Chapter 2
Hesitant Fuzzy Multiple Criteria Decision Analysis Based on TODIM

Abstract The TODIM is a valuable technique for solving classical MCDM problems in case of considering the decision maker's psychological behavior. One main goal of this chapter is to introduce the measured functions-based hesitant fuzzy TODIM technique to deal with the behavioral MCDM problem under hesitant fuzzy environments. The main advantages of this technique are that (1) it can handle the MCDM problems in which the ratings of alternatives with respect to each criterion are represented by HFEs or IVHFEs and (2) it can take the decision maker's psychological behavior into account. Another aim of this chapter is to present the hesitant trapezoidal fuzzy TODIM method with a closeness index-based ranking approach to handle MCGDM problems in which decision data is expressed as comparative linguistic expressions based on HTrFNs. This proposed method first transforms comparative linguistic expressions into HTrFNs for carrying out computing with word processes. Then, a closeness index-based ranking method is proposed for comparing the magnitude of HTrFNs. By using such a ranking method, the dominance values of alternatives over others for each expert are calculated. Next, a nonlinear programming model is established to derive the dominance values of alternatives over others for the group and correspondingly the optimal ranking order of alternatives is determined.

The classical TODIM method originally proposed by Gomes and Lima (1991, 1992) is a discrete multiple criteria decision analysis method based on prospect theory (Kahneman and Tversky 1979) and has been proven to be a valuable tool for solving the classical MCDM problems in case of considering the decision maker's psychological behavior. In the classical TODIM approach, the prospect value function is first built to measure the dominance degree of each alternative over the others, which reflects the decision maker's behavioral characteristics such as reference dependence and loss aversion; and then the ranking orders of alternatives can be obtained by calculating the overall prospect value of each alternative.

The classical TODIM method has been extensively applied in various fields of decision making, such as the selection of the destination of natural gas (Gomes et al. 2009), the evaluation of residential properties (Gomes and Rangel 2009), and the oil

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spill response problem (Passos et al. 2014), etc. Considering the fact that in some real-world situations the relationships among criteria are interdependent, Gomes et al. (2013) developed a method combining Choquet integral and the classical TODIM to handle the MCDM problems with criteria interactions. Owing to the fact that in many situations crisp data are inadequate or insufficient to model the real-world decision making problems, the fuzzy set and its extensions are more appropriate to model human judgments. This realization has motivated many researchers to extend the classical TODIM method for dealing with the MCDM problems under various fuzzy environments. For instance, considering the decision data assessed by TFNs or TrFNs, Krohling and de Souza (2012) developed a fuzzy extension of TODIM (named F-TODIM) for solving the fuzzy MCDM problems. Fan et al. (2013) proposed another extension of TODIM (named H-TODIM) to deal with the hybrid MCDM problems with three forms of criteria values (crisp numbers, interval numbers and fuzzy numbers). More recently, Lourenzutti and Krohling (2013) also presented a generalization of the TODIM method (named IF-RTODIM) which considers intuitionistic fuzzy information and an underlying random vector.

Although the existing TODIM methods can solve effectively the classical MCDM problems or fuzzy MCDM problems in case of considering the decision maker's psychological behavior, they fail to handle such MCDM problems under hesitant fuzzy environment. The MCDM problems with HFEs and/or IVHFEs have recently received increasing attentions and many corresponding MCDM methods (Farhadinia 2013; Liao and Xu 2013; Xu and Zhang 2013; Zhang 2013) have also been developed, but none of them can be used to solve the hesitant fuzzy MCDM problems in case of considering the decision maker's psychological behavior. To this end, Zhang and Xu (2014a) extended the classical TODIM method to solve the hesitant fuzzy MCDM problems in case of considering the decision maker's psychological behavior. In this approach, two novel ranking functions are developed for comparing the magnitude of HFEs and IVHFEs, which are more reasonable and effective compared with the existing ranking functions. Then, the prospect values of each alternative related to the others are calculated based on novel ranking functions and distance measures. By aggregating these prospect values, the overall prospect value of each alternative is further obtained and the ranking of alternatives is also obtained. Finally, Zhang and Xu (2014a) provided a decision making problem that concerns the evaluation and ranking of the service quality among domestic airlines to illustrate the validity and applicability of this approach.

On the other hand, Zhang et al. (2016) proposed a new concept of HTrFN which is an extension of HFE and is well enough to represent the uncertainty and vagueness of comparative linguistic expressions. The HTrFNs benefited from the superiority of both TrFNs and HFEs can be directly applied in MCDM and MCGDM. To handle the MCGDM problems in which the decision data are expressed by comparative linguistic expressions based on HTrFNs, Zhang and Liu (2016) developed a hesitant trapezoidal fuzzy TODIM method with a closeness index-based ranking approach. This proposed method first transforms comparative linguistic expressions into HTrFNs for carrying out computing with word processes. Then, a closeness index-based ranking method is proposed for comparing the magnitude of HTrFNs. By using the closeness index-based ranking method of

HTrFNs, the gain and loss of each alternative relative to the others are identified. Next, the dominance values of alternatives over others for each expert are calculated. Furthermore, a nonlinear programming model is established to derive the dominance values of alternatives over others for the group and correspondingly the optimal ranking order of alternatives is determined. At length, the proposed method is implemented in an evaluation problem of the service quality of airlines in order to demonstrate its decision making process and its applicability.

2.1 Description of the Classical TODIM Method

The classical TODIM method is to measure the dominance degree of each alternative over the others by establishing a prospect value function based on prospect theory (Kahneman and Tversky 1979). The main advantage of the classical TODIM method is its capability of capturing the decision maker's psychological behavior. It is worth mentioning that the classical TODIM method can only be suitable to deal with the classical MCDM problems in which the criteria values and the weights of criteria are in the format of crisp numbers. The algorithm (Algorithm 2.1) of the classical TODIM approach is introduced as follows (Zhang and Xu 2014a; Fan et al. 2013):

Step 1. Identify the decision matrix $\Re = (x_{ij})_{m\times n}$, and normalize it into $\bar{\Re} = (\bar{x}_{ij})_{m\times n}$ by using Eq. (1.3), where both x_{ij} and $\bar{x}_{ij}$ ($i \in \{1,2,\ldots,m\}, j \in \{1,2,\ldots,n\}$) are crisp numbers.

Step 2. Determine the relative weight w_{jR} of the criterion C_j to the reference criterion C_R according to the following expression:

$$w_{jR} = w_j / w_R, \quad j \in \{1,2,\ldots,n\} \tag{2.1}$$

where w_j is the weight value of the criterion C_j and $w_R = \max_{j=1}^{n}\{w_j\}$.

Step 3. Calculate the prospect values of the alternatives A_ξ ($\xi \in \{1,2,\ldots,m\}$) over the alternatives A_ζ ($\zeta \in \{1,2,\ldots,m\}$) using the following expression:

$$\mathscr{H}(A_\xi, A_\zeta) = \sum_{j=1}^{n} \mathscr{D}_j(A_\xi, A_\zeta), \quad \forall(\xi,\zeta) \tag{2.2}$$

where

$$\mathscr{D}_j(A_\xi, A_\zeta) = \begin{cases} \sqrt{w_{jR}\left(\bar{x}_{\xi j} - \bar{x}_{\zeta j}\right) \Big/ \sum_{j=1}^{n} w_{jR}}, & \text{if } \bar{x}_{\xi j} - \bar{x}_{\zeta j} > 0 \\ 0, & \text{if } \bar{x}_{\xi j} - \bar{x}_{\zeta j} = 0 \\ -\frac{1}{\theta}\sqrt{\left(\sum_{j=1}^{n} w_{jR}\right)\left(\bar{x}_{\zeta j} - \bar{x}_{\xi j}\right) \Big/ w_{jR}}, & \text{if } \bar{x}_{\xi j} - \bar{x}_{\zeta j} < 0 \end{cases} \tag{2.3}$$

The term $\mathscr{D}_j(A_\xi, A_\zeta)$ represents the contribution of the criterion C_j to the function $\mathscr{H}(A_\xi, A_\zeta)$ when comparing the alternative A_ξ with the alternative A_ζ. The parameter θ represents the attenuation factor of the losses, which can be tuned according to the problem at hand. In Eq. (2.3), three cases can occur:

(1) if $\bar{x}_{\xi j} - \bar{x}_{\zeta j} > 0$, then $\mathscr{D}_j(A_\xi, A_\zeta)$ represents a gain;
(2) if $\bar{x}_{\xi j} - \bar{x}_{\zeta j} = 0$, then $\mathscr{D}_j(A_\xi, A_\zeta)$ represents a nil;
(3) if $\bar{x}_{\xi j} - \bar{x}_{\zeta j} < 0$, then $\mathscr{D}_j(A_\xi, A_\zeta)$ represents a loss.

Step 4. Calculate the overall prospect values of the alternatives A_ξ ($\xi \in \{1, 2, \ldots, m\}$) according to the following expression:

$$\mathscr{Q}(A_\xi) = \frac{\sum_{\zeta=1}^{m} \mathscr{H}(A_\xi, A_\zeta) - \min_\xi \left\{ \sum_{\zeta=1}^{m} \mathscr{H}(A_\xi, A_\zeta) \right\}}{\max_\xi \left\{ \sum_{\zeta=1}^{m} \mathscr{H}(A_\xi, A_\zeta) \right\} - \min_\xi \left\{ \sum_{\zeta=1}^{m} \mathscr{H}(A_\xi, A_\zeta) \right\}} \tag{2.4}$$

Step 5. Rank all the alternatives by comparing their overall prospect values $\mathscr{Q}(A_\xi)$ ($\xi \in \{1, 2, \ldots, m\}$).

It is easily noted that the main idea of the classical TODIM method is to construct a prospect value function for measuring the dominance degree of each alternative over the others by comparing the criteria values. That is to say, to address the MCDM problem with hesitant fuzzy data by using the TODIM-based method, it is necessary to develop an effective ranking method for comparing the magnitudes of hesitant fuzzy data. To this end, Zhang and Xu (2014a) proposed novel ranking methods for HFEs and IVHFEs.

2.2 Ranking Functions Related to HFEs and IVHFEs

In practical fuzzy decision making process, the ranking of fuzzy information plays an important role in solving the fuzzy MCDM problems; while the ranking method is essentially based on the ranking function of fuzzy information which maps fuzzy information into crisp number. In general, the ranking methods of fuzzy information can be classified into two categories: the algorithmic ranking approaches and the non-algorithmic ranking approaches. In the non-algorithmic ranking approaches of fuzzy information, the ranking results are achieved in only one step, such as Xia and Xu (2011)'s ranking method, Farhadinia (2013)'s ranking method; while the ranking results obtained by the algorithmic ranking approaches of fuzzy information usually need to perform several steps, for instance, Liao and Xu (2013)'s ranking method.

In the sequel, we first review the existing ranking functions of HFEs, and then present a comparative study of these existing ranking functions. Furthermore, a novel ranking function developed by Zhang and Xu (2014a) is introduced for comparing the magnitude of HFEs, and its extension for comparing the magnitude of IVHFEs is also presented.

2.2.1 The Existing Ranking Functions of HFEs

Firstly, we review Xia and Xu (2011)'s ranking method which is referred to the non-algorithmic ranking approach of HFEs, as follows:

Definition 2.1 (Xia and Xu 2011). For a HFE $h = H\{\gamma^{\lambda} | \lambda = 1, 2, \ldots, \#h\}$, the score function of h is defined as follows:

$$s(h) = \frac{\gamma^1 + \gamma^2 + \cdots + \gamma^{\#h}}{\#h} \tag{2.5}$$

Then, for two HFEs h_1 and h_2, it is obtained that:

(1) if $s(h_1) > s(h_2)$, then h_1 is superior to h_2, denoted by $h_1 \succ_s h_2$;
(2) if $s(h_1) = s(h_2)$, then h_1 is indifferent to h_2, denoted by $h_1 \sim_s h_2$;
(3) if $s(h_1) < s(h_2)$, then h_1 is inferior to h_2, denoted by $h_1 \prec_s h_2$.

Example 2.1 (Zhang and Xu 2014a). Given three HFEs $h_1 = H\{0.3, 0.5\}$, $h_2 = H\{0.4\}$ and $h_3 = H\{0.2, 0.4, 0.6\}$, we need to compare the magnitude of these three HFEs.

According to the score function of HFE, we can obtain:

$$s(h_1) = 0.4, \quad s(h_2) = 0.4, \quad s(h_3) = 0.4.$$

Thus, we get $h_1 \sim_s h_2 \sim_s h_3$ by using Xia and Xu (2011)'s ranking method.

It is easy to notice that there yield indistinguishable HFEs when using the ranking function $s(h)$. In other words, the score-based ranking of HFEs is invalid in this situation and therefore should be improved. This realization has motivated Farhadinia (2013) to propose an improved score function and the corresponding ranking approach for HFEs as follows:

Assumption 2.1 (Farhadinia 2013). (1) The arrangement of elements in a HFE h is in an increasing order; (2) for two HFEs h_1 and h_2 with $\#h_1 \neq \#h_2$, we should extend the shorter one by adding the maximum element until them have the same length.

Remark 2.1 We point out that Assumption 2.1 is one special case of Definition 1.3, i.e., the special situation where the decision maker is optimist or risk-seeking.

Definition 2.2 (Farhadinia 2013). For a HFE $h = H\{\gamma^{\lambda} | \lambda = 1, 2, \ldots, \#h\}$, an improved score function $Is(h)$ of h is defined as follows:

$$Is(h) = \frac{\sum_{\lambda=1}^{\#h} f(\lambda)\gamma^{\lambda}}{\sum_{\lambda=1}^{\#h} f(\lambda)} \tag{2.6}$$

where $f(\lambda)$ $(\lambda = 1, 2, \ldots, \#h)$ is a positive-valued monotonic increasing sequence of index λ.

For convenience, Farhadinia (2013) suggested $f(\lambda) = \lambda$, and Eq. (2.6) is transformed into the following equation:

$$Is(h) = \frac{2\sum_{\lambda=1}^{\#h} \lambda\gamma^{\lambda}}{\#h \times (\#h + 1)} \tag{2.7}$$

Then, under Assumption 2.1, for two HFEs h_1 and h_2, we have:

(1) if $Is(h_1) > Is(h_2)$, then h_1 is superior to h_2, denoted by $h_1 \succ_{Is} h_2$;
(2) if $Is(h_1) = Is(h_2)$, then h_1 is indifferent to h_2, denoted by $h_1 \sim_{Is} h_2$;
(3) if $Is(h_1) < Is(h_2)$, then h_1 is inferior to h_2, denoted by $h_1 \prec_{Is} h_2$.

Example 2.2 (Zhang and Xu 2014a). For three HFEs shown in Example 2.1, according to Assumption 2.1 we extend the shorter one by adding the maximum element until these three HFEs have the same length. Namely, h_1 is extended to $h_1 = H\{0.3, 0.5, 0.5\}$ and h_2 is extended to $h_2 = H\{0.4, 0.4, 0.4\}$. Using the improved score function we can obtain:

$$Is(h_1) = 0.383, \quad Is(h_2) = 0.4, \quad Is(h_3) = 0.467.$$

Thus, according to Farhadinia (2013)'s ranking method, we can get $h_3 \succ_{Is} h_2 \succ_{Is} h_1$.

Obviously, the comparison of results of Examples 2.1 and 2.2 shows that the improved score function $Is(h)$ is more effective in ranking than the score function $s(h)$. However, we also notice that in the definition of $Is(h)$, the elements being an increasing order in h have the increasing importance, which is not consistent with the definition of the HFE (i.e. Definition 1.1) in which the importance weights of all elements in a HFE are the same.

