Functors and Natural Transformations

S. Roman, *An Introduction to the Language of Category Theory*, Compact Textbooks in Mathematics, DOI 10.1007/978-3-319-41917-6_2, © The Author(s) 2016

Let us now take a closer look at functors, beginning with some additional examples.

Examples of Functors

We have already discussed the power set functor and the forgetful functor. Let us consider some other examples of functors.

Example 26

1) For a given positive integer n, we can define a *matrix functor* F_n: **CRng** $\Rightarrow$ **Grp** sending a commutative ring R to the general linear group $GL_n(R)$ of nonsingular $n \times n$ matrices over R. Each ring homomorphism $f: R \to S$ is sent to the map that works elementwise on the entries of a matrix.
2) Another functor G: **CRng** $\Rightarrow$ **Grp** is defined by setting $GR = R^*$, the group of units of R and $Gf = f\big|_{R^*}$ for any ring homomorphism $f: R \to S$. This makes sense since a ring homomorphism maps units to units. □

Example 27

If P is a poset, then a nonempty subset D of P is a **down-set** if $d \in D$ and $x \leq d$ imply that $x \in D$. Let **Poset** be the category of all posets. Define the **down-set functor** $\mathcal{O}$: **Poset** $\Rightarrow$ **Poset** as follows. A poset P is sent to the family $\mathcal{O}(P)$ of all down-sets in P, ordered by set inclusion. If $f: P \to Q$ is a monotone map, then the inverse image of a down-set in Q is a down-set in P and so we may take $\mathcal{O}(f): \mathcal{O}(Q) \to \mathcal{O}(P)$ to be the induced inverse map f^{-1}. Since

$$1_P^{-1} = 1_{\mathcal{O}(P)}$$

and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

it follows that $\mathcal{O}$ is a contravariant functor on **Poset**. □

Example 28

Let $A \in \mathcal{C}$ and consider the comma category $(\mathcal{C} \to A)$ of arrows entering A. Each object of $(\mathcal{C} \to A)$ is an ordered pair $(X, f: X \to A)$, as X ranges over the objects of $\mathcal{C}$. The **domain functor** $F: (\mathcal{C} \to A) \Rightarrow \mathcal{C}$ sends an object $(X, f: X \to A)$ to its domain X and a morphism

$$\overline{\alpha}: (X, f: X \to A) \to (Y, g: Y \to A)$$

which is a map $\alpha: X \to Y$ satisfying

$$g \circ \alpha = f$$

to the underlying morphism α. Thus $F\overline{\alpha} = \alpha$. We leave it to you to show that F is indeed a functor. □

Example 29

Here is a functor tongue-twister. Let $\mathcal{C}$ be a category. We can define a functor $F: \mathcal{C} \Rightarrow$ **SmCat** that takes an object $A \in \mathcal{C}$ to the comma category $(\mathcal{C} \to A) \in$ **SmCat**, with target object A. For this reason, we might call the functor F an **target functor** (a nonstandard term). A morphism $f: A \to A'$ between target objects in $\mathcal{C}$ must map under F to a *functor*, that is,

$$f: A \to A' \quad \underset{F}{\Rightarrow} \quad Ff: (\mathcal{C} \to A) \Rightarrow (\mathcal{C} \to A')$$

between the relevant comma categories. As shown on the left in Figure 20, the object portion of Ff must take an object $(C, \alpha: C \to A)$ in $(\mathcal{C} \to A)$ to an object in $(\mathcal{C} \to A')$. We take

$$Ff[(C, \alpha: C \to A)] = (C, f \circ \alpha: C \to A')$$

and so Ff is essentially the "follow by f" map $f^{\leftarrow}$ on objects.

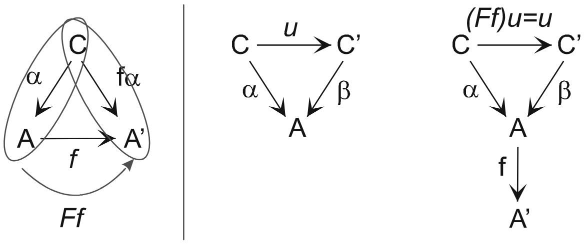

Figure 20

As to the arrow part, as shown on the right in Figure 20, recall that a morphism

$$\overline{u}: (C, \alpha: C \to A) \to (C', \beta: C' \to A)$$

in $(\mathcal{C} \to A)$ comes from a qualifying morphism $u: C \to C'$, that is, a morphism for which

$$\beta \circ u = \alpha$$

Now, Ff must take $\overline{u}$ to a morphism

$$(Ff)(\overline{u})\colon (C, f\circ\alpha\colon C\to A')\to (C', f\circ\beta\colon C'\to A')$$

But

$$(f\circ\beta)\circ u = f\circ\alpha$$

implies that u is also qualifying for the pair

$$P=((C, f\circ\alpha\colon C\to A'),(C', f\circ\beta\colon C'\to A'))$$

so we can take

$$(Ff)(\overline{u})=\overline{u}$$

where the overbar on the right means give u the domain and codomain in P.

We will leave it to you to show that Ff is indeed a functor and then that F is also a functor! □

Example 30

Let $\mathcal{C}$ be a category with binary products. We define the **squaring functor** as follows. For each object A, fix a product $(A\times A, \rho_1, \rho_2)$ of A with itself. Let $F\colon \mathcal{C}\Rightarrow\mathcal{C}$ send A to $A\times A$.

For a morphism $f\colon A\to B$ in $\mathcal{C}$, we want to define an appropriate morphism

$$Ff\colon (A\times A, \rho_1, \rho_2)\to (B\times B, \sigma_1, \sigma_2)$$

This clearly calls for the mediating morphism trick. So we need a couple of maps: one from $A\times A$ to B_1 and one from $A\times A$ to B_2.

The two compositions $f\circ\rho_i\colon A\times A\to B_i$ for $i=1,2$ will do the trick. Specifically, there is a unique mediating morphism

$$\theta_f : A\times A\to B\times B$$

as shown in Figure 21,

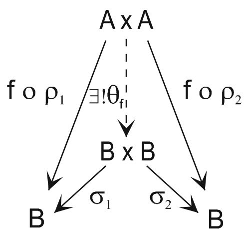

Figure 21

for which

$$\sigma_i \circ \theta_f = f \circ \rho_i$$

Let $Ff = \theta_f$. Then Ff is uniquely defined by the conditions

$$\sigma_i \circ Ff = f \circ \rho_i (i = 1, 2)$$

It is clear that $F1_A = 1_A$, because we have fixed a single product for each pair of objects. Also, if f: $A \rightarrow B$, g: $B \rightarrow C$ and the product for C is $(C \times C, \tau_1, \tau_2)$, then

$$\tau_i \circ (Fg \circ Ff) = g \circ \sigma_i \circ Ff = g \circ f \circ \rho_i = \tau_i \circ F(g \circ f)$$

for all i and so $Fg \circ Ff = F(g \circ f)$. Thus, F is a covariant functor on $\mathcal{C}$. □

■ **Example 31**

Let $\mathcal{C}$ be a category with binary products. To define a **product functor**

$$F: \mathcal{C} \times \mathcal{C} \Rightarrow \mathcal{C}$$

we must assume that for every pair (X, Y) of objects in $\mathcal{C}$, we have selected a product

$$(X \times Y, \zeta_1, \zeta_2)$$

The product functor F takes an object (A_1, A_2) to its chosen product $(A_1 \times A_2, \alpha_1, \alpha_2)$ and a morphism

$$(f_1, f_2): (A_1, A_2) \longrightarrow (B_1, B_2)$$

to the product morphism

$$f_1 \times f_2: (A_1 \times A_2, \alpha_1, \alpha_2) \longrightarrow (B_1 \times B_2, \beta_1, \beta_2) \tag{32}$$

recall that $f_1 \times f_2$ is defined as the unique morphism satisfying the conditions

$$\beta_1 \circ (f_1 \times f_2) = f_1 \circ \alpha_1 \quad \text{and} \quad \beta_2 \circ (f_1 \times f_2) = f_2 \circ \alpha_2$$

To see that F is a functor, we must first show that $1_{A_1} \times 1_{A_2}$ is the identity $1_{A_1 \times A_2}$ on $A_1 \times A_2$ and for this, we use (32). Since

$$\alpha_1 \circ 1_{A_1 \times A_2} = 1_{A_1} \circ \alpha_1 \quad \text{and} \quad \alpha_2 \circ 1_{A_1 \times A_2} = 1_{A_2} \circ \alpha_2$$

the uniqueness condition implies that

$$F(1_A, 1_B) = 1_A \times 1_B = 1_{A \times B}$$

As to composition, suppose that

$$g_1 \times g_2 \colon (B_1 \times B_2, \beta_1, \beta_2) \to (C_1 \times C_2, \gamma_1, \gamma_2)$$

Then

$$F[(g_1, g_2) \circ (f_1, f_2)] = F[(g_1 \circ f_1, g_2 \circ f_2)] = (g_1 \circ f_1) \times (g_2 \circ f_2)$$

Hence, by definition, the map $h = F[(g_1, g_2) \circ (f_1, f_2)]$ is the *unique* map for which

$$\gamma_1 \circ h = (g_1 \circ f_1) \circ \alpha_1 \quad \text{and} \quad \gamma_2 \circ h = (g_2 \circ f_2) \circ \alpha_2$$

The uniqueness conditions implies that we need only show that the map

$$k = F[(g_1, g_2)] \circ F[(f_1, f_2)] = (g_1 \times g_2) \circ (f_1 \times f_2)$$

also satisfies these equations, that is, that

$$\gamma_1 \circ [(g_1 \times g_2) \circ (f_1 \times f_2)] = (g_1 \circ f_1) \circ \alpha_1$$

and

$$\gamma_2 \circ [(g_1 \times g_2) \circ (f_1 \times f_2)] = (g_2 \circ f_2) \circ \alpha_2$$

As to the first of these equations, we have

$$\begin{aligned} \gamma_1 \circ [(g_1 \times g_2) \circ (f_1 \times f_2)] &= (g_1 \circ \beta_1) \circ (f_1 \times f_2) \\ &= g_1 \circ (\beta_1 \circ (f_1 \times f_2)) \\ &= g_1 \circ (f_1 \circ \alpha_1) \end{aligned}$$

as desired. The second equation is proved similarly. □

We have saved the most important example of a functor (at least from the perspective of category theory itself) for last.

