

The passage is followed by a detailed description of the main proof idea in the habilitation thesis. Indeed, the text of §3 of the published thesis follows closely the sketch in **BTIZ**. The series **BTIZ** contains, though, at least three more paragraphs in comparison to the published habilitation thesis. §4 on page 171 is titled “The unprovability of TI to ε_0 in pure number theory,” with the addition “resp. demonstration of unprovability through the use of the valuation previously applied in the consistency proof.” §5 appears three times, on page 199 as “The domain without $\supset$ (and $\neg$) in LI and LK treated by the method of main thread,” developed fully on pages 199–210. On page 207 we find §5 a second time, with the title “The train of thought.” As mentioned, the short page 160 tells that pages 161 to 198.2 contain the elaboration for the habilitation thesis, but the text read originally “page 161–210,” the latter then cancelled and 198.2 written above. Thus, there seems to have been a serious plan of including the materials of pages 199–210 in the thesis, cancelled at some later occasion. The title of §5 is given on page 222 as “Sharpened formulation,” the same as the title of §6 on page 225. Finally, on page 229 there is §7 with the title “Main valuation.” Why these attempts at carrying the result on transfinite induction through in LI instead of LK failed, is still unclear.

The details of the series **BTIZ** are as follows: There was a first, sustained and continuous push with **BTIZ** that begins with page 133. The next page 134 is dated 19.V.39. All pages from there on up to page 160 are preserved, with the date 27.V. on page 155. Now there is the gap, with just the four extant pages 169–172 (page 170 dated 23.VI.39). Next follows a continuous suite, from 199 (11.VII.39) to 244 (242 dated 23.VII.). The last-mentioned page has the comment: “My thoughts don’t seem to be moving anymore,” and page 243 continues: “End with it for the time being.”

In sum, there are 80 stenographic pages between 19.V.39 and 23.VII.39 in what could be called **BTIZ-I**. They are continuous, except for the gap 161–198 that, by grace, is covered by the habilitation thesis. The typed thesis must have been prepared in August and September, before Gentzen’s military service began.

A second installment of **BTIZ** begins under Gentzen’s convalescence in a military hospital, with thoughts written on little slips of paper and numbered from 245.1 to 245.22, the extant pages being 245.9–10, 15–16, and 21–22. These are dated IV.42 and described as “Lazarettideen.” There are a few additional pages, numbered 246.1–2 and 247.1–3, with the dates IV.42 for 246 and IX.42 for 247. The former is an impressive table of 12 reduction schemes for transfinite induction. This ensemble of pages can be designated as **BTIZ-II**.

A third attempt at the proof theory of intuitionistic arithmetic, **BTIZ-III**, takes place between III.43 and VII.43, with 28 consecutive stenographic pages numbered 248–275, all of them preserved.

In sum, the preserved pages of **BTIZ** are 133–160, 169–172, and 199–275. The material should amount to some two hundred printed pages. The main difficulty with these pages is that they build on earlier developments, the lost pages **BTIZ** 1–132, that are occasionally referred to in the extant pages.

Why was it important for Gentzen to find a consistency proof within an intuitionistic sequent calculus? First, work on **BTIZ** took place in direct continuation to the *Habilitationsschrift*. There Gentzen makes it clear that not all of the treatment of transfinite induction is carried through intuitionistically. Thus, the main motivation of **BTIZ** seems to be that a consistency proof within intuitionistic sequent calculus would be a prerequisite for a thoroughly intuitionistic treatment of transfinite induction.

The particular problem Gentzen was wrestling with in **BTIZ** concerned contraction, the analogue in sequent calculus of a multiple use of an assumption that gets closed in natural deduction. When a cut is eliminated, contraction is a problem, as discussed above in I.1. When in this situation a cut or analogously a non-normality is resolved, there occurs typically a multiplication of some part of a derivation. The problem is to find a measure of the complexity of derivations that does not increase with such multiplication. One solution is offered in Siders (2015).

This is not the place to detail out the possible reasons for the thrice-failed proof theory of intuitionistic arithmetic, but one thing can be said: There is a very neatly written series from August 1944, with the signum **WKRd** that stands for something like: *Widerspruchsfreiheit, Kalküle und Reduktionen, Durchsicht* (Consistency, overview of calculi and reductions). It is a clear indication of Gentzen's belief, namely: a suitable formulation of a logical calculus should bring success in the proof theory of intuitionistic arithmetic, in the intuitionistic treatment of transfinite induction, and also in the consistency problem of analysis. Unfortunately, a similar overview **WTZd** about transfinite induction, placed originally with the Putbus notes, got removed at some point.

3. Consistency of analysis

The series **WA** contains what is left of Gentzen's attempts at proving the consistency of analysis. The first preserved collection of notes has been written after the 1938 new proof of the consistency of arithmetic was finished in May. His article (1938) with the new proof ends with some discussion of the extension of transfinite induction beyond the ordinal ε_0 :

In the proof of consistency [of arithmetic] and its eventual extensions, the question is anyway about an initial part, a “section” of the II number class, even if it is already one that goes relatively far. For a proof of the consistency of analysis, it would probably

be an initial part that goes considerably further.

This idea seems to have been a rather widely endorsed one. For example, Gödel expressed in his stenographic notebooks repeatedly his belief in the possibility of a proof of the consistency of analysis through an extension of Gentzen's transfinite induction.

What is left of the series **WA** has been written in two stretches: A first one is from 11.VIII.38 to 8.XII.38, with the pages 77–80 (11.VIII.), 111–116 (29.IX.), 121–128 (1.–11.X.), 135–146 (–8.XI.), 169–172 (22.XI.–3.XII.), and 177–180 (8.XII.); altogether 38 pages.

Pages 77–80 of **WA** contain a suggestion for a realizability interpretation of intuitionistic arithmetic, where functionals over ordinal numbers act as realizers. Gentzen's letter to Bernays of 17 July 1936 mentions that he is studying the system of Church, i.e., λ -calculus, “because of the relations to transfinite numbers.” He considers in the preserved pages only implication introduction and elimination and a rule of induction. To the assumptions are assigned some variables of appropriate functional type. Whenever there is an implication introduction, there is a functional abstraction over some function Γ , written with the bound variable underlined as in $\underline{\eta}\Gamma(\eta)$. Whenever there is an implication elimination with an argument β , a corresponding functional application $\Gamma(\beta)$ is written:

$\underline{\eta}\Gamma(\eta)$ It means: the expression that follows considered as a function of η . We have naturally $(\underline{\eta}\Gamma(\eta))\beta = \Gamma(\beta)$. The type results by itself.

Whenever there is a step of the induction rule, an omega-power of the term is taken. An example derivation on page **WA**78 has the associated ordinal function terms written with a blue pencil next to each formula in the natural derivation. Gentzen's aim was, naturally, to show that the reduction of a derivation leads to a reduction of the associated ordinal term.

Gentzen's ordinal realizability for natural arithmetic derivations comes quite close to the Curry-Howard correspondence of typed λ -calculus.

A second series of notes with the signum **WA** follows between May and September 1943. The pages that have remained are 183–184 (11.V.43), 187–197, 198.01, 198.02, 198.5–198.11 (9.VI.–1.VIII.), 209–222 (3.VIII.–15.IX.), and 225–254 (–27.IX.43); altogether 56 pages.

The first page of this second series of **WA** is 183, with the title *Allgemeine Gedanken. Analogien und vermutete Zusammenhänge* (General thoughts. Analogies and presumed connections). The text proper begins with the boxed date 11.V.43.: “Promenade thoughts in connection with **WTZ** 150.”

The extant pages of the series **WA** amount to some two hundred printed pages. There is in addition to the **WA**-pages an 11-page stencil copy of a typewritten manuscript with the signum **WAdd** that stands for *Consistency*

of *analysis, overview* (or summary of overview, for the double-d). The title is *From **WA** the most modern part* and the date II.45. It is the last known writing of Gentzen's, a stenographic overview of the notes of the consistency problem of analysis, from where the preserved notes end, so from page 255 on and "cleared through page 339" as it states. The text was transcribed around 1948 by Hellmuth Kneser and H. Urban (presumably a pseudonym for Hans Rohrbach, see Menzler-Trott 2007, p. 277). The original stenography is lost. Rohrbach showed the transcription to some people such as Bernays, but nothing definitive could be said on its basis. Counting to page 339, the **WA**-notes would amount to a length of some five hundred printed pages. There were in addition the pages of the series **WTZ**, at least 210 pages on consistency proofs by transfinite induction that are often referred to in **WA**, but nothing is left of it to help prospective readers.

Menzler-Trott's biography contains descriptions of the attempts at recovering Gentzen's papers from the time in Prague (pp. 263–5). Maximilian Pinl writes in a letter that he had been able to ascertain "that Mr. Gentzen had left behind a luggage with precious manuscripts." A letter from Heinrich Scholz to Gentzen's mother reports of the account of Pinl in the following words: "Your son up to his separation from him had been of an exceptional mental activity and occupied himself without break with the consistency of analysis." How likely is it that Gentzen would have found a proof of consistency? The summary of February 1945 gives no indication of such a proof and after that month, the time window to Gentzen's arrest and imprisonment on 7 May is very narrow. It remains to be seen what further transcription and study of the series **WA** leads to in this respect.

4. Foundations and philosophy of mathematics

As said, Gentzen's idea was to write a popular book on 'mathematical foundational research,' with the motto cited above, *Spannend wie ein Kriminalroman!*, exciting like a detective story. There are mainly fragmentary notes for the book. Some of these complement things known from elsewhere, say the remarks on normalization and arithmetic.

The notes for the series **BG**, "Buch über die Grundlagenforschung," are written on all sorts of slips of paper over several years. The remarks begin in 1939 with a list of contents and some outlines, but most of them stem from April and September 1942. These fragmentary notes amount altogether to twenty-odd pages and are found here as the last item. The plan of April 1939 has the following chapters:

Chapter I: (The basic concepts of proof theory)

§1. The structure of mathematical propositions.

§2. The formalization of mathematical propositions.

The ways of inference of mathematics

Formalization of these

(Resp. in general a *chapter* on: *reduction of the basic concepts etc.* Examples of formalized proofs. Even difficult ones. Remark: Any illiterate can control the correctness).

(Proof methods, axioms.) Equivalence of functions and predicates. Such that.

Chapter II: Mathematical logic.

Here the most important theorems. Such as: prenex normal form. (Possibly) my dissertation Hauptsatz.

As well as: double-negation theorem in a general formulation.

1. part: propositional logic. (Possibly: the propositional calculus. Here what is calculus-like).

2. part: predicate logic. Possibly two chapters.

Chapter III: (Axiomatics.) (The axioms, ways of inference and methods of concept formation of specific parts of mathematics.)

Herein: the structure of theories. Recursive theories, equivalence between theories. The form of analysis. The form of geometries. Set theory. Form of proof theory.

Possibly a chapter IV on: “The incompleteness theorems of proof theory.” Theorem of Gödel, possibly Skolem, (Church), Tarski.