Drawing on the mean-variance model in statistics, Liao and Xu (2013) developed a score-variance model to rank HFEs. The concept of variance function $Var(h)$ of HFEs is introduced as follows:

Definition 2.3 (Liao and Xu 2013). For a HFE $h = H\{\gamma^1, \gamma^2, \ldots, \gamma^{\#h}\}$, the variance function $Var(h)$ of h is defined as follows:

$$Var(h) = \frac{\sqrt{\sum_{\gamma^{\xi}, \gamma^{\zeta} \in h} (\gamma^{\xi} - \gamma^{\zeta})^2}}{\#h} \tag{2.8}$$

It is easy to see that $Var(h)$ reflects the variance value among all possible values in h. Based on $s(h)$ and $Var(h)$, a score-variance model is introduced to rank the HFEs:

Definition 2.4 (Liao and Xu 2013). Let h_1 and h_2 be two HFEs, the $s(h_1)$ and $s(h_2)$ be the scores of h_1 and h_2, respectively, $Var(h_1)$ and $Var(h_2)$ be the deviation values of h_1 and h_2, respectively. Then, it is concluded that

(1) if $s(h_1) < s(h_2)$, then $h_1 \prec_s h_2$;

(2) if $s(h_1) = s(h_2)$, then $\begin{cases} Var(h_1) < Var(h_2) \Rightarrow h_1 \succ_{sv} h_2 \\ Var(h_1) = Var(h_2) \Rightarrow h_1 \sim_{sv} h_2 \, ; \\ Var(h_1) > Var(h_2) \Rightarrow h_1 \prec_{sv} h_2 \end{cases}$

(3) if $s(h_1) > s(h_2)$, then $h_1 \succ_s h_2$.

Example 2.3 (Zhang and Xu 2014a). For three HFEs displayed in Example 2.1, by Eqs. (2.5) and (2.8) it is easy to obtain:

$$\begin{aligned} s(h_1) = 0.4, \quad & s(h_2) = 0.4, \quad s(h_3) = 0.4, \\ Var(h_1) = 0.1, \quad & Var(h_2) = 0, \quad Var(h_3) = 0.1633. \end{aligned}$$

According to Liao and Xu (2013)'s ranking approach, we can obtain the ranking $h_3 \prec_{sv} h_1 \prec_{sv} h_2$, which is different from the result $(h_1 \prec_{Is} h_2 \prec_{Is} h_3)$ obtained by Farhadinia (2013)'s ranking approach.

With the help of Examples 2.1, 2.2 and 2.3, it is easily showed that Liao and Xu (2013)'s ranking approach is superior to Xia and Xu (2011)'s ranking approach and Farhadinia (2013)'s ranking approach.

Although Liao and Xu (2013)'s ranking approach can be applied to HFEs and seems to be effective, this approach which is referred to the algorithm ranking approach makes the process of decision making more time-consuming. Because when we utilize this approach to deal with the hesitant fuzzy MCDM problems, the process of decision making is required to be divided into several steps and it is necessary to add other rules for obtaining the best alternative.

Bearing this fact in mind, Zhang and Xu (2014a) developed a novel ranking function which is referred to the non-algorithmic ranking approach for comparing the HFEs.

2.2.2 The Proposed Ranking Functions

Definition 2.5 (Zhang and Xu 2014a). For a HFE $h = H\{\gamma^1, \gamma^2, \ldots, \gamma^{\#h}\}$, a new ranking function $Hr_\delta(h)$ of h is defined as follows:

$$Hr_\delta(h) = \left(\frac{(\gamma^1)^\delta + (\gamma^2)^\delta + \cdots + (\gamma^{\#h})^\delta}{\#h} \right)^{1/\delta} \tag{2.9}$$

where δ $(0 < \delta \leq 1)$ is a parameter determined by the decision maker, which can be tuned according to the practical situation.

In particular, if $\delta = 1$, then the new ranking function $Hr_{\delta=1}(h) = (\gamma^1 + \gamma^2 + \cdots + \gamma^{\#h}) / \#h$, which is reduced to the score function of HFEs developed by Xia and Xu (2011).

Meanwhile, Zhang and Xu (2014a) also discussed some properties of the new ranking function $Hr_\delta(h)$ as below:

Proposition 2.1 (Zhang and Xu 2014a). For any HFE $h = H\{\gamma^1, \gamma^2, \ldots, \gamma^{\#h}\}$, the proposed ranking function $Hr_\delta(h) \in [0, 1]$.

Proof Let $\gamma^+ = \max_{\lambda=1}^{\#h}\{\gamma^\lambda\}$ and $\gamma^- = \min_{\lambda=1}^{\#h}\{\gamma^\lambda\}$, because $0 < \delta \leq 1$ and $\gamma^\lambda \in [0, 1]$ $(\lambda \in \{1, 2, \ldots, \#h\})$, then

$$\begin{aligned} Hr_\delta(h) &= \left(\left((\gamma^1)^\delta + (\gamma^2)^\delta + \cdots + (\gamma^{\#h})^\delta\right) \big/ \#h\right)^{1/\delta} \\ &\leq \left(\left((\gamma^+)^\delta + (\gamma^+)^\delta + \cdots + (\gamma^+)^\delta\right) \big/ \#h\right)^{1/\delta} = \gamma^+ \leq 1, \end{aligned}$$

$$\begin{aligned} Hr_\delta(h) &= \left(\left((\gamma^1)^\delta + (\gamma^2)^\delta + \cdots + (\gamma^{\#h})^\delta\right) \big/ \#h\right)^{1/\delta} \\ &\geq \left(\left((\gamma^-)^\delta + (\gamma^-)^\delta + \cdots + (\gamma^-)^\delta\right) \big/ \#h\right)^{1/\delta} = \gamma^- \geq 0. \end{aligned}$$

Obviously, $0 \leq Hr_\delta(h) \leq 1$, which completes the proof (Zhang and Xu 2014a). □

Proposition 2.2 (Zhang and Xu 2014a). For a single-valued HFE $h = H\{\gamma\}$, the proposed ranking function $Hr_\delta(h) = \gamma$. In particular, if h is the hesitant empty element, i.e., $h = H\{0\}$, then $Hr_\delta(h) = 0$; if h is the hesitant full element, i.e., $h = H\{1\}$, then $Hr_\delta(h) = 1$.

Proposition 2.3 (Zhang and Xu 2014a). For two HFEs h_1 and h_2 having the same length, and the arrangement of elements in these two HFEs are in increasing orders, if $h_1 \leq h_2$, then $Hr_\delta(h_1) \leq Hr_\delta(h_2)$.

Proof For two HFEs $h_i = H\{\gamma_i^\lambda | \lambda = 1, 2, \ldots, \#h_i\}$ $(i = 1, 2)$, because they have the same length (i.e., $\#h = \#h_1 = \#h_2$) and the arrangement of elements in these two HFEs are in increasing orders. According to Definition 1.4, if $h_1 \leq h_2$, then $\gamma_1^\lambda \leq \gamma_2^\lambda$ $(\lambda \in \{1, 2, \ldots, \#h\})$. Owing to $0 < \delta \leq 1$, then $(\gamma_1^\lambda)^\delta \leq (\gamma_2^\lambda)^\delta$ $(\lambda \in \{1, 2, \ldots, \#h\})$. Apparently, we have

$$\left(\frac{(\gamma_1^1)^\delta + (\gamma_1^2)^\delta + \cdots + (\gamma_1^{\#h})^\delta}{\#h}\right)^{\frac{1}{\delta}} \leq \left(\frac{(\gamma_2^1)^\delta + (\gamma_2^2)^\delta + \cdots + (\gamma_2^{\#h})^\delta}{\#h}\right)^{\frac{1}{\delta}}$$

Namely, $Hr_\delta(h_1) \leq Hr_\delta(h_2)$, which completes the proof (Zhang and Xu 2014a). □

On the basis of the ranking function proposed by Definition 2.5, Zhang and Xu (2014a) introduced the following ranking method for HFEs which is referred to the non-algorithmic ranking approach:

Definition 2.6 (Zhang and Xu 2014a). For two HFEs h_1 and h_2, $Hr_\delta(h_1)$ and $Hr_\delta(h_2)$ are the new ranking functions of h_1 and h_2, respectively, then:

(1) if $Hr_\delta(h_1) > Hr_\delta(h_2)$, then h_1 is superior to h_2, denoted by $h_1 \succ_{Hr} h_2$;
(2) if $Hr_\delta(h_1) = Hr_\delta(h_2)$, then h_1 is indifferent to h_2, denoted by $h_1 \sim_{Hr} h_2$;
(3) if $Hr_\delta(h_1) < Hr_\delta(h_2)$, then h_1 is inferior to h_2, denoted by $h_1 \prec_{Hr} h_2$.

Example 2.4 (Zhang and Xu 2014a). For three HFEs being shown in Example 2.1, according to Definitions 2.5 and 2.6 it is easy to obtain the ranking results of these HFEs which are listed in Table 2.1.

In order to provide a synthetic view of the comparison results, we put all the results of the ranking of alternatives with different approaches into Table 2.2 (Zhang and Xu 2014a).

As shown in Table 2.2, the ranking order of HFEs obtained by Zhang and Xu (2014a)'s ranking method is the same as the result obtained by Liao and Xu (2013)'s ranking approach, but not consistent with the results obtained by Xia and Xu (2011)'s ranking approach and Farhadinia (2013)'s ranking approach. The main reason is that Xia and Xu (2011)'s ranking approach only considers the average value of all elements in HFEs and is just the special case of Zhang and Xu (2014a)'s ranking method, which cannot distinguish these three HFEs; While Farhadinia (2013)'s ranking approach is not only based on Assumption 2.1 which is an especial case of Definition 1.3, but also adopts the index $f(\lambda)$, which is unreasonable. Although Liao and Xu (2013)'s ranking approach is consistent with Zhang and Xu (2014a)'s ranking method, this approach referred to the algorithmic ranking approach makes the process of decision making more time-consuming. Therefore, we can confidently say that Zhang and Xu (2014a)'s ranking method referred to the

Table 2.1 The calculation results obtained by Definitions 2.5 and 2.6

Zhang and Xu (2014a)'s ranking method	h_1	h_2	h_3	The ranking orders
$Hr_{\delta=0.001}(h)$	0.3873	0.4	0.3635	$h_3 \prec_{Hr} h_1 \prec_{Hr} h_2$
$Hr_{\delta=0.01}(h)$	0.3874	0.4	0.3638	$h_3 \prec_{Hr} h_1 \prec_{Hr} h_2$
$Hr_{\delta=0.1}(h)$	0.3886	0.4	0.3672	$h_3 \prec_{Hr} h_1 \prec_{Hr} h_2$

Table 2.2 The comparison results of rankings of HFEs

The ranking approach	Ranking orders of HFEs
Xia and Xu (2011)'s ranking approach	$h_1 \sim_s h_2 \sim_s h_3$
Farhadinia (2013)'s ranking approach	$h_1 \prec_{Is} h_2 \prec_{Is} h_3$
Liao and Xu (2013)'s ranking approach	$h_3 \prec_{sv} h_1 \prec_{sv} h_2$
Zhang and Xu (2014a)'s ranking method	$h_3 \prec_{Hr} h_1 \prec_{Hr} h_2$

non-algorithmic ranking approach is much superior to Xia and Xu (2011)'s ranking approach, Liao and Xu (2013)'s ranking approach and Farhadinia (2013)'s ranking approach, which implies that Zhang and Xu (2014a)'s ranking method can provide a more useful technique than the previous ones to efficiently assist the decision maker.

Analogously, a new ranking method for IVHFEs is introduced as below:

Definition 2.7 (Zhang and Xu 2014a). For an IVHFE $\tilde{h} = \tilde{H}\{[\gamma^{1L}, \gamma^{1U}], [\gamma^{2L}, \gamma^{2U}], \ldots, [\gamma^{\#\tilde{h}L}, \gamma^{\#\tilde{h}U}]\}$, a new ranking function $Ir_\delta(\tilde{h})$ of $\tilde{h}$ is defined as follows:

$$Ir_\delta(\tilde{h}) = \left(\frac{(\gamma^{1L})^\delta + (\gamma^{2L})^\delta + \cdots + (\gamma^{\#\tilde{h}L})^\delta}{\#\tilde{h}}\right)^{1/\delta} + \left(\frac{(\gamma^{1U})^\delta + (\gamma^{2U})^\delta + \cdots + (\gamma^{\#\tilde{h}U})^\delta}{\#\tilde{h}}\right)^{1/\delta} \tag{2.10}$$

where δ $(0 < \delta \le 1)$ is a parameter determined by the decision maker, which can be tuned according to the real-life decision problem at hand.

Based on the ranking function of IVHFEs, it is easy to give the following ranking method for IVHFEs, which is also referred to the non-algorithmic ranking approach:

Definition 2.8 (Zhang and Xu 2014a). For two IVHFEs $\tilde{h}_1$ and $\tilde{h}_2$, $Ir_\delta(\tilde{h}_1)$ and $Ir_\delta(\tilde{h}_2)$ are the new ranking functions of $\tilde{h}_1$ and $\tilde{h}_2$, respectively, then:

(1) if $Ir_\delta(\tilde{h}_1) > Ir_\delta(\tilde{h}_2)$, then $\tilde{h}_1$ is superior to $\tilde{h}_2$, denoted by $\tilde{h}_1 \succ_{Ir} \tilde{h}_2$;
(2) if $Ir_\delta(\tilde{h}_1) = Ir_\delta(\tilde{h}_2)$, then $\tilde{h}_1$ is indifferent to $\tilde{h}_2$, denoted by $\tilde{h}_1 \sim_{Ir} \tilde{h}_2$;
(3) if $Ir_\delta(\tilde{h}_1) < Ir_\delta(\tilde{h}_2)$, then $\tilde{h}_1$ is inferior to $\tilde{h}_2$, denoted by $\tilde{h}_1 \prec_{Ir} \tilde{h}_2$.

Next, on the basis of the proposed ranking functions and the ranking methods of HFEs and IVHFEs, Zhang and Xu (2014a) extended the classical TODIM method to solve the MCDM problems in which the criteria values are denoted by two different formats (i.e., HFEs or IVHFEs).

2.3 The Ranking Functions-Based Hesitant Fuzzy TODIM Approach

The MCDM problem is to identify the desirable compromise solution from a set of feasible alternatives which are assessed based on a set of conflicting criteria. Let $\boldsymbol{A} = \{A_1, A_2, \ldots, A_m\}$ be a finite alternative set and $\boldsymbol{C} = \{C_1, C_2, \ldots, C_n\}$ be a finite criteria set. The classical MCDM problem can be expressed in a decision matrix $\Re = (x_{ij})_{m\times n}$ whose elements x_{ij} indicate the ratings of the alternatives

$A_i(i \in \{1, 2, \ldots, m\})$ with respect to the criteria $C_j(j \in \{1, 2, \ldots, n\})$. In this chapter, we extend the decision matrix $\Re$ to the hesitant fuzzy decision matrix by considering the MCDM under hesitant fuzzy environment. Three scenarios are described in the decision matrix $\Re$ as follows (Zhang and Xu 2014a):

(1) All the criteria values in $\Re$ are expressed in the format of HFEs, i.e.,

$$x_{ij} = h_{ij} = H\{\gamma^1, \gamma^2, \ldots, \gamma^{\# h_{ij}}\}_{ij}$$

In this case, x_{ij} should be interpreted that the degree to which the alternative A_i satisfies the criterion C_j is some possible values between 0 and 1.