▪ Example 33

One of the most important classes of functors are the *hom functors*, shown in Figure 22. Let $\mathcal{C}$ be a category and let $A \in \mathcal{C}$. We refer to A as the **source object** for the hom functor.

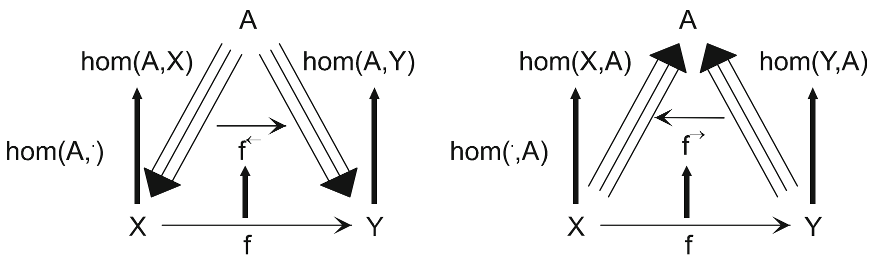

Figure 22

The **covariant hom functor**

$$\hom(A, \ \cdot \ ) : \mathcal{C} \Rightarrow \textbf{set}$$

sends an object $X \in \mathcal{C}$ to the hom-set of all morphisms from the source object A to X,

$$\hom(A, \ \cdot \ )(X) = \hom(A, X)$$

and it sends a morphism $f: X \to Y$ to the "follow by f" map,

$$\hom(A, \ \cdot \ )f = f^{\leftarrow}$$

Thus,

$$f^{\leftarrow}: \hom(A, X) \to \hom(A, Y)$$

is defined by

$$f^{\leftarrow}\tau = f \circ \tau$$

for any $\tau: A \to X$. This functor is covariant precisely because

$$(g \circ f)^{\leftarrow} = g^{\leftarrow} \circ f^{\leftarrow}$$

Covariant hom functors are also called **covariant representable functors**.

Dually, the **contravariant hom functor**

$$\hom(\ \cdot \ , \ A): \mathcal{C} \Rightarrow \textbf{set}$$

is defined by

$$\hom(\ \cdot \ , \ A)(X) = \hom(X, A)$$

for all $X \in \mathcal{C}$ and

$$\hom(\ \cdot \ , \ A)(f) = f^{\rightarrow}$$

where $f^{\rightarrow}$ is the "preceed by f" map,

$$f^{\rightarrow}\tau = \tau \circ f$$

for any $\tau: Y \to A$. This functor is contravariant precisely because

$$(g \circ f)^{\rightarrow} = f^{\rightarrow} \circ g^{\rightarrow}$$

Contravariant hom functors are also called **contravariant representable functors**. □

Morphisms of Functors: Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We would like to form a new category, denoted by $\mathcal{D}^{\mathcal{C}}$, whose objects are the *functors* from $\mathcal{C}$ to $\mathcal{D}$. But what about the morphisms between functors?

Consider a pair of parallel covariant functors $F, G\colon \mathcal{C} \Rightarrow \mathcal{D}$, as shown in Figure 23. As discussed earlier in the book, we think of F and G as mapping one-arrow diagrams.

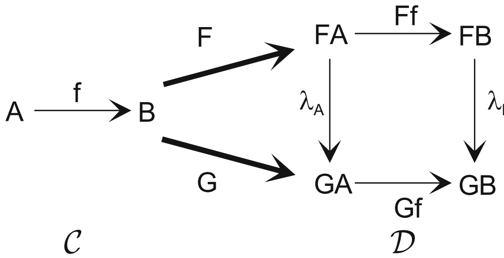

Figure 23 *A natural transformation*

A structure-preserving map between F and G is a "map" between the image one-arrow diagrams

$$FA \xrightarrow{Ff} FB \quad \text{and} \quad Gf\colon GA \xrightarrow{Gf} GB$$

As shown in Figure 23, this is accomplished by a *family* of morphisms in $\mathcal{D}$

$$\lambda = \left\{\lambda_A\colon FA \to GA \;\middle|\; A \in \mathcal{D}\right\}$$

for which the square in Figure 23 commutes, that is,

$$Gf \circ \lambda_A = \lambda_B \circ Ff$$

The family λ is called a *natural transformation* from F to G.

Definition

Let $F, G\colon \mathcal{C} \Rightarrow \mathcal{D}$ be parallel functors of the same type (both covariant or both contravariant). A **natural transformation** *from F to G, denoted by $\lambda\colon F \stackrel{\cdot}{\to} G$ or $\{\lambda_A\}\colon F \stackrel{\cdot}{\to} G$ is a family of morphisms in $\mathcal{D}$*

$$\lambda = \left\{\lambda_A\colon FA \to GA \;\middle|\; A \in \mathcal{D}\right\}$$

for which the appropriate square in Figure 24 commutes. Specifically, if F and G are covariant, as shown on the left in Figure 24, then

$$\lambda_B \circ Ff = Gf \circ \lambda_A$$

for any $f: A \to B$ *in* $\mathcal{C}$ *and if* F *and* G *are contravariant, as shown on the right in Figure 24, then*

$$\lambda_A \circ Ff = Gf \circ \lambda_B$$

for any $f: A \to B$ *in* $\mathcal{C}$. *Each morphism* λ_A *is called a* **component** *of* λ. *It is customary to say that* λ_A *is* **natural in** A **from** F *to* G. *We denote the class of natural transformations from* F *to* G *by* $\mathrm{Nat}(F, G)$. *We also use the notation* $\lambda(A)$ *for* λ_A *when it is more convenient.* □

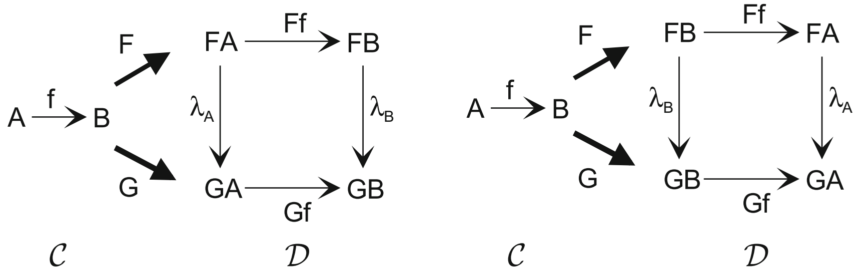

Figure 24 *Natural transformations*

Some authors refer to the function $\lambda: A \mapsto \lambda_A$ that maps an object $A \in \mathcal{C}$ to the component λ_A as a natural transformation as well.

Intuitively Speaking

Intuitively speaking, we can think of a natural transformation as follows. If $f: A \to B$ is a morphism in $\mathcal{C}$, then let us think of Ff and Gf as two different *versions* of f. Then the natural transformation condition

$$\lambda_B \circ Ff = Gf \circ \lambda_A$$

is a kind of **commutativity rule**, for it says that we can swap λ (actually, an appropriate component of λ) with one version of f provided we change the version of f.

An Example

Let us do an example.

▪ **Example 34 (The determinant)**

Fix a positive integer n. As shown in Figure 25, consider two parallel functors G, U: **CRng** $\Rightarrow$ **Grp** defined as follows. The functor G sends a ring R to the general linear group $GL_n(R)$ and a morphism $f: R \to S$ to the map f applied elementwise to the elements of a matrix, which we denote by f_e,

$$G: R \mapsto GL_n(R), \quad G: (f: R \to S) \mapsto (f_e: GL_n(R) \to GL_n(S))$$

The functor U sends a ring R to its group R^* of units and a ring map f to the restricted map $f_u: R^* \to S^*$, which makes sense since a ring map sends units to units,

$$U: R \mapsto R^*, \quad U: (f: R \to S) \mapsto (f: R^* \to S^*)$$

So we have two versions of the ring map f: apply f to matrices elementwise and apply f to units. Can you think of some operation $\{\lambda_A\}$ that "commutes" with these two versions of f, that is, for which

$$\lambda_S \circ f_e = f_u \circ \lambda_R$$

for $f: R \to S$?

Well, the determinant does not care whether it is applied before or after a ring map f, more precisely, before f_u or after f_e, in symbols,

$$\det(f_e A) = f_u(\det(A))$$

Thus,

$$\det_S \circ f_e = f \circ \det_R$$

which says that $\{\det_R \mid R \in \mathbf{CRng}\}$ is *natural* in R. □

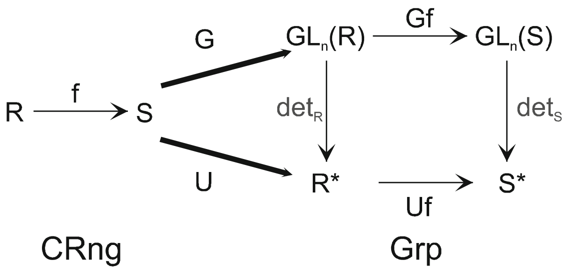

Figure 25 *The determinant is natural*

We will do some additional examples in a moment.

Composition of Natural Transformations

Natural transformations can be composed by composing corresponding components. In particular, if $\lambda: F \xrightarrow{\cdot} G$ and $\mu: G \xrightarrow{\cdot} H$ are natural transformations, then the composition $\mu \circ \lambda: F \xrightarrow{\cdot} H$ is defined by

$$(\mu \circ \lambda)_A = \mu_A \circ \lambda_A$$

2

We leave it to the reader to show that the composition of natural transformations is a natural transformation. Also, the identity natural transformation $1: F \dot{\to} F$ is defined by specifying that

$$1(A) = 1_{FA}$$

Natural Isomorphisms

Suppose that $\lambda: F \dot{\to} G$ is a natural transformation from F to G, where each component $\lambda_A: FA \approx FB$ is an isomorphism. The condition that λ is natural is

$$\lambda_B \circ Ff = Gf \circ \lambda_A$$

Applying λ_B^{-1} on the left and λ_A^{-1} on the right gives

$$Ff \circ \lambda_A^{-1} = \lambda_B^{-1} \circ Gf$$

Thus, the family $\mu = \{\lambda_A^{-1} \mid A \in \mathcal{C}\}$ is a natural transformation from G to F, that is, $\mu: G \dot{\to} F$. Moreover, $\mu \circ \lambda = 1$ and $\lambda \circ \mu = 1$ are the respective identity natural isomorphisms (each component is an identity morphism).