Chapter V: The problem of infinity

The later notes relate usually to this structure, either through a chapter number or by topic. In the end, there are miscellaneous notes from before the book plan, and on the reverse of the last page the following unrelated cancelled text, written during Gentzen’s convalescence in 1942:

have not caused any essential recovery have caused a certain, though not yet essential recovery. only a modest

Considering the unfortunate fact that it is objectively difficult to demonstrate an illness of this kind, and that I have to refer to good faith in several points of my statement, I hereby assure with emphasis that I have given all the information according to my best knowledge and conscience.

Gerhard Gentzen.

5. Three lost items

(A) *A treatise on quantum-mechanical tunneling*: After the finished doctoral thesis, Gentzen prepared for an exam that would give him the license to teach at high schools. Gentzen's application to the exam is dated 26 May, 1933. The date had to do with his thesis, the manuscript of which must have been ready and handed in at that date. The doctoral thesis was one of the two essays that had to be prepared by the candidates before the examination proper. The other essay had the title *Elektronenbahnen in axialsymmetrischen Feldern unter Anwendung auf kosmische Probleme* (Electron trajectories in axially symmetric fields in relation to applications in cosmic problems). Thus, the "scientific examination" for a high school teacher's position seems to have been a rather demanding undertaking. The exam itself was conducted on 13–16 November in Göttingen and qualified Gentzen to teach mathematics, physics, and applied mathematics. His wish was to continue research, but there was no financing for that. A teacher's job would work as a security.

The documents about the examination have been found in 2009, at the Niedersächsische Hauptstaatsarchiv in Hannover. It appears that Gentzen, in need of copies of his thesis manuscript for other applications, obtained a permission to change it to an offprint of his first article, on Hertz systems. In his announcement to the examination of 26 May 1933, he asked to have the second work from "the theory of electricity." The manuscript on physics is not found among the papers, but there is a report prepared by Hertha Sponer who acted as the physics examiner. A form dated 12 July 1933 was sent to Gentzen and gave the title of the physics essay, with Sponer's name under it, the essay to be delivered by 13 September. Sponer had obtained the habilitation (permission to teach at a university) in 1932, as one of the three first women in physics in Germany the other two being Lise Meitner and Hedwig Kohn. She was nominated extraordinary professor the year after.

Quantum-mechanical tunneling was explained in 1928 in a work by Ralph Fowler and Lothar Nordheim, the latter Hilbert's physics assistant in Göttingen. Electron microscopy is based on this phenomenon. The title of Gentzen's essay suggests quantum-mechanical tunneling, but what the intended application to cosmic radiation was, I have not been able to find out. Whether Nordheim initially had a role in Gentzen's examination paper is not known, but being Jewish, he was anyway excluded from any such role by the spring of 1933 and instead emigrated the year after.

Sponer's specialty in the early 1930s was the theory of molecular spectra treated by methods of quantum mechanics, on which she published a two-volume book in 1935–36. Unable to pursue as a woman a scientific career in Nazi Germany, she emigrated in 1934 first to Norway and then to the United States. No obvious direct link to Gentzen's manuscript can be made from

her published papers that became very few after the Nazi takeover. In her evaluation, she writes that “the candidate proved to be quite familiar with the mathematical presentation, whereas the purely physical side remains somewhat distant.” A separate statement is as follows:

The work

Electron trajectories in axially symmetric fields in relation to applications in cosmic problems

has been carried through by Mr. Gentzen in a delightful way. He has gone through the literature with a very good understanding and good critique. These works are written in part in a way that is little transparent, and so he had to give a simpler and more understandable presentation. Since he was able to do this well and since the work contains even independent considerations, it can obtain the predicate “With Distinction.”

H. Spöner

(B) *Gentzen’s lost lecture on Kepler’s laws*: In 1943, Gentzen gave a lecture on Kepler’s laws, to prove his ability as a teacher for the German university of Prague. Hans Rohrbach’s evaluation of this lecture contains (Menzler-Trott, p. 235):

He began his presentation with a reference to the local connections of his topic, since Kepler had worked many years in Prague and the necessary material for the development of the laws of the planets was present in the observations of Tycho de Brahe. His historical overview of the efforts of the human spirit since antiquity to describe the movements of the planets made its connection here. He closed part 1 of the lecture with an interesting demonstration that the theory of epicycles of Ptolemy is mathematically equivalent to Kepler’s First Law. Part 2 of the lecture was dedicated to the theoretical derivation of Kepler’s Laws from Newton’s general law of gravitation.

It is known from Menzler’s biography that Gentzen had a great interest in astronomy as a youth. Had he perhaps made in the second part of his lecture the same work as Feynman, as found in the book *Feynman’s Lost Lecture: The Motion of Planets Around the Sun*, namely a geometrical derivation of Kepler’s laws?

(C) *A report on the V-2 missile project*: Next to mathematical lectures, Gentzen’s tasks in Prague also included work in the calculation office of Hans Rohrbach at the mathematics institute, intended by the latter to save him from front service. Mysteriously, there is among the Putbus notes a large

folded millimeter paper, the grid measuring 450 by 300 mm, with what seem like ballistic curves drawn on the millimeter side, numbered 79, and a table of values and calculations and shorthand text on the reverse white page 80. It appears that Gentzen's time from late 1943 to the Summer of 1944 was mostly taken by this work. Menzler-Trott's biography tells (p. 238–243) that the computations were done for the V-2 rocket development at the ill-famed Peenemünde rocket test site and the concentration camp rocket production site at Mittelbau-Dora. Some three thousand V-2 rockets were targeted between September 1944 and March 1945 against civil population mainly in Britain, though they led to more victims, an incredible twenty thousand, at the production site; still, Rohrbach's letters, even in the 1980s, show no regret at having participated in and advanced this Nazi war crime and at having made Gentzen do the same (as on p. 240):

I could, through a corresponding proposal, bring Gentzen to my mathematical institute in Prague, where I had already begun to carry out computational works for the V-weapon manufacture in Peenemünde. I could release Gentzen for this and transfer to him the direction of the workgroup, which I had already formed. At the time these were young girls from the upper classes of the higher schools, who with certain mathematical-statistical computing works, to which they would be trained, must carry out assistance work, also a sort of military action [*eine Art Kriegseinsatz*, contribution to the war]. Gentzen performed his task well and worked out good results for Peenemünde; the problems would be posed from there.

The two manuscript pages 79 and 80 are clearly written for some extensive report. What the latter is and whether there were more authors is unknown. If a document was finished, it is likely that it is preserved somewhere; in the spring of 1945, there was a race between the Soviets and western Allied forces to recover the remaining V-2 rockets and paraphernalia, personnel, and documents pertaining to it. The V-2 rocket was the first one to reach outer space, an altitude up to over 200 km, with an enormous military potential in connection with the development of the atomic bomb. An operation of the American forces brought Wernher von Braun and many other Nazi warfare criminals to the United States in 1945, together with over a hundred rockets for test launches. An account of the race from the Soviet side, under whose control Peenemünde and Dora finally fell, is found in Boris Chertok's *Rockets and People: Creating a Rocket Industry*. A great number of German rocket scientists and technicians got moved to the Soviet Union and continued the development of ballistic missiles. Michael Neufeld's *The Rocket and the Reich: Peenemünde and the Coming of the Ballistic Missile Era* gives a comprehensive presentation of this initial phase of rocketry.

6. GENTZEN'S GENIUS

Eckart Menzler's Gentzen biography bears the title *Logic's Lost Genius*. In a review that Bill Tait wrote, he plays on the double meaning of that phrase (Tait 2010, pp. 274–75):

Certainly logic lost a genius with Gentzen's early death in the sense that, whatever he might have made of the consistency problem for analysis, he was certain to have contributed much more of value to logic had he lived longer and regained his health. . . . Is the meaning of the title rather (or also) that Gentzen, perhaps in his effort to solve his career problem, lost himself? . . . I like the ambiguity.

There is a clear sense in which Gentzen's work bears the hallmark of genius: he began with simple, basic things and made no mistakes on the way. I wrote in my review *Gentzen's proof systems* (2012, p. 314): "It is of course some kind of an arithmetic necessity that simple new things are few in number and therefore hard to find. Whenever there is such an idea, success follows if things are in the right order from the start." To ask what the logical form of actual inference in mathematics is, is just such a simple idea. In a short time, from about February to September 1932, Gentzen had distilled out of his question a pure form of logical inference, and even a single error could have led him completely astray. The normalization result for intuitionistic natural deduction followed in a few months' time, then the interpretation of classical arithmetic in intuitionistic arithmetic in early 1933, the development of sequent calculus with the cut elimination theorem and a detailed written account in the doctoral thesis by May, the last in a period of some three months.

Things went similarly with some other problems: The consistency of arithmetic by transfinite induction up to ε_0 got figured out in some four months around 1935–36, with a clear view of the basic ideas of ordinal proof theory as witnessed by the correspondence with Bernays, and finally the great work on transfinite induction in the *Habilitationsschrift* that was done between early 1939 and June.

I have mentioned occasionally aspects of Gentzen's published papers that appear to be motivated by reasons of exposition. He wanted to keep his papers accessible even outside the circle of the few fellow logicians. His research had a similar vein: Results were one thing, what one presented in research papers another. This much at least becomes clear from the manuscripts in this collection and in the two remaining series **WA** and **BTIZ**. Some of the unpublished results were things he just left behind, such as the proof of normalization for natural deduction, others topics he planned to discuss on a suitable occasion, of which there were just a few. Where could one present

one's results, where to publish them? Two papers came out in 1938 in Heinrich Scholz' series *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften*, a publication made possible through Scholz' cunning (see Menzler-Trott 2007, pp. 85–86).

Gentzen was able to close himself from the rest of the world. His shorthand notes are confined strictly to logic and foundations, with an occasional personal note of joy and optimism in the style of “Hurrah! Es stimmt!” or disappointment at things not working out the right way. He was able to fantasize, from 1939 on well into the 1940s, about writing a popular book on foundational research.

Gentzen had in 1945 behind him a period of three years from the convalescence of the spring of 1942. Apparently, no conclusive results came from these years, but just loose ends. Had his brilliant mind gone broken, or were the problems too hard? In the evaluation of his test lecture for the position as Docent in Prague, Rohrbach included the remark (see Menzler-Trott, p. 237): “This judgement is not affected by the current slowness while lecturing of Dr. Gentzen caused by nervous disease.” I hope that these Gentzen manuscripts and others to come can lead to deepened insights into his way of looking at logic and foundations of mathematics, at times in a form that is well developed, at other times as just flashes into the anatomy of a unique mind's workings.

PART II: AN OVERVIEW OF THE SHORTHAND NOTES

1. GENTZEN'S SERIES OF STENOGRAPHIC MANUSCRIPTS

Except for the early thesis manuscript, the correspondence with Bernays and Heyting, and the summary of the series **WA** of February 1945, all the source materials are written in the unified shorthand. The last-mentioned item was originally written in shorthand, but it got transcribed and typewritten by 1948 (see Menzler-Trott 2007, p. 277). The original has not been found.

Here follows an alphabetical list of series of notes I have found mentioned in Gentzen's manuscripts. Then I summarize the contents of the items in this collection.

The number of different stenographic series from which pages have been preserved is about 12. There are in addition loose pages and notes, with no signum, and about 9 series to which references can be found in the preserved pages but of which nothing is left.