(2) Consider some practical situations that it is somewhat difficult for the decision maker to assign exact values for the membership degrees of certain elements to a given set, but an interval number may be assigned. In this case, all criteria values in $\Re$ take the form of IVHFEs, i.e.,

$$x_{ij} = \tilde{h}_{ij} = \tilde{H}\{[\tilde{\gamma}^{1L}, \tilde{\gamma}^{1U}], [\tilde{\gamma}^{2L}, \tilde{\gamma}^{2U}], \ldots, [\tilde{\gamma}^{\# \tilde{h}_{ij} L}, \tilde{\gamma}^{\# \tilde{h}_{ij} U}]\}_{ij}$$

The criterion value x_{ij} indicates that the degree to which the alternative A_i satisfies the criterion C_j is several possible intervals belonging to the interval [0, 1].

(3) The criteria values in the decision matrix $\Re$ are denoted by two different formats: HFEs and IVHFEs. In this case, the set of criteria $\boldsymbol{C}$ can be divided into two subsets $\boldsymbol{C}^{HF}$ and $\boldsymbol{C}^{IVHF}$, representing the criteria whose values are in the formats of HFEs and IVHFEs, respectively. Let $\boldsymbol{C}^{HF} = \{C_1, C_2, \ldots, C_{j_1}\}$, $\boldsymbol{C}^{IVHF} = \{C_{j_1+1}, C_{j_1+2}, \ldots, C_n\}$, then $\boldsymbol{C}^{HF} \cup \boldsymbol{C}^{IVHF} = \boldsymbol{C}$ and $\boldsymbol{C}^{HF} \cap \boldsymbol{C}^{IVHF} = \varnothing$. If $C_j \in \boldsymbol{C}^{HF}$, then x_{ij} is a HFE denoted by $x_{ij} = h_{ij} = H\{\gamma^1, \gamma^2, \ldots, \gamma^{\# h_{ij}}\}_{ij}$; while if $C_j \in \boldsymbol{C}^{IVHF}$, then x_{ij} is an IVHFE denoted by $\tilde{H}\{[\tilde{\gamma}^{1L}, \tilde{\gamma}^{1U}], [\tilde{\gamma}^{2L}, \tilde{\gamma}^{2U}], \ldots, [\tilde{\gamma}^{\# \tilde{h}_{ij} L}, \tilde{\gamma}^{\# \tilde{h}_{ij} U}]\}_{ij}$.

It is noticed that Scenarios (1) and (2) are the special cases of Scenario (3), and in this chapter we only take Scenario (3) into account. In practical MCDM problems, there usually exist benefit criteria (the larger the better) and cost criteria (the smaller the better). Meanwhile, the dimensions and measurements of criteria values are often different since the natures of these criteria are different. In order to ensure the compatibility between criteria values, the criteria values must be converted into a compatible scale (or dimensionless indices). Consequently, to eliminate the effect of different physical dimensions and measurements in the final decision results, in this chapter the criteria values of the cost type are transformed into the criteria values of the benefit type by normalizing the hesitant fuzzy decision matrix $\Re = (x_{ij})_{m \times n}$ to yield a corresponding normalized hesitant fuzzy decision matrix $\bar{\Re} = (\bar{x}_{ij})_{m \times n}$, using the following Eqs. (2.11) and (2.12):

(a) If $C_j \in \boldsymbol{C}^{HF}$, then the criterion value x_{ij} is a HFE h_{ij}. It can be normalized as follows (Zhu et al. 2012):

$$\bar{x}_{ij} = \begin{cases} h_{ij}, & \mathit{if}\ C_j \in \left(\boldsymbol{C}_I \cap \boldsymbol{C}^{HF}\right) \\ \left(h_{ij}\right)^c, & \mathit{if}\ C_j \in \left(\boldsymbol{C}_{II} \cap \boldsymbol{C}^{HF}\right) \end{cases} \tag{2.11}$$

(b) If $C_j \in \boldsymbol{C}^{IVHF}$, then the criterion value x_{ij} is an IVHFE $\tilde{h}_{ij}$. We can normalize it as follows (Zhang and Xu 2014a):

$$\bar{x}_{ij} = \begin{cases} \tilde{h}_{ij}, & \mathit{if}\ C_j \in \left(\boldsymbol{C}_I \cap \boldsymbol{C}^{IVHF}\right) \\ \left(\tilde{h}_{ij}\right)^c, & \mathit{if}\ C_j \in \left(\boldsymbol{C}_{II} \cap \boldsymbol{C}^{IVHF}\right) \end{cases} \tag{2.12}$$

To solve the aforementioned MCDM problem in case of considering the decision maker's psychological behavior, Zhang and Xu (2014a) developed a novel ranking functions-based hesitant fuzzy TODIM method. We call it the HF-TODIM method for convenience. Similar to the steps of the classical TODIM approach, in the HF-TODIM method, we first need to normalize the original decision matrix by using Eqs. (2.11) and (2.12). Then, we proceed to measure the prospect value of each alternative over the others by constructing the prospect value function based on prospect theory (Kahneman and Tversky 1979).

For this, we first identify a reference criterion and calculate the relative weight of each criterion to the reference criterion. Usually, the criterion with the highest weight can be regarded as the reference criterion and then the relative weight w_{jR} of the criterion C_j to the reference criterion C_R can be obtained by Eq. (2.1). Afterwards, based on the novel ranking functions $Hr_\delta(h)$ and $Ir_\delta(\tilde{h})$, we can compare with the magnitudes of the ratings of alternatives regarding each criterion which are represented by the HFE or IVHFE. Furthermore, analogous to Eq. (2.2), we can calculate the gain and loss of the alternative A_ξ over the alternative A_ζ concerning the criterion C_j.

(1) If $C_j \in \boldsymbol{C}^{HF}$, the criteria values x_{ij} $(i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, j_1\})$ are expressed by HFEs, i.e., $x_{ij} = h_{ij}$, then the gain and loss of the alternative A_ξ over the alternative A_ζ concerning the criterion C_j can be obtained by the following expression (Zhang and Xu 2014a):

$$\mathscr{D}_j^h(A_\xi, A_\zeta) = \begin{cases} \sqrt{\dfrac{w_{jR}\, d(h_{\xi j}, h_{\zeta j})}{\sum_{j=1}^{n} w_{jR}}}, & \mathit{if}\ Hr_\delta\left(h_{\xi j}\right) - Hr_\delta\left(h_{\zeta j}\right) > 0 \\ 0, & \mathit{if}\ Hr_\delta\left(h_{\xi j}\right) - Hr_\delta\left(h_{\zeta j}\right) = 0 \\ -\dfrac{1}{\theta}\sqrt{\dfrac{\left(\sum_{j=1}^{n} w_{jR}\right) d\left(h_{\xi j}, h_{\zeta j}\right)}{w_{jR}}}, & \mathit{if}\ Hr_\delta\left(h_{\xi j}\right) - Hr_\delta\left(h_{\zeta j}\right) < 0 \end{cases} \tag{2.13}$$

where the parameter θ represents the attenuation factor of the losses, and the $d(\tilde{h}_{\xi j}, \tilde{h}_{\zeta j})$ denotes the distance between the HFEs $h_{\xi j}$ and $h_{\zeta j}$ using Eq. (1.2).

(2) If $C_j \in \boldsymbol{C}^{IVHF}$, the criteria values $x_{ij}(i \in \{1, 2, \ldots, m\}, j \in \{j_1 + 1, j_1 + 2, \ldots, n\})$ are expressed by IVHFEs, i.e., $x_{ij} = \tilde{h}_{ij}$, then the gain and loss of the alternative A_ξ over the alternative A_ζ concerning the criterion C_j is obtained by the following expression (Zhang and Xu 2014a):

$$\mathscr{D}_j^h(A_\xi, A_\zeta) = \begin{cases} \sqrt{\frac{w_{jR} d(\tilde{h}_{\xi j}, \tilde{h}_{\zeta j})}{\sum_{j=1}^n w_{jR}}}, & if\ Ir_\delta(\tilde{h}_{\xi j}) - Ir_\delta(\tilde{h}_{\zeta j}) > 0 \\ 0, & if\ Ir_\delta(\tilde{h}_{\xi j}) - Ir_\delta(\tilde{h}_{\zeta j}) = 0 \\ -\frac{1}{\theta}\sqrt{\frac{\left(\sum_{j=1}^n w_{jR}\right) d(\tilde{h}_{\xi j}, \tilde{h}_{\zeta j})}{w_{jR}}}, & if\ Ir_\delta(\tilde{h}_{\xi j}) - Ir_\delta(\tilde{h}_{\zeta j}) < 0 \end{cases} \tag{2.14}$$

where the $d(\tilde{h}_{\xi j}, \tilde{h}_{\zeta j})$ denotes the distance between the IVHFEs $\tilde{h}_{\xi j}$ and $\tilde{h}_{\zeta j}$ using Eq. (1.29).

It is easily observed from Eq. (2.13) that there exist three cases:

(1) If $Hr_\delta(h_{\xi j}) - Hr_\delta(h_{\zeta j}) > 0$, then $\mathscr{D}_j^h(A_\xi, A_\zeta)$ represents a gain;
(2) If $Hr_\delta(h_{\xi j}) - Hr_\delta(h_{\zeta j}) = 0$, then $\mathscr{D}_j^h(A_\xi, A_\zeta)$ represents a nil;
(3) If $Hr_\delta(h_{\xi j}) - Hr_\delta(h_{\zeta j}) < 0$, then $\mathscr{D}_j^h(A_\xi, A_\zeta)$ represents a loss.

Likewise, there exist three cases in Eq. (2.14):

(1) If $Ir_\delta(\tilde{h}_{\xi j}) - Ir_\delta(\tilde{h}_{\zeta j}) > 0$, then $\mathscr{D}_j^h(A_\xi, A_\zeta)$ represents a gain;
(2) If $Ir_\delta(\tilde{h}_{\xi j}) - Ir_\delta(\tilde{h}_{\zeta j}) = 0$, then $\mathscr{D}_j^h(A_\xi, A_\zeta)$ represents a nil;
(3) If $Ir_\delta(\tilde{h}_{\xi j}) - Ir_\delta(\tilde{h}_{\zeta j}) < 0$, then $\mathscr{D}_j^h(A_\xi, A_\zeta)$ represents a loss.

By aggregating $\mathscr{D}_j^h(A_\xi, A_\zeta)$ with each criterion C_j, the prospect value of the alternative A_ξ over the alternative A_ζ can be obtained as follows (Zhang and Xu 2014a):

$$\mathscr{H}^h(A_\xi, A_\zeta) = \sum_{j=1}^n \mathscr{D}_j^h(A_\xi, A_\zeta), \quad \forall(\xi, \zeta) \tag{2.15}$$

At length, we calculate the overall prospect value of the alternatives A_ξ $(\xi \in \{1, 2, \ldots, m\})$ according to the following expression (Zhang and Xu 2014a):

$$\mathscr{Q}^h(A_\xi) = \frac{\sum_{\zeta=1}^m \mathscr{H}^h(A_\xi, A_\zeta) - \min_\xi\left\{\sum_{\zeta=1}^m \mathscr{H}^h(A_\xi, A_\zeta)\right\}}{\max_\xi\left\{\sum_{\zeta=1}^m \mathscr{H}^h(A_\xi, A_\zeta)\right\} - \min_\xi\left\{\sum_{\zeta=1}^m \mathscr{H}^h(A_\xi, A_\zeta)\right\}} \tag{2.16}$$

Obviously, $0 \le \mathcal{Q}^h(A_\xi) \le 1$, and the greater the $\mathcal{Q}^h(A_\xi)$ is, the better the alternative A_ξ will be. Therefore, we can determine the ranking of all alternatives according to the increasing orders of the overall prospect values of the alternatives A_ξ $(\xi \in \{1, 2, \ldots, m\})$, and select the desirable alternative(s) from the alternative set $\boldsymbol{A} = \{A_1, A_2, \ldots, A_m\}$.

Based on the above models and analysis, an algorithm (Algorithm 2.2) for the HF-TODIM approach is presented as follows (Zhang and Xu 2014a):

Step 1. Identify the original decision matrix $\Re = \left(x_{ij}\right)_{m\times n}$, and obtain the normalized decision matrix $\bar{\Re} = \left(\bar{x}_{ij}\right)_{m\times n}$ by Eqs. (2.11) and (2.12).

Step 2. Determine the reference criterion C_R, and calculate the relative weight w_{jR} of the criterion C_j to the reference criterion C_R using Eq. (2.2).

Step 3. Calculate the gain and loss of the alternative A_ξ over the alternative A_ζ concerning each criterion C_j using Eqs. (2.13) and (2.14), respectively.

Step 4. Calculate the prospect value of the alternative A_ξ over the alternative A_ζ using Eq. (2.15).

Step 5. Calculate the overall prospect values of the alternatives A_ξ $(\xi \in \{1, 2, \ldots, m\})$ using Eq. (2.16).

Step 6. The ranking of all alternatives is generated according to the increasing orders of the overall prospect values of alternatives and the desirable alternative from $\boldsymbol{A} = \{A_1, A_2, \ldots, A_m\}$ is determined.

Remark 2.2 It is worth pointing out that in the HF-TODIM model, the decision data take the forms of HFEs and IVHFEs. Whereas, in the classical TODIM model (Gomes and Lima 1991, 1992), the F-TODIM model (Krohling and de Souza 2012), the H-TODIM model (Fan et al. 2013) and the IF-RTODIM model (Lourenzutti and Krohling 2013), the decision data are represented by crisp numbers, fuzzy numbers, hybrid types (crisp numbers, intervals numbers and fuzzy numbers) and IFNs, respectively. Apparently, these aforementioned models cannot deal directly with the decision making problems where the decision information takes the forms of HFEs and IVHFEs. On the other hand, the existing hesitant fuzzy MCDM methods (Farhadinia 2013; Liao and Xu 2013; Xu and Zhang 2013; Zhang 2013) are based on the strict assumption regarding the complete rationality of the decision maker and thus fail to capture the decision maker's psychological behavior, while the HF-TODIM approach which is based on prospect theory can fully consider the decision maker's behavior in the hesitant fuzzy MCDM process.

2.4 Illustration Example Based on the Evaluation Problem of Service Quality

To demonstrate the applicability and the implementation process of the HF-TODIM approach, Zhang and Xu (2014a) presented a practical decision making problem that concerns the evaluation and ranking of the service quality among domestic airlines [adapted from Liou et al. (2011), Liao and Xu (2013)].

2.4.1 *Description*

Due to the development of high-speed railroad, the domestic airline marketing has faced a stronger challenge in Taiwan. More and more airlines have attempted to attract customers by reducing price. Unfortunately, they soon found that there was a no-win situation and only service quality is the critical and fundamental element to survive in this highly competitive domestic market. In order to improve the service quality of domestic airline, the civil aviation administration of Taiwan (CAAT) wants to know which airline is the best in Taiwan and then calls for the others to learn from it. Thus, the CAAT constructs a committee to investigate the four major domestic airlines, which are UNI Air (A_1), Transasia (A_2), Mandarin (A_3), and Daily Air (A_4), and four major criteria are given based on the research of Liou et al. (2011) to evaluate these four domestic airlines. These four main criteria are: Booking and ticketing service (C_1), Check-in and boarding process (C_2), Cabin service (C_3), and Responsiveness (C_4); a detailed description of the four criteria are given in Table 2.3 (Liou et al. 2011).