Conversely, if $\lambda: F \dot{\to} G$ and $\mu: G \dot{\to} F$ are natural transformations for which $\mu \circ \lambda = 1$ and $\lambda \circ \mu = 1$, then λ_A is an isomorphism for all $A \in \mathcal{C}$.

■ **Theorem 35**

Let $\lambda: F \dot{\to} G$ be a natural transformation. The following are equivalent:
1) Each component of λ is an isomorphism.
2) There is a natural transformation $\mu: G \dot{\to} F$ for which

$$\mu \circ \lambda = 1 \quad \textit{and} \quad \lambda \circ \mu = 1$$

where 1 is the appropriate natural isomorphism all of whose components are identity morphisms.

When these statements hold, we say that λ is a **natural isomorphism** *and that F and G are* **naturally isomorphic**, *written $\lambda: F \dot{\approx} G$ or $F \dot{\approx} G$. When F and G are set-valued, we use the notation $\dot{\leftrightarrow}$. in place of $\dot{\approx}$, since the components are bijections in this case.* □

More Examples of Natural Transformations

Let us consider some additional examples of natural transformations.

■ **Example 36 (The double-dual)**

Let **Vect** be the category of vector spaces over a field k, with linear maps. We need a little vector space theory for this example. As you probably know, the **dual space** V^* of a vector space V is the family of linear functionals on V. Hence, the **double-dual space** V^{**} is the family of linear functionals on V^*.

For example, if $v \in V$, then the **evaluation** at v map $\overline{v}\colon V^* \to k$ defined by

$$\overline{v}(f) = f(v)$$

for all $f \in V^*$ belongs to the double dual V^{**}. Let us set

$$\epsilon_V\colon V \to V^{**}, \quad \epsilon_V(v) = \overline{v}$$

The **operator adjoint** $\tau^{\to}\colon W^* \to V^*$ of a linear map $\tau\colon V \to W$ is defined by

$$\tau^{\to}(f) = f \circ \tau$$

Therefore, the **second adjoint** $\tau^{\to\to}\colon V^{**} \to W^{**}$ is given by

$$\tau^{\to\to}(\alpha) = \alpha \circ \tau^{\to}$$

for $\alpha \in V^{**}$.

In this case, we would like to find two versions of a linear map $\tau\colon V \to W$ that commute with evaluation.

We begin by looking at $\epsilon_W \circ \tau$. If $v \in V$, then

$$(\epsilon_W \circ \tau)(v) = \epsilon_W(\tau v) = \overline{\tau v}$$

Now we want to massage this until evaluation pops out the front. Applying $\overline{\tau v}$ to $f \in V^*$ gives

$$\overline{\tau v}(f) = f(\tau(v)) = \overline{v}(f \circ \tau) = \overline{v}(\tau^{\to}(f)) = (\overline{v} \circ \tau^{\to})(f)$$

and so

$$\overline{\tau v} = \overline{v} \circ \tau^{\to}$$

Thus,

$$(\epsilon_W \circ \tau)(v) = \overline{\tau v} = \overline{v} \circ \tau^{\to} = \tau^{\to\to}(\overline{v}) = \tau^{\to\to}(\epsilon_V(v)) = (\tau^{\to\to} \circ \epsilon_V)(v)$$

and we finally arrive at

$$\epsilon_W \circ \tau = \tau^{\to\to} \circ \epsilon_V \tag{37}$$

We can now put this in the language of natural transformations. Define a functor F: **Vect** $\Rightarrow$ **Vect** that takes a vector space V to its double dual V^{**} and a linear map $\tau\colon V \to W$ to its double adjoint,

$$F\colon V \mapsto V^{**} \quad \text{and} \quad F\colon \tau \mapsto \tau^{\to\to}$$

Then (37) can be written as

$$F\tau \circ \epsilon_V = \epsilon_W \circ I\tau$$

where I is the identity functor on **Vect**. Thus, as shown in Figure 26, the family $\{\epsilon_V \mid V \in \mathbf{Vect}\}$ is natural in V.

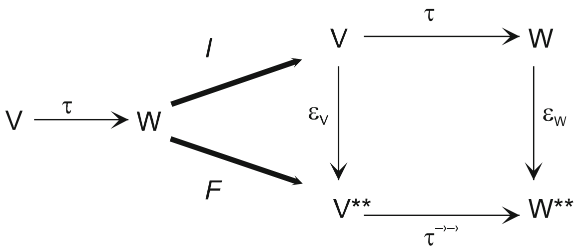

Figure 26

This example is a little more abstruse than the determinant example. The determinant example says that we can apply a ring map either before or after taking the determinant. This example says that we can either follow a linear map τ by evaluation or preceed its *second adjoint* $\tau^{\rightarrow\rightarrow}$ by evaluation. Whoever would have guessed that?

We leave it to you to show that the family is a natural *isomorphism* when restricted to the category of *finite-dimensional* vector spaces. □

■ Example 38 (The Riesz map)

For the category **Vect**, the **dual functor** G is defined by

$$GV = V^* \quad \text{and} \quad G\tau = \tau^{\rightarrow}$$

In examining the relationship between vector spaces and their duals, it is immediately clear that there cannot be a natural transformation from the identity functor on **Vect** to the dual functor on **Vect** because the identity functor is *covariant* but the dual functor is *contravariant*.

On the other hand, there is an important (and basis free) natural transformation for finite-dimensional *inner product* spaces. Let **FinInner** be the category of finite-dimensional real inner product spaces, with unitary transformations. A linear transformation $\sigma\colon V \to W$ is **unitary** if it is a bijection and

$$\langle \sigma u, v \rangle = \left\langle u, \sigma^{-1} v \right\rangle$$

The background we need here is the Riesz representation theorem. Define the **Riesz map** $R_V\colon V \to V^*$ by

$$R_V(v)(x) = \langle v, x \rangle$$

In words, $R_V(v)$ is "inner product with v." Because V is finite-dimensional, the Riesz representation theorem says that R_V is an isomorphism and so each element of V^* has the form $R_V(v) = \langle v, \cdot \rangle$ for a unique $v \in V$.

$$\begin{tikzcd} V \arrow[r, "\tau"] \arrow[d, "R_V"'] & W \arrow[d, "R_W"] \\ V^* \arrow[r, "(\tau^{-1})^{\rightarrow}"'] & W^* \end{tikzcd}$$

Figure 27

In an effort to find a commutativity rule involving the Riesz maps, we write for any $v \in V$,

$$\begin{aligned}(R_W \circ \tau)(v) &= R_W(\tau v) \\ &= \langle \tau v, \cdot \rangle \\ &= \left\langle v, \tau^{-1} \cdot \right\rangle \\ &= \langle v, \cdot \rangle \circ \tau^{-1} \\ &= R_V(v) \circ \tau^{-1} \\ &= (\tau^{-1})^{\rightarrow}(R_V(v)) \\ &= \left((\tau^{-1})^{\rightarrow} \circ R_V\right)(v)\end{aligned}$$

and so

$$R_W \circ \tau = \left(\tau^{-1}\right)^{\rightarrow} \circ R_V$$

This prompts us to make the following definition. Define the **Riesz functor** G by

$$GV = V^* \quad \text{and} \quad G(\tau) = \left(\tau^{-1}\right)^{\rightarrow}$$

where $\tau: V \rightarrow W$ is unitary. Then

$$R_W \circ \tau = G\tau \circ R_V$$

and so the family $\{R_V \mid V \in \textbf{FinInner}\}$ is natural. In words, we can swap the Riesz maps with τ and $(\tau^{-1})^{\rightarrow}$.

■ Example 39 (The coordinate map)

Let k be a field. For each nontrivial vector space V over k, choose an ordered basis $\mathcal{B}_V$. Choose the standard basis $\mathcal{E}_n$ for the vector spaces k^n. Let $\textbf{FinVectB}^*$ be the category whose objects are the ordered pairs $(V, \mathcal{B}_V)$. We will write V_n to denote the fact that V has dimension n.

The morphisms $\tau: (V_n, \mathcal{B}_V) \rightarrow (W_m, \mathcal{B}_W)$ are just the usual linear transformations $\tau: V_n \rightarrow W_m$.

Now, the coordinate map is defined by

$$\phi_{(V,\mathcal{B}_V)}\colon (V_n, \mathcal{B}_V) \to (k^n, \mathcal{E}_n), \quad \phi_{(V,\mathcal{B}_V)}(v) = [v]_{\mathcal{B}_V}$$

where $[v]_{\mathcal{B}_V}$ is the coordinate matrix of v with respect to $\mathcal{B}_V$ is an isomorphism.

The coordinate map can be used to define the matrix representation $[\tau]_{\mathcal{B},\mathcal{C}}$ of a linear map $\tau\colon V \to W$ with respect to a pair of ordered bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively. Recall that this matrix satisfies the equation

$$[\tau v]_{\mathcal{B}_W} = [\tau]_{\mathcal{B}_V,\mathcal{B}_W}[v]_{\mathcal{B}_V}$$

where $[x]_{\mathcal{B}}$ denotes the coordinate matrix of x with respect to $\mathcal{B}$. In terms of coordinate maps, this can be written

$$\phi_{(W,\mathcal{B}_W)}(\tau v) = [\tau]_{\mathcal{B}_V,\mathcal{B}_W}\phi_{(V,\mathcal{B}_V)}(v)$$

or equivalently in terms of the matrix multiplication operator,

$$\phi_{(W,\mathcal{B}_W)} \circ \tau = [\tau]_{\mathcal{B}_V,\mathcal{B}_W} \circ \phi_{(V,\mathcal{B}_V)} \tag{40}$$

Now it is time for some functors, one being the identity functor I on **FinVectB***. The other functor G is the **matrix representation functor** defined by

$$G(V_n, \mathcal{B}_V) = (k^n, \mathcal{E}_n)$$

and

$$G\tau = [\tau]_{\mathcal{B}_V,\mathcal{B}_W}$$

(We leave it to you to check that this is a covariant functor.)