The pages are often written on four-page sheets, with the sheets numbered consecutively, and pages in a sheet in the style of 96.1 to 96.4. At times, pages are written at the back of proofs or manuscripts for other articles, or on any piece of paper that was at hand, such as a "Luftwachezettel," a form used for reporting the allied airplanes that flew over Brunswick where Gentzen was stationed between September 1939 and his nervous breakdown around the turn of 1941–42.

Alphabetical list of Gentzen's series of stenographic notes:

1. AL. This stands for *Aussagenlogik*, propositional logic. The preserved pages are 133 (II.37), 137–139 (X.42), and 141–142 (IV.43).

In the beginning of the first item in this collection, Gentzen mentions "my calculus **D** (in the book about the propositional calculus etc.);" that may be the initial part of **AL**.

2. Allerlei Bemerkungen. This series of "miscellaneous remarks" is mentioned in **SLF** 17, but next to the passages there dated as X.33–V.34, nothing is left.

3. BG. This stands for *Buch über Grundlagenforschung*, the book on mathematical foundational research that Gentzen fancied composing from the spring or summer of 1939 on. It is described in Section I.5.4 above.

4. BTIZ. This stands for *Beweistheorie der intuitionistischen Zahlentheorie*, proof theory of intuitionistic number theory, as described in Section I.5.2 above.

5. BZ. This stands for *Beweistheorie der Zahlentheorie*, proof theory of number theory. The extant pages are 1–6 (VIII.34–XI.34) and 9–12 (I.35–III.35).

6. D. This could stand for *Deduktion*, as the first page of the first item in

this collection mentions “my calculus **D** (in the book about the propositional calculus etc).” **D** got changed into **INH** in October 1932, at page 92.2. The note on *Five different forms of natural calculi* of 23 September 1932 begins as the series **PLF**1, a later addition. In the corner of the page, references to pages **D**85.3 and **DB**131 have been added. The remaining pages of the series **D** are: 92–96, each a four-page sheet, and 97.3–4.

7. DB. The signum **DB** seems to indicate a later continuation of the series **D**, probably after the first installment of **INH** had been finished. As mentioned under **D**, **PLF** 1 contains the reference **DB** 131. A reference to page **D** 138 has been added later on page 17 of **INH**, likely a **DB**-page.

8. Funktionenkalkül. References from the fall of 1931 to pages 20 and 25 of this lost series are found in **VOR**.

9. GRO. The series **INH**, pp. 26 and 30, refers to pages 27–28 of this otherwise unknown series and **BZ** to page 28. The last page of the handwritten thesis manuscript contains reduction schemes that suggest a possible reading of the letters GR as *Gipfelreduktion*, reduction of peaks.

10. HÜG. This series refers obviously to the normalization procedure, in which local peaks (*Hügel*) get removed, but nothing is left of it.

11. INH. The abbreviation stands for *Die formale Erfassung des Begriffs der inhaltlichen Richtigkeit in der reinen Zahlentheorie, Verhältnis zum Widerspruchsfreiheitsbeweis*, The formal conception of the notion of contentful correctness in pure number theory, relation to proof of consistency. This 36 page series has been bound together with covers, the only one of its kind among the preserved stenographic notes.

12. MK. This series is mentioned in **WA**, pages 252.1 and 254.1, and in **WAdd** that refers to **MK** 62–64. Gentzen speaks in this context of “Modellherleitungen,” example derivations, and the letter K seems to indicate a rule of inference that corresponds to a comprehension axiom, the latter the topic of pages **WA** 177–180 of December 1938.

13. PLF. Also abbreviated as **PL**, the former for *Formen der Prädikatenlogik* and the latter for just *Prädikatenlogik*. This is a collection of various short texts as detailed out in Section II.2.14 below.

14. RED. This is a renaming and possible addition to pages 97.1–4 of the series **D**. The extant pages are **RED** 3–4, placed in between pages of **INH**.

15. SLF. This stands for *Formen der Stufenlogik*, or Forms of type theory (“logic of types,” literally). The pages are few, from **SLF**15 to **SLF**18.

16. SW. The series **SLF** has a reference to **SW** that should stand for *Stufenlogik Widerspruchsfreiheit*, consistency of type theory, the topic of Gentzen’s paper published in 1936. Nothing is left of this series.

17. VOR. I think this stands for *Vorstudien* or a related word, preliminary studies. The extant pages are 13, 14.1–14.4, and 14–20.

18. WA. The signum **WA** stands for *Widerspruchsfreiheit Analysis*, though the **A** could in the earlier stages also stand for *Arithmetik*. A description is found in Section I.5.3. **WAdd** is an overview of the series **WA** from page 255 to 339, written by Gentzen in shorthand in February 1945 and preserved in a 13-page typewritten transcript.

19. WAV. This signum is connected to the early pages of **WA**, and here the **A** stands for *Arithmetik*, and the whole for something like *Widerspruchsfreiheit Arithmetik Veröffentlichung*, consistency of arithmetic publication. Ten pages are left of this series, up to page 86.

20. WKR. This stands for *Widerspruchsfreiheit, Kalküle und Reduktionen*, Consistency, calculi and reductions. There is an eight pages long, very clearly written overview of the series that stems from August 1944, with the signum **WK Rd.** The series **WKR** itself has had at least 210 pages. There is in addition a reference to a series **WKRLG** on page 265 of **BTIZ**, clearly an account of the groundsequent calculi *LIG* and *LKG*.

21. WTZ. This stands for *Widerspruchsfreiheit transfinite Zahlen*. There is a note of VIII.44 with the signum **WTZd** cancelled, so there has been an extensive series **WTZ** up to at least 210 pages, as well as an overview of it. Not a single page of **WTZ**, the series that signals the birth of ordinal proof theory, has remained.

2. THE ITEMS IN THIS COLLECTION

The following summaries are intended to explain to some extent the central contents of each of the items. For one who likes to discover things on one's own, it is best to turn directly to the original papers. The list as well as the papers in this collection follow an approximate chronological order of earliest appearance, approximate because the series overlap, have gaps, and so on.

1. Reduction of number-theoretic problems to predicate logic: 1931

The first item bears the long title *Reduction of number-theoretic problems to the decision problem of the lower functional calculus*. It is an early separate observation from the fall of 1931 that relates directly to the series **VOR**: The decision problem of arithmetic reduces to that of pure predicate logic by the translation from a language with arithmetic operations into one with relations. Gentzen's conclusion was the one Church and Turing proved to be the case in 1936, namely that predicate logic is not a decidable theory,

for if it were, also arithmetic would be decidable. Contrary to, e.g., Jacques Herbrand, Gentzen did not think that likely.

2. Replacement of functions by predicates: late 1931

The topic of this early series is the expression of mathematical operations in the language of pure logic, in particular, the expression of arithmetic with additional relations such as $\sigma(x, y, z)$ (“the sum of x and y is z ”). The systematic development starts from page 14 that has the title *Replacement of functions by relations*.

The extant pages of the series **VOR** are 13, 14, 14.1–14.4, and 15 to 20. **VOR** proper seems to begin with page 14. The pages 14.1, 14.2, and 14.3 have been written earlier, in September 1931, and added with a new numeration in between pages 14 and 15. It is my guess that **VOR** stands for *Vorstudien*, preliminary studies that Gentzen conducted on logical and mathematical notions before starting with his consistency program in early 1932. References are made in **VOR** to pages 20 and 25 of a series called **Funktionenkalkül**, calculus of functions, but no pages of such a series have survived.

Page 13 seems to precede the main contents of **VOR** proper without being connected to the rest. Similarly, the reverse of page 14.4 is cancelled; it contains axioms for natural numbers, namely the equality axioms, the schematic replacement axiom for equals, and uniqueness axioms for functions. The reverse of page 14.3 is the unrelated short text about the origins of mathematical concepts, item 3 below.

The pages of **VOR** pertinent to the theme of functions and relations are here given in approximate chronological order, first the pages 14.1–14.3 with the title *Functions instead of relations* with which page 14.1 begins, then the pages 14–20 with the title *Replacement of functions by predicates*, from page 14. The insertion of pages 14.1–14.3 between 14 and 15 would create a discontinuity in the text, in the middle of a phrase.

Gentzen uses the word *Zählausdruck* (‘counting expression’) for an expression of predicate logic, or equivalently first-order expression, a common practice of the time. I have used the latter in translation.

The topic of turning functions into relations is seen already in item **1**, and it is also used in the printed doctoral thesis written in 1933, when the classical sequent calculus *LK* is extended by arithmetic, the rule of induction excluded. The problem of consistency is easy, because the formulation is in terms of relations rather than functions, i.e., within pure predicate logic to which the cut elimination theorem can be applied.

The series **BZ**, pages 9–11 from January to March 1935, refers to **VOR** 14–19 as “old investigations” and gives comparisons with the treatment of the description operation ι . In fact, the preserved **VOR**-pages are found between pages 10 and 11 of **BZ**.

3. Correspondence in the beginning of mathematics: end of 1931

The topic of this one-page text is how “primitive man began to exercise mathematics.” It relates Gentzen’s view of the nature of the basic concepts of mathematics, natural numbers in particular, to those of Russell and Hilbert. He finds that they are sufficiently clear in their own right, and that “it is not necessary to define them, as sets of the respective objects in reality that have been made to correspond to them, as in Russell, or as the signs that have been introduced for them, as in Hilbert.”

4. Five different forms of natural calculi: IX.32

This remarkable short document contains the rules of natural deduction. The text has lots of references to alternative formulations, up to dozens of different conditions on formal proofs that Gentzen had explored. A final form of the rules can be found in the thesis manuscript written towards the end of 1932, item 6 below.

The *Five different forms* is here followed by remarks that are found at the beginning of the manuscript **INH**. The second page has the date 9.X.32. This short piece is saved only because **INH** proper happens to begin on page 92.2 of the four-page sheet 92 of the series **D**.

Also the *Five different forms* refers to pages of **D**, up to page 84, and it is dated 23.IX.32. The preserved page and a half of **D**, namely 92.1 to 92.2, have been written just two weeks later, and Gentzen is trying to figure out how to deal with the rule of negation introduction, *RA* in his notation, and with induction, *VI* in his notation (for *Vollständige Induktion*). There are many references to earlier pages of **D**, and transfinite induction is also mentioned. This was a stage in which the development of natural deduction was still tied to the attempt at proving the consistency of arithmetic.

4bis. Two fragments on normalization: IX–X.32

The previous item begins at the middle of a page. Its top has a short text on normalization, quite inconclusive but of interest nevertheless. A longer similar passage is found in the beginning of **INH** as described above. It gives some indication of Gentzen’s way to the normalization of derivations in natural deduction.

5. Formal conception of correctness in arithmetic I: X.32

Gentzen’s manuscript with the signum **INH**, from the blue folder, is titled *Die formale Erfassung des Begriffs der inhaltlichen Richtigkeit in der reinen Zahlentheorie, Verhältnis zum Widerspruchsfreiheitsbeweis*. My translation is: “The formal conception of the notion of contentful correctness in pure number theory, relation to proof of consistency.” The stenographic manuscript has about legal size pages (*Kanzleiformat* in German), 36 in number,

bound together with what are now somewhat rusted metal clips. It is the only such single bound piece in the Gentzen papers.