After the survey about passengers' importance degrees and perceptions for the service criteria done by Liou et al. (2011), they found that the cabin service is

Table 2.3 Criteria for evaluating domestic airlines

Criteria	Description of criteria
Booking and ticketing service C_1	Booking and ticketing service involves conveniences of booking or buying ticket, promptness of booking or buying ticket, courtesy of booking or buying ticket
Check-in and boarding process C_2	Check-in and boarding process consists of convenience check-in, efficient check-in, courtesy of employee, clarity of announcement and so on
Cabin service C_3	Cabin service can be divided into cabin safety demonstration, variety of newspapers and magazines, courtesy of flight attendants, flight attendant willing to help, clean and comfortable interior, in-flight facilities, and captain's announcement
Responsiveness C_4	Responsiveness consists of fair waiting-list call, handing of delayed flight, complaint handing, and missing baggage handling

Table 2.4 Hesitant fuzzy decision matrix

	C_1	C_2
A_1	$\tilde{H}$ {[0.4, 0.55], [0.7, 0.9]}	H {0.6, 0.8}
A_2	$\tilde{H}$ {[0.5, 0.6], [0.8, 0.9]}	H {0.5, 0.8, 0.9}
A_3	$\tilde{H}$ {[0.3, 0.4], [0.5, 0.65], [0.8,0.9]}	H {0.6, 0.7, 0.95}
A_4	$\tilde{H}$ {[0.4, 0.5], [0.6, 0.7]}	H {0.7, 0.9}
	C_3	C_4
A_1	$\tilde{H}$ {[0.35, 0.4], [0.5, 0.65], [0.8, 0.95]}	H {0.4, 0.5, 0.9}
A_2	$\tilde{H}$ {[0.4, 0.5], [0.65, 0.9]}	H {0.4, 0.6, 0.7}
A_3	$\tilde{H}$ {[0.3,0.45], [0.6,0.8]}	H {0.5, 0.8}
A_4	$\tilde{H}$ {[0.2, 0.4], [0.6, 0.7]}	H {0.4, 0.5}

considered the most important factor of service quality, which can be interpreted easily because the cabin service occupies more of a passenger's travelling time than other aspects. Meanwhile, the booking and ticketing service is less important due to the fact that the work is mainly done by computers. Therefore, the weight vector of the criteria is $\boldsymbol{w} = (0.15, 0.25, 0.35, 0.25)^T$, which is consistent with the result of the survey done by Liou et al. (2011). Suppose that the committee gives the criteria values by using HFEs and IVHFEs, and then the hesitant fuzzy decision matrix is presented in Table 2.4 (Zhang and Xu 2014a).

2.4.2 Decision Making Model

In the following, we use the HF-TODIM decision model to solve the decision making problem mentioned in Sect. 2.4.1. The solution process and the computation results are summarized as follows (Zhang and Xu 2014a):

Firstly, we take the criterion C_3 as the reference criterion because the cabin service (C_3) is considered the most important factor of service quality. Thus, the weight of the reference criterion is $w_R = 0.35$. Meanwhile, we take $\theta = 1$, which means that the losses will contribute with their real value to the global value (Gomes and Rangel 2009).

Secondly, based on the novel ranking functions $Hr_\delta(h)$ and $Ir_\delta(\tilde{h})$, we compare with the magnitudes of the ratings of alternatives regarding each criterion which are represented by HFEs or IVHFEs. Here we take $\delta = 0.1$. As a result, the superior-inferior table can be obtained and is listed in Table 2.5 (Zhang and Xu 2014a). The top-left cell ${}_{1/2}\mathrm{I}_1$ in Table 2.5 means that for the booking and ticketing service C_1, UNI Air A_1 is inferior to Transasia A_2, because the criterion value of A_1 for C_1 is equal to $\tilde{h}_{11} = \tilde{H}$ {[0.4, 0.55], [0.7, 0.9]}, and the criterion value of A_2 for C_1 is equal to $\tilde{h}_{21} = \tilde{H}$ {[0.5, 0.6], [0.8, 0.9]}, according to the new ranking method

Table 2.5 Superior-inferior table for six pairs of alternatives and four criteria

	A_1/A_2	A_1/A_3	A_1/A_4	A_2/A_3	A_2/A_4	A_3/A_4
C_1	$_{1/2}\mathrm{I}_1$	$_{1/3}\mathrm{S}_1$	$_{1/4}\mathrm{S}_1$	$_{2/3}\mathrm{S}_1$	$_{2/4}\mathrm{S}_1$	$_{3/4}\mathrm{S}_1$
C_2	$_{1/2}\mathrm{I}_2$	$_{1/3}\mathrm{I}_2$	$_{1/4}\mathrm{I}_2$	$_{2/3}\mathrm{I}_2$	$_{2/4}\mathrm{I}_2$	$_{3/4}\mathrm{S}_2$
C_3	$_{1/2}\mathrm{S}_3$	$_{1/3}\mathrm{S}_3$	$_{1/4}\mathrm{S}_3$	$_{2/3}\mathrm{S}_3$	$_{2/4}\mathrm{S}_3$	$_{3/4}\mathrm{S}_3$
C_4	$_{1/2}\mathrm{S}_4$	$_{1/3}\mathrm{I}_4$	$_{1/4}\mathrm{S}_4$	$_{2/3}\mathrm{I}_4$	$_{2/4}\mathrm{S}_4$	$_{3/4}\mathrm{S}_4$

Note: '*S*' *denotes* "*superior to*", '*I*' *denotes* "*inferior to*", "*E*" *denotes* "*equal to*"

Table 2.6 Hesitant fuzzy heterogeneous decision matrix

	C_1	C_2
A_1	$\tilde{H}$ {[0.4, 0.55], [0.4, 0.55], [0.7, 0.9]}	$H\{0.6, 0.6, 0.8\}$
A_2	$\tilde{H}$ {[0.5, 0.6], [0.5, 0.6], [0.8, 0.9]}	$H\{0.5, 0.8, 0.9\}$
A_3	$\tilde{H}$ {[0.3, 0.4], [0.5, 0.65], [0.8, 0.9]}	$H\{0.6, 0.7, 0.95\}$
A_4	$\tilde{H}$ {[0.4, 0.5], [0.4, 0.5], [0.6, 0.7]}	$H\{0.7, 0.7, 0.9\}$
	C_3	C_4
A_1	$\tilde{H}$ {[0.35, 0.4], [0.5, 0.65], [0.8, 0.95]}	$H\{0.4, 0.5, 0.9\}$
A_2	$\tilde{H}$ {[0.4, 0.5], [0.4, 0.5], [0.65, 0.9]}	$H\{0.4, 0.6, 0.7\}$
A_3	$\tilde{H}$ {[0.3, 0.45], [0.3, 0.45], [0.6, 0.8]}	$H\{0.5, 0.5, 0.8\}$
A_4	$\tilde{H}$ {[0.2, 0.4], [0.2, 0.4], [0.6, 0.7]}	$H\{0.4, 0.4, 0.5\}$

of IVHFEs, it is easy to obtain $Ir_{0.1}(\tilde{h}_{11}) = 1.2369 < Ir_{0.1}(\tilde{h}_{21}) = 1.3706$. Similar logic is used to determine the remaining entries in Table 2.5.

Considering that the numbers of values in different HFEs and IVHFEs in Table 2.4 are different, in order to accurately calculate their distances, we should extend the shorter one until both of them have the same length. According to the regulations mentioned in Definitions 1.3 and 1.10, we assume that the experts are pessimistic in the above decision making problem, and normalize the decision data in Table 2.4 by adding the minimal values as listed in Table 2.6 (Zhang and Xu 2014a).

Then, based on the analysis results in Table 2.5, using Eqs. (2.13) and (2.14) we can calculate the gains and losses of each alternative over the others concerning each criterion, listed in Tables 2.7, 2.8, 2.9 and 2.10 (Zhang and Xu 2014a).

Furthermore, by aggregating the gains and losses of the alternative A_ξ over the alternative A_ζ concerning each criterion C_j using Eq. (2.15), we can obtain the prospect value of each alternative over the others, which are listed in Table 2.11 (Zhang and Xu 2014a).

Thirdly, based on the data in Table 2.11, the overall prospect values of the alternatives are obtained by Eq. (2.16), namely, $\mathscr{Q}^h(A_1) = 0.9390$, $\mathscr{Q}^h(A_2) = 0.9233$, $\mathscr{Q}^h(A_3) = 1.0$, and $\mathscr{Q}^h(A_4) = 0.0$. Finally, according to the overall values, the ranking of the four domestic airlines is determined, i.e., $A_3 \succ A_1 \succ A_2 \succ A_4$. Apparently, A_3 (Mandarin) is the most desirable domestic airline.

Table 2.7 Gains and losses of each alternative over the others regarding the criterion C_1

	A_1	A_2	A_3	A_4
A_1	0	−0.7136	0.1237	0.1198
A_2	0.1070	0	0.1326	0.1456
A_3	−0.8249	−0.8842	0	0.1493
A_4	−0.7989	−0.9710	−0.9953	0

Table 2.8 Gains and losses of each alternative over the others regarding the criterion C_2

	A_1	A_2	A_3	A_4
A_1	0	−0.7521	−0.6452	−0.6325
A_2	0.1881	0	−0.5886	−0.7186
A_3	0.1613	0.1471	0	0.1270
A_4	0.1581	0.1797	−0.5081	0

Table 2.9 Gains and losses of each alternative over the others regarding the criterion C_3

	A_1	A_2	A_3	A_4
A_1	0	0.1944	0.2343	0.2744
A_2	−0.5555	0	0.1663	0.2322
A_3	−0.6693	−0.4753	0	0.1635
A_4	−0.7839	−0.6636	−0.4761	0

Table 2.10 Gains and losses of each alternative over the others regarding the criterion C_4

	A_1	A_2	A_3	A_4
A_1	0	0.1797	−0.5715	0.2440
A_2	−0.7186	0	−0.6325	0.2021
A_3	0.1429	0.1581	0	0.2188
A_4	−0.9758	−0.8082	−0.8752	0

Table 2.11 The prospect value of each alternative over the others

	A_1	A_2	A_3	A_4
A_1	0	−1.0916	−0.8587	0.0057
A_2	−0.9791	0	−0.9222	−0.1387
A_3	−1.1900	−1.0543	0	0.6586
A_4	−2.4005	−2.2631	−2.8457	0

2.4.3 Sensitivity Analysis of Parameters

The sensitivity analysis is usually performed by modifying the parameter θ (i.e., the attenuation factor of the losses). By increasing or decreasing the values of the parameter θ, we recalculate the ranking orders of alternatives.

By changing θ from 1 to 4, we can obtain the change results of ranking orders of alternatives, listed in Table 2.12 (Zhang and Xu 2014a).

From the sensitivity analysis results presented in Table 2.12, we notice that the ranking orders of alternatives are not sensitive to the value of θ. In other words, in

Table 2.12 Ranking orders of alternatives with different values of θ

Different values of θ	Ranking orders of alternatives
$\theta = 1$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\theta = 2$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\theta = 3$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\theta = 4$	$A_3 \succ A_1 \succ A_2 \succ A_4$

Table 2.13 Ranking orders of alternatives with different values of δ

Different values of δ	Ranking orders of alternatives
$\delta = 0.001$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\delta = 0.01$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$\delta = 0.1$	$A_3 \succ A_1 \succ A_2 \succ A_4$

spite of the alteration in the value of the parameter θ, the obtained rankings are usually consistent.

In addition, it is worth pointing out that the above sensitivity analysis is based on $\delta = 0.1$. In the following, we do the sensitivity analysis about the parameter δ. We assume $\theta = 1$, the sensitivity analysis is performed by modifying (increasing or decreasing) the value of δ, and recalculating the ranking orders of alternatives with different values of δ. By changing δ from 0.001 to 0.1, we can obtain the change results of the ranking orders of alternatives. These calculation results are listed in Table 2.13 (Zhang and Xu 2014a).

Based on the results of sensitivity analysis presented in Table 2.13, it is easy to see that the rankings of alternatives are not sensitive to the value of the parameter δ. That is to say, in spite of the alteration in the values of δ, the obtained rankings are often consistent.

2.4.4 Comparative Analysis and Discussions

A comparative study was conducted by Zhang and Xu (2014a) to validate the results of the HF-TODIM method with those from another approach. With the analysis on the same decision making problem mentioned in Sect. 2.4.1, the HF-TOPSIS approach proposed by Xu and Zhang (2013) was selected to facilitate the comparative analysis. However, it is worth pointing out that the HF-TOPSIS approach is just suitable to deal with the MCDM problems with HFEs but fail to handle the above problem where the criteria values are measured in two different formats, i.e., HFEs and IVHFEs. Therefore, Zhang and Xu (2014a) presented a modified HF-TOPSIS approach to tackle appropriately the MCDM problems in which the criteria values take the form of HFEs or IVHFEs and applied it to the decision making problem mentioned in Sect. 2.4.1.

The modified HF-TOPSIS method starts with the determination of the hesitant fuzzy heterogeneous PIS A^+ and the hesitant fuzzy heterogeneous NIS A^-, which are defined as follows:

(1) if $C_j \in \boldsymbol{C}^{HF}$, then $x_{ij} = h_{ij}$ $(i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, j_1\})$. Thus, we have

$$A^{HF+} = \left(H\{\max_{i=1}^m \gamma_{ij}^1, \max_{i=1}^m \gamma_{ij}^2, \ldots, \max_{i=1}^m \gamma_{ij}^{\#h_{ij}}\} | C_j \in \boldsymbol{C}^{HF}\right) \tag{2.17}$$

$$A^{HF-} = \left(H\{\min_{i=1}^m \gamma_{ij}^1, \min_{i=1}^m \gamma_{ij}^2, \ldots, \min_{i=1}^m \gamma_{ij}^{\#h_{ij}}\} | C_j \in \boldsymbol{C}^{HF}\right) \tag{2.18}$$

(2) if $C_j \in \boldsymbol{C}^{IVHF}$, then $x_{ij} = \tilde{h}_{ij}$ $(i \in \{1, 2, \ldots, m\}, j \in \{j_1 + 1, j_1 + 2, \ldots, n\})$, and we have

$$A^{IVHF+} = \left(\tilde{H}\{\max_{i=1}^m \tilde{\gamma}_{ij}^1, \max_{i=1}^m \tilde{\gamma}_{ij}^2, \ldots, \max_{i=1}^m \tilde{\gamma}_{ij}^{\#\tilde{h}_{ij}}\} | C_j \in \boldsymbol{C}^{IVHF}\right) \tag{2.19}$$

$$A^{IVHF-} = \left(\tilde{H}\{\min_{i=1}^m \tilde{\gamma}_{ij}^1, \min_{i=1}^m \tilde{\gamma}_{ij}^2, \ldots, \min_{i=1}^m \tilde{\gamma}_{ij}^{\#\tilde{h}_{ij}}\} | C_j \in \boldsymbol{C}^{IVHF}\right) \tag{2.20}$$

Obviously, $A^+ = A^{HF+} \cup A^{IVHF+}$ and $A^- = A^{HF-} \cup A^{IVHF-}$.

Thus, the corresponding hesitant fuzzy heterogeneous PIS and NIS in the above decision making problem can be obtained by Eqs. (2.17)–(2.20) and are listed in Table 2.14 (Zhang and Xu 2014a).