Then (40) becomes

$$\phi_{(W,\mathcal{B}_W)} \circ I\tau = G\tau \circ \phi_{(V,\mathcal{B}_V)}$$

which shows (see Figure 28) that the family $\left\{\phi_{(V,\mathcal{B}_V)} \mid V \in \textbf{FinVectB}^*\right\}$ is natural in V.

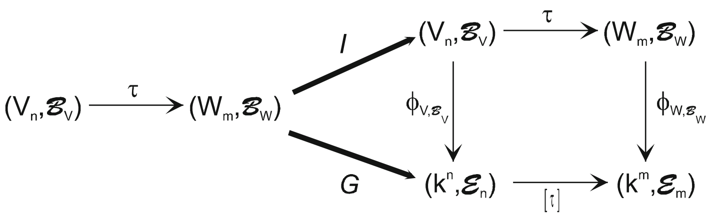

Figure 28 *The coordinate maps are natural*

In words, to swap factors in the composition $\phi_{(W,\mathcal{B}_W)} \circ \tau$, replace τ by its matrix representation. □

Example 41 **(Arrow part of functor is natural transformation between hom functors)** Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ be a functor and let $A \in \mathcal{C}$. If $f\colon X \to Y$ and $g\colon A \to X$ in $\mathcal{C}$, then

$$F(f \circ g) = Ff \circ Fg$$

which can also be written in the form

$$F(f^{\leftarrow}(g)) = \big((Ff)^{\leftarrow} \circ F\big)(g)$$

and so

$$F \circ f^{\leftarrow} = (Ff)^{\leftarrow} \circ F$$

This shows that the square in Figure 29 commutes.

$$\begin{tikzcd}
 & \mathrm{hom}_{\mathcal{C}}(A,X) \arrow[rr, "\mathrm{hom}_{\mathcal{C}}(A,\bullet)(f)=f^{\leftarrow}"] \arrow[d, "F"'] & & \mathrm{hom}_{\mathcal{C}}(A,Y) \arrow[d, "F"] \\
f{:}X \to Y \arrow[ur, "\mathrm{hom}_{\mathcal{C}}(A,\bullet)"] \arrow[r, "\mathrm{hom}_{\mathcal{D}}(FA,\bullet)"'] & \mathrm{hom}_{\mathcal{D}}(FA,FX) \arrow[rr, "\mathrm{hom}_{\mathcal{D}}(FA,\bullet)(Ff)=(Ff)^{\leftarrow}"'] & & \mathrm{hom}_{\mathcal{D}}(FA,FY)
\end{tikzcd}$$

Figure 29

It follows that the arrow parts

$$F_X\colon \mathrm{hom}_{\mathcal{C}}(A,X) \to \mathrm{hom}_{\mathcal{D}}(A,FX)$$

of the functor F actually form a natural transformation from the hom functor $\mathrm{hom}_{\mathcal{C}}(A,\cdot)$ with source A to the hom functor $\mathrm{hom}_{\mathcal{D}}(FA,F\cdot)$ with source FA, in symbols

$$F\colon \mathrm{hom}_{\mathcal{C}}(A,\ \cdot\ ) \xrightarrow{\cdot} \mathrm{hom}_{\mathcal{D}}(FA, F\ \cdot\ )$$

We will refer to this natural transformation by the name **arrow part** of F. □

$$
\begin{tikzcd}
 & \mathrm{hom}_{\mathcal{C}}(B,X) \arrow[r, "h^{\leftarrow}"] \arrow[dd, "f^{\rightarrow}"'] \arrow[ddr, phantom, "f:A\to B"] & \mathrm{hom}_{\mathcal{C}}(B,Y) \arrow[dd, "f^{\rightarrow}"] \\
h:X\to Y \arrow[ur, "\mathrm{hom}_{\mathcal{C}}(B,.)"] \arrow[dr, "\mathrm{hom}_{\mathcal{D}}(A,.)"'] & & \\
 & \mathrm{hom}_{\mathcal{D}}(A,X) \arrow[r, "h^{\leftarrow}"'] & \mathrm{hom}_{\mathcal{D}}(A,Y)
\end{tikzcd}
$$

Figure 30

Example 42 (Any morphism defines a natural transformation between hom functors)
Let $\mathcal{C}$ be a category and let f, h and α be morphisms in $\mathcal{C}$ for which the composition exists

$$h \circ \alpha \circ f$$

Because composition is associative, this composition can be written in two ways using the hom functors as follows

$$f^{\rightarrow}(h^{\leftarrow}(\alpha)) = h \circ \alpha \circ f = h^{\leftarrow}(f^{\rightarrow}(\alpha))$$

and so

$$f^{\rightarrow} \circ h^{\leftarrow} = h^{\leftarrow} \circ f^{\rightarrow}$$

Thus, the square in Figure 30 commutes and so the morphism $f: A \to B$ defines a natural transformation

$$\{f^{\rightarrow}\}: \mathrm{hom}_{\mathcal{C}}(B, \,\cdot\,) \dot{\to} \mathrm{hom}_{\mathcal{C}}(A, \,\cdot\,)$$

where each component is $f^{\rightarrow}$ (applied to the appropriate domain). We will see a bit later that all natural transformations between hom functors have this form. □

Natural Isomorphisms and Full Faithfulness

It is not surprising that a natural isomorphism of functors preserves fullness and faithfulness. We leave proof of the following as an exercise.

Theorem 43

1) *Let $F \dot{\approx} G$ be naturally isomorphic functors.*
   a) *F is faithful if and only if G is faithful.*
   b) *F is full if and only if G is full.*
   In particular, if $F \dot{\approx} I_{\mathcal{C}}$, then F is fully faithful.
2) *Let $F: \mathcal{C} \Rightarrow D$ and $G: \mathcal{D} \Rightarrow \mathcal{C}$ be functors.*
   a) *If $G \circ F$ is faithful, then F is faithful.*
   b) *If $G \circ F$ is full, then G is full.*

In particular, if

$$G \circ F \dot{\approx} I_{\mathcal{C}} \quad \textit{and} \quad F \circ G \dot{\approx} I_{\mathcal{D}}$$

then F *and* G *are fully faithful.* □

Functor Categories

As mentioned earlier, if $\mathcal{C}$ and $\mathcal{D}$ are categories, we would like to form the category $\mathcal{D}^{\mathcal{C}}$, whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are the natural transformations between functors. The only problem is that our definition of category requires that each hom-set be a *set*, but the class of natural transformations between two functors need not be a set. This issue can be resolved by requiring $\mathcal{C}$ to be a small category, that is, by requiring that **Obj**$(\mathcal{C})$ be a set. From now on, when we use the functor category $\mathcal{D}^{\mathcal{C}}$, it is with the tacit assumption that $\mathcal{C}$ is small.

Example 44

Let **2** be the category whose objects are 0 and 1 and whose morphisms are 1_0, 1_1 and $01\colon 0 \to 1$. Then each functor $F\colon \mathbf{2} \Rightarrow \mathcal{D}$ essentially just selects an arrow $F(01)\colon F(0) \to F(1)$ of $\mathcal{D}$. Moreover, a natural transformation $\{\lambda_0, \lambda_1\}\colon F \dot{\to} G$ is a pair of morphisms in $\mathcal{D}$, as shown in Figure 31.

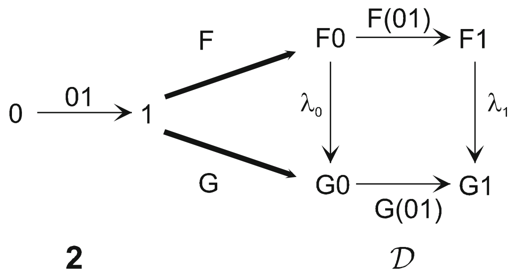

Figure 31

Hence, the functor category $\mathcal{D}^{\mathbf{2}}$, whose objects are the functors $F\colon \mathbf{2} \Rightarrow \mathcal{D}$ and whose morphism are the natural transformations $\{\lambda_0, \lambda_1\}\colon F \dot{\to} G$ between functors is just the category $\mathcal{D}^{\to}$ of arrows of $\mathcal{D}$. □

The Category of Diagrams

If $\mathcal{C}$ is a small category, the family of all diagrams $J\colon \mathcal{J} \Rightarrow \mathcal{C}$ in $\mathcal{C}$ over a particular index category $\mathcal{J}$ form the objects of a new category $\mathbf{dia}_{\mathcal{J}}(\mathcal{C})$. The morphisms $f\colon F \Rightarrow G$ from diagram $J\colon \mathcal{J} \Rightarrow \mathcal{C}$ to diagram $G\colon \mathcal{J} \Rightarrow \mathcal{C}$ are simply the natural transformations from J to G.

We can draw a picture of a morphism

$$\{\lambda_n\}: \mathbb{D} \stackrel{\cdot}{\to} \mathbb{E}$$

from diagram $\mathbb{D}$ to diagram $\mathbb{E}$ as shown in Figure 32. For this to represent a morphism, the square must be commutative.

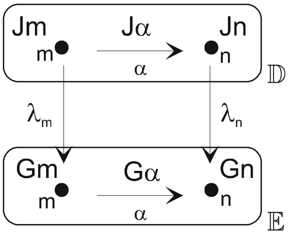

Figure 32

The category of diagrams will prove to be quite useful to us when we discuss universality later in the book.

Natural Equivalence

Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *naturally equivalent* if there are antiparallel covariant functors $F: \mathcal{C} \Rightarrow \mathcal{D}$ and $G: \mathcal{D} \Rightarrow \mathcal{C}$ for which the compositions $F \circ G$ and $G \circ F$ are naturally isomorphic to the corresponding identity functors $I_\mathcal{D}$ and $I_\mathcal{C}$. There is also a similar concept for contravariant functors.