INH has been written between October 1932 and January 1935. The first 20 pages are from between early October and the 10th of November 1932. They record, usually with a couple of days' intervals, the progress of his thinking. Then follow a few pages dated "should be about February 33" and titled "Ultraintuitionistic criticism of the ways of inference. Finitism." The next entries are in April and June 1933. The rest is written in the second half of 1934, especially October and November, as well as a final part in January 1935.

The main question of **INH** is what the sense of a consistency proof for arithmetic could be. By April 1933, Gentzen had come to very clear ideas about three kinds of possible consistency proofs: 1. A "purely-formal" one (*rein-formal*), 2. A "semi-contentful" one (*halbinhaltlich*), 3. A "contentful" one (*inhaltlich*, sometimes even *ganz inhaltlich*).

By April 1933, Gentzen was sure that a formal proof requires a transfinite induction, something he had surmised the previous fall. The second proof was to use an ordinary double induction, and the same for the third.

Gentzen reasoned initially as follows, on the basis of a contentful notion of *correctness*: The axioms of arithmetic are obviously correct. The rules of logic maintain correctness. Therefore all things provable are correct. He then asked, well knowing that such a proof cannot be presented within arithmetic: "Where is the Gödel-point?"

Gentzen had fixed by 23 September 1932 the system of natural deduction, a rule of induction included, and clearly saw that formal derivations by the logical rules can be normalized. The detailed proof of normalization was produced later, perhaps by early 1933. All derivations ended with the closing of the open assumptions by implication introductions, even an aspect of his printed thesis that has gone largely unnoticed. Then, the conclusion of the derivation builds up into a longest formula that has all the formulas in the derivation as subformulas, provided that there are no "inner hillocks." If this held for arithmetic, its consistency would be an immediate consequence: Take a presumed derivation of $1 \neq 2$, and you see that it cannot have a normal derivation.

Gentzen could not get hold of the multiplication of parts of derivations in the reduction steps connected to instances of the rule of induction. Even detour conversions on implication proved difficult, but there the derivation grew broader in conversion. With the rule of induction, it grew in height when a step of induction was replaced by a numerical induction up to some number n . He thought of using the maximum number of nested inductions as the main parameter, and the overall number of inductions as a secondary parameter, but saw on the other hand that a proof should not be formalizable within Peano arithmetic.

Gentzen tried first the reduction (i.e., normalization) of derivations for what he called *direct* derivations. These are derivations with no assumptions, but just instances of the arithmetic axioms as topformulas. The rules would be those for conjunction, universal quantification, and induction. Only the last one has assumptions that have to be explained away. One idea was to use “levels” such that there is a ground level, say the base case of an uppermost induction in a derivation. Reduction of the induction removes the assumptions and the subderivation to the uppermost induction becomes direct. None of his attempts worked properly, and could not work.

Of the three possible proofs, the significance of the first, purely-formal one was not clear. What would be accomplished with it? The third, contentful one was trivial, so the second, semi-contentful one remained. The main idea was to define a notion of *correctness* for formulas, and derivations, based on the notion of reducibility. Correctness is a, as we would now say, semantic notion, and the idea of a proof of consistency would be that formal derivability produces results correct in the defined sense. The difference to a purely-formal proof by transfinite induction is clear now: The latter is a purely syntactic proof.

The central point in the second proof was the definition of correctness. This notion is well known from Gentzen’s original proof of consistency as submitted for publication on 11 August 1935. There a reduction procedure for *sequents* is given. These have the form $\Gamma \rightarrow C$. The sequent is reduced to simpler forms starting from C . At a certain point, it happens that there is just an equation at right. Assume it turned out false. Now the task is to reduce the antecedent part so that it also turns out to contain a false equation. Then, in whatever way one succeeds in producing a false equation at right, it should be possible to produce a false equation at left. Suppose this succeeds. Then, if all formulas in the antecedent are correct, so is the formula in the succedent. The proof is finished by showing that derivable sequents are of the said kind, “reduce to endform” as Gentzen said. The reduction procedure is an attempt at producing a *counterexample* to $\Gamma \rightarrow C$, even if Gentzen does not use that terminology. It inspired Gödel’s Dialectica-interpretation. Gödel had studied Gentzen’s original proof in the fall of 1935 in Princeton.

Some letters indicate that Gentzen had the consistency proof ready by the end of 1934. By that time, **INH** had been superseded by another series with the signum **WTZ**, obviously for something like *Widerspruchsfreiheit transfinite Zahlen* (Consistency transfinite numbers). The late 1934 parts of **INH** contain already plans and outlines and details for a paper to be published that can be compared to corresponding parts of the 1935 paper. Then, in the other direction, one can understand parts of **INH** by studying the 1935 paper.

There is another series in the Gentzen papers that is useful for the evalua-

tion of the latter parts of **INH**. It has the signum **BZ**, for *Beweistheorie der Zahlentheorie* (Proof theory of number theory), and extends from August 1934 to March 1935. Nine out of eleven pages have been preserved. These refer to another series **WAV** of which ten pages are extant. The title of the latter should stand for *Widerspruchsfreiheit der Arithmetik Veröffentlichung* (Consistency of arithmetic publication). A large part of **BZ** has to do with the production of a prenex normal form for formulas in sequents, and the same holds for **WAV**.

6. Investigations into logical inference: 1932–33

This item is an early manuscript version of Gentzen's doctoral thesis *Untersuchungen über das logische Schliessen*. I found a photocopy of the manuscript during a visit to Erlangen in February 2005. It was among the documents about Gentzen that Christian Thiel has gathered. An accompanying letter indicated that it had been sent to him by Eckart Menzler in 1988 and that the original belongs to the Paul Bernays collection at the ETH-Zurich. From their web-pages one finds that there is indeed such a document, numbered *Hs.* 974:271 and given as 41 pages in length and with the, somewhat misleading, indication that the first 12 pages have been handed in as a dissertation. One also recognizes the photographic reproduction of part of a page from Gentzen's manuscript on p. *xiii* of Szabo's edition of Gentzen's *Collected Papers* (North-Holland, 1969). Page 9 of the manuscript contains the standard detour conversion schemes of natural deduction. The introductory chapter closes with a summary of Gentzen's main results in the planned thesis, ending with: "The consistency of arithmetic will be proved." The next part consists of a long, detailed proof of normalization for intuitionistic natural deduction. There follow short summaries of an intuitionistic sequent calculus, with remarks that relate to attempts at proving the consistency of arithmetic, a multisuccedent classical calculus, and a final part on translations between the new calculi and axiomatic logic. The composition of the thesis underwent changes that are indicated by the order of the material and by marginal additions.

The manuscript is written in the "Sütterlin Schrift," an obsolete form of handwriting that is not easy to read, but the 14 pages of the proof of normalization are a later addition written in a less formal hand he had in other occasions used for the printer. The proof is found in English translation in *The Bulletin of Symbolic Logic*, vol. 14 (2008), pp. 245–257.

A comparison to the printed thesis is useful because of additions and deletions, as with the inversion principle in chapter I, §5 where the manuscript clearly states that the reducibility of derivations through a match of *I*- and *E*-rules aims at decidability and consistency results, thus revealing Gentzen's original idea for a consistency proof of arithmetic. I have made such comparisons in my *Gentzen's proof systems: byproducts in a work of*

genius (2012). More on this topic is found in Negri and von Plato (2015).

Gentzen's original plan for the thesis seems to have been the following: First set up the calculus of natural deduction, a rule of induction included (chapter I). Next show that the calculus is equivalent to an axiomatic calculus (II). Then show normalization for intuitionistic natural deduction (III). In the next step, show that classical arithmetic, and especially the question of its consistency, reduces to intuitionistic arithmetic (IV). Finally, extend the subformula property of natural deduction to the system of intuitionistic arithmetic, to prove the consistency of the latter (V).

Gentzen's grand plan for a doctoral thesis failed, as he found out some time around the turn of the year 1932–33, because the system of arithmetic does not have, and cannot have, a subformula property. He took out the chapter on the translation of classical into intuitionistic arithmetic, sent it first to Heyting, then to *Mathematische Annalen* in mid-March 1933, and developed classical and intuitionistic sequent calculi for pure logic to be presented as a doctoral thesis. This latter cannot have taken more than a few months, because the thesis was handed in on 26 May 1933. The finished thesis has not been preserved in Göttingen or elsewhere, but the published version should be the same, save for possible minor corrections.

In addition to the first introductory chapter that presents the system of natural deduction and the third on normalization, the handwritten manuscript contains the very first sketches towards an intuitionistic sequent calculus, a brief section on a multisuccedent classical sequent calculus, and translations between natural deduction and an axiomatic system of logic, the topic of the originally planned chapter II. These translations are different from those in the published thesis in which Gentzen started from an axiomatic calculus in which all the standard connectives and quantifiers are present, as in Hilbert (1931). Axioms are given there separately for each connective, so that an intuitionistic calculus can be obtained by leaving out the law of double negation. Derivations in such a calculus are first translated into intuitionistic natural deduction, then to sequent calculus, then to the axiomatic calculus, to make a full circle. The last of these translations is rather complicated. Afterwards the analogous translations for the corresponding classical calculi are outlined.

In the manuscript, the axiomatic calculus is that of Hilbert and Ackermann (1928) that contains only disjunction, negation, and universal quantification. Gentzen gives a classical system of natural deduction, called *N2*, that has rules for conjunction, negation, and universal quantification. Contrary to the published thesis, a direct translation of derivations in *N2* to derivations in axiomatic logic is given. This translation seems to be at the origin of the discovery of sequent calculus, but no trace of it can be seen in the published thesis. The translation is presented in detail in von Plato (2014).

Letters from Gentzen to Heyting from the winter and spring of 1933 show that the latter had received a manuscript with the translation from classical to intuitionistic arithmetic in January or early February 1933. Heyting was corresponding at the same time also with Gödel, and a letter shows that Gödel reacted rather strongly to the news that someone else had done the thing (see Gödel's *Collected Works*, vol. V, p. 70). In the end, Gentzen withdrew his paper from publication, but Bernays kept a set of galley proofs through which the work was preserved and finally published in English translation in 1969 and in the German original in 1974. A page of corrections in the manuscript of the thesis is left where there once was a planned fourth chapter, all corrections done in the proofs. It can be gathered that the text of the article came directly from that chapter.

As said, the chapter on normalization is a later addition to the manuscript, distinguished by a standard handwriting style that can be read without greater difficulties. It contains also notational and terminological changes as compared to the first chapter, such as the use of horseshoe implication in place of the earlier arrow.

7. Reduction of classical to intuitionistic logic: I.33

This remarkable short note contains a proof transformation of derivations in classical natural deduction for arithmetic to ones in arithmetic with minimal logic in the fragment with $\supset$, $\neg$, and $\forall$. The idea is that double negation elimination reduces to components in the fragment until it reaches equations for which it holds intuitionistically. The procedure is presented very clearly in Bernays (1935a), so it can be concluded that Gentzen had explained the transformation to him. The note suggests, incidentally, that the normalization theorem was proved by January 1933, at least for the fragment in question.

