation to a system of algebraic equations for $n = 2$, and taking into account that there are $\lambda = 0.5$, $A_1 = A_2 = 0.5$, we obtain the system (2.14) in this case in the form of

$$\left(1 - \frac{1}{4}K_{1,1}\right)\tilde{\varphi}(x_1) - \frac{1}{4}K_{1,2}\tilde{\varphi}(x_2) = f_1$$

$$-\frac{1}{4}K_{2,1}\tilde{\varphi}(x_1) + \left(1 - \frac{1}{4}K_{2,2}\right)\tilde{\varphi}(x_2) = f_2 \ldots$$

According to the above, for x_1 and x_2 are taken Gauss abscissas for the interval [0, 1],

$x_1 = 0.2113$, $x_2 = 0.7887$. Calculating the value of $K_{\iota,\kappa} = K(x_i, x_k)$ and $f_i = f(x_i)$ and substituting them into the system of Eq. (2.14), we obtain.

$$\begin{cases} 0.7386\tilde{\varphi}(x_1) - 0.2954\tilde{\varphi}(x_2) = 0.4434 \\ -0.2954\tilde{\varphi}(x_1) + 0.5343\tilde{\varphi}(x_2) = 0.2384 \end{cases}$$
$$\tilde{\varphi}(x_1) = 0.9997; \quad \tilde{\varphi}(x_2) = 0.9990$$

Therefore, an approximate solution of Eq. (2.14) in any other points in the interval (0.1) is given by the following equation

$$\tilde{\varphi}(x_1) = \frac{1}{4}\left(\mathrm{e}^{0.2113x}0.9997 + 0.999\mathrm{e}^{0.7887x}\right) + 1 - \frac{1}{2x}\left(\mathrm{e}^{x} - 1\right)$$

We can, for example, calculate the values: $\tilde{\varphi}(x = 0) = 0.997$; $\tilde{\varphi}(x = 1) = 0.9991$.

In other words, an approximate solution of almost coincides with the exact solution.

The following Examples 2.9, 2.10, and 2.11 have a twofold purpose. First, they are directly related to the problems of material creep at high temperatures, due to the fact that the creep function contains the Arrhenius law (exponentially increasing function of dimensionless temperature). Second, in dealing with these examples we use the numerical methods for solving Volterra equations of the second kind, which will be used in subsequent chapters of this book in solving practical engineering creep problems. At the same time, this chapter provides a qualitative assessment of solution of integral equations. For example, it appears that the Galerkin method provides a very conservative estimate of the solution (the lower bound).

Example 2.9 Data: The IE $\varphi(x) - \lambda \int_a^x K(x,s)\varphi(s)\mathrm{d}s = f(x)$ with the degenerate kernel $K(x,y) = \sum_{i=1}^{n} a_i(x)b_i(y)$ is:

$$K_N(\theta, s) = \sum_{n=1}^{N} a_n(\theta)b_n(s) \quad \alpha_1 = 0.33; \quad \alpha_2 = 0.99$$
$$a_n(\theta) = A_1(1 - \mathrm{e}^{-\alpha_1\theta}) + A_2(1 - \mathrm{e}^{-\alpha_2\theta})$$
$$b_n(s) = \left[\exp\left(\frac{s}{1 + 0.1s}\right)\right]\left[\mathrm{e}^{-s} + \mathrm{e}^{-2s}\right]$$

Then the contracting-mapping principle implies that if $|\lambda| < \left(\int_a^b \int_a^b K^2(x,s)\,\varphi(s)\mathrm{d}s\mathrm{d}x\right)^{-1/2}$. The solution of the degenerate integral Eq. (1.14) is uniformly convergent in this case, since

$$|\lambda = -1| < \left(\int_{\mathrm{a}}^{\mathrm{b}}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{K}^2(\mathrm{x},\mathrm{s})\varphi(\mathrm{s})\mathrm{d}s\mathrm{d}x\right)^{-1/2}$$

$$= \left(\int_0^1\int_0^1 \sum_{i=1}^{2}\{(1-\mathrm{e}^{-\alpha_1 x})+(1-\mathrm{e}^{-\alpha_2 x})\}\left\{\left[\exp\left(\frac{s}{1+0.1s}\right)\right][\mathrm{e}^{-s}+\mathrm{e}^{-2s}]\right\}\mathrm{d}s\mathrm{d}x\right)^{-1/2} = 3.34$$