Definition

1) *Two categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are* **naturally equivalent** *if there are covariant functors* $F: \mathcal{C} \Rightarrow \mathcal{D}$ *and* $G: \mathcal{D} \Rightarrow \mathcal{C}$ *for which*

$$F \circ G \stackrel{\cdot}{\approx} I_\mathcal{D} \quad \textit{and} \quad G \circ F \stackrel{\cdot}{\approx} I_\mathcal{C}$$

where $I_\mathcal{D}$ *and* $I_\mathcal{C}$ *are identity functors.*

2) *Two categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are* **dually equivalent** (*or* **dual**) *if there are contravariant functors* $F: \mathcal{C} \Rightarrow \mathcal{D}$ *and* $G: \mathcal{D} \Rightarrow \mathcal{C}$ *for which*

$$F \circ G \stackrel{\cdot}{\approx} I_\mathcal{D} \quad \textit{and} \quad G \circ F \stackrel{\cdot}{\approx} I_\mathcal{C}$$

where $I_\mathcal{D}$ *and* $I_\mathcal{C}$ *are identity functors.* □

Note that the functors in this definition are fully faithful.

Example 45

Let us show that the category $\mathbf{FinVect}^*$ of nonzero finite-dimensional vector spaces over a field k and the matrix category $\mathbf{Matr}_k$ are naturally equivalent. We assume that for each vector space

V in **FinVect***, an ordered basis $\mathcal{B}_V$ is chosen and that for the vector spaces k^n, the chosen basis is the standard basis $\mathcal{E}_n$.

The dimension functor dim: **FinVect**$^* \Rightarrow$ **Matr**$_k$ sends V to its dimension and sends each linear transformation $\tau\colon V_n \to W_m$ to the $m \times n$ matrix $[\tau]$ of τ with respect to the chosen ordered bases for V_n and W_m,

$$V_n \xrightarrow{\tau} V_m \quad \overset{\text{dim}}{\Rightarrow} \quad n \xrightarrow{[\tau]} m$$

To see that dim is a functor, note that $\dim(1_V)$ is the identity matrix and if $\tau\colon U \to V$ and $\sigma\colon V \to W$ then

$$\dim(\sigma\tau) = [\sigma\tau]_{\mathcal{B}_U,\mathcal{B}_W} = [\sigma]_{\mathcal{B}_V,\mathcal{B}_W}[\tau]_{\mathcal{B}_U,\mathcal{B}_V} = \dim(\sigma)\dim(\tau)$$

In the other direction, consider the map

$$\exp\colon \mathbf{Matr}_k \Rightarrow \mathbf{FinVect}^*$$

that takes a positive integer n to the vector space k^n and an $m \times n$ matrix $M\colon n \to m$ to the multiplication by M map, denoted by μ_M:

$$n \xrightarrow{M} m \quad \overset{\exp}{\Rightarrow} \quad k^n \xrightarrow{\mu_M} k^m$$

Since $\mu_I = 1_V$ and $\mu_{MN} = \mu_M\,\mu_N$, it follows that exp is also a functor.

The composition dim $\circ$ exp: **Matr**$_k \Rightarrow$ **Matr**$_k$ is the identity functor, since for any positive integer n,

$$\dim \circ \exp(n) = \dim(k^n) = n$$

and for any $m \times n$ matrix M,

$$\dim \circ \exp(M) = \dim(\mu_M) = [\mu_M]_{\mathcal{E}_m,\mathcal{E}_n} = M$$

The composition exp $\circ$ dim is the matrix representation functor, since

$$\exp \circ \dim(V_n) = \exp(n) = k^n$$

and for $\tau\colon V \to W$,

$$\exp \circ \dim(\tau) = exp\left([\tau]_{\mathcal{B}_V,\mathcal{B}_W}\right) = \mu_{[\tau]}$$

Thus, dim $\circ$ exp *is* the identity functor whereas exp $\circ$ dim, while not equal to the identity functor, is naturally isomorphic to the identity functor. Hence, **FinVect*** and **Matr**$_k$ are equivalent categories. □

Natural Transformations Between Hom Functors

Let us speak about natural transformations between hom functors. Let $\mathcal{C}$ be a small category. Recall that for each $A \in \mathcal{C}$, the covariant hom functor

$$\hom_{\mathcal{C}}(A, \cdot): \mathcal{C} \Rightarrow \textbf{set}$$

with source A is defined by

$$\hom_{\mathcal{C}}(A, \cdot)(X) = \hom_{\mathcal{C}}(A, X)$$

and for each $f: X \to Y$ in $\mathcal{C}$,

$$\hom(A, \cdot)f = f_A^{\leftarrow}$$

where we have used subscripts to remind us to which domain the "follow by f" map applies.

Figure 33 shows the diagram for a natural transformation λ between two hom functors.

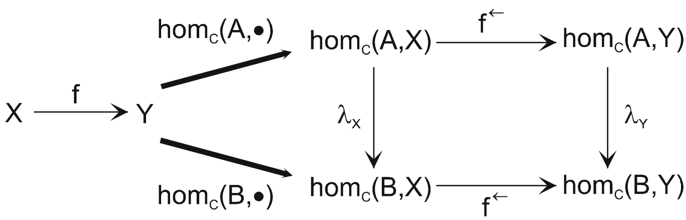

Figure 33

The naturalness condition is the commutativity of the square, that is,

$$f_B^{\leftarrow} \circ \lambda_X = \lambda_Y \circ f_A^{\leftarrow} \tag{46}$$

Taking $X = A$ and applying this to the identity 1_A gives

$$\left(f_B^{\leftarrow} \circ \lambda_A\right)(1_A) = \left(\lambda_Y \circ f_A^{\leftarrow}\right)(1_A)$$

or

$$f_B^{\leftarrow}(\lambda_A(1_A)) = \lambda_Y(f \circ 1_A)$$

or (replacing Y by X),

$$f \circ (\lambda_A(1_A)) = \lambda_X(f)$$

or finally,

$$\lambda_X(f) = [\lambda_A(1_A)]^{\rightarrow}(f)$$

for all $f\colon A \to X$. Hence,

$$\lambda_X = [\lambda_A(1_A)]^{\rightarrow}$$

for all $f\colon A \to X$. Thus, *all* natural transformations have the form

$$\lambda = \{h_X^{\rightarrow} \mid X \in \mathcal{C}\}$$

where

$$h = \lambda_A(1_A) \in \hom_{\mathcal{C}}(B, A)$$

Note that all of the components $h_X^{\rightarrow}$ of λ do the same thing (preceed by h) but to different domains, so they are different morphisms.

Conversely, if $h \in \hom_{\mathcal{C}}(B, A)$, then the family $\{h^{\rightarrow}\}$ is natural from $\hom_{\mathcal{C}}(A, \cdot)$ to $\hom_{\mathcal{C}}(B, \cdot)$ because for any $g\colon A \to X$,

$$f_B^{\leftarrow} \circ h_X^{\leftarrow}(g) = f \circ (g \circ h_X) \quad \text{and} \quad h_X^{\rightarrow} \circ f_A^{\leftarrow}(g) = (f \circ g) \circ h_X$$

which are equal precisely because composition is associative. Thus, we have completely characterized the natural transformations between hom functors.

Theorem 47

Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}$. Then the natural transformations

$$\lambda\colon \hom_{\mathcal{C}}(A, \ \cdot) \stackrel{\cdot}{\to} \hom_{\mathcal{C}}(B, \ \cdot\ )$$

between hom functors are precisely the families

$$\lambda = \{h_X^{\rightarrow} \mid X \in \mathcal{C}\}$$

as h varies over the set $\hom_{\mathcal{C}}(B, A)$, *where for $X \in \mathcal{C}$, the X-component of λ,*

$$h_X^{\rightarrow}\colon \hom_{\mathcal{C}}(A, X) \stackrel{\cdot}{\to} \hom_{\mathcal{C}}(B, X)$$

is "preceed by h on $\hom_{\mathcal{C}}(A, X)$." □

The Yoneda Embedding

Now that we understand the nature of natural transformations between hom functors, we can define a rather important *contravariant* functor, called the *Yoneda embedding*

$$y\colon \mathcal{C} \Rightarrow \mathbf{Set}^{\mathcal{C}}$$

as follows. To each object $A \in \mathcal{C}$, we associate the covariant hom functor $\hom_{\mathcal{C}}(A, \cdot)$ with source A. Thus, the object part of y is

$$y(A) = \hom_{\mathcal{C}}(A,\ \cdot\ )$$

The arrow part of y maps a morphism $h\colon B \to A$ to a natural transformation between hom functors and Theorem 47 gives us the "natural" choice

$$y(h) = \{h^{\rightarrow}\}\colon \hom_{\mathcal{C}}(A,\ \cdot\ ) \dot{\rightarrow} \hom_{\mathcal{C}}(B,\ \cdot\ )$$

To see that y actually is a contravariant functor, note that

$$y(1_A) = \{1_A^{\rightarrow}\}$$

is the identity natural transformation and that

$$y(g \circ h) = (g \circ h)^{\rightarrow} = h^{\rightarrow} \circ g^{\rightarrow} = y(g) \circ y(h)$$

It is customary to view y as a *covariant* functor,

$$y\colon \mathcal{C}^{op} \Rightarrow \mathbf{Set}^{\mathcal{C}}$$

from the opposite category $\mathcal{C}^{op}$ to the functor category $\mathbf{Set}^{\mathcal{C}}$, or equivalently, as a covariant functor

$$y\colon \mathcal{C} \Rightarrow \mathbf{Set}^{\mathcal{C}^{op}}$$

However, lest all of these opposite categories give you a headache, we will leave the functor y alone and live with its contravariance.