In order to measure the distance between the alternative A_i and the hesitant fuzzy heterogeneous PIS A^+ as well as the hesitant fuzzy heterogeneous NIS A^-, respectively, we adopt the hesitant fuzzy Euclidean distance proposed by Xu and Xia (2011b) and the interval-valued hesitant fuzzy Euclidean distance proposed by Chen et al. (2013b). Thus, for the criterion C_j, the distances between the alternative A_i and the hesitant fuzzy heterogeneous PIS A^+ as well as the hesitant fuzzy heterogeneous NIS A^-, respectively, are calculated by the following equations (Zhang and Xu 2014a):

Table 2.14 The hesitant fuzzy heterogeneous PIS and NIS

	A^+	A^-
C_1	$\tilde{H}$ {[0.5, 0.6], [0.5, 0.65], [0.8, 0.9]}	$\tilde{H}$ {[0.3, 0.4], [0.4, 0.5], [0.6, 0.7]}
C_2	H {0.7, 0.8, 0.95}	H {0.5, 0.6, 0.8}
C_3	$\tilde{H}$ {[0.4, 0.5], [0.5, 0.65], [0.8, 0.95]}	$\tilde{H}$ {[0.2, 0.4], [0.2, 0.4], [0.6, 0.7]}
C_4	H {0.5, 0.6, 0.9}	H {0.4, 0.4, 0.5}

(1) if $C_j \in \boldsymbol{C}^{HF}$, then we obtain

$$d_{ij}^{+} = d\left(h_{ij}, h_j^{+}\right) = \sqrt{\sum\nolimits_{\lambda=1}^{\#h_{ij}} (\gamma_{ij}^{\lambda} - (\lambda_j^{\lambda})^{+})^2 \Big/ \#h_{ij}} \tag{2.21}$$

and

$$d_{ij}^{-} = d\left(h_{ij}, h_j^{-}\right) = \sqrt{\sum\nolimits_{\lambda=1}^{\#h_{ij}} (\gamma_{ij}^{\lambda} - (\lambda_j^{\lambda})^{-})^2 \Big/ \#h_{ij}} \tag{2.22}$$

(2) if $C_j \in \boldsymbol{C}^{IVHF}$, then we obtain

$$d_{ij}^{+} = d\left(\tilde{h}_{ij}, \tilde{h}_j^{+}\right) = \sqrt{\sum\nolimits_{\lambda=1}^{\#\tilde{h}_{ij}} \begin{pmatrix} (\tilde{\gamma}_{ij}^{\lambda L} - (\tilde{\gamma}_j^{\lambda L})^{+})^2 \\ + (\tilde{\gamma}_{ij}^{\lambda U} - (\tilde{\gamma}_j^{\lambda U})^{+})^2 \end{pmatrix} \Big/ (2\#\tilde{h}_{ij})} \tag{2.23}$$

and

$$d_{ij}^{-} = d\left(\tilde{h}_{ij}, \tilde{h}_j^{-}\right) = \sqrt{\sum\nolimits_{\lambda=1}^{\#\tilde{h}_{ij}} \begin{pmatrix} (\tilde{\gamma}_{ij}^{\lambda L} - (\tilde{\gamma}_j^{\lambda L})^{-})^2 \\ + (\tilde{\gamma}_{ij}^{\lambda U} - (\tilde{\gamma}_j^{\lambda U})^{-})^2 \end{pmatrix} \Big/ (2\#\tilde{h}_{ij})} \tag{2.24}$$

For all criteria, the weighted distances between the alternative A_i $(i \in \{1, 2, \ldots, m\})$ and the hesitant fuzzy heterogeneous PIS A^{+} as well as the hesitant fuzzy heterogeneous NIS A^{-}, respectively, are derived from the following equations:

$$\begin{aligned} d^{hh}(A_i, A^{+}) = {} & \sum_{j=1}^{j_1} w_j \sqrt{\sum\nolimits_{\lambda=1}^{\#h_{ij}} (\gamma_{ij}^{\lambda} - (\lambda_j^{\lambda})^{+})^2 \Big/ \#h_{ij}} \\ & + \sum_{j=j_1+1}^{n} w_j \sqrt{\sum\nolimits_{\lambda=1}^{\#\tilde{h}_{ij}} \begin{pmatrix} (\tilde{\gamma}_{ij}^{\lambda L} - (\tilde{\gamma}_j^{\lambda L})^{+})^2 \\ + (\tilde{\gamma}_{ij}^{\lambda U} - (\tilde{\gamma}_j^{\lambda U})^{+})^2 \end{pmatrix} \Big/ (2\#\tilde{h}_{ij})} \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} d^{hh}(A_i, A^{-}) = {} & \sum_{j=1}^{j_1} w_j \sqrt{\sum\nolimits_{\lambda=1}^{\#h_{ij}} (\gamma_{ij}^{\lambda} - (\lambda_j^{\lambda})^{-})^2 \Big/ \#h_{ij}} \\ & + \sum_{j=j_1+1}^{n} w_j \sqrt{\sum\nolimits_{\lambda=1}^{\#\tilde{h}_{ij}} \begin{pmatrix} (\tilde{\gamma}_{ij}^{\lambda L} - (\tilde{\gamma}_j^{\lambda L})^{-})^2 \\ + (\tilde{\gamma}_{ij}^{\lambda U} - (\tilde{\gamma}_j^{\lambda U})^{-})^2 \end{pmatrix} \Big/ (2\#\tilde{h}_{ij})} \end{aligned} \tag{2.26}$$

Table 2.15 Closeness indices and the ranking of alternatives

	$d^{hh}(A_i, A^+)$	$d^{hh}(A_i, A^-)$	$CI^{hh}(A_i)$	Ranking
$\mathbf{A_1}$	**0.0953**	**0.1235**	**0.5645**	**1**
A_2	0.0978	0.1179	0.5467	2
A_3	0.1439	0.0975	0.4039	4
A_4	0.1392	0.1436	0.5076	3

The relative closeness index of the alternative A_i to the hesitant fuzzy heterogeneous PIS A^+ is defined as the following formula:

$$CI^{hh}(A_i) = \frac{d^{hh}(A_i, A^-)}{d^{hh}(A_i, A^+) + d^{hh}(A_i, A^-)} \tag{2.27}$$

Using Eqs. (2.25)–(2.27), the distances $d^{hh}(A_i, A^+)$ and $d^{hh}(A_i, A^-)$, and the relative closeness index $CI^{hh}(A_i)$ in the above problem can be obtained, respectively. The results are presented in Table 2.15 (Zhang and Xu 2014a), together with the corresponding rankings on the basis of $CI^{hh}(A_i)$.

It is easy to see that the optimal order for these four major domestic airlines is $A_1 \succ A_2 \succ A_4 \succ A_3$, and thus the UNI Air ($A_1$) is the most desirable domestic airline.

To provide a better view of the comparison results, we put the results of the ranking of alternatives obtained by the HF-TODIM and the modified HF-TOPSIS methods into Fig. 2.1 (Zhang and Xu 2014a).

From Fig. 2.1, it is easily observed that the ranking orders of alternatives obtained by these two techniques are remarkable different. Using the HF-TODIM technique, the best suitable alternative in the above decision making problem is A_3, while using the modified HF-TOPSIS technique the best alternative is A_1. The main reason is that the HF-TODIM technique, which can take into account the decision maker's behavior in the decision making process, can yield more persuasive results which is more in line with the decision maker's actual experience, while the modified HF-TOPSIS technique, which is based on the strict assumption regarding

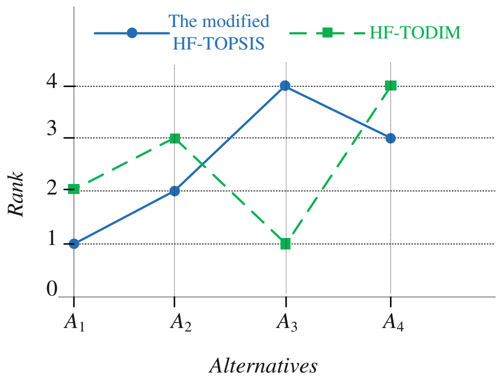

Fig. 2.1 The pictorial representation of the HF-TOPSIS and HF-TODIM rankings

complete rationality of the decision maker, fails to take the decision maker's psychological behavior into account. Obviously, for the practical decision making problems in which the decision maker's psychological behavior should be taken fully into account, compared with the modified HF-TOPSIS approach, the HF-TODIM method can obtain a better final decision result since it effectively captures the decision maker's psychological behavior.

2.5 Extension of the Developed Approach for Handling the MCGDM Problems with HTrFNs

2.5.1 The Concept of HTrFN

In this section, the concept of TrFN is first reviewed. Based on the concepts and operational laws of TrFNs and HFEs, Zhang et al. (2016) presented a new concept of the HTrFN which is good enough to represent the vagueness of the HFLTS.

Definition 2.9 (Zadeh 1975). A fuzzy number $\tilde{\alpha} = Tr(a, b, c, d)$ is said to be a TrFN if its membership function is given as follows:

$$\mu_{\tilde{\alpha}}(t) = \begin{cases} (t-a)/(b-a), & (a \leq t < b) \\ 1, & (b \leq t \leq c) \\ (d-t)/(d-c), & (c < t \leq d) \\ 0, & otherwise \end{cases} \tag{2.28}$$

where the closed interval $[b, c]$, a and d are the mode, low and upper limits of $\tilde{\alpha}$, respectively.

Remark 2.3 (Zhang et al. 2016). It is noted that a TrFN $\tilde{\alpha} = Tr(a, b, c, d)$ is reduced to a TFN if $b = c$. A TrFN $\tilde{\alpha} = Tr(a, b, c, d)$ is reduced to a real number if $a = b = c = d$. A TrFN $\tilde{\alpha} = Tr(a, b, c, d)$ is the normalized TrFN if $a \geq 0$ and $d \leq 1$. Thus, the TrFN $\tilde{1} = Tr(1, 1, 1, 1)$ is the maximal normalized TrFN which is called the positive ideal TrFN, while the TrFN $\tilde{0} = Tr(0, 0, 0, 0)$ is the minimal normalized TrFN which is called the negative ideal TrFN. Usually, the TrFN is well enough to capture the vagueness of linguistic terms, and the relationships between linguistic term set with seven-point rating scale and the TrFNs are shown in Table 2.16.

Definition 2.10 (Zhang et al. 2016). Let T be a fixed set, a HTrFS $\mathscr{H}$ on T is defined as:

$$\mathscr{H} = \{ < t, h_{\mathscr{H}}(t) > | t \in T \} \tag{2.29}$$

where $h_{\mathscr{H}}(t)$ is a set of different normalized TrFNs, representing the possible membership degrees of the element $t \in T$ to $\mathscr{H}$.

Table 2.16 Linguistic terms and the corresponding TrFNs

Ratings	Abbreviation	TrFNs
l_0: Very poor	VP	Tr(0.0, 0.0, 0.1, 0.2)
l_1: Poor	P	Tr(0.1, 0.2, 0.2, 0.3)
l_2: Medium poor	MP	Tr(0.2, 0.3, 0.4, 0.5)
l_3: Fair	F	Tr(0.4, 0.5, 0.5, 0.6)
l_4: Medium good	MG	Tr(0.5, 0.6, 0.7, 0.8)
l_5: Good	G	Tr(0.7, 0.8, 0.8, 0.9)
l_6: Very good	VG	Tr(0.8, 0.9, 1.0, 1.0)

For convenience, $h_{\mathscr{H}}(t)$ is called a HTrFN denoted by $\hbar = \{\tilde{\alpha}^1, \tilde{\alpha}^2, \ldots, \tilde{\alpha}^{\#\hbar}\}$ where the $\tilde{\alpha}^\lambda = Tr(a^\lambda, b^\lambda, c^\lambda, d^\lambda)(\lambda = 1, 2, \ldots, \#\hbar)$ is a normalized TrFN and $\#\hbar$ is the number of all TrFNs in $\hbar$. If $\#\hbar = 1$, then the HTrFN $\hbar$ is reduced a TrFN. If $b^\lambda = c^\lambda(\lambda = 1, 2, \ldots, \#\hbar)$, the HTrFN $\hbar$ is reduced to a hesitant triangular fuzzy number (Zhao et al. 2014).

Example 2.5 (Zhang and Liu 2016). Let $T = \{t_1, t_2, t_3\}$, and let

$$\begin{aligned}
\hbar_{\mathscr{H}}(t_1) &= \{Tr(0.2, 0.3, 0.4, 0.5), Tr(0.3, 0.4, 0.5, 0.6), Tr(0.35, 0.4, 0.45, 0.5)\}, \\
\hbar_{\mathscr{H}}(t_2) &= \{Tr(0.1, 0.2, 0.3, 0.5), Tr(0.3, 0.4, 0.4, 0.6)\}, \\
\hbar_{\mathscr{H}}(t_3) &= \{Tr(0.2, 0.4, 0.5, 0.6), Tr(0.1, 0.3, 0.4, 0.6)\}
\end{aligned}$$

be three HTrFNs of $t_i\ (i = 1, 2, 3)$ to a set $h_{\mathscr{H}}(t)$. Thus, $\mathscr{H}$ can be called an HTrFS which is denoted as:

$$\mathscr{H} = \left\{ \begin{array}{l} \langle t_1, \{Tr(0.2, 0.3, 0.4, 0.5), Tr(0.3, 0.4, 0.5, 0.6), Tr(0.35, 0.4, 0.45, 0.5)\}\rangle, \\ \langle t_2, \{Tr(0.1, 0.2, 0.3, 0.5), Tr(0.3, 0.4, 0.4, 0.6)\}\rangle, \\ \langle t_3, \{Tr(0.2, 0.4, 0.5, 0.6), Tr(0.1, 0.3, 0.4, 0.6)\}\rangle \end{array} \right\}.$$

According to the definition of HTrFNs, it is easy to note that HTrFNs are suitable to capture and represent the uncertainty and vagueness of the HFLTS (please see Chap. 6 for the concept of HFLTS).

Example 2.6 (Zhang et al. 2016). Let $L = \{l_0, l_1, l_2, l_3, l_4, l_5, l_6\}$ be a linguistic term set, the linguistic terms and the corresponding TrFNs are shown in Table 2.16. Given two HFLTSs $hl_1 = HL\{l_0, l_1, l_2\}$ and $hl_2 = HL\{l_1, l_2, l_3\}$, their semantics can be captured by the following two HTrFNs (see Fig. 2.2):

$$\begin{aligned}
\hbar_1 &= \left\{ \begin{array}{l} Tr(0.0, 0.0, 0.1, 0.2), Tr(0.1, 0.2, 0.2, 0.3), \\ Tr(0.2, 0.3, 0.4, 0.5) \end{array} \right\} \Leftrightarrow hl_1; \\
\hbar_2 &= \left\{ \begin{array}{l} Tr(0.1, 0.2, 0.2, 0.3), Tr(0.2, 0.3, 0.4, 0.5), \\ Tr(0.4, 0.5, 0.5, 0.6) \end{array} \right\} \Leftrightarrow hl_2;
\end{aligned}$$

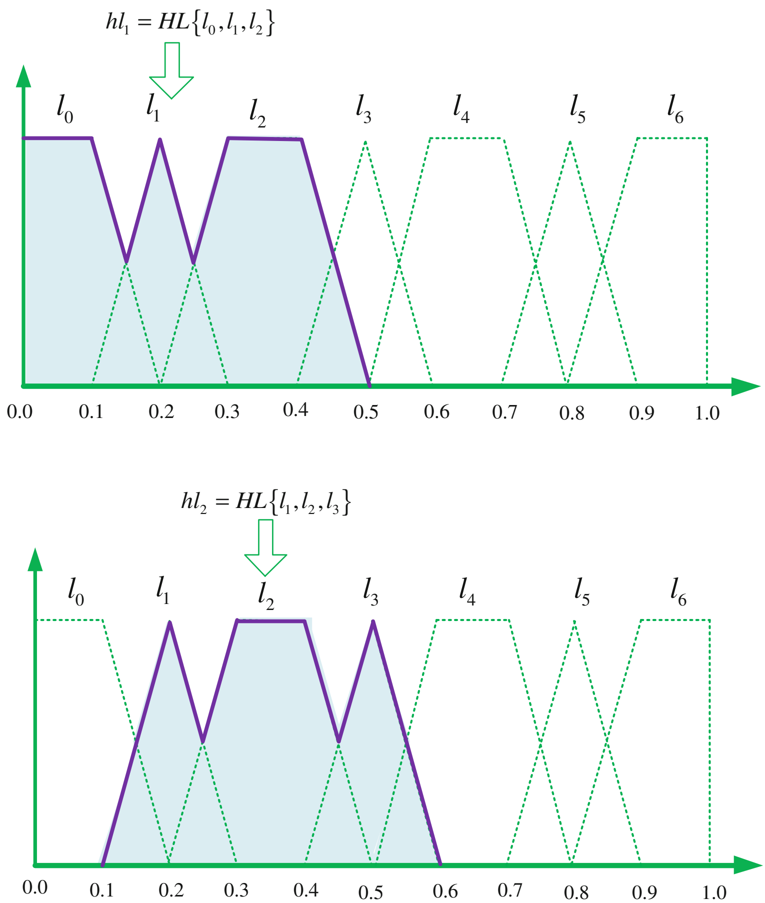

Fig. 2.2 The HFLTSs and the corresponding HTrFNs

As far as we know, Liu and Rodríguez (2014) suggested the use of TrFNs to caputre the vagueness of HFLTSs, and the TrFNs envelopes of hl_1 and hl_2 are obtained as below:

$$env_F(hl_1) = Tr(0, 0, 0.239, 0.5), \quad env_F(hl_2) = Tr(0.1, 0.28, 0.42, 0.6).$$

To provide a better view of the comparison results, we put the results obtained by Zhang et al. (2016), Liu and Rodríguez (2014) in Fig. 2.3.