The approximate solution of integral equation is as follows:

$$\tilde{\varphi}(x) + \int_0^x K_N(x,s)\tilde{\varphi}(s)\mathrm{d}s = f(x)$$

The corresponding dummy differential equations are as follows:

$$\frac{\mathrm{d}\psi_{11}(x)}{\mathrm{d}x} = a_1(x)b_1(x) = (1-\mathrm{e}^{-\alpha_1\theta})\left[\exp\left(\frac{x}{1+0.1x}\right)\right]\mathrm{e}^{-x}$$

$$\frac{\mathrm{d}\psi_{12}(x)}{\mathrm{d}x} = a_1(x)b_2(x) = (1-\mathrm{e}^{-\alpha_1\theta})\left[\exp\left(\frac{x}{1+0.1x}\right)\right]\mathrm{e}^{-2x}$$

$$\frac{\mathrm{d}\psi_{21}(x)}{\mathrm{d}x} = a_2(x)b_1(x) = (1-\mathrm{e}^{-\alpha_2\theta})\left[\exp\left(\frac{x}{1+0.1x}\right)\right]\mathrm{e}^{-x}$$

$$\frac{\mathrm{d}\psi_{22}(x)}{\mathrm{d}x} = a_2(x)b_2(x) = (1-\mathrm{e}^{-\alpha_2\theta})\left[\exp\left(\frac{x}{1+0.1x}\right)\right]\mathrm{e}^{-2x}$$

$$\frac{\mathrm{d}\vartheta_1(x)}{\mathrm{d}x} = f(x)b_1(x) = x\mathrm{e}^{-0.15x}\left[\exp\left(\frac{x}{1+0.1x}\right)\right]\mathrm{e}^{-x}$$

$$\frac{\mathrm{d}\vartheta_2(x)}{\mathrm{d}x} = f(x)b_2(x) = x\mathrm{e}^{-0.15x}\left[\exp\left(\frac{x}{1+0.1x}\right)\right]\mathrm{e}^{-2x}$$

Solutions are (using POLYMATH software):

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0	0	1.0	1.0
2	$y1 = \psi_{11}$	0	0	0.1417315	0.1417315
3	$y2 = \psi_{12}$	0	0	0.0760957	0.0760957
4	$y3 = \psi_{21}$	0	0	0.0760957	0.0760957

(continued)

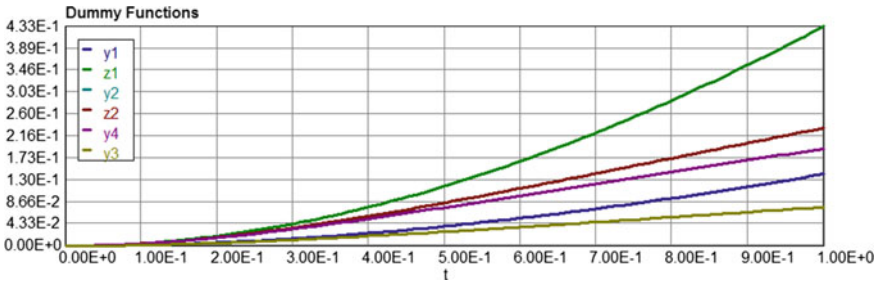

Fig. 2.6 Dummy equation

(continued)

	Variable	Initial value	Minimal value	Maximal value	Final value
5	$y4 = \psi_{22}$	0	0	0.1911314	0.1911314
6	$z1 = \vartheta_1(x)$	0	0	0.4327624	0.4327624
7	$z2 = \vartheta_2(x)$	0	0	0.232214	0.232214