8. CV of candidate Gerhard Gentzen: V.33

Gentzen wrote a short CV to accompany his doctoral thesis, dated 26 May 1933, that contains useful remarks about his early work, including that he has “been occupied with this field in detail independently.” It is found together with the documents about his state exam of November 1933, at the Niedersächsische Hauptstaatsarchiv in Hanover (Hann. 144, Acc. 65/85, Nr. 2289).

9. Letters to Heyting: II.33–IV.34

There are four known letters from Gentzen to Heyting, beginning with one of 25 February 1933 that concerns Gentzen (1933), the manuscript on the interpretation of classical arithmetic in intuitionistic arithmetic. The second letter announces the withdrawal of this paper from publication. Gentzen's

last letter is from 16 April 1934, and there is in addition a postcard signed by Gentzen and Heinrich Scholz, sent during Gentzen's visit to Scholz in Münster, 28 June 1936.

There is a logical thread in the correspondence that can be felt, even in the absence of replies on Heyting's part: In the end of the second letter, Gentzen mentions that he has a decision procedure for intuitionistic propositional logic. In the next letter, he explains to Heyting that the decision procedure comes from "a quite general theorem about intuitionistic and classical propositional and predicate logic," which we know to be the cut elimination theorem. Then he indicates an application to underviability results in predicate logic, clearly with Heyting's (1930) article in mind. Gentzen's last letter answers "affirmatively to your question," namely, that the disjunction property of intuitionistic logic follows from Gentzen's theorem.

The letters that Gentzen sent to Heyting contain announcements of results that are not always found elsewhere in his work, most importantly, underviability results in intuitionistic predicate calculus, especially the double negation shift that was earlier obtained through counterexamples that use intuitionistic real numbers, as in Brouwer (1928) and Heyting (1930). These are discussed in some detail in Section I.4.5 above.

The letters are found in the Heyting collection at the *Rijksarchief* in Haarlem. The Heyting-Johansson correspondence discussed above in Section I.5 is also found there.

10. Formal conception of correctness in arithmetic II: II.33–X.34

This is the second part of **INH**, the details of which have been described above.

11. Proof theory of number theory: VIII.34–III.35

There are ten extant pages of the series **BZ**, for *Beweistheorie der Zahlentheorie*. These pages contain mainly a conversion of derivations in the classical sequent calculus *LK* so that all formulas are in prenex normal form. The extant pages **BZ** are found in the violet folder after the series **INH**. They are **BZ** 1–6 (VIII.34–XI.34) and 9–12 (I.35–III.35). The series is not a record of continuous work. The first page has the dates VIII.34 and X.34. The latter date records work on the substitution of existential quantifiers by functions which is one of the central concerns of the series **VOR**. The extant pages of **VOR** are, in fact, found between pp. 10 and 11 of **BZ**. Pages 1–4 of **BZ** are dedicated to the production of sequent calculus derivations in which all formulas have a prenex normal form. It refers to **WAV** for a detailed treatment and the pertinent pages of the latter are preserved in between sheets of **BZ**.

Letters by Gentzen indicate that he had obtained a first proof of the consistency of arithmetic by about October 1934, if not even in the preceding

spring.

12. Consistency of arithmetic, for publication: X.34

Gentzen's stenographic manuscripts contain ten pages of a series **WAV**, written around October 1934. The signum stands for something like *Widerspruchsfreiheit der Arithmetik, Veröffentlichung* (consistency of arithmetic, publication), and the contents consist of preparatory notes for the writing of the long paper Gentzen submitted for publication in August 1935, with a major revision submitted in February 1936. The preserved pages are 55–56, 77–80, and 83–86. These are quite polished and readable and amount to 15 printed pages.

The series **BZ** contains material similar to **WAV** 55–56, 77, and 83–86, namely, the transformation of derivations in the classical sequent calculus *LK* into ones in which all formulas appear in prenex normal form. The idea was to delimit the “transfinite part” of arithmetic into the quantifier inferences, separated and appearing below a decidable and finitistic propositional part in the style of Gentzen's midsequent theorem. In the end, Gentzen chose to use in the paper he wrote in the spring of 1935 a natural calculus in sequent calculus style, mainly for reasons of exposition, to make the work accessible to a reader without logical training.

The most remarkable part of **WAV** are the pages 78–80. They give a three-page outline for a consistency proof of arithmetic, similar to the 1935 proof that is based on the notion of reduction of sequents, but done for a classical sequent calculus. It is referred to as “the second proof of correctness (*LK* consistency proof)” (p. 78), the first being a proof with a single-succedent calculus rather similar to the 1935 version. The crucial point of the proof is that reducibility is maintained under the application of the *mix rule*, i.e., the multicut rule of the doctoral dissertation.

The dissertation contained a form of sequent calculus that Gentzen used often, even if it was not explicitly singled out as such, namely what he called a calculus with *Grundsequenzen* (groundsequent calculus). It can be given in a single-succedent form, designated *LIG*, and a “symmetric” multisuccedent form, designated *LKG*. The idea is to replace half of the logical rules by logical groundsequents by which derivations can start. There is even a variant in which all classical propositional rules are substituted by groundsequents, as on p. 55 of **WAV**. By the “sharpened *Hauptsatz*,” cuts can be permuted to such sequents, but not completely eliminated. However, the cuts are harmless because they maintain the subformula property. Cut elimination is somewhat simplified by the use of groundsequents: It never happens that a cut formula is principal in both premisses of cut.

13. Correspondence with Paul Bernays: IV.34–VII.39

The correspondence with Bernays consists of 17 letters, 13 of them by Gent-

zen, and is at its most significant with the letters that concern Gentzen's original proof of the consistency of arithmetic. Bernays was in Princeton at this time, and his part of the correspondence of this period has not been preserved; The reader has to make inferences and guesses about a missing half of the letters. It seems that Gentzen understood quite well the remarks made by Bernays, and also by Gödel and von Neumann who were in Princeton at the same time. At one point, on 4 November, Gentzen sent a four-page tightly and very orderly written letter to Princeton where Bernays, possibly with the help of Gödel, had taken up the central problem of Gentzen's proof, as can be gathered from Gentzen's answer:

I have considered all these aspects already myself, including the geometrical image of branching line segments. You are quite right that the finiteness of even a single reduction path for the sequent $\Gamma, \Delta \rightarrow C$ can get grounded on the finiteness of a whole series of different reduction paths for $D, \Delta \rightarrow C$. But this does nothing for my proof idea!

The letters contain many interesting details about Gentzen's later work, up to the summer of 1939. For this period, four letters on the part of Bernays have been preserved, even if some others by both clearly have disappeared. In a letter of 13 June 1938, Bernays expresses dissatisfaction with the proof of Gentzen's lemma 3.4.3 of the 1938 proof of consistency, and suggests an alternative, but at this stage, only minor changes were possible in the paper that was in print. One of the ideas of Bernays, from a letter dated 9 May 1938, is to look at the reduction procedure of sequents of Gentzen's original 1935 proof as a non-formal calculus with an infinitistic rule and with consistency as cut elimination for the said calculus, in the style known otherwise from Kurt Schütte's work (1951). There is also discussion, in connection with the preparation of the second volume of the *Grundlagen der Mathematik* in September–November 1937, about Hilbert's ε -substitution method that Bernays endorsed and Gentzen instead found “not worth the effort.”

Gentzen's last preserved letter to Bernays is from 16 June 1939. It is an answer to a lost letter on the part of Bernays, dated 31 May 1939. The contents of the letter appear somewhat mysterious, but Bernays refers to this exchange in a letter to Gödel of 28 September the same year (see Gödel 2003, p. 126): He tells that Gentzen has extended the double-negation interpretation to simple type theory. In Gentzen's letter, instead of a translation of the logical language into the fragment with $\&$, $\neg$, and $\forall$, double negations are added in front of each subformula of a formula in the way of the Kolmogorov translation, known to some extent to Gentzen at this point. The effect is that the law of double negation, $\neg\neg A \supset A$, is provable for each A , even for the comprehension axiom as Gentzen notes (in the formulation of page 360 of his 1936 paper on the consistency of type theory). Gödel replies to

Bernays in late December with the remark that the intuitionistic interpretation is of no special interest (ibid, p. 130). Here Gödel's opinion is based on his conviction that intuitionistic logic fails to be properly constructive, witness his 1941 article "In what sense is intuitionistic logic constructive?"

A certain human touch is found in Bernays' worries about Gentzen's delicate psychological balance, with Gentzen telling about periods of depression – always writing to his "most respected Professor Bernays," deposed many years earlier by the Nazis, and Bernays responding to his "dear Mr. Gentzen."

14. The forms of type theory: VII.34–X.37

The signum **SLF** stands for the series *Formen der Stufenlogik* or *Stufenlogische Formen*, forms of type theory. The preserved pages are 15 to 18, the first with the date VII.34 and some additional comments dated IX.35. That is one month after Gentzen had submitted his paper *Die Widerspruchsfreiheit der Stufenlogik* (The consistency of type theory) for publication. The upper part of the first preserved page is a short unrelated text. The series proper seems to start with the title *General thoughts about the axioms of type theory* on page 15.

There are two brief notes that come with this series, one a reproduction of Carnap's formulation of the axiom of choice, another a remark by Bernays, noted in 1933, by which the axiom of choice is needed in analysis, for example to show that a bounded set has a point of convergence.

15. Predicate logic: 1935–IV.43

The violet folder contains a few pages from the series **PLF**, for *Prädikatenlogische Formen*, and some additional short texts on different aspects of predicate logic: Pages 1–2 and 6–9 belong to **PLF**, the first two pages containing the *Five different forms of natural calculi*, item 4 above. Pages 6–9 are a single sheet, with page 6 a discussion about the role of disjunction, item 15A, and page 7 blank. Item 15B, about decidability in predicate logic of February 1937, is on a separate piece of paper, with item 1 of September 1931 on the other side. Item 15C on decidability and the cut theorem of April 1938 is written at the back of item 7 of January 1933. Pages 8 and 9 contain the attempt at a "natural introduction of the calculus *LK*," i.e., the multisuccedent sequent calculus. It ends with a reference to the multisuccedent sequent calculus for intuitionistic predicate logic that is found in the series **AL**, here placed under 15E.

15A. Unnaturalnesses of inferences in formal logic: 1935

The topic of this note is the perplexity created by the introduction rule for disjunction: from A to infer $A \vee B$. Why would one weaken the claim in such a way? One answer Gentzen gives is that there can be a disjunctive

assumption, as in $A \vee B$. If A leads to C and B to D , then $A \vee B$ leads to $C \vee D$. The discussion ends with: “The apparently ‘natural’ conception of $\vee$ touches on many impurities that are left aside in the conception used in formal logic. Here $\mathfrak{A} \vee \mathfrak{B}$ means simply: At least one of the propositions holds. Thereby the inference OI is obvious.”

15B. Decidability in predicate logic: II.37

Gentzen asks first what it would mean if, say, the negation of Fermat’s theorem were provable from the axioms of elementary arithmetic: There would be a counterexample that follows by logical inferences and no induction. Gentzen then mentions his attempts at solving special cases of the decision problem in number theory by cut elimination. Of decidability in the cases of existential propositions, he notes that such “is understandably not possible, because there need not obtain any connection between the simplicity of the axioms and the length of construction of an existing object with a determinate property. . . In any case, one does not know anything about this. With the number-theoretic axioms also presupposed, one cannot know whether by ever new computations, a number arises in the end with a determinate computable property. This can take an eternity.”