Theorem 48

Let $\mathcal{C}$ be a category. The contravariant functor $y\colon \mathcal{C} \Rightarrow \mathbf{Set}^{\mathcal{C}}$ defined by

$$y(A) = \hom_{\mathcal{C}}(A,\ \cdot\ ) \quad \textit{and} \quad y(h) = \{h^{\rightarrow}\}\colon \hom_{\mathcal{C}}(A,\ \cdot\ ) \dot{\rightarrow} \hom_{\mathcal{C}}(B,\ \cdot\ )$$

for all $A \in \mathcal{C}$ and all $h \in \hom_{\mathcal{C}}(B, A)$ is a contravariant embedding of $\mathcal{C}$ into the functor category $\mathbf{Set}^{\mathcal{C}}$, *called the* **Yoneda embedding** *of $\mathcal{C}$ in* $\mathbf{Set}^{\mathcal{C}}$.

Proof

We must show that y is an embedding, that is, that the object part of y is injective and that the local arrow parts of y are bijective. The object part of y maps A to $\hom_{\mathcal{C}}(A, \cdot)$ and since $\hom_{\mathcal{C}}(A, \cdot)$ and $\hom_{\mathcal{C}}(B, \cdot)$ are distinct for distinct objects A and B, the object part of y is injective.

To get the local arrow part of y, we fix $A, B \in \mathcal{C}$ to get the map

$$y_{A,B}\colon \hom_{\mathcal{C}}(B, A) \to \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), \hom_{\mathcal{C}}(B,\ \cdot\ ))$$

given by

$$y_{A,B}(h) = \{h_X^{\rightarrow} \mid X \in \mathcal{C}\}\colon \hom_{\mathcal{C}}(A,\ \cdot\ ) \dot{\rightarrow} \hom_{\mathcal{C}}(B,\ \cdot\ )$$

We have already proven that $y_{A,B}$ is surjective, that is, that all natural transformations from $\hom_{\mathcal{C}}(A,\cdot)$ to $\hom_{\mathcal{C}}(B,\cdot)$ have the form $\{h^{\rightarrow}\}$. As to injectivity, if $y_{A,B}(h) = y_{A,B}(k)$ for $h,k\colon B \to A$, then

$$\{h_X^{\rightarrow} \mid X \in C\} = \{k_X^{\rightarrow} \mid X \in C\}$$

In particular, for the components associated with $X = A$, we can apply them to 1_A to get

$$1_A \circ h = 1_A \circ k$$

and so $h = k$. Thus, the local arrow parts of *y* are injective and the Yoneda embedding is indeed an embedding. □

The Yoneda embedding states that any category $\mathcal{C}$ can be (contravariantly) embedded in the functor category $\mathbf{Set}^{\mathcal{C}}$ of set-valued functors on $\mathcal{C}$. In other words, each object $A \in \mathcal{C}$ can be *represented* as a hom functor $\hom_{\mathcal{C}}(A,\cdot)$ and each morphism as a natural transformation between hom functors.

To help remember the contravariant Yoneda embedding, we can also think of it as the **source embedding**. Specifically an object $A \in \mathcal{C}$ is used as the *source* of the hom functor $\hom_{\mathcal{C}}(A,\cdot)$ and a morphism $h\colon B \to A$ is used to *change* the source from A to B, since the embedding is contravariant. But to change the source from A to B, we must *preceed* by h.

■ Example 49

It is said that the Yoneda embedding is a vast generalization of Cayley's theorem of group theory. Cayley's theorem says that any group G can be embedded in a permutation group. Specifically, for $a \in G$, **right translation** by a is defined by

$$\rho_a\colon G \to G, \quad \rho_a(g) = ga$$

Cayley's theorem says that the map

$$\rho\colon G \to S_G, \quad \rho(a) = \rho_a$$

is an embedding of G into the permutation group S_G.

Now recall that the group G can be thought of as a category $\mathcal{G}$ with just one object, namely G itself. Moreover, each element $a \in G$ is a morphism $a\colon G \to G$ and composition of morphisms is the group product of elements.

Since $\mathcal{G}$ has only one object, it has only one hom functor

$$\hom(G,\ \cdot\ )\colon \mathcal{G} \Rightarrow \mathbf{Set}$$

defined by

$$\hom(G,\ \cdot\ )G = \hom(G,G) = U(G)$$

where $U(G)$ is the underlying set of G (that is, G thought of simply as a set) and for $a \in G$,

$$\mathrm{hom}(G, \cdot\,)a = a^{\leftarrow}$$

The contravariant Yoneda embedding $y\colon \mathcal{G} \to \mathbf{Set}^{\mathcal{G}}$ is

$$y(G) = \mathrm{hom}(G, \cdot\,), \quad y(a) = \{a^{\rightarrow}\}$$

where

$$a^{\rightarrow}(b) = ba = \rho_a(b)$$

for all $b \in G$ and so

$$y(G) = \mathrm{hom}(G, \cdot\,), \quad y(a) = \{\rho_a\}$$

for all $a \in G$.

But in this case, since there is only one object G, their is only one component in the family $\{\rho_a\}$ and so the arrow part of y is essentially just the Cayley embedding ρ. □

Yoneda's Lemma

Yoneda's lemma examines the nature of natural transformations from hom-set functors to *arbitrary* set valued functors. With reference to Figure 34, let $A \in \mathcal{C}$ and consider the hom functor with source A,

$$\mathrm{hom}_{\mathcal{C}}(A, \cdot\,)\colon \mathcal{C} \Rightarrow \mathbf{Set}$$

and any set-valued functor $H\colon \mathcal{C} \Rightarrow \mathbf{Set}$.

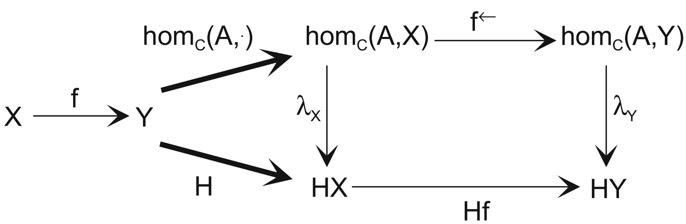

Figure 34

If

$$\lambda\colon \mathrm{hom}_{\mathcal{C}}(A, \cdot\,) \dot{\to} H$$

is a natural transformation, then for any $f\colon X \to Y$,

$$\lambda_Y \circ f^{\leftarrow} = Hf \circ \lambda_X$$

As in the special case we discussed earlier, we take $X = A$ and apply this to 1_A to get (after replacing Y by X),

$$\lambda_X(g) = (Hg)[\lambda_A(1_A)] \tag{50}$$

for any g: $A \to X$. To simplify the notation, let

$$a_A = \lambda_A(1_A) \in H(A)$$

and so

$$\lambda_X(g) = (Hg)a_A$$

Let us refer to the element $a_A \in H(A)$, which *completely characterizes* the natural transformation λ as the **Yoneda representative** of λ.

On the other hand, for any element $a \in HA$, the components λ_X defined by

$$\lambda_X(g) = (Hg)_a \tag{51}$$

for all $X \in \mathcal{C}$ and for all $g \in \hom_{\mathcal{C}}(A, X)$ form a natural transformation $\lambda = \{\lambda_X\}$, because for any $f: X \to Y$,

$$\begin{aligned}(\lambda_Y \circ f^{\leftarrow})g &= \lambda_Y(f \circ g) \\ &= H(f \circ g)a \\ &= (Hf \circ Hg)a \\ &= Hf \circ [\lambda_X(g)]\end{aligned}$$

and so

$$\lambda_Y \circ f^{\leftarrow} = Hf \circ \lambda_X$$

Note that by taking $X = A$ and $g = 1_A$ in (51), we get

$$a = \lambda_A(1_A)$$

Thus, any element of $H(A)$ is the Yoneda representative for some natural transformation λ from $\hom_{\mathcal{C}}(A, \cdot)$ to H.

■ **Theorem 52 (Yoneda lemma, part 1)**
*Let $\mathcal{C}$ be a category and let $H: \mathcal{C} \Rightarrow$ **Set** be a set-valued functor.*
1) *The natural transformations*

$$\lambda: \hom_{\mathcal{C}}(A, \cdot) \stackrel{\cdot}{\to} H$$

are precisely the maps defined by

$$\lambda_X(g) = (Hg)a$$

for all $g: A \to X$*, where* $a \in H(A)$*. The connection between* λ *and its Yoneda representative* a *is given by*

$$a = \lambda_A(1_A)$$

2) *The* **Yoneda representative map**

$$\phi = \phi_{H,A}: \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H) \to H(A)$$

defined by

$$\phi(\lambda) = \lambda_A(1_A)$$

is a bijection. It follows that the class $\mathrm{Nat}(\hom_{\mathcal{C}}(A, \cdot), H)$ *is a set.*

3) *When* $H = \hom_{\mathcal{C}}(B, \cdot)$ *is also a hom functor, the natural transformations*

$$\lambda: \hom_{\mathcal{C}}(A,\ \cdot\ ) \stackrel{\cdot}{\to} \hom_{\mathcal{C}}(B,\ \cdot\ )$$

are precisely the families

$$\lambda = \{\lambda^{\to}\}$$

as h *varies over the set* $\hom_{\mathcal{C}}(B, A)$.