Differential equations

1 $d(y1)/d(t) = (\exp((t)/(1.0 + 0.1 * t))) * (\exp(-(1 * t)))\text{^}1 * (1 - \exp(-0.33 * (t - 0))) * (t\text{^}0)$
2 $d(z1)/d(t) = (t\text{^}1) * (\exp(-1 * (t)))\text{^}1 * (\exp(-0.15 * t))\text{^}1 * (\exp((t)/(1.0 + 0.1 * t)))$
3 $d(y2)/d(t) = (\exp((t)/(1.0 + 0.1 * t))) * (\exp(-(2 * t)))\text{^}1 * (1 - \exp(-0.33 * (t - 0))) * (t\text{^}0)$
4 $d(z2)/d(t) = (\exp((t)/(1.0 + 0.1 * t))) * (t) * (\exp(-0.15 * t)) * (\exp(-2 * t))$
5 $d(y4)/d(t) = (\exp((t)/(1.0 + 0.1 * t))) * (\exp(-(2 * t)))\text{^}1 * (1 - \exp(-0.99 * (t - 0)))$
6 $d(y3)/d(t) = (\exp((t)/(1.0 + 0.1 * t))) * (\exp(-(2 * t)))\text{^}1 * (1 - \exp(-0.33 * (t - 0))) * (t\text{^}0)$

See Fig. 2.6.
Model: $y1 = a * (1 - \exp(-1 * t))$

Variable	Initial guess	Value
a	1.0	0.153

Model: $y2 = a * (1 - \exp(-1 * t))$

Variable	Initial guess	Value
a	1.0	0.0934

Model: $y3 = a * (1 - \exp(-1 * t))$

Variable	Initial guess	Value
a	1.0	0.0934

Model: $y4 = a * (1 - \exp(-1 * t))$

Variable	Initial guess	Value
a	1.0	0.243

Model: $z1 = a * (1 - \exp(-1 * t))$

Variable	Initial guess	Value
a	1.0	0.466

Model: $z2 = a * (1 - \exp(-1 * t))$

Variable	Initial guess	Value
a	1.0	0.285

$$f_1 = 0.466 \int_0^1 [1 - e^{-0.33s}][1 - e^{-s}]ds = 0.0322 \quad \alpha_1 = 0.33; \quad \alpha_2 = 0.99$$

$$f_2 = 0.285 \int_0^1 [1 - e^{-s}][1 - e^{-0.99s}]ds = 0.0476$$

$$b_{11} = 1 + 0.153 \int_0^1 A_1(1 - e^{-0.33s})[1 - e^{-s}]ds = 1.01$$

$$b_{12} = 0.0934 \int_0^1 A_1(1 - e^{-0.33s})[1 - e^{-s}]ds = 0.00646$$

$$b_{21} = 0.0934 \int_0^{10} A_2(1 - e^{-0.99s})[1 - e^{-s}]ds = 0.0156$$

$$b_{22} = 1 + 0.243 \int_0^{10} A_2(1 - e^{-0.99s})[1 - e^{-s}]ds = 1.0406$$

1.01	0.00646	0.0322
0.0156	1.0406	0.378

Linear Equations Solution

	Variable	Value
1	$x1 = A_1$	0.0295606
2	$x2 = A_2$	0.3628088

The equations

[1] $1.01 \cdot x1 + 0.00646 \cdot x2 = 0.0322$

[2] $0.0156 \cdot x1 + 1.0406 \cdot x2 = 0.378$

$$\sigma_1(\theta) = f(\theta) - \left[0.0296(1 - \mathrm{e}^{-0.33\theta}) + 0.363(1 - \mathrm{e}^{-0.99\theta})\right]$$
$$\sigma_1(\theta = 1) = 0.86 - \left[0.0296(1 - \mathrm{e}^{-0.33}) + 0.363(1 - \mathrm{e}^{-0.99})\right] = 0.624$$
$$\sigma_{10}(\theta = 10) = 2.23 - \left[0.0296(1 - \mathrm{e}^{-3.3}) + 0.363(1 - \mathrm{e}^{-9.9})\right] = 1.838$$

Example 2.10 Data: Successive approximation ($b(s)$—Dirichlet series)