15C. Decidability and the cut theorem: IV.38

This item has been added in the bottom part of the second page of item 7. It contains, similarly to the preceding one, brief comments on Gentzen’s hope, and subsequent failure by and large, to obtain decidability results in predicate logic through the cut elimination theorem.

15D. Natural introduction of the calculus $LK?$: IV.38

This item occupies pages 8 and 9 of the series **PLF**. The topic is to introduce the classical multisuccedent sequent calculus LK in a pedagogical way. The result is not particularly satisfying; as Gentzen notes: “A natural introduction of LK is possible to some extent, along the above path, but then one cannot yet recognize its advantage. I.e., its advantage does not depend on its being accessible in a natural way, but that is a more arbitrary side issue.”

15E. Formulation of intuitionistic logic with symmetric sequents: IV.43

Pages 141–142 of the series **AL** from IV.43 contain the details of a multisuccedent intuitionistic sequent calculus, the quantifiers included, with exactly the right kind of restrictions in rules for $R\supset$, $R\neg$, and $R\forall$. The multisuccedent intuitionistic sequent calculus was reinvented by Maehara in 1954.

16. Propositional logic: II.37–X.42

Very little is left of the series **AL**, a mere six pages between **AL**133 and

142. There is a short note on the classical propositional calculus *LK*, and a substantial attempt at a decision method for intuitionistic propositional logic, and the multisuccedent intuitionistic sequent calculus for predicate logic on pages 141–142, placed in the previous item.

16A. Completeness of the propositional calculus *LK*: II.37

On page 133 of **AL** is found a text with the date II.37 that is about the completeness of the propositional calculus of *LK* “after a reading of Kalmár.” The work in question is obviously Kalmár’s (1935) article. Kalmár shows that if any unprovable formula is added to classical propositional logic, the calculus becomes inconsistent. He gives, among others, a very careful formulation of the deduction theorem. Gentzen’s sequent calculus would be a natural setting for most of Kalmár’s results, as Kalmár acknowledges.

16B. Correctness and completeness for positive propositional logic: X.42

Pages 137–139 contain the attempt at a decision procedure dated X.42, and described in detail in I.5.1 above.

17. Foundational research in mathematics: IV.39–42

This item is described in some detail in Section I.5.4 above. At the back of the last page of these notes, Gentzen had sketched a statement about his illness in 1942.

3. PRACTICAL REMARKS ON THE MANUSCRIPTS

The stenographic manuscripts are found in two folders, one blue, the other violet. Each cover has a glued slip of paper, the blue one with the text *Buch: „Die mathematische Grundlagenforschung“* and the violet one with *Aussagenlogik*, cancelled with a light line. The violet folder contains in addition the name “Dr. Gerhard Gentzen” and the following text on a smaller paper:

VIII.44. This folder contains what has still some value from **PL**, **AL**, Consistency of number theory, **BZ**. Not duplicated.

For Putbus.

As well as **WKRd**, [cancelled: **WTZd**]

WA earlier.

The contents are as follows:

The blue folder:

Book: Mathematical Foundational Research (**BG**)

Proof theory of intuitionistic number theory (**BTIZ**)

The violet folder:

The forms of type theory (**SLF**)

Consistency of analysis (**WA**)

Propositional logic (**AL**)

The forms of predicate logic (**PLF**)

Formal conception of the notion of correctness in number theory (**INH**)

Proof theory of number theory (**BZ** and **WAV**)

The pages are not in a numerical order, but mixed somewhat for reasons of continuity of argument, as in the case of **BZ** and **WAV**.

Stenographic writing has been used for various purposes. The main use has been to write down what was said, say in a dictation or lecture. Thus, the idea is just to reproduce a spoken language in written form. Stenographic writing for one's own purposes must have another reason. With Gentzen's research notes, the increase in writing speed in itself should not have been a consideration of importance. Rather, it seems that once the habit of writing in shorthand is acquired, the slowness of longhand writing would be felt as a disturbing element in the rendering of thoughts in a written form. Another aspect of stenography is that longhand writing takes much more space. With all the long series of manuscripts that Gentzen wrote, compactness may have played a role. He seems to have kept the piles of papers in a cellar, as indicated for example on page 245 of the series **BTIZ**.

The idea of shorthand writing is old, and such systems were used already in Antiquity. In the cursive form of shorthand, words are written by and large as continuous lines. It was developed by Franz Gabelsberger, and his system of writing was the prevailing one in the German-speaking world from its creation in the 1830s to 1924. Logicians such as Paul Bernays and Kurt Gödel used it constantly for notes and for the preparation of lecture series, say. Another competing shorthand notation, developed from Gabelsberger, was the Stolze-Schrey system, used for example by Rudolf Carnap and Hermann Weyl.

Many variants of Gabelsberger shorthand exist in languages other than German, even one in Finnish as I happen to remember from my childhood: The mail brought every now and then to our home a –very thin!– bulletin of which I could read only the title, the rest was a mystery: *Pikakirjoitus-lehti* (Finnish for Shorthand Bulletin). Sweden has its own system, but the Swedish-language minority of Finland has retained the original Gabelsberger writing, perhaps a unique relict in the world. It is upheld by a tiny society with the name *Föreningen Gabelsberger*, and the main task is to record what is said in Swedish at parliamentary sessions.

There were too many variants of shorthand in Germany, and therefore a reformation was forced on the competing schools of stenography that led to a compromise result in 1924. The multitude of systems was replaced by a “unified shorthand” (*einheitliche Kurzschrift*) that retained most of the consonant writing of Gabelsberger but changed the mode of indicating the vowels. The shorthand is still taught at least in Germany, and used by secretaries and journalists among others.

It is difficult to say if there is more than a casual connection, the brevity of expression, between interest in symbolic logic and in shorthand, say some common ideal of rationality. Gentzen was obviously completely at home with them. It is known that his mother had taught shorthand writing, both German and English. In addition to shorthand, Gentzen wrote in three different styles plus typewriting. One was the formal handwriting that was part of the school education, called Sütterlin-script, as in the handwritten thesis manuscript. A sample can be found in the frontmatter of Szabo’s edition of Gentzen’s *Collected Works* that contains a photographic reproduction of part of page 2 of the manuscript. Sometimes marginal remarks are written in a rather childish-looking and clumsy, but otherwise readable hand. When writing for a printer, Gentzen seems to have used either a typewritten manuscript, or a very readable though somewhat irregular handwriting as in the chapter on normalization that was later added to the thesis manuscript. Even here, formulas are usually written with the Sütterlin alphabet. The writing of formal expressions uses also in shorthand mainly the Sütterlin. A printer would set such in Fraktur; well, there are hardly any printers left as the typesetting task has been relegated to us authors, and I have followed what was once the standard practice.

A special aspect of logic is the use of metasymbols, *Mitteilungszeichen* in German, or “signs for communication.” Gentzen’s thesis manuscript, item 6 below, begins with the following explanation:

§1. *Short explanation of some ways of notation.*

I shall distinguish between (*cf.* H.A. [Hilbert-Ackermann] pages 51–52):

Object variables, variables for short. E.g. $a, b, c \dots$ } *Objects.*
Specific objects, e.g. 1, 2, 3 ...

Predicate variables, e.g. $A(\ , \ , \), G(\), P.$ } *Predicates.*
Specific predicates, e.g. $< , =$ } (In H.A.: logical functions)

Gentzen writes the letters in a Latin style, sometimes in block letters, at other times in cursive. The example proofs that follow in the introductory chapter are also written in a Latin style, first with X, Y, Z as propositional letters, then F as a predicate letter. When Gentzen comes to explaining the

tree structure of derivations, he starts writing formulas in Sütterlin, rendered as Fraktur in a corresponding passage of the printed version of the thesis.

In the printed thesis, we have the following divisions and notation (p. 179): There are signs that are divided into “determinate” ones such as signs for numbers, arithmetic and logical operations, and specific predicates and relations. Parentheses, commas, and the sequent arrow are “auxiliary signs.” The signs for object and predicate variables are as in the handwritten version. Finally, “German and Greek letters serve us as ‘signs for communication,’ so they are not signs of formalized logic, but variables for its consideration.”

The Sütterlin handwriting, designed on commission by Ludwig Sütterlin and taken into use in 1915, is much harder to read than the printed Fraktur; Even if printing does not do justice to the cursive nature of the former, it will give some idea of the difference, as in *Sütterlin* and *Fraktur*. In the former, many lower case letters are hard to distinguish, as in *meinen* and *nennen*. Even some capitals are very close to each other, such as *I* that stands for capital I, not J, the latter being *J*.¹ In Fraktur, letters are close to each other in just a few cases, as with capital C and E that are printed as *C* and *E*, respectively. The main thing to keep an eye on are the variables *x, y, z* that are printed in Fraktur as *x, y, z*, for the Sütterlin *x, y, z*. There is at the end of this introduction a list of letters in today’s standard German alphabet, in Sütterlin, and in Fraktur. At places, Gentzen’s writing of a cursive Latin *F* and Sütterlin *f* come very close to each other, but in the end, no difficulty is created if these are mixed.

In deciphering formulas in the shorthand notes and in deciding how they should be printed, the handwritten thesis and its printed version are the best guide. For example, in the series **WA**, there is a function that seems to be denoted by *sc*, but it could also be an upper case Sütterlin *ℵ* that would be printed as *ℵ*, written very small, or even a Greek *ℵ*. At one point, Gentzen changed the notation, writing an upper case letter on top of the *s*. The same letter is found in the word *Schlüssen* on the first page of chapter III of the thesis manuscript, the one that does not use Sütterlin. Then, by the context of use, this *Sc* stands for something like “Schnitt-charakter.”

The manuscripts have lots of underlined passages. It could be that this is in part a side-effect of stenography, meant to highlight topics and concepts that perhaps would not otherwise be as easy to detect as in longhand writing and in a printed text. On the *other* hand, even Gentzen’s *published* papers use a lot of *emphasis in italics* that may *seem* as exaggerated as *here*. The manuscripts have also often titles and other passages highlighted by a coloured underlining, usually red, blue, or green. The scans I have used have a good resolution and show minute details of the manuscripts. Copies of these scans are kept at the Philosophisches Archiv der Universität Konstanz.

¹ For similar typographical reasons, Gentzen’s intuitionistic calculi *NI* and *LI* have been almost always rendered erroneously as *NJ* and *LJ* in English.

The manuscripts have what are obviously later additions, sometimes seen from an added date. At other times, Gentzen indicated later additions by freely drawn triangles that I have reproduced ▷ like this ◁. Most manuscript pages contain series and page numberings in a thick red pencil. Sometimes there is a letter d, apparently for “Durchsicht,” overview, written with a green pencil. Yet another usage is a thick vertical black line in the center of a page or passage. It could be a deletion, but it could also just indicate that Gentzen had read the passage and summarized it somewhere else. Shorter cancellations are usually clear, several light horizontal lines drawn over the text. There are also parentheses out of the normal usage. They may highlight a side issue. Finally, there are vertical marginal lines of various types, say, a double violet line. What all these emendations to the plain text mean is not easy to say. I have in each case read the text and made my own decisions.