It can be easily seen from Fig. 2.3 that the HTrFNs take the semantic of each linguistic term of the HFLTSs into account. While in Liu and Rodríguez (2014), the TrFNs were used to represent the semantic of HFLTS, which is relative complex because the TrFNs are obtained by aggregating the fuzzy membership functions of the linguistic terms of the HFLTS using the OWA aggregation operator.

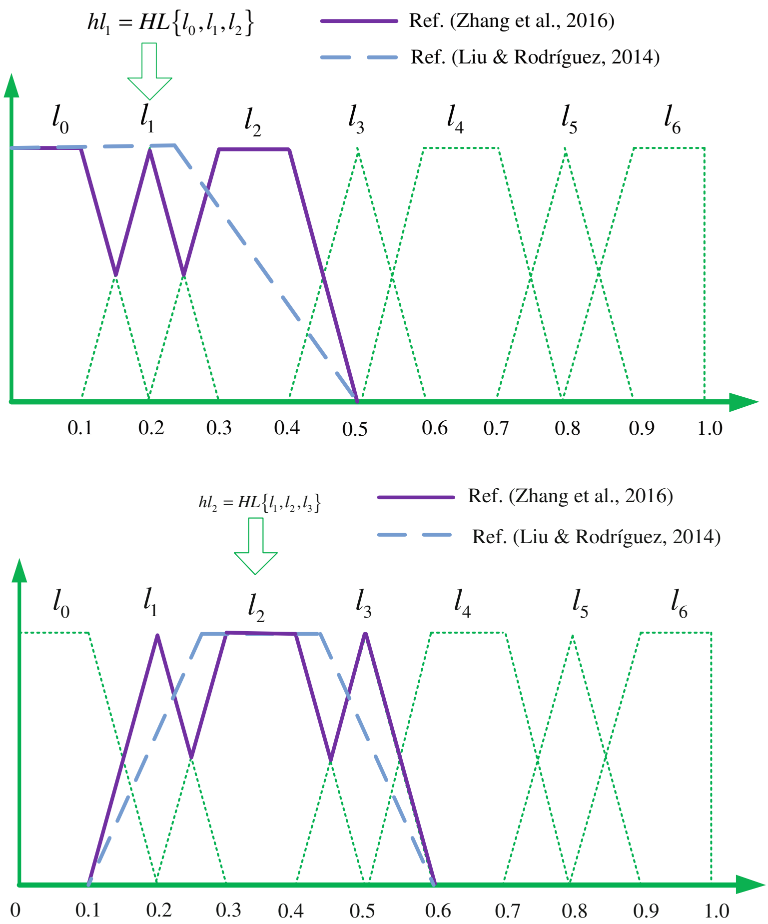

Fig. 2.3 The results obtained by two different methods

Inspired by the operations on HFEs and TrFNs, the basic operations of HTrFNs are introduced as follows:

Definition 2.11 (Zhang et al. 2016). Let $\hbar$, $\hbar_1$ and $\hbar_2$ be three HTrFNs, the basic operations of HTrFNs are defined as:

(1) $\theta\hbar = \cup_{\tilde{\alpha}\in\hbar}\left\{Tr\begin{pmatrix}1-(1-a)^{\theta}, 1-(1-b)^{\theta}, \\ 1-(1-c)^{\theta}, 1-(1-d)^{\theta}\end{pmatrix}\right\}$ $(\theta > 0)$;

(2) $\hbar^{\theta} = \cup_{\tilde{\alpha}\in\hbar}\left\{Tr\left(a^{\theta}, b^{\theta}, c^{\theta}, d^{\theta}\right)\right\}$ $(\theta > 0)$;

(3) $\hbar_1 \oplus \hbar_2 = \cup_{\tilde{\alpha}_1\in h_1, \tilde{\alpha}_2\in\hbar_2}\left\{Tr\begin{bmatrix}a_1 + a_2 - a_1 \times a_2,\ b_1 + b_2 - b_1 \times b_2, \\ c_1 + c_2 - c_1 \times c_2,\ d_1 + d_2 - d_1 \times d_2\end{bmatrix}\right\}$;

(4) $\hbar_1 \otimes \hbar_2 = \cup_{\tilde{\alpha}_1\in\hbar_1, \tilde{\alpha}_2\in\hbar_2}\{Tr(a_1 \times a_2,\ b_1 \times b_2,\ c_1 \times c_2,\ d_1 \times d_2)\}$.

Proposition 2.4 (Zhang et al. 2016). Let $\hbar$, $\hbar_1$ and $\hbar_2$ be three HTrFNs, then

(1) $\hbar_1 \oplus \hbar_2 = \hbar_2 \oplus \hbar_1$;
(2) $\hbar_1 \otimes h_2 = \hbar_2 \otimes \hbar_1$;
(3) $\theta(\hbar_1 \oplus \hbar_2) = \theta\hbar_1 \oplus \theta\hbar_2$ $(\theta > 0)$;
(4) $(\hbar_1 \otimes \hbar_2)^{\theta} = \hbar_1^{\theta} \otimes \hbar_2^{\theta}$ $(\theta > 0)$;
(5) $(\theta_1 + \theta_2)\hbar = \theta_1\hbar \oplus \theta_2\hbar$ $(\theta_1, \theta_2 > 0)$;
(6) $\hbar^{(\theta_1 + \theta_2)} = \hbar^{\theta_1} \otimes h^{\theta_2}$ $(\theta_1, \theta_2 > 0)$.

According to Definition 2.11, it is not hard to obtain the conclusions in Proposition 2.4 (Proof is omitted).

It is worthwhile to point out that the number of TrFNs in different HTrFNs may be different. In such cases, we should extend the shorter one until both of them have the same length when we compare them. To extend the shorter one, the best way is to add some TrFNs in it. Inspired by the similar approaches in the references (Liu and Wang 2007; Xu and Xia 2011a; Xu and Zhang 2013), we extend the shorter one by adding the TrFN in it which mainly depends on the DMs' risk preferences. The optimists anticipate the desirable outcomes and may add the maximum TrFN, while the pessimists expect the unfavorable outcomes and may add the minimum TrFN. Here we employ the sign distance method proposed by Abbasbandy and Asady (2006) to compare the magnitude of TrFNs and further to identify the maximum TrFN or minimum TrFN.

Definition 2.12 (Zhang and Liu 2016). Let $\hbar_j = \{\tilde{\alpha}_j^1, \tilde{\alpha}_j^2, \ldots, \tilde{\alpha}_j^{\#\hbar_j}\}$ $(j = 1, 2)$ be two HTrFNs, and $\#\hbar_1 = \#\hbar_2 = \#\hbar$, then a nature quasi-ordering on HTrFNs is defined as follows.

$$\hbar_1 \le \hbar_2 \text{ if and only if } \tilde{\alpha}_1^{\lambda} \le \tilde{\alpha}_2^{\lambda} \quad (\lambda = 1, 2, \ldots, \#\hbar).$$

It is easily observed from Definition 2.12 that the HTrFN $\hbar^+ = \{Tr(1,1,1,1), \ldots, Tr(1,1,1,1)\}$ is the biggest HTrFN and the HTrFN $\hbar^- = \{Tr(0,0,0,0), \ldots, Tr(0,0,0,0)\}$ is the smallest HTrFN, respectively. We also call $\hbar^+$ the positive ideal HTrFN and $\hbar^-$ the negative ideal HTrFN, respectively.

Zhang et al. (2016) developed the hesitant trapezoidal Hamming distance for HTrFNs as follows:

Definition 2.13 (Zhang et al. 2016). Given two HTrFNs $\hbar_j = \{\tilde{\alpha}_j^1, \tilde{\alpha}_j^2, \ldots, \tilde{\alpha}_j^{\#\hbar_j}\}$ $(j = 1, 2)$ with $\#\hbar_1 = \#\hbar_2 = \#\hbar$, the hesitant trapezoidal Hamming distance between them is defined as follows:

$$d(\hbar_1, \hbar_2) = \frac{1}{6\#\hbar} \sum_{\lambda=1}^{\#\hbar} \begin{pmatrix} |a_1^{\lambda} - a_2^{\lambda}| + 2|b_1^{\lambda} - b_2^{\lambda}| \\ + 2|c_1^{\lambda} - c_2^{\lambda}| + |d_1^{\lambda} - d_2^{\lambda}| \end{pmatrix} \tag{2.30}$$

Example 2.7 (Zhang and Liu 2016). For two HTrFNs

$$\hbar_1 = \{Tr(0.1, 0.2, 0.3, 0.5), Tr(0.3, 0.4, 0.4, 0.6)\},$$
$$\hbar_2 = \{Tr(0.1, 0.3, 0.4, 0.6), Tr(0.2, 0.4, 0.5, 0.6)\},$$

the following result based on Definition 2.13 is obtained:

$$\begin{aligned} d(\hbar_1, \hbar_2) &= \frac{1}{6 \times 2}\begin{pmatrix} |0.1 - 0.1| + 2|0.2 - 0.3| + 2|0.3 - 0.4| + |0.5 - 0.6| + \\ |0.3 - 0.2| + 2|0.4 - 0.4| + 2|0.4 - 0.5| + |0.6 - 0.6| \end{pmatrix} \\ &= 0.0667. \end{aligned}$$

Then, the distance between the HTrFN $\hbar = \{\tilde{\alpha}^1, \tilde{\alpha}^2, \ldots, \tilde{\alpha}^{\#\hbar}\}$ and the positive ideal HTrFN $\hbar^+$ can be calculated as follows:

$$\begin{aligned} d(\hbar, \hbar^+) &= \frac{1}{6\#\hbar}\sum_{\lambda=1}^{\#\hbar}(1 - a^\lambda + 2(1 - b^\lambda) + 2(1 - c^\lambda) + 1 - d^\lambda) \\ &= \frac{1}{6\#\hbar}\sum_{\lambda=1}^{\#\hbar}(6 - a^\lambda - 2b^\lambda - 2c^\lambda - d^\lambda) \end{aligned} \tag{2.31}$$

and the distance between the HTrFN $\hbar$ and the negative ideal HTrFN $\hbar^-$ can be computed as:

$$d(\hbar, \hbar^-) = \frac{1}{6\#\hbar}\sum_{\lambda=1}^{\#\hbar}(a^\lambda + 2b^\lambda + 2c^\lambda + d^\lambda) \tag{2.32}$$

In general, the smaller the distance $d(\hbar, \hbar^+)$ is, the bigger the HTrFN $\hbar$ is; and the larger the distance $d(\hbar, \hbar^-)$ is, the bigger the HTrFN $\hbar$ is. Motivated by the idea of TOPSIS method (Hwang and Yoon 1981), the closeness index for the HTrFN is developed as follows.

Definition 2.14 (Zhang and Liu 2016). Let $\hbar = \{\tilde{\alpha}^1, \tilde{\alpha}^2, \ldots, \tilde{\alpha}^{\#\hbar}\}$ be a HTrFN, $\hbar^+$ be the positive ideal HTrFN and $\hbar^-$ be the negative ideal HTrFN; then the closeness index of $\hbar$ can be defined as:

$$\begin{aligned} \varphi(\hbar) &= \frac{d(\hbar, \hbar^-)}{d(\hbar, \hbar^-) + d(\hbar, \hbar^+)} = \frac{\sum_{\lambda=1}^{\#\hbar}(a^\lambda + 2b^\lambda + 2c^\lambda + d^\lambda)}{\sum_{\lambda=1}^{\#\hbar}(6 - a^\lambda - 2b^\lambda - 2c^\lambda - d^\lambda) + \sum_{\lambda=1}^{\#\hbar}(a^\lambda + 2b^\lambda + 2c^\lambda + d^\lambda)} \\ &= \frac{1}{6\#\hbar}\sum_{\lambda=1}^{\#\hbar}(a^\lambda + 2b^\lambda + 2c^\lambda + d^\lambda) \end{aligned} \tag{2.33}$$

Obviously, if $\hbar = \hbar^-$, then $\varphi(\hbar) = 0$; while if $\hbar = \hbar^+$, then $\varphi(\hbar) = 1$.

Based on the closeness indices of HTrFNs, a comparison law for HTrFNs is introduced.

Definition 2.15 (Zhang and Liu 2016). Given two HTrFNs $\hbar_j = \{\tilde{\alpha}_j^1, \tilde{\alpha}_j^2, \ldots, \tilde{\alpha}_j^{\#\hbar_j}\}$ $(j = 1, 2)$, $\varphi(\hbar_1)$ and $\varphi(\hbar_2)$ be the closeness indices of $\hbar_1$ and $\hbar_2$, respectively, then

(1) if $\varphi(\hbar_1) < \varphi(\hbar_2)$, then $\hbar_1 \prec \hbar_2$;
(2) if $\varphi(\hbar_1) > \varphi(\hbar_2)$, then $\hbar_1 \succ \hbar_2$;
(3) if $\varphi(\hbar_1) = \varphi(\hbar_2)$, then $\hbar_1 \sim \hbar_2$.

Example 2.8 (Zhang and Liu 2016). For two HTrFNs

$$\hbar_1 = \{Tr(0.1, 0.2, 0.3, 0.5), Tr(0.3, 0.4, 0.4, 0.6)\},$$
$$\hbar_2 = \{Tr(0.1, 0.3, 0.4, 0.6), Tr(0.2, 0.4, 0.5, 0.6)\},$$

the following result based on Definition 2.14 is obtained:

$$\varphi(\hbar_1) = \frac{1}{12}(0.1 + 0.4 + 0.6 + 0.5 + 0.3 + 0.8 + 0.8 + 0.6) = 0.2563,$$
$$\varphi(\hbar_2) = \frac{1}{12}(0.1 + 0.6 + 0.8 + 0.6 + 0.2 + 0.8 + 1.0 + 0.6) = 0.3912.$$

According to Definition 2.15, it is observed that $\varphi(h_1) < \varphi(\hbar_2)$, i.e., $\hbar_1 \prec \hbar_2$.