The IE $\varphi(x) - \lambda \int_a^x K(x,s)\varphi(s)\mathrm{d}s = f(x)$ with the degenerate kernel $K(x,y) = \sum_{i=1}^{n} a_i(x)b_i(y)$ is:

$$a(x) = (1 - \mathrm{e}^{-\alpha_1 x});\ b(s) = \mathrm{e}^{-\alpha_1 s} + \mathrm{e}^{-2\alpha_1 s} + \mathrm{e}^{-3\alpha_1 s} + \ldots \alpha_1 = 0.33$$
$$K_N(x,s) = (1 - \mathrm{e}^{-\alpha_1 x})(e^{-\alpha_1 s} + \mathrm{e}^{-2\alpha_1 s} + \mathrm{e}^{-3\alpha_1 s} + \ldots) = (1 - \mathrm{e}^{-\alpha_1 x})[(\mathrm{e}^{\alpha_1 s})/(1 - \mathrm{e}^{-\alpha_1 s})]$$
$$\tilde{\varphi}(x) + \int_0^x K_N(x,s)\tilde{\varphi}(s)\mathrm{d}s = f(x) x \in [0,1] \alpha_1 = 0.33 \quad 0 < x < 1$$
$$\tilde{\varphi}_0(x) = f(x) = x\mathrm{e}^{-0.15x}$$
$$\tilde{\varphi}_1(x) = \int_0^x K_N(x,s)\tilde{\varphi}_0(s)\mathrm{d}s$$
$$= \int_0^x (1 - \mathrm{e}^{-\alpha_1 x})[\mathrm{e}^{-\alpha_1 s} + \mathrm{e}^{-\alpha_1 2s} + \mathrm{e}^{-\alpha_1 3s} + \ldots]\left[\exp\left(\frac{s}{1 + 0.1s}\right)\right] s\mathrm{e}^{-0.15s}\mathrm{d}s$$
$$\text{Dummy equation: } \frac{\mathrm{d}y}{\mathrm{d}s} = \left[\frac{\mathrm{e}^{-(0.15+\alpha_1)s}}{1 - \mathrm{e}^{-\alpha_1 s}}\right]\left[\exp\left(\frac{s}{1 + 0.1s}\right)\right][s]$$
$$\text{Solution : } \tilde{\varphi}_1 = (1 - \mathrm{e}^{-\alpha_1 x})(-0.211 + 40.2x)$$
$$\text{1st approximation } \varphi_1(x) = x\mathrm{e}^{-0.15x} - (1 - \mathrm{e}^{-\alpha_1 x})(-0.211 + 40.2x)$$
$$\max|\varphi_0(x) - \varphi_1(x)| = |\varphi_0(x = 1) - \varphi_1(x = 1)| = 0.26$$

Differential equations

$$8 \quad \mathrm{d}(y6)/\mathrm{d}(t) = (\exp((t)/(1.0 + 0.1 * t)))\text{^}1 * t * \exp(-(0.48 * t))\text{^}1/(1 - \exp(-(0.33 * t))\text{^}1)$$

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0.001	0.001	1.0	1.0
7	$y6$	0	0	4.188095	4.188095

Model: $y6 = a0 + a1 * t$ $\varphi_{11} = -0.211 + 4.2x$

Variable	Value
$a0$	−0.2109213
$a1$	4.200333

2nd approximation:

$$\tilde{\varphi}_2(x) = \int_0^x K_N(x,s)\tilde{\varphi}_1(s)\mathrm{d}s$$

$$= (1-\mathrm{e}^{-\alpha_1 x})\int_0^x \left[\exp\left(\frac{s}{1+0.1s}\right)\right]\left[\frac{\mathrm{e}^{-(\alpha_1)s}}{1-\mathrm{e}^{-\alpha_1 s}}\right](1-\mathrm{e}^{-\alpha_1 s})(-0.211+4.2\mathrm{s})\mathrm{d}s$$

Dummy equation: $\dfrac{\mathrm{dy}}{\mathrm{d}s} = \left[\dfrac{\mathrm{e}^{-(\alpha_1)s}}{1-\mathrm{e}^{-\alpha_1 s}}\right]\left[\exp\left(\dfrac{s}{1+0.1s}\right)\right](1-\mathrm{e}^{-\alpha_1 s})(-0.211+40.2\mathrm{s})$