4. MANUSCRIPT ILLUSTRATIONS

The illustrations begin with a list of the German alphabet in three different forms, followed by a scan of a table from the book *Lebendige Sprache: Ein Wegweiser in die Muttersprache* by Hans Schorer and Albert Wiechmann (Verlar Moritz Diesterweg, 10th printing 1959). This table comes much closer to written Sütterlin than the type L^AT_EX offers. The book is my own copy that I used at the *Deutsche Schule Helsinki*, found among old papers in my childhood home several years after my rediscovery of Gentzen’s thesis manuscript in 2005. By about 1960, the teaching of Sütterlin had been abandoned since two decades, but older people used it, as on page 91 of *Lebendige Sprache* that contains a “letter from aunt Gertrud” printed in a Sütterlin emulation, so children would have to learn to read it.

There follow various illustrations of Gentzen’s manuscripts:

The first of these illustrations, on page 88, is from the series **WAV**, lower part of page 78 from late 1934. It is written with a sharp pencil in a very clear hand. The bottom of the paper is somewhat damaged and a word in the middle of the line not clearly readable. What is said in the sentence is clear anyway and can be read from the facing page.

Page 90 is from the series **BTIZ**, page 245.10 written in April 1942. The text is unified shorthand written with a dull pencil on a cardboard-like paper. The bottom has a later addition, as indicated by the ▷triangles◁. The German transcription is given on the facing page.

Page 92 is from the handwritten version of Gentzen’s thesis, written in the Sütterlin-script possibly towards the end of 1932. There is a facing transcription.

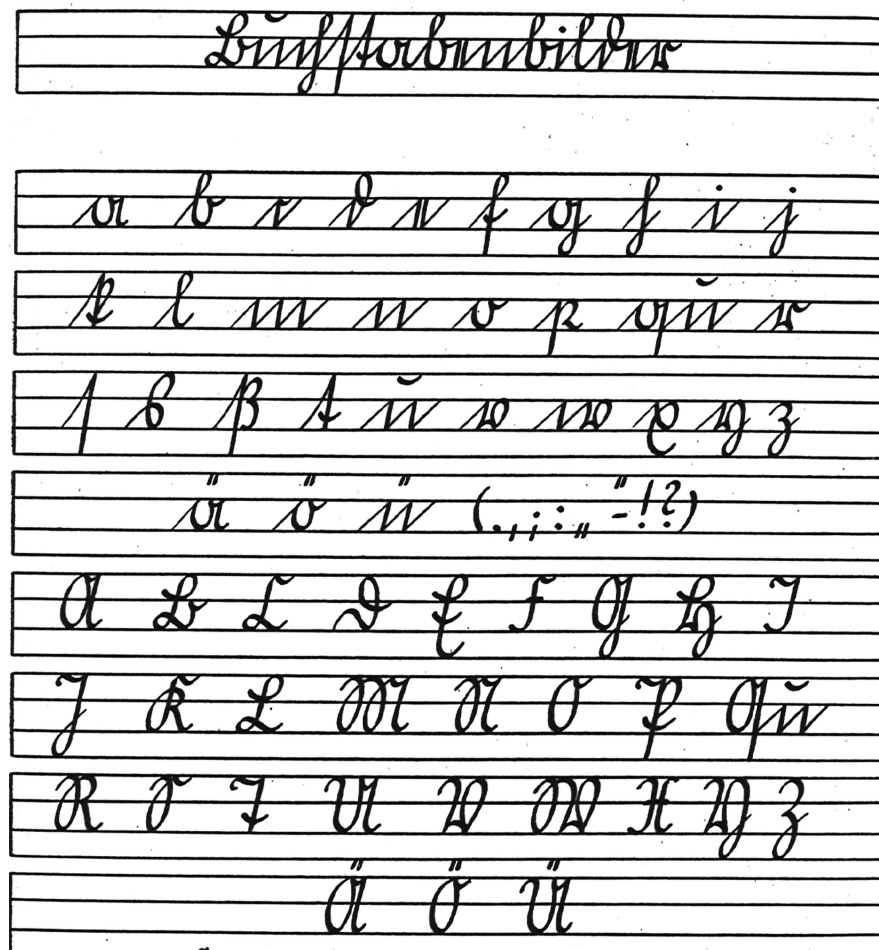
Page 94 is also from the handwritten thesis manuscript, from the chapter on normalization some time in early 1933. It is written in a readable hand-

writing that Gentzen seems to have used when not writing in shorthand or in the formal Sütterlin-script. A transcription would be hardly needed.

THE GERMAN ALPHABET IN LATIN, SÜTTERLIN, AND FRAKTUR TYPE

The following table gives the alphabet in today's upper case standard Latin (Lat), Sütterlin ($\mathfrak{U}$), Fraktur ($\mathfrak{F}$), and lower case Latin (lat), Sütterlin ($\mathfrak{u}$), Fraktur ($\mathfrak{f}$). There are two variants of lower case s in Sütterlin, one as in the above $\mathfrak{u}$, and a second $\mathfrak{h}$ that is used in the end of words, as well as a symbol for double s, similar to today's German β . Letters with umlauts have two short vertical lines resp. two dots above.

Lat	$\mathfrak{U}$	$\mathfrak{F}$	lat	$\mathfrak{u}$	$\mathfrak{f}$
A	$\mathfrak{A}$	$\mathfrak{A}$	a	$\mathfrak{a}$	$\mathfrak{a}$
B	$\mathfrak{B}$	$\mathfrak{B}$	b	$\mathfrak{b}$	$\mathfrak{b}$
C	$\mathfrak{C}$	$\mathfrak{C}$	c	$\mathfrak{c}$	$\mathfrak{c}$
D	$\mathfrak{D}$	$\mathfrak{D}$	d	$\mathfrak{d}$	$\mathfrak{d}$
E	$\mathfrak{E}$	$\mathfrak{E}$	e	$\mathfrak{e}$	$\mathfrak{e}$
F	$\mathfrak{F}$	$\mathfrak{F}$	f	$\mathfrak{f}$	$\mathfrak{f}$
G	$\mathfrak{G}$	$\mathfrak{G}$	g	$\mathfrak{g}$	$\mathfrak{g}$
H	$\mathfrak{H}$	$\mathfrak{H}$	h	$\mathfrak{h}$	$\mathfrak{h}$
I	$\mathfrak{I}$	$\mathfrak{I}$	i	$\mathfrak{i}$	$\mathfrak{i}$
J	$\mathfrak{J}$	$\mathfrak{J}$	j	$\mathfrak{j}$	$\mathfrak{j}$
K	$\mathfrak{K}$	$\mathfrak{K}$	k	$\mathfrak{k}$	$\mathfrak{k}$
L	$\mathfrak{L}$	$\mathfrak{L}$	l	$\mathfrak{l}$	$\mathfrak{l}$
M	$\mathfrak{M}$	$\mathfrak{M}$	m	$\mathfrak{m}$	$\mathfrak{m}$
N	$\mathfrak{N}$	$\mathfrak{N}$	n	$\mathfrak{n}$	$\mathfrak{n}$
O	$\mathfrak{O}$	$\mathfrak{O}$	o	$\mathfrak{o}$	$\mathfrak{o}$
P	$\mathfrak{P}$	$\mathfrak{P}$	p	$\mathfrak{p}$	$\mathfrak{p}$
Q	$\mathfrak{Q}$	$\mathfrak{Q}$	q	$\mathfrak{q}$	$\mathfrak{q}$
R	$\mathfrak{R}$	$\mathfrak{R}$	r	$\mathfrak{r}$	$\mathfrak{r}$
S	$\mathfrak{S}$	$\mathfrak{S}$	s	$\mathfrak{s}$	$\mathfrak{s}$
T	$\mathfrak{T}$	$\mathfrak{T}$	t	$\mathfrak{t}$	$\mathfrak{t}$
U	$\mathfrak{U}$	$\mathfrak{U}$	u	$\mathfrak{u}$	$\mathfrak{u}$
V	$\mathfrak{V}$	$\mathfrak{V}$	v	$\mathfrak{v}$	$\mathfrak{v}$
W	$\mathfrak{W}$	$\mathfrak{W}$	w	$\mathfrak{w}$	$\mathfrak{w}$
X	$\mathfrak{X}$	$\mathfrak{X}$	x	$\mathfrak{x}$	$\mathfrak{x}$
Y	$\mathfrak{Y}$	$\mathfrak{Y}$	y	$\mathfrak{y}$	$\mathfrak{y}$
Z	$\mathfrak{Z}$	$\mathfrak{Z}$	z	$\mathfrak{z}$	$\mathfrak{z}$



The Sütterlin alphabet was still taught at the *Deutsche Schule Helsinki* around 1960. This illustration with the title *Buchstabenbilder* and beautiful handwritten Sütterlin letters is from the author's personal copy of the book of readings *Lebendige Sprache. Ein Wegweiser in die Muttersprache* (part 2 for the third and fourth school-years), written by Hans Schorer and Albert Wiechmann and published by Verlag Moritz Diesterweg (tenth printing, 1959).

9. Richtigkeits
§(7.) Der 2. (Widerspruchsfreiheits) beweis.

LK WfBeweis

Nach dem 3. Ersetzungssatz können wir uns beschränken auf den symmetrischen Kalkül ohne transfinites Vorkommen von $\&$, $\vee$, $\neg$ ($\supset$ kommt überhaupt nicht vor.)

Der Richtigkeitsbeweis kann auch ohne besondere Schwierigkeiten auf die Zulassbarkeit dieser Verknüpfungen ausgedehnt werden, der 3. Ersetzungssatz stellt keinen wesentlichen Teil desselben dar. Doch wird das seit Vorwegnahme der Widerspruchsfreiheitsbeweis selbst [cancelled: klarer] übersichtlicher, mehr auf das Wesentliche beschränkt.

Der ganze Beweis verläuft analog zum 1. Richtigkeitsbeweis, formal [cancelled: noch] einfacher als jener. Wir beginnen wieder mit der Erklärung eines Begriffs „reduzierbar“ für Sequenzen.

[Cancelled: Dieser stimmt für Sequenzen mit einer Hinterformel mit dem alten überein. Ist also nur eine Verallgemeinerung derselben.] (nein! Durch die Vermehrungsmöglichkeit von Hinterformeln wesentlicher Unterschied!) Eine Sequenz mit freie Variable ist reduzierbar, wenn jede durch Einsetzen beliebiger Zahlen für die freie variable entstehende Sequenz und anschließendes Ersetzen aller Terme durch ihre Werte in dem nachfolgenden Sinn reduzierbar ist.