■ Proof

We have already proved parts 1) and 3). For part 2), since

$$\lambda_X(g) = (Hg)[\lambda_A(1_A)]$$

it is clear that $\lambda_A(1_A) = \phi(\lambda)$ uniquely determines λ and so ϕ is injective. We have already seen that it is surjective. □

There is another part to Yoneda's lemma, which describes the naturalness of the families $\{\phi_A | A \in \mathcal{C}\}$, where H is fixed and $\{\phi_{A,H} | H \in \mathbf{Set}^{\mathcal{C}}\}$, where A is fixed

■ Theorem 53 (Yoneda lemma, part 2)

Let $\mathcal{C}$ *be a category. The family of Yoneda representative maps*

$$\{\phi_{H,A}: \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H) \approx H(A) \mid A \in \mathcal{C}, H \in \mathbf{Set}^{\mathcal{C}}\}$$

is natural in both H *and* A*, as shown in the commutative diagram of Figure 35.*

$$
\begin{tikzcd}
 & \mathrm{Nat}(\hom_C(A,\cdot),H) \arrow[dl, "\phi_{K,A}\circ\lambda^{\leftarrow}"'] \arrow[d, "\phi_{H,A}"] \arrow[dr, "\phi_{H,B}\circ f^{\to\to}"] & \\
KA \arrow[r, phantom, "(\lambda: H \dot{\to} K)"] & HA \arrow[l, "\lambda_A"] \arrow[r, "Hf"'] \arrow[r, phantom, "(f: A\to B)"] & HB
\end{tikzcd}
$$

Figure 35

In particular:

1) *For a fixed* $H\colon \mathcal{C} \Rightarrow$ **Set**, *the family of bijections*

$$\{\phi_A\colon \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H) \leftrightarrow HA \mid A \in \mathcal{C}\}$$

is natural in A, as shown in Figure 36.

$$
\begin{tikzcd}
 & & \mathrm{Nat}(\hom_C(A,\cdot),H) \arrow[r, "f^{\to\to}"] \arrow[dd, "\phi_A"'] & \mathrm{Nat}(\hom_C(B,\cdot),H) \arrow[dd, "\phi_B"] \\
A \arrow[r, "f"] & B \arrow[ur, "F"] \arrow[dr, "H"'] & & \\
 & & HA \arrow[r, "Hf"'] & HB
\end{tikzcd}
$$

Figure 36

Specifically, define a functor $F\colon \mathcal{C} \Rightarrow$ **Set** *as follows. If* $A \in \mathcal{C}$ *then*

$$F(A) = \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H)$$

Also, if $f\colon A \to B$ *in* $\mathcal{C}$, *then*

$$F(f)\colon \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H) \to \mathrm{Nat}(\hom_{\mathcal{C}}(B,\ \cdot\ ), H)$$

is defined by

$$F(f) = f^{\to\to}$$

that is, if $\lambda = \{\lambda_X \mid X \in \mathcal{C}\} \in \mathrm{Nat}(\hom_{\mathcal{C}}(A,\cdot), H)$, *then*

$$f^{\to\to}(\lambda) = \{f^{\to\to}(\lambda_X) \mid X \in \mathcal{C}\} = \{\lambda_X \circ f^{\to} \mid X \in \mathcal{C}\}$$

Then

$$\phi_A\colon F \stackrel{\cdot}{\to} H$$

2) *If $\mathcal{C}$ is a small category, then for a fixed $A \in \mathcal{C}$, the family of bijections*

$$\{\phi_H\colon \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H) \leftrightarrow HA \mid H \in \mathbf{Set}^{\mathcal{C}}\}$$

is natural in H, as shown in Figure 37.

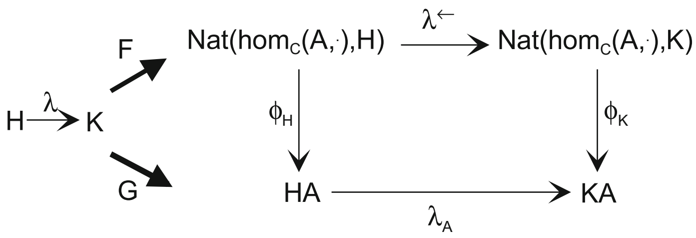

Figure 37

Specifically, define functors $F, G\colon \mathbf{Set}^{\mathcal{C}} \Rightarrow \mathbf{Set}$ as follows. For any $H \in \mathbf{Set}^{\mathcal{C}}$, let

$$F(H) = \mathrm{Nat}(\hom_{\mathcal{C}}(A,\ \cdot\ ), H) \quad \textit{and} \quad G(H) = HA$$

and for any $\lambda\colon H \stackrel{\cdot}{\to} K$ in $\mathbf{Set}^{\mathcal{C}}$,

$$F(\lambda) = \lambda^{\leftarrow} \quad \textit{and} \quad G(\lambda) = \lambda_A$$

where $\lambda^{\leftarrow}(\mu) = \lambda \circ \mu$. Then

$$\phi_H\colon F \stackrel{\cdot}{\to} G$$

▪ Proof

For part 1), we must first show that if

$$\{\lambda_X\}\colon \hom_{\mathcal{C}}(A,\ \cdot\ ) \stackrel{\cdot}{\to} H$$

then

$$\{\lambda_X \circ f^{\to}\}\colon \hom_{\mathcal{C}}(B,\ \cdot\ ) \stackrel{\cdot}{\to} H$$

that is, we must show that for all $g\colon X \to Y$,

$$Hg \circ \lambda_X \circ f^{\to} = \lambda_Y \circ f^{\to} \circ g^{\leftarrow}$$

But the condition that λ is natural is

$$Hg \circ \lambda_X = \lambda_Y \circ g^{\leftarrow}$$

and the result follows by applying $f^{\to}$ and noting that $g^{\leftarrow} \circ f^{\to} = f^{\to} \circ g^{\leftarrow}$.

Now, the naturalness of $\phi_A\colon F \dot{\to} H$ is

$$Hf \circ \phi_A = \phi_B \circ Ff$$

for all $f\colon A \to B$ in $\mathcal{C}$. This is equivalent to

$$Hf \circ \phi_A(\lambda) = \phi_B \circ Ff(\lambda)$$

for all $\lambda \in \mathrm{Nat}(\hom_{\mathcal{C}}(A, \cdot), H)$ and this is equivalent to

$$Hf[\lambda_A(1_A)] = \phi_B(\lambda \circ f^{\to})$$

But

$$\phi_B(\lambda \circ f^{\to}) = (\lambda \circ f^{\to})_B 1_B = \lambda_B \circ f^{\to} 1_B = \lambda_B(f)$$

and the Yoneda lemma implies that

$$\lambda_B(f) = Hf[\lambda_A 1_A]$$

as desired.

For part 2), the naturalness condition we wish to verify is

$$\lambda_A \circ \phi_H = \phi_K \circ \lambda^{\leftarrow}$$

But for $\mu \in \mathrm{Nat}(\hom_{\mathcal{C}}(A, \cdot), H)$,

$$\lambda_A \circ \phi_H(\mu) = \lambda_A[\mu_A(1_A)]$$

and

$$\phi_K \circ \lambda^{\leftarrow}(\mu) = \phi_K(\lambda \circ \mu) = (\lambda \circ \mu)_A(1_A) = \lambda_A[\mu_A(1_A)] \qquad \square$$

Exercises

1. Show that contravariant functors are also covariant functors.
2. Is the forgetful functor from **Grp** to **Set** full? Is it faithful?

3. Let G be a group and let G' be the commutator subgroup, that is, the subgroup generated by all commutators $aba^{-1}b^{-1}$, where $a, b \in G$. Then G' is a normal subgroup of G and G/G' is abelian. Let F: **Grp** $\Rightarrow$ **AbGrp** send G to G/G' and send $\sigma\colon G \to H$ to the map $F\sigma\colon G/G' \to H/H'$ defined by

$$(F\sigma)(aG') = (\sigma a)H'$$

   a) Show that this defines a functor.
   b) Modify the function F slightly so that it maps into **Grp** and find a natural transformation from the identity functor I: **Grp** $\Rightarrow$ **Grp** to F.
4. Find two distinct covariant functors from **Grp** to **Grp** both of whose object maps are the identity.
5. For the category **Grp**, map each group to its commutator subgroup $C(G)$ and each homomorphism $f\colon G \to H$ to its restriction $Cf\colon C(G) \to C(H)$. Show that C is a functor.
6. Show that the hom functor

$$\hom_{\mathcal{C}}(A, \cdot)\colon \mathcal{C} \Rightarrow \mathbf{Set}$$

   preserves monics, that is, if α: $C \to D$ is monic in $\mathcal{C}$, then

$$\alpha^{\leftarrow}\colon \hom_{\mathcal{C}}(A, C) \to hom_{\mathcal{C}}(A, D)$$

   is also monic.
7. Describe the arrow part of the hom functor $\hom_{\mathcal{C}}(B, \cdot)$ and the naturalness condition.
8. Let $\mathcal{C}$ be a category with binary products. Fix a product for each pair of objects in $\mathcal{C}$. If $f\colon A \to A'$ and $g\colon B \to B'$, then define the product $f \times g\colon A \times B \to A' \times B'$ as the unique mediating morphism from the cone

$$\left(A \times B, f \circ \rho_{A\times B,1}\colon A \times B \to A', g \circ \rho_{A\times B,2}\colon A \times B \to B'\right)$$

   to the product $A' \times B'$. In other symbols,

$$f \times g = \begin{pmatrix} f \circ \rho_{A\times B,1} \\ g \circ \rho_{A\times B,2} \end{pmatrix}$$

   a) Prove that

$$1_A \times 1_B = 1_{A\times B}$$

   b) If f: $A \to A'$ and g: $B \to B'$ and f': $A' \to A''$ and g': $B' \to B''$ prove that

$$(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \circ g)$$

   or in different notation

$$\begin{pmatrix} f' \circ f \circ \rho_{A\times B,1} \\ g' \circ g \circ \rho_{A\times B,2} \end{pmatrix} = \begin{pmatrix} f' \circ \rho_{A'\times B',1} \\ g' \circ \rho_{A'\times B',2} \end{pmatrix} \circ \begin{pmatrix} f \circ \rho_{A\times B,1} \\ g \circ \rho_{A\times B,2} \end{pmatrix}$$

9. Let $\mathcal{C}$ be a category with binary products. Let A be an object in $\mathcal{C}$ and fix a product $C \times A$ for every object C in $\mathcal{C}$. Define the **product functor**

$$- \times A \colon \mathcal{C} \Rightarrow \mathcal{C}$$

sending C to $C \times A$ and $f\colon C \to D$ to the product morphism

$$f \times 1_A \stackrel{\text{def}}{=} \begin{pmatrix} f \circ \rho_{C\times A,1} \\ 1_A \circ \rho_{C\times A,2} \end{pmatrix} : C \times A \to D \times A$$

where $\begin{pmatrix} x_{XA} \\ y_{XB} \end{pmatrix}$ denotes the mediating morphism for a cone with legs x and y over the diagram $\{A, B\}$. Prove that this does define a functor.