Solution : $\tilde{\varphi}_2 = (1-\mathrm{e}^{-\alpha_1 x})(0.04-0.676x+3.183x^2)$

$$\varphi_{\mathrm{tot}}(x) = x\mathrm{e}^{-0.15x} - (1-\mathrm{e}^{-\alpha_1 x})(-0.211+40.2x) + (1-\mathrm{e}^{-\alpha_1 x})(0.04-0.676x+3.183x^2)$$

$$= x\mathrm{e}^{-0.15x} - (1-\mathrm{e}^{-\alpha_1 x})(-0.171+30.524x-3.183x^2)$$

$$max|\varphi_1(x)-\varphi_2(x)| = |\varphi_1(x=1)-\varphi_2(x=1)| = 0.0478 \rightarrow \text{small}$$

$$\varphi_{\mathrm{tot}}^{(2)}(x=1) = 0.812$$

See Fig. 2.7.

Model: $y5 = a0 + a1 * t + a2 * t^2$ $\varphi_{22} = 0.04 - 0.676x + 3.183x^2$

Variable	Value
$a0$	0.0405221
$a1$	−0.6765522
$a2$	3.182798

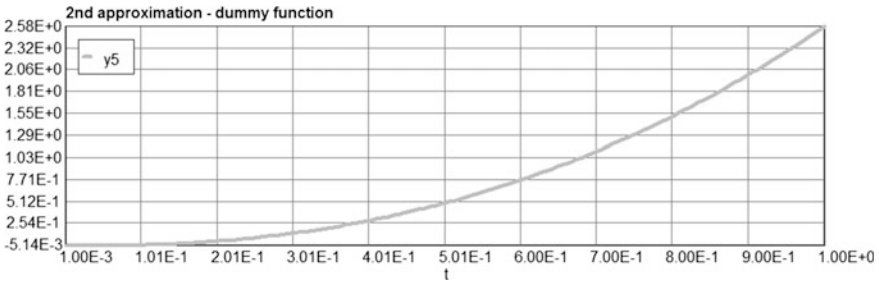

Fig. 2.7 Dummy equation (second approximation)

See Fig. 2.8.

Example 2.11 Data: Successive approximation (Spectra Method)

The IE $\varphi(x) - \lambda \int_a^x K(x,s)\varphi(s)\mathrm{d}s = f(x)$ with the degenerate kernel $K(x,y) = \sum_{i=1}^{n} a_i(x)b_i(y)$ is:

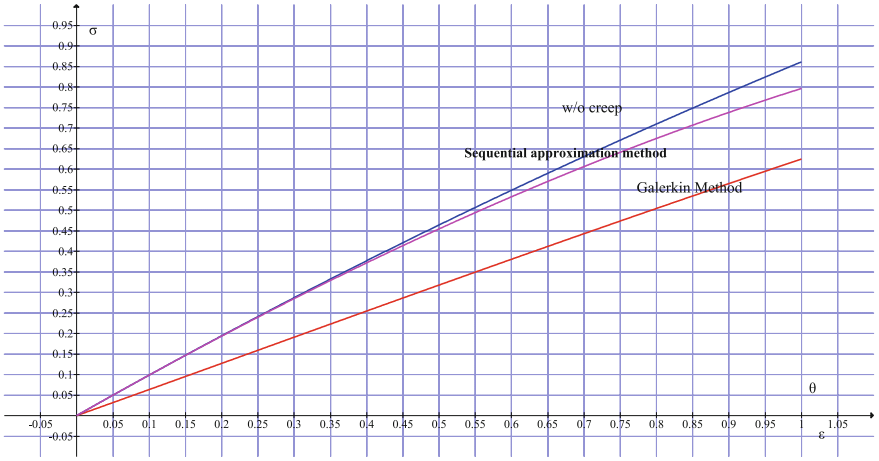

Fig. 2.8 Stress–Temperature/Strain diagram

$$\tilde{\varphi}(x)+\int_0^x K_N(x,s,\alpha)\tilde{\varphi}(s)\mathrm{d}s=f(x)x\in[0,1]0<\alpha<\infty$$