Reduzierbarkeit für Sequenzen ohne freie Variable:

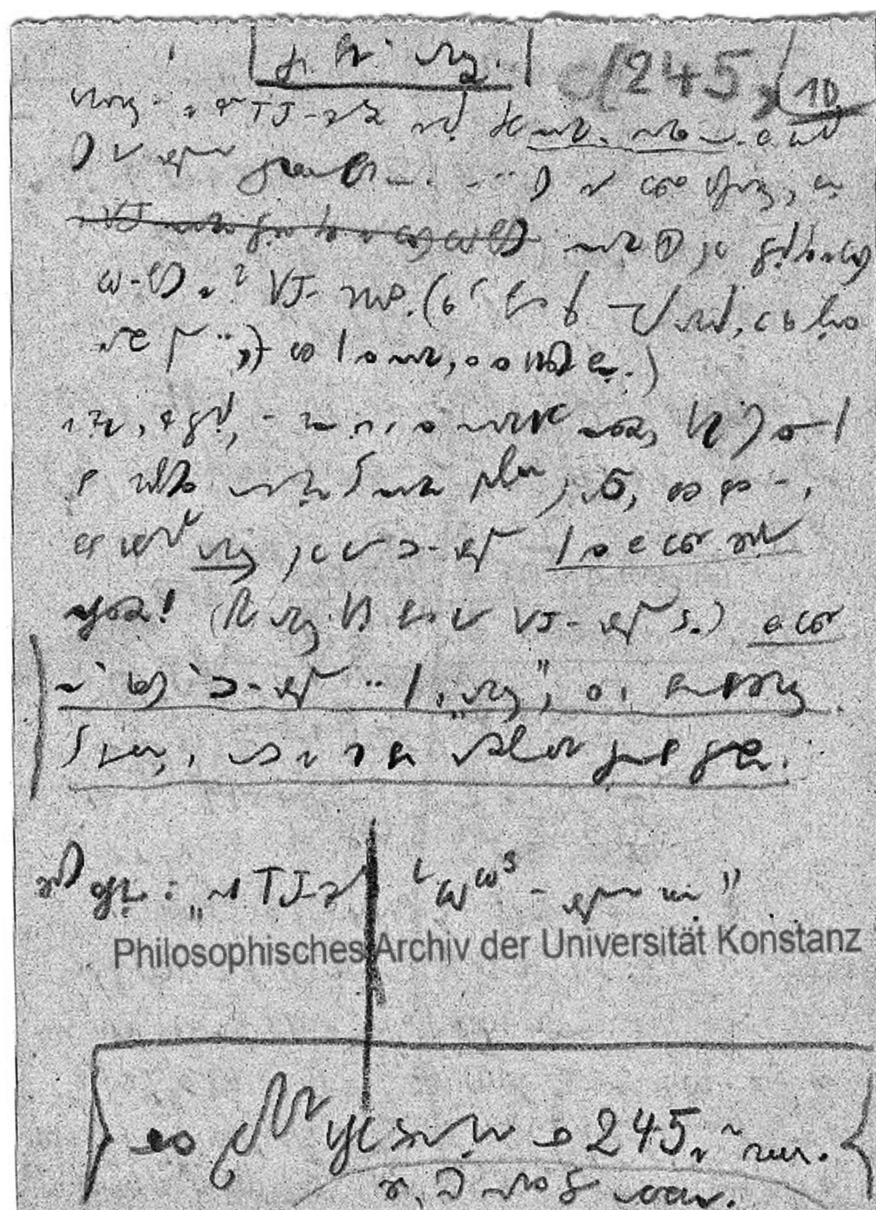
Zunächst „Reduktion einer Sequenz“:

Freie Reduktionen: Man ersetzt eine Hinterformel der Gestalt $\forall x Fx$ [written above: oder Vorderformel $\exists x Fx$] durch $F\nu$ mit beliebig gewähltem ν , Zahl.

[To the right of previous: Etwa gleich nur für Sequenzen mit transfiniten Bestandteilen erlaubt. So dass finite Sequenzen ohne weiteres $\supset$ nur reduzierbar sind, wenn sie richtig sind.]

Gebundene Reduktionen: Man ersetzt eine Hinterformel $\exists x Fx$ beziehungsweise Vorderformel $\forall x Fx$ durch $F\nu$ mit einem bestimmten, irgendwoher gegebenen ν .

Termreduktionen: Ein Term ohne freie Variable wird durch den betreffenden Zahlenwert, nach Definitionen verdünnt [?], ersetzt. (Nach oben verlegt, wohl nötig wegen z.B. EE , AB .)



Page 245.10 from the series BTIZ.

Zur Frage der Vermehrungen

245.10

Bemerkung: In diesen TI-Herleitungen kommt jede Annahme nur einmal vor. Das bleibt auch bei Reduktionen zwangsläufig so. Es ist auch keine wesentliche Beschränkung, denn [cancelled: die VI-Annahmen stecken ja in Wahrheit ω -fach] Annahme ① steckt ja in Wahrheit ω -fach in dem VI-Ergebnis. (Sie wird freilich erst so weit vermehrt, wenn sie offenes Land geworden ist, d.h. nicht als Annahme, sondern als Tatsache dann.)

Ich glaube, dies stimmt, und man kann es als normativ ansehen, braucht sich also nicht durch mehrfaches Vorkommen von Annahmen irreführen zu lassen, d.h. dieses und die dadurch bedingte Vermehrung, wie bei $\supset$ -Reduktion nicht als das Wesentliche hierbei anzusehen! (Echte Vermehrungen treten freilich bei VI-Reduktion auf.) Das Wesentliche an der Besonderheit der $\supset$ -Reduktion ist nicht die „Vermehrung“, sondern die Durcheinanderschachtung von Teilen, die vorher in ganz anderer Reihenfolge zueinander standen.

Hiernach Satzteil [?]: „Natürliche TI-Herleitung bis ω^{ω^3} -Reduktionsstellen“

▷ Alles Unwichtige bzw. Übernommene aus 245 in den Keller.

Hier nur noch einiges vielleicht Verwendbare. <

IV. ~~Annahme und die Folge:~~

Obwohl eine logische Überlegenheit im Fallbül VIJ bewiesen ist, so gilt ab einem solchen Beweis für ein, in diesem nur Überlegenheit von ihr zu kommen, allzufalls mit anderen Variablen.

Der Beweis hierfür ist ~~offenbar~~ fertig, ~~weil~~ für einen Fallbül N2J, welcher von N1J nur die Zeichen UE, UB, AE, AB, R enthält, aber dieselbe folgende Fortsetzung enthält:

Im Falle der Logikbeweise einer logischen Überlegenheit im Klappstein aus dem Funktionalkül läßt sich zurückführen auf die Folge der Logikbeweise einer (anderen) logischen Überlegenheit im Fallbül N1J.

(Dies hängt mit III zusammen.)

V. Im Widerspruchsforsicht der Axiomatik wird bewiesen; Dabei wird der Angriff der unmittelbaren Folgen von notwendigen Aussagen bewiesen, ferner von einer Fall der Folge von Widerspruchsforsicht.

Der Beweis ist also nicht intuitionistisch. Vielleicht läßt sich das Tertium non datur angreifen.

IV. ~~Vermutet wird der~~ Satz:

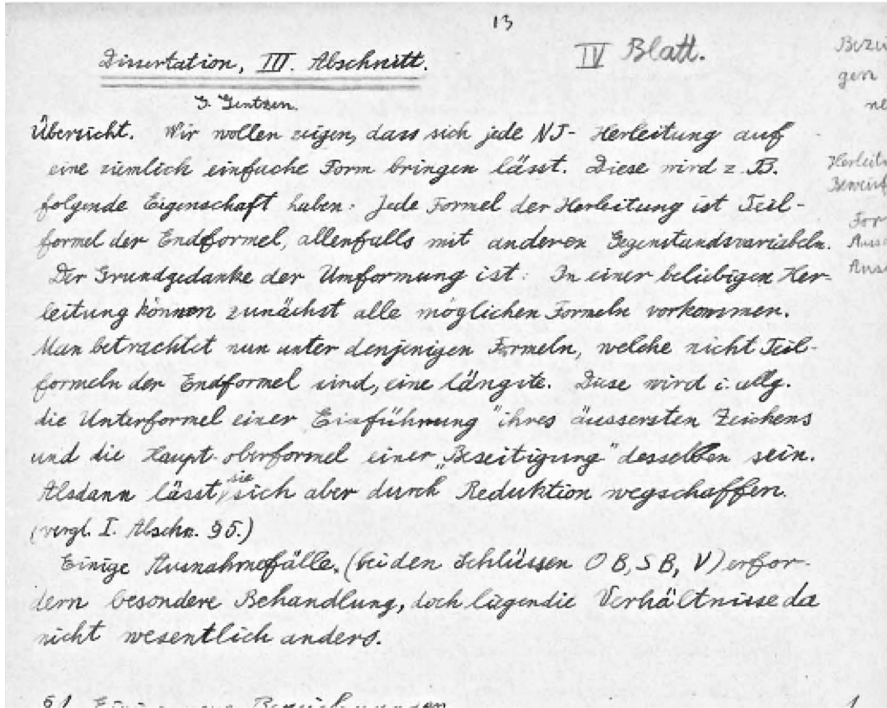
Wenn eine logische Aussage im Kalkül $N1I$ beweisbar ist, so gibt es einen solchen Beweis für sie, in welchem nur Teilaussagen von ihr vorkommen, allenfalls mit anderen Variablen.

Der Beweis hierfür ist fertig [cancelled: noch nicht fertig ausser für einen Teilkalkül $N2I$, welcher von $N1I$ nur die Schlüsse UE, UB, AE, AB, R enthält, aber durch folgende Tatsache Bedeutung besitzt:

Die Frage der Beweisbarkeit einer logischen Aussage im klassischen „engeren Funktionenkalkül lässt sich zurückführen auf die Frage der Beweisbarkeit einer (anderen) logischen Aussage im Kalkül $N1I$. (dies hängt mit III zusammen). End of cancellation]

V. Die Widerspruchsfreiheit der Arithmetik wird bewiesen; dabei wird der Begriff der unendlichen Folge von natürlichen Zahlen benutzt, ferner an einer Stelle der Satz von ausgeschlossenen dritten.

Der Beweis ist also nicht intuitionistisch. Vielleicht lässt sich doch tertium non datur wegschaffen.



Part of page 13 from the early handwritten version of Gentzen's thesis.

Dissertation, III. Abschnitt

G. Gentzen

Übersicht. Wir wollen zeigen, dass sich jede NI -Herleitung auf eine ziemlich einfache Form bringen lässt. Diese wird z.B. folgende Eigenschaft haben: Jede Formel der Herleitung ist Teilformel der Endformel, allerdings mit anderen Gegenstandsvariablen. Der Grundgedanke der Umformung ist: In einer beliebigen Herleitung können zunächst alle möglichen Formeln vorkommen. Man betrachtet nun unter denjenigen Formeln, welche nicht Teilformeln der Endformel sind, eine längste. Diese wird i.allg. die Unterformel einer „Einführung“ ihres äussersten Zeichens und die Haupt-oberformel einer „Beseitigung“ desselben sein. Alsdann lässt sie sich aber durch Reduktion wegschaffen. (vergl. I. Abschn. §5.)

Einige Ausnahmefälle (bei den Schlüssen OB, SB, V) erfordern besondere Behandlung, doch liegen die Verhältnisse da nicht wesentlich anders.

Saved from the Cellar

Gerhard Gentzen's Shorthand Notes on Logic and
Foundations of Mathematics

von Plato, J.

2017, X, 315 p. 9 illus., Hardcover

ISBN: 978-3-319-42119-3