10. Let $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be categories. A functor $H\colon \mathcal{B} \times \mathcal{C} \Rightarrow \mathcal{D}$ from the product $\mathcal{B} \times \mathcal{C}$ into a category $\mathcal{D}$ is called a **functor of two variables** or a **bifunctor**. If $B \in \mathcal{B}$, the map H_B is defined as follows: H_B takes an object C of $\mathcal{C}$ to $H(B,C)$ and takes a morphism $g\colon C \to C'$ to $H(1_{B,\, g})$, where 1_B is the identity morphism on B. For any object C in $\mathcal{C}$, the map H_C is defined analogously.
    a) Show that $H_B\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $H_C\colon \mathcal{B} \Rightarrow \mathcal{D}$ are functors.
    b) Show that if $f\colon B \to B'$ and $g\colon C \to C'$, then

$$H_{B'}(g) \circ H_C(f) = H_{C'}(f) \circ H_B(g)$$

    c) Suppose that for each object C in $\mathcal{C}$, there is a functor $G_C\colon \mathcal{B} \Rightarrow \mathcal{D}$ and for each object B in $\mathcal{B}$, there is a functor $F_B\colon \mathcal{C} \Rightarrow \mathcal{D}$. Under what conditions is there a bifunctor $H\colon \mathcal{B} \times \mathcal{D} \Rightarrow \mathcal{D}$ for which $H_B = F_B$ and $H_C = G_C$?

11. Prove the following statements:
    1) Let $F \dot{\approx} G$ be naturally isomorphic functors.
        a) F is faithful if and only if G is faithful.
        b) F is full if and only if G is full.
        In particular, if $F \dot{\approx} I_{\mathcal{C}}$, then F is fully faithful.
    2) Let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ be functors.
        a) If $G \circ F$ is faithful, then F is faithful.
        b) If $G \circ F$ is full, then G is full.
        In particular, if

$$G \circ F \dot{\approx} I_{\mathcal{C}} \quad \text{and} \quad F \circ G \dot{\approx} I_{\mathcal{D}}$$

        then F and G are fully faithful.

12. Let $F,G\colon \mathcal{C} \Rightarrow \mathcal{D}$ and let $\lambda(C)\colon F \dot{\to} G$ be a natural transformation from F to G.
    a) Let $H\colon \mathcal{E} \Rightarrow \mathcal{C}$. Let $\lambda H\colon \mathbf{Obj}(\mathcal{E}) \to \mathbf{Obj}(\mathcal{C})$ be defined by
    $$\lambda H(E) = \lambda(H(E))$$
    Show that $\lambda H\colon FH \dot{\to} GH$ is a natural transformation.
    b) Let $H\lambda\colon \mathbf{Obj}(\mathcal{E}) \to \mathbf{Obj}(\mathcal{C})$ be defined by
    $$H\lambda(C) = H(\lambda(E))$$
    Show that $H\lambda\colon HF \dot{\to} HG$ is a natural transformation.
13. Let S be a nonempty set. A group with operators S or an S-group is a group G together with a homomorphism $\sigma\colon S \to \mathrm{End}(G)$ where $\mathrm{End}(G)$ is the group of endomorphisms of G. Let M be a monoid, thought of as a category with one object, where each element of M is a morphism. Show that the objects in the functor category $\mathbf{Grp}^M$ are the groups with operators M.
14. Let A be an abelian group. The **torsion subgroup** A^t of A is the set of elements of A that have finite order. The **torsion functor** $G\colon \mathbf{AbGrp} \Rightarrow \mathbf{AbGrp}$ is defined by
    $$GA = A^t$$
    and for a group homomorphism $f\colon A \to B$
    $$Gf = f\,|_{A^t}\colon A^t \to B^t$$
    which makes sense since a group homomorphism maps torsion elements to torsion elements.
    a) Show that G is indeed a functor.
    b) Find a natural transformation from the torsion functor G to the identity functor I.
15. Let **FinSet** be the category of all finite sets and let **FinOrd** be the category of all finite ordinal numbers. The inclusion map I: **FinOrd** $\Rightarrow$ **FinSet** is a functor, with $I(f) = f$, as a set function. We define another functor Card: **FinSet** $\Rightarrow$ **FinOrd** as follows. $\mathrm{Card}(S)$ is the unique finite ordinal that is equipotent to S. For the maps, for each finite set S, we fix a bijection θ_S from S to $\mathrm{Card}(S)$, where if n is a finite ordinal then $\theta_n = 1$ (the identity). Then for a set function $f\colon S \to T$, the map $\mathrm{Card}(f)\colon \mathrm{Card}(S) \to \mathrm{Card}(T)$ is defined as
    $$Card(f) = \theta_T \circ f \circ \theta_S^{-1}$$
    a) Show that $\mathrm{Card} \circ I$: **FinOrd** $\Rightarrow$ **FinOrd** is the identity functor.
    b) Show that $I \circ \mathrm{Card}$: **FinSet** $\Rightarrow$ **FinSet** is not the identity functor, but is naturally isomorphic to the identity functor. Thus, **FinOrd** and **FinSet** are equivalent categories.

16. Let S be a fixed set. Consider the map F that sends each set X to $X^S \times S$, where X^S is the set of all functions from S to a set X. Show that F is the object map of a functor F from **Set** to **Set**. Find a natural transformation from F to the identity functor on **Set**.
17. Let $F, G: \mathcal{C} \Rightarrow \mathcal{P}$ be functors from a category $\mathcal{C}$ to a preorder $\mathcal{P}$.
    a) Describe necessary and sufficient conditions under which there is a natural transformation from F to G.
    b) Prove that if $\mathcal{P}$ and $\mathcal{Q}$ are preorders, then the functor category $\mathcal{Q}^{\mathcal{P}}$ is also a preorder.
18. Verify that the functor category $\mathcal{D}^{\mathcal{C}}$ is a category.
19. Prove that the functor category $\mathcal{D}^2$ is essentially the category of arrows $\mathcal{D}^{\rightarrow}$ of $\mathcal{D}$.
20. Show that the map that sends each group to its center cannot be the object map of a functor from **Grp** to **AbGrp**. *Hint*: Consider the triangle formed by $S_2 \rightarrow S_3 \rightarrow S_2$.
21. A **pointed set** is a pair $S_* = (S, s)$ where $s \in S$. Less formally, a pointed set is just a set that contains a specially designated element. To simplify the notation, we let $*$ denote this element. Let $\mathbf{Set}_*$ be the category whose objects are pointed sets and whose morphisms are all set functions $f: A_* \rightarrow B_*$ for which $f(*) = *$. These are called **pointed functions**. Let $\mathbf{Set}_0$ be the category whose objects are sets and whose morphisms are *partial* set functions $f: A \rightarrow B$, that is, the domain of f is a (possibly empty) subset of A. Prove that $\mathbf{Set}_*$ and $\mathbf{Set}_0$ are isomorphic.
22. If S is a set and $s \in S$, we define the ordered pair (S, s) to be a **set with base point** (or a **set with distinguished element**). Let $\mathbf{Set}_*$ be the category of pointed sets (see the previous exercise) and let $\mathcal{C}$ be the category of all sets with base point. Show that the map $F : \mathbf{Set}_* \Rightarrow \mathcal{C}$ sending S_* to $(S_*, *)$ and sending $f: S_* \rightarrow T_*$ to itself is a functor. Is it an isomorphism?
23. a) Let $F, G: \mathcal{C} \Rightarrow \mathcal{D}$ and let

$$\lambda = \{\lambda_C\}: F \dot{\rightarrow} G$$

Then if $H: \mathcal{D} \Rightarrow \mathcal{E}$, then the composition

$$H \circ \lambda_C: FC \rightarrow HGC$$

makes sense. Show that the family

$$H\lambda = \{H \circ \lambda_C \mid C \in \mathcal{C}\}$$

is natural from HF to HG.

b) If $K: \mathcal{B} \Rightarrow \mathcal{C}$, then for each $B \in \mathcal{B}$,

$$\lambda_{KB}: FKB \rightarrow GKB$$

This is a form of “composition” of K followed by λ. Show that the family

$$\lambda H = \{\lambda_{HB} \mid B \in \mathcal{B}\}$$

is natural from FH to GH.

c) Let $F, G\colon \mathcal{A} \Rightarrow \mathcal{B}$ and $H, K\colon \mathcal{B} \Rightarrow \mathcal{C}$ and let $\alpha\colon F \stackrel{\cdot}{\to} G$ and $\beta\colon H \stackrel{\cdot}{\to} K$. Show that

$$K\alpha_A \circ \beta_{FA} = \beta_{GA} \circ H\alpha_A$$

The **Godement product** $\beta * \alpha$ is defined by

$$(\beta * a)_A := K\alpha_A \circ \beta_{FA} = \beta_{GA} \circ H\alpha A$$

Show that this defines a natural transformation from *HF* to *KG*. Show that the products $H\lambda$ and λH are special cases of the Godement product.

24. Prove the contravariant case of Yoneda's lemma.
25. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ be a covariant functor. Show that the natural transformations

$$\lambda_{A,B} = \{\lambda_{A,B}(\cdot)\}\colon \hom_{\mathcal{C}}(\,\cdot\,,\ A) \stackrel{\cdot}{\to} \hom_{\mathcal{D}}(F\,\cdot\,, B)$$

between contravariant functors have the form

$$\lambda_{A,B}(X)f = (Ff)^{\to} g = g \circ Ff$$

for all $f\colon X \to A$, where $g \in \hom_{\mathcal{D}}(FA, B)$. In this case, $g = \lambda_{A,B}(A)1_A$ and so

$$\lambda_{A,B}(X)f = (Ff)^{\to}\lambda_{A,B}(A)1_A = \lambda_{A,B}(A)1_A \circ Ff$$

http://www.springer.com/978-3-319-41916-9

An Introduction to the Language of Category Theory
Roman, S.
2017, XII, 169 p. 176 illus., 5 illus. in color., Softcover
ISBN: 978-3-319-41916-9
A product of Birkhäuser Basel