$$\tilde{\varphi}_0(x)=f(x)=x\mathrm{e}^{-0.15x}$$

$$\tilde{\varphi}_1(x)=\int_0^x K_N(x,s)\tilde{\varphi}_0(s)\mathrm{d}s=\int_0^x\int_0^\infty F(\alpha)K_N(x,s)\tilde{\varphi}_0(s)\mathrm{d}s\mathrm{d}\alpha$$

$$=\int_0^\infty \mathrm{e}^{-\alpha}(1-\mathrm{e}^{-\alpha x})\mathrm{d}\alpha\int_0^x[\mathrm{e}^{-s}+\mathrm{e}^{-2s}+\mathrm{e}^{-3s}+\ldots]\left[\exp\left(\frac{s}{1+0.1s}\right)\right]s\mathrm{e}^{-0.15s}\mathrm{d}s$$

$$\text{Dummy equation: }\frac{\mathrm{d}y}{\mathrm{d}s}=\left[\frac{\mathrm{e}^{-(1.15)s}}{1-\mathrm{e}^{-s}}\right]\left[\exp\left(\frac{s}{1+0.1s}\right)\right][s]$$

$$\text{Solution : }\tilde{\varphi}_1=\int_0^\infty \mathrm{e}^{-\alpha}(1-\mathrm{e}^{-\alpha x})(-0.0244+10.147x)\mathrm{d}\alpha$$

$$\text{1st approximation }\varphi_1(x)=x\mathrm{e}^{-0.15x}-\frac{x}{x+1}(-0.0244+10.147x)$$

$$\varphi_{\mathrm{tot}}^{(1)}(x)=0.86-0.56=0.3$$

2nd approximation:

$$\tilde{\varphi}_2(x)=\int_0^x K_N(x,s)\tilde{\varphi}_1(s)\mathrm{d}s$$

$$=\int_0^\infty \mathrm{e}^{-\alpha}(1-\mathrm{e}^{-\alpha x})\mathrm{d}\alpha\int_0^x\left[\exp\left(\frac{s}{1+0.1s}\right)\right]\left[\frac{\mathrm{e}^{-s}}{1-\mathrm{e}^{-s}}\right](-0.0244+1.147s)\frac{s}{s+1}\mathrm{d}s$$

$$\text{Dummy equation: }\frac{\mathrm{d}y}{\mathrm{d}s}=\left[\frac{\mathrm{e}^{-s}}{1-\mathrm{e}^{-s}}\right]\left[\exp\left(\frac{s}{1+0.1s}\right)\right](\frac{s}{s+1})(-0.0244+10.147\mathrm{s})$$

$$\text{Solution : }\tilde{\varphi}_2=\left(\frac{x}{x+1}\right)(-0.00466+0.0338x+0.409x^2)$$

$$\varphi_{\mathrm{tot}}(x)=x\mathrm{e}^{-0.15x}-\left(\frac{x}{x+1}\right)[(-0.0244+10.147x)-(-0.00466+0.0338x+0.409x^2)]$$

$$=x\mathrm{e}^{-0.15x}-\left(\frac{x}{x+1}\right)(-0.02+10.113x-0.409x^2)$$

$$\varphi_{\mathrm{tot}}^{(2)}(x=1)=0.518$$

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0.001	0.001	1.	1.
4	$y3$	0	0	1.140278	1.140278

Differential equations

6 $\mathrm{d}(y3)/\mathrm{d}(t)=t*(\exp((t)/(1.0+0.1*t)))*(\exp(-(1.15*t)))\wedge 1/(1-\exp(-1*(t-0)))$

Model: $y3 = a0 + a1 * t$

Variable	Value
$a0$	−0.0244
$a1$	1.147

2nd approximation:

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0.001	0.001	1.0	1.0
	$y5$	0	−0.0002337	0.4346849	0.4346849

Differential equations

7 $\mathrm{d}(y5)/\mathrm{d}(t) = ((\exp(t/(1.0 + 0.1 * t)))) * (t/(1 + t)) *$
$(\exp(-(1 * t))) * (-0.0244 + 1.147 * t) * (1/(1 - (\exp(-(1 * t)))))$

See Fig. 2.9.