2.5.2 Hesitant Trapezoidal Fuzzy TODIM Decision Analysis Method

Consider a decision environment based on HTrFNs for the MCGDM problems in which the criteria values of alternatives take the form of comparative linguistic expressions (please see Chap. 6 for the concept of comparative linguistic expressions). Let $\boldsymbol{A} = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of m $(m \geq 2)$ feasible alternatives, $\boldsymbol{C} = \{C_1, C_2, \ldots, C_n\}$ be a finite set of criteria, and $\boldsymbol{E} = \{e_1, e_2, \ldots, e_g\}$ be a group of experts. The criteria values of the alternative $A_i \in \boldsymbol{A}$ with respect to the criterion $C_j \in \boldsymbol{C}$ provided by the expert $e_k \in \boldsymbol{E}$ can be represented by comparative linguistic expressions ll_{ij}^k. Usually, the semantics of ll_{ij}^k can be captured by a HTrFN $\hbar_{ij}^k = \{\tilde{\alpha}_{ij}^{k(1)}, \tilde{\alpha}_{ij}^{k(2)}, \ldots, \tilde{\alpha}_{ij}^{k(\#\hbar_{ij}^k)}\}$. Thus, the MCGDM problem is expressed in the matrix format $R^k = (\hbar_{ij}^k)_{m \times n}$ $(e_k \in \boldsymbol{E})$. We also denote the weighting vector of criteria by $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^T$, where w_j is the weight of the criterion C_j, satisfying the normalization condition: $\sum_{j=1}^n w_j = 1$ and $w_j \geq 0$. Meanwhile, we denote the weighting vector of experts by $\boldsymbol{\varpi} = (\varpi_1, \varpi_2, \ldots, \varpi_g)^T$, where ϖ_k is the weight of the expert e_k, satisfying the normalization condition: $\sum_{k=1}^g \varpi_k = 1$ and $\varpi_k \geq 0$. In this chapter, the weights of criteria are completely known in advance and the weights of experts are completely unknown or partially known. Let Γ be a set of the known weight information of experts, and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$ can be constructed by the following forms (Zhang and Xu 2014b, 2015):

(1) a weak ranking: $\Gamma_1 = \{\varpi_i \geq \varpi_j\}$;
(2) a strict ranking: $\Gamma_2 = \{\varpi_i - \varpi_j \geq \kappa_{ij}\}$ $(\kappa_{ij} > 0)$;
(3) a ranking of differences: $\Gamma_3 = \{\varpi_i - \varpi_j \geq \varpi_\xi - \varpi_\zeta\}$ $(i \neq j \neq \xi \neq \zeta)$;
(4) a ranking with multiples: $\Gamma_4 = \{\varpi_i \geq \kappa_{ij}\varpi_j\}$ $(0 \leq \kappa_{ij} \leq 1)$;
(5) an interval form: $\Gamma_5 = \{\kappa_i \leq \varpi_i \leq \kappa_i + \varepsilon_i\}$ $(0 \leq \kappa_i \leq \kappa_i + \varepsilon_i \leq 1)$.

To deal effectively with the above MCGDM problem, Zhang and Liu (2016) developed a hesitant trapezoidal fuzzy TODIM method. The focus of the proposed method is to measure the dominance degree of each alternative over the others by constructing the prospect value function based on prospect theory. For this purpose, we need to identify the reference criterion and calculate the relative weight of each criterion to the reference criterion. According to the idea of the classical TODIM, the criterion with the highest weight is usually regarded as the reference criterion C_R, namely,

$$C_R = \{C_j : \max_{j=1}^{n} w_j\} \tag{2.34}$$

Then, the relative weight w_{jR} of the criterion $C_j \in \boldsymbol{C}$ to the reference criterion C_R can be obtained by the following equation:

$$w_{jR} = w_j / w_R, \quad j \in \{1, 2, \ldots, n\} \tag{2.35}$$

where w_R is the weight of the reference criterion C_R.

Furthermore, by employing the closeness index-based ranking method of HTrFNs we compare with the magnitude of the criteria values of alternatives with respect to each criterion. Then, for the expert $e_k \in \boldsymbol{E}$, the dominance value of the alternative $A_\xi \in \boldsymbol{A}$ over the alternative $A_\zeta \in \boldsymbol{A}$ concerning the criterion $C_j \in \boldsymbol{C}$ can be calculated by using the following expression (Zhang and Liu 2016):

$$\mathscr{D}_j^k(A_\xi, A_\zeta) = \begin{cases} \sqrt{\dfrac{w_{jR}(\varphi(\hslash_{\xi j}^k) - \varphi(\hslash_{\zeta j}^k))}{\sum_{j=1}^{n} w_{jR}}}, & \text{if } \varphi(\hslash_{\xi j}^k) - \varphi(\hslash_{\zeta j}^k) > 0 \\ 0, & \text{if } \varphi(\hslash_{\xi j}^k) - \varphi(\hslash_{\zeta j}^k) = 0 \\ -\dfrac{1}{\theta}\sqrt{\dfrac{(\sum_{j=1}^{n} w_{jR})(\varphi(\hslash_{\zeta j}^k) - \varphi(\hslash_{\xi j}^k))}{w_{jR}}}, & \text{if } \varphi(\hslash_{\xi j}^k) - \varphi(\hslash_{\zeta j}^k) < 0 \end{cases} \tag{2.36}$$

where $\varphi(\hslash_{\xi j}^k)$ and $\varphi(\hslash_{\zeta j}^k)$ are respectively the closeness indices of the criteria values $\hslash_{\xi j}^k$ and $\hslash_{\zeta j}^k$, and the parameter $\theta \in [1, 10]$ represents the attenuation factor of the losses.

From Eq. (2.36), it is easily observed that:

(1) if $\varphi(\hslash_{\xi j}^k) - \varphi(\hslash_{\zeta j}^k) > 0$, then $\mathscr{D}_j^k(A_\xi, A_\zeta)$ represents a gain;
(2) if $\varphi(\hslash_{\xi j}^k) - \varphi(\hslash_{\zeta j}^k) = 0$, then $\mathscr{D}_j^k(A_\xi, A_\zeta)$ represents a nil;
(3) if $\varphi(\hslash_{\xi j}^k) - \varphi(\hslash_{\zeta j}^k) < 0$, then $\mathscr{D}_j^k(A_\xi, A_\zeta)$ represents a loss.

For the expert $e_k \in \boldsymbol{E}$, the collective dominance value of the alternative $A_\xi \in \boldsymbol{A}$ over the alternative $A_\zeta \in \boldsymbol{A}$ can be obtained as follows (Zhang and Liu 2016):

$$\mathscr{H}^k(A_\xi, A_\zeta) = \sum_{j=1}^{n} \mathscr{D}_j^k(A_\xi, A_\zeta) \tag{2.37}$$

The overall dominance value of the alternative $A_\xi \in \boldsymbol{A}$ for the expert $e_k \in \boldsymbol{E}$ can be obtained by the following equation and listed in Table 2.17 (Zhang and Liu 2016):

$$\mathscr{Q}^k(A_\xi) = \frac{\sum_{\zeta=1}^m \mathscr{H}^k(A_\xi, A_\zeta) - \min_{\xi=1}^m \left\{ \sum_{\zeta=1}^m \mathscr{H}^k(A_\xi, A_\zeta) \right\}}{\max_{\xi=1}^m \left\{ \sum_{\zeta=1}^m \mathscr{H}^k(A_\xi, A_\zeta) \right\} - \min_{\xi=1}^m \left\{ \sum_{\zeta=1}^m \mathscr{H}^k(A_\xi, A_\zeta) \right\}} \tag{2.38}$$

After obtaining the overall dominance value $\mathscr{Q}^k(A_\xi)$, we need to determine the overall dominance value for the group which is represented by $\mathscr{Q}^*(A_\xi)$ $(\xi \in \{1, 2, \ldots, m\})$. Based on the decision data in Table 2.17, we further establish a nonlinear programming model to calculate $\mathscr{Q}^*(A_\xi)$ $(\xi \in \{1, 2, \ldots, m\})$ for the group as follows (Zhang and Liu 2016):

$$\begin{cases} \min \quad Z = \sum\limits_{\xi=1}^{m} \sum\limits_{k=1}^{g} \varpi_k \left| \mathscr{Q}^k(A_\xi) - \mathscr{Q}^*(A_\xi) \right| \\ s.t. \quad \sum\limits_{k=1}^{g} \varpi_k = 1, \ \varpi_k \geq 0, \ \ k \in \{1, 2, \ldots, g\} \end{cases} \tag{MOD-2.1}$$

To solve the model (MOD-2.1), let

$$\eta_i^k = \frac{1}{2}\left(\left| \mathscr{Q}^k(A_\xi) - \mathscr{Q}^*(A_\xi) \right| + \left(\mathscr{Q}^k(A_\xi) - \mathscr{Q}^*(A_\xi) \right) \right) \tag{2.39}$$

and

$$\rho_i^k = \frac{1}{2}\left(\left| \mathscr{Q}^k(A_\xi) - \mathscr{Q}^*(A_\xi) \right| - \left(\mathscr{Q}^k(A_\xi) - \mathscr{Q}^*(A_\xi) \right) \right) \tag{2.40}$$

Then, the optimal model (MOD-2.1) is transformed into the following optimal model (Zhang and Liu 2016):

Table 2.17 The overall dominance values of alternatives for each expert

Alternatives	Experts			
	e_1	e_2	...	e_g
A_1	$\mathscr{Q}^1(A_1)$	$\mathscr{Q}^2(A_1)$	…	$\mathscr{Q}^g(A_1)$
A_2	$\mathscr{Q}^1(A_2)$	$\mathscr{Q}^2(A_2)$	…	$\mathscr{Q}^g(A_2)$
...	…	…	…	…
A_m	$\mathscr{Q}^1(A_m)$	$\mathscr{Q}^2(A_m)$	…	$\mathscr{Q}^g(A_m)$

$$\begin{cases} \min \quad Z = \sum_{i=1}^{m} \sum_{k=1}^{g} \varpi_k (\eta_i^k + \rho_i^k) \\ s.t. \\ \qquad \mathcal{Q}^k(A_\xi) - \mathcal{Q}^*(A_\xi) - \eta_i^k + \rho_i^k = 0 \quad \begin{pmatrix} i \in \{1,2,\ldots,m\}, \\ k \in \{1,2,\ldots,g\} \end{pmatrix} \\ \qquad \eta_i^k \geq 0, \quad \rho_i^k \geq 0, \quad \eta_i^k \rho_i^k = 0 \quad \begin{pmatrix} i \in \{1,2,\ldots,m\}, \\ k \in \{1,2,\ldots,g\} \end{pmatrix} \\ \qquad \sum_{k=1}^{g} \varpi_k = 1, \ \varpi_k \geq 0, \quad k \in \{1,2,\ldots,g\} \end{cases} \tag{MOD-2.2}$$

It is observed that the model (MOD-2.2) is a linear programming model and can be easily executed by using the MATLAB 7.4.0 or LINGO 11.0. By solving this model, we get the $\mathcal{Q}^*(A_\xi)$ $(\xi \in \{1,2,\ldots,m\})$ and $\boldsymbol{\varpi} = (\varpi_1, \varpi_2, \ldots, \varpi_g)^T$.

In addition, there are real-world situations that the weights of experts are not completely unknown but partially known. For these cases, based on the set of the known weight information of experts Γ, we construct the following optimization model to get $\mathcal{Q}^*(A_\xi)$ $(\xi \in \{1,2,\ldots,m\})$ and $\boldsymbol{\varpi} = (\varpi_1, \varpi_2, \ldots, \varpi_g)^T$ (Zhang and Liu 2016):

$$\begin{cases} \min \quad Z = \sum_{i=1}^{m} \sum_{k=1}^{g} \varpi_k (\eta_i^k + \rho_i^k) \\ s.t. \\ \qquad \mathcal{Q}^k(A_\xi) - \mathcal{Q}^*(A_\xi) - \eta_i^k + \rho_i^k = 0 \quad \begin{pmatrix} i \in \{1,2,\ldots,m\}, \\ k \in \{1,2,\ldots,g\} \end{pmatrix} \\ \qquad \eta_i^k \geq 0, \quad \rho_i^k \geq 0, \quad \eta_i^k \rho_i^k = 0 \quad \begin{pmatrix} i \in \{1,2,\ldots,m\}, \\ k \in \{1,2,\ldots,g\} \end{pmatrix} \\ \qquad (\varpi_1, \varpi_2, \ldots, \varpi_g) \in \Gamma \end{cases} \tag{MOD-2.3}$$

where Γ is a set of constraint conditions that the expert weights ϖ_k $(k \in \{1,2,\ldots,g\})$ should satisfy according to the requirements in the real-world situations.

Obviously, the greater the value of $\mathcal{Q}^*(A_\xi)$ $(\xi \in \{1,2,\ldots,m\})$ is, the better the alternative A_ξ will be. Therefore, we can determine the ranking order of alternatives according to the increasing orders of $\mathcal{Q}^*(A_\xi)$ $(\xi \in \{1,2,\ldots,m\})$, and select the best alternative from the alternative set $\boldsymbol{A}$. According to the above analysis, the steps of the proposed method are summarized as follows (Algorithm 2.3):

Step 1. Identify the criteria values of alternatives on criteria under each expert and the weights of criteria, respectively.

Step 2. For each expert, we employ Eq. (2.36) to calculate the dominance value of the alternative $A_\xi \in \boldsymbol{A}$ over the alternative $A_\zeta \in \boldsymbol{A}$ concerning the criterion $C_j \in \boldsymbol{C}$.

Step 3. For each expert, we use Eq. (2.37) to compute the collective dominance value of the alternative $A_{\xi} \in \boldsymbol{A}$ over the alternative $A_{\zeta} \in \boldsymbol{A}$.

Step 4. We utilize Eq. (2.38) to determine the overall dominance value of the alternative $A_{\xi} \in \boldsymbol{A}$ for the expert $e_k \in \boldsymbol{E}$.

Step 5. If the weights of experts are completely unknown, according to the model (MOD-2.2) we construct a linear programming model to determine the overall dominance value of the alternative $A_{\xi} \in \boldsymbol{A}$ for the group; if the weights of experts are partially known, based on the model (MOD-2.3) we construct an optimal model to determine the overall dominance value of the alternative $A_{\xi} \in \boldsymbol{A}$ for the group.

Step 6. Rank the alternatives by comparing the magnitudes of the overall dominance value of the alternative $A_{\xi} \in \boldsymbol{A}$ for the group.

2.5.3 Case Illustration

To demonstrate the decision process and the applicability of the proposed approach, Zhang and Liu (2016) modified the real-life MCDM problem introduced in Sect. 2.4, and assumed that the CAA invites a committee including three experts (e_1, e_2, e_3) to investigate four major Taiwan airlines according to their four criteria. The weight vector of the criteria is $\boldsymbol{w} = (0.2, 0.25, 0.35, 0.2)^{\mathrm{T}}$. The weight vector of the experts is given as follows:

$$\Gamma = \left\{ \begin{array}{l} \varpi_3 \geq \varpi_1, 0.15 \leq \varpi_2 - \varpi_1 \leq 0.25,\ \varpi_1 + \varpi_3 \geq \varpi_2, 0.2 \leq \varpi_2 \leq 0.35, \\ \varpi_1 + \varpi_2 + \varpi_3 = 1, \varpi_1 \geq 0, \varpi_2 \geq 0, \varpi_3 \geq 0 \end{array} \right\}$$

All experts employ the linguistic terms or comparison linguistic expressions to provide the assessment values of alternatives with respect to each criterion as shown in Table 2.18.