Model: $y5 = a0 + a1 * t + a2 * t\text{^}2$

Variable	Value
$a0$	−0.00466
$a1$	0.0338
$a2$	0.409

$$\tilde{\varphi}(x) + \int_0^x K_N(x, s, \alpha)\tilde{\varphi}(s)\mathrm{d}s = f(x) x \in [0, 1] \quad 0 < \alpha < \infty$$

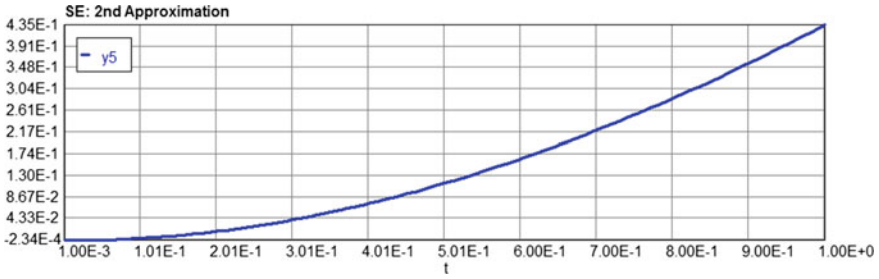

Fig. 2.9 Dummy Equation (second approximation—successive approximation)

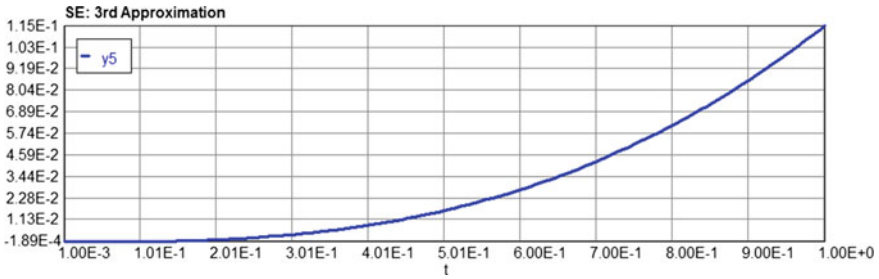

Fig. 2.10 Figure 1.9 Dummy Equation (3nd approximation—successive approximation)

3rd approximation:

$$\tilde{\varphi}_3(x) = \int_0^x K_N(x,s)\tilde{\varphi}_2(s)\mathrm{d}s =$$

$$\text{Dummy equation: } \frac{\mathrm{d}y}{\mathrm{d}s} = [\frac{\mathrm{e}^{-s}}{1-\mathrm{e}^{-s}}][\exp(\frac{s}{1+0.1s})](\frac{s}{s+1})(-0.00466+0.0338x+0.409x^2)$$

$$\text{Solution}: \tilde{\varphi}_3 = (\frac{x}{x+1})(0.00466-0.0604x+0.167x^2)$$

$$\varphi_{\text{tot}}(\mathrm{x}) = x\mathrm{e}^{-0.15x} - (\frac{x}{x+1})[(-0.02+1.113x-0.409x^2)+(0.00466-0.0604x+0.167x^2)] =$$

$$= x\mathrm{e}^{-0.15x} - (\frac{x}{x+1})(-0.0153+1.05x-0.242x^2)$$

$$\varphi^{(3)}_{\text{tot}}(x=1) = 0.464 \approx 0.518(110.6\%)$$

Calculated values of DEQ variables

	Variable	Initial value	Minimal value	Maximal value	Final value
1	t	0.001	0.001	1.0	1.0
	$y5$	0	−0.0001892	0.1149818	0.1149818

Differential equations

$$7 \quad \mathrm{d}(y5)/\mathrm{d}(t) = ((\exp(t/(1.0+0.1*t)))) * (t/(1+t)) * (\exp(-(1*t))) * (-0.00466+0.0338*t+0.409*t\text{^}2) * (1/(1-(\exp(-(1*t)))))$$

See Fig. 2.10.
Model: $y5 = a0 + a1 * t + a2 * t\text{^}2$

Variable	Value
$a0$	0.00466
$a1$	−0.0604
$a2$	0.167

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