The top-left cell "*Between MG and G*" in Table 2.18 indicates that the degree to which the alternative A_1 (UNI Air) satisfies the criterion C_1 (Booking and ticketing service) is *between Medium Good and Good*. The others in Table 2.18 have the similar meanings. Then, comparative linguistic expressions are transformed into HTrFNs which are listed in Table 2.19.

In what follows, we employ the proposed hesitant trapezoidal fuzzy TODIM method to solve the above group decision making problem. According to the decision steps of the proposed method, we first take the criterion C_3 as the reference criterion and thus the weight of the reference criterion is $w_R = 0.35$. Based on the closeness index-based ranking method of HTrFNs, we compare with the

Table 2.18 The linguistic criteria values of alternatives for each expert

Experts	Alternatives	Criteria			
		C_1	C_2	C_3	C_4
e_1	A_1	Between MG and G	MG	Between MG and G	At most MP
	A_2	MG	Between MP and F	MP	Between F and MG
	A_3	Between MP and F	Between P and MP	At least MG	MG
	A_4	MG	F	G	Between P and MP
e_2	A_1	Between F and MG	MP	P	G
	A_2	Between G and VG	At least MG	F	F
	A_3	G	MG	MP	MP
	A_4	MP	G	F	Between P and MP
e_3	A_1	At least MG	F	At most MP	MG
	A_2	F	MG	At least G	P
	A_3	Between F and G	Between MP and F	Between P and MP	At least G
	A_4	MG	G	Between F and G	G

magnitudes of the criteria values and obtain the superior-inferior table under the expert e_1 as in Table 2.20 (Zhang and Liu 2016).

The top-left cell "$_{1/2}S_1$" in Table 2.20 indicates that for the expert e_1 and under the criterion C_1 the alternative A_1 is superior to the alternative A_2 because of $\varphi(\hbar_{11}^1) = 0.725 > \varphi(\hbar_{21}^1) = 0.65$. Similar logic is used to determine the remaining entries in Table 2.20. Without loss of generality, we take the value of the parameter θ as 3 and the dominance value of the alternative $A_\xi \in \boldsymbol{A}$ over the alternative $A_\zeta \in \boldsymbol{A}$ under the criterion $C_j \in \boldsymbol{C}$ can be calculated by using Eq. (2.36). The calculated results are listed in Tables 2.21, 2.22, 2.23 and 2.24 (Zhang and Liu 2016).

Furthermore, by aggregating the gains and losses of the alternative $A_\xi \in \boldsymbol{A}$ over the alternative $A_\zeta \in \boldsymbol{A}$ under the criterion $C_j \in \boldsymbol{C}$ using Eq. (2.37), we can obtain the weighted dominance value of each alternative over the others, listed in Table 2.25.

Using Eq. (2.38), the overall dominance values of alternatives for the expert e_1 are obtained as follows:

$$\mathcal{Q}^1(A_1) = 0.6150, \quad \mathcal{Q}^1(A_2) = 1.0, \quad \mathcal{Q}^1(A_3) = 0.0, \quad \mathcal{Q}^1(A_4) = 0.5892.$$

Analogously, we can also calculate the overall dominance values of alternatives for the expert e_2 as:

Table 2.19 The hesitant criteria values of alternatives

Experts	Alternatives	Criteria	
		C_1	C_2
e_1	A_1	{Tr(0.5, 0.6, 0.7, 0.8), Tr(0.7, 0.8, 0.8, 0.9)}	Tr(0.5, 0.6, 0.7, 0.8)
	A_2	Tr(0.5, 0.6, 0.7, 0.8)	{Tr(0.2, 0.3, 0.4, 0.5), Tr(0.4, 0.5, 0.5, 0.6)}
	A_3	{Tr(0.2, 0.3, 0.4, 0.5), Tr(0.4, 0.5, 0.5, 0.6)}	{Tr(0.1, 0.2, 0.2, 0.3), Tr(0.2, 0.3, 0.4, 0.5)}
	A_4	Tr(0.5, 0.6, 0.7, 0.8)	Tr(0.4, 0.5, 0.5, 0.6)
e_2	A_1	{Tr(0.4, 0.5, 0.5, 0.6), Tr(0.5, 0.6, 0.7, 0.8)}	Tr(0.2, 0.3, 0.4, 0.5)
	A_2	{Tr(0.7, 0.8, 0.8, 0.9), Tr(0.8, 0.9, 1.0, 1.0)}	{Tr(0.5, 0.6, 0.7, 0.8), Tr(0.7, 0.8, 0.8, 0.9), Tr(0.8, 0.9, 1.0, 1.0)}
	A_3	Tr(0.7, 0.8, 0.8, 0.9)	Tr(0.5, 0.6, 0.7, 0.8)
	A_4	Tr(0.2, 0.3, 0.4, 0.5)	Tr(0.7, 0.8, 0.8, 0.9)
e_3	A_1	{Tr(0.5, 0.6, 0.7, 0.8), Tr(0.7, 0.8, 0.8, 0.9), Tr(0.8, 0.9, 1.0, 1.0)}	Tr(0.4, 0.5, 0.5, 0.6)
	A_2	Tr(0.4, 0.5, 0.5, 0.6)	Tr(0.5, 0.6, 0.7, 0.8)
	A_3	{Tr(0.4, 0.5, 0.5, 0.6), Tr(0.5, 0.6, 0.7, 0.8), Tr(0.7, 0.8, 0.8, 0.9)}	{Tr(0.2, 0.3, 0.4, 0.5), Tr(0.4, 0.5, 0.5, 0.6)}
	A_4	Tr(0.5, 0.6, 0.7, 0.8)	Tr(0.7, 0.8, 0.8, 0.9)
		C_3	C_4
e_1	A_1	{Tr(0.5, 0.6, 0.7, 0.8), Tr(0.7, 0.8, 0.8, 0.9)}	{Tr(0.0, 0.0, 0.1, 0.2), Tr(0.1, 0.2, 0.2, 0.3), Tr(0.2, 0.3, 0.4, 0.5)}
	A_2	Tr(0.7, 0.8, 0.8, 0.9)	{Tr(0.4, 0.5, 0.5, 0.6), Tr(0.5, 0.6, 0.7, 0.8)}
	A_3	{Tr(0.5, 0.6, 0.7, 0.8), Tr(0.7, 0.8, 0.8, 0.9), Tr(0.8, 0.9, 1.0, 1.0)}	Tr(0.5, 0.6, 0.7, 0.8)
	A_4	Tr(0.7, 0.8, 0.8, 0.9)	{Tr(0.1, 0.2, 0.2, 0.3), Tr(0.2, 0.3, 0.4, 0.5)}
e_2	A_1	Tr(0.1, 0.2, 0.2, 0.3)	Tr(0.7, 0.8, 0.8, 0.9)
	A_2	Tr(0.4, 0.5, 0.5, 0.6)	Tr(0.4, 0.5, 0.5, 0.6)
	A_3	Tr(0.2, 0.3, 0.4, 0.5)	Tr(0.2, 0.3, 0.4, 0.5)
	A_4	Tr(0.4, 0.5, 0.5, 0.6)	{Tr(0.4, 0.5, 0.5, 0.6), Tr(0.5, 0.6, 0.7, 0.8)}
e_3	A_1	{Tr(0.0, 0.0, 0.1, 0.2), Tr(0.1, 0.2, 0.2, 0.3), Tr(0.2, 0.3, 0.4, 0.5)}	Tr(0.5, 0.6, 0.7, 0.8)
	A_2	{Tr(0.7, 0.8, 0.8, 0.9), Tr(0.8, 0.9, 1.0, 1.0)}	Tr(0.1, 0.2, 0.2, 0.3)
	A_3	{Tr(0.4, 0.5, 0.5, 0.6), Tr(0.5, 0.6, 0.7, 0.8)}	{Tr(0.7, 0.8, 0.8, 0.9), Tr(0.8, 0.9, 1.0, 1.0)}
	A_4	{Tr(0.4, 0.5, 0.5, 0.6), Tr(0.5, 0.6, 0.7, 0.8), Tr(0.7, 0.8, 0.8, 0.9)}	Tr(0.7, 0.8, 0.8, 0.9)

Table 2.20 A superior-inferior table over alternatives with criteria for the expert e_1

	A_1/A_2	A_1/A_3	A_1/A_4	A_2/A_3	A_2/A_4	A_3/A_4
C_1	${}_{1/2}S_1$	${}_{1/3}S_1$	${}_{1/4}S_1$	${}_{2/3}S_1$	${}_{2/4}E_1$	${}_{3/4}I_1$
C_2	${}_{1/2}S_2$	${}_{1/3}S_2$	${}_{1/4}S_2$	${}_{2/3}S_2$	${}_{2/4}I_2$	${}_{3/4}I_2$
C_3	${}_{1/2}I_3$	${}_{1/3}I_3$	${}_{1/4}I_3$	${}_{2/3}S_3$	${}_{2/4}E_3$	${}_{3/4}I_3$
C_4	${}_{1/2}I_4$	${}_{1/3}I_4$	${}_{1/4}I_4$	${}_{2/3}I_4$	${}_{2/4}S_4$	${}_{3/4}S_4$

Table 2.21 Gains and losses of alternatives over the others for the criterion C_1 and the expert e_1

	A_1	A_2	A_3	A_4
A_1	0	0.1225	0.2449	0.1225
A_2	−0.2041	0	0.2121	0
A_3	−0.4082	−0.3536	0	−0.3536
A_4	−0.2041	0	0.2121	0

Table 2.22 Gains and losses of alternatives over the others for the criterion C_3 and the expert e_1

	A_1	A_2	A_3	A_4
A_1	0	−0.1543	−0.1484	−0.1543
A_2	0.1620	0	0.0443	0
A_3	0.1559	−0.0422	0	−0.0422
A_4	0.1620	0	0.0443	0

Table 2.23 Gains and losses of alternatives over the others for the criterion C_2 and the expert e_1

	A_1	A_2	A_3	A_4
A_1	0	0.2372	0.3062	0.1936
A_2	−0.3162	0	0.1936	−0.1826
A_3	−0.4082	−0.2582	0	−0.3162
A_4	−0.2582	0.1369	0.2372	0

Table 2.24 Gains and losses of alternatives over the others for the criterion C_4 and the expert e_1

	A_1	A_2	A_3	A_4
A_1	0	−0.4530	−0.4969	−0.1964
A_2	0.2718	0	−0.2041	0.2449
A_3	0.2981	0.1225	0	0.2739
A_4	0.1178	−0.4082	−0.4564	0

Table 2.25 Weighted dominance values of alternatives over others for the expert e_1

	A_1	A_2	A_3	A_4
A_1	0	−0.2476	−0.0942	−0.0346
A_2	−0.0865	0	0.2459	0.0623
A_3	−0.3624	−0.5315	0	−0.4381
A_4	−0.1825	−0.2713	0.0372	0

$$\mathscr{Q}^2(A_1) = 0.0,\ \ \mathscr{Q}^2(A_2) = 1.0,\ \ \mathscr{Q}^2(A_3) = 0.0218,\ \ \mathscr{Q}^2(A_4) = 0.3991,$$

and the overall dominance values of alternatives for the expert e_3 as follows:

$$\mathscr{Q}^3(A_1) = 0.0210,\ \ \mathscr{Q}^3(A_2) = 0.0,\ \ \mathscr{Q}^3(A_3) = 0.3669,\ \ \mathscr{Q}^3(A_4) = 1.0.$$

According to the model (MOD-2.3), we construct the following optimal model (Zhang and Liu 2016):

$$\begin{cases} \min \quad Z = \sum_{i=1}^{4}\sum_{k=1}^{3} \varpi_k(\eta_i^k + \rho_i^k) \\ s.t. \\ \qquad 0.615 - \mathscr{Q}^*(A_1) - \eta_1^1 + \rho_1^1 = 0; \quad 0.0 - \mathscr{Q}^*(A_1) - \eta_1^2 + \rho_1^2 = 0; \\ \qquad 0.021 - \mathscr{Q}^*(A_1) - \eta_1^3 + \rho_1^3 = 0; \quad 1.0 - \mathscr{Q}^*(A_2) - \eta_2^1 + \rho_2^1 = 0; \\ \qquad 1.0 - \mathscr{Q}^*(A_2) - \eta_2^2 + \rho_2^2 = 0; \quad 0.0 - \mathscr{Q}^*(A_2) - \eta_2^3 + \rho_2^3 = 0; \\ \qquad 0.0 - \mathscr{Q}^*(A_3) - \eta_3^1 + \rho_3^1 = 0; \quad 0.0218 - \mathscr{Q}^*(A_3) - \eta_3^2 + \rho_3^2 = 0; \\ \qquad 0.3669 - \mathscr{Q}^*(A_3) - \eta_3^3 + \rho_3^3 = 0; \quad 0.5892 - \mathscr{Q}^*(A_4) - \eta_4^1 + \rho_4^1 = 0; \\ \qquad 0.3991 - \mathscr{Q}^*(A_4) - \eta_4^2 + \rho_4^2 = 0; \quad 1.0 - \mathscr{Q}^*(A_4) - \eta_4^3 + \rho_4^3 = 0; \\ \qquad \varpi_3 \geq \varpi_1, 0.15 \leq \varpi_2 - \varpi_1 \leq 0.25,\ \varpi_1 + \varpi_3 \geq \varpi_2, 0.2 \leq \varpi_2 \leq 0.35 \\ \qquad \sum_{k=1}^{3} \varpi_k = 1,\ \varpi_k \geq 0,\ \ k \in \{1,2,3\} \\ \qquad \eta_i^k \geq 0,\ \rho_i^k \geq 0,\ \eta_i^k \rho_i^k = 0;\ \ i \in \{1,2,3,4\}, k \in \{1,2,3\} \end{cases}$$

By solving the above model, the weights of experts and the overall dominance values of the alternatives for the group can be obtained as follows (Zhang and Liu 2016):

$$\begin{aligned} &\varpi_1 = 0.20,\ \ \varpi_2 = 0.35,\ \ \varpi_3 = 0.45 \\ &\mathscr{Q}^*(A_1) = 0.0210,\ \mathscr{Q}^*(A_2) = 1.0000, \\ &\mathscr{Q}^*(A_3) = 0.0218,\ \ \mathscr{Q}^*(A_4) = 0.5892. \end{aligned}$$

Apparently, the ranking of alternatives is obtained as $A_2 \succ A_4 \succ A_3 \succ A_1$, and the best alternative is A_2.

2.6 Conclusions

The classical TODIM method is a helpful tool for solving the classical MCDM problems in case of considering the decision maker's psychological behavior, but it cannot be used to directly handle the MCDM problems with fuzzy information. This chapter introduced a HF-TODIM method developed by Zhang and Xu (2014a) for solving the MCDM problems with hesitant fuzzy information in case of considering the decision maker's psychological behavior. The main advantages of the HF-TODIM approach are that (1) it can handle the MCDM problems in which the ratings of alternatives with respect to each criterion are represented by HFEs or IVHFEs and (2) it can take the decision maker's psychological behavior into account. On the other hand, this chapter also introduced a hesitant trapezoidal fuzzy TODIM approach developed by Zhang and Liu (2016) to handle the MCGDM problems in which the decision data are expressed as comparative linguistic expressions based on HTrFNs. Both the HF-TODIM method and the hesitant trapezoidal fuzzy TODIM approach can be further extended to deal with the MCDM or MCGDM problems with interdependent criteria, and can also be expected to be applicable to other similar decision making problems, such as performance evaluation, supply chain management, risk investment, etc.

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