Basics of Mechanics of Micropolar Shells

Victor Eremeyev and Holm Altenbach

Abstract The chapter is devoted to the introduction to the nonlinear theory of micropolar shells called also six-parametric shell theory. Within the theory a shell is described as a deformable directed material surface each point of which has six degrees of freedom (DOF), i.e. three translational and three rotational DOF. In other words the shell kinematics coincides with the kinematics of a two-dimensional (2D) micropolar or Cosserat body. Here we present the basic equations of the micropolar shell theory including variational statements, compatibility conditions, etc.

1 Introduction

Theory of plates and shells is one of the oldest branches of mechanics. First scientific publications in the field belong to Euler who published paper in 1767, to the paper by J. Bernoulli in 1789, see the bibliography collected by Jemielita (2001) and the first chapter in this book. Nowadays there are theories of plates and shells related with the names by Kirchhoff, Love, Cosserat, Timoshenko, Reissner, Mindlin, Koiter, Naghdi, Donnell, Vekua and many others. So, many models may be considered as refinement or extension of the classical Kirchhoff-Love theory of shells. Nowadays, various refined 2D theories are implemented in commercial FEM software and widely used in contemporary engineering practice. Nevertheless, the refinement of plate and shell

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theories are still required in various directions, see brief discussion by Eremeyev and Pietraszkiewicz (2014). In particular, the following directions in the field of thin-walled structures are very actual:

- refinement of 2D governing relations for better accuracy;
- application to modelling of new materials and phenomena;
- development of efficient numerical tools.

For various methods of derivations of plates and shells equations we refer to Ambartsumyan (1970), Naghdi (1972), Goldenveizer (1976), Reissner (1985), Novozhilov et al. (1991), Libai and Simmonds (1998), Ciarlet (1997, 2000), Wang et al. (2000), Tovstik and Smirnov (2001), Kabrits et al. (2002), Reddy (2003), Kreja (2007), Amabili (2008), Carrera et al. (2011), Jaiani (2011). The current state of the art in the field can be found in recent paper collections and reviews (Pietraszkiewicz and Szymczak 2005; Jaiani and Podio-Guidugli 2008; Pietraszkiewicz and Kreja 2010; Altenbach and Eremeyev 2011c; Altenbach and Mikhasev 2014; Alijani and Amabili 2014; Pietraszkiewicz and Górski 2014).

In this chapter we discuss the nonlinear micropolar shell theory using the direct approach and its applications. Within the direct approach the basic governing equations are derived for a 2D continuum. The discussed model coincides kinematically with the general resultant nonlinear six-parameter theory of shells derived using the through-the-thickness integrations of the motion equations of the nonlinear elasticity. The basics of this theory is presented in Libai and Simmonds (1983, 1998), Chróścielewski et al. (2004a), Eremeyev and Zubov (2008), Lebedev et al. (2010), Eremeyev et al. (2013), Altenbach and Eremeyev (2013b), Pietraszkiewicz (2015). Within the micropolar shell theory the kinematics of the shell is determined by two kinematically independent fields of translations and rotations. The surface stress and couple stress tensors are introduced in the theory. Each point of the micropolar shell base surface has six degrees of freedom as in rigid body dynamics. This means that the drilling moment is taken in account. The advantage of the six-parameter shell model is the correct description of multifolded shells, of interaction of a shell with a rigid body, etc., see Konopińska and Pietraszkiewicz (2007), Pietraszkiewicz and Konopińska (2011), Pietraszkiewicz and Konopińska (2015) and the references therein. The full micropolar kinematics may be important for proper modelling of piezoelectric or piezomagnetic shells since electromagnetics fields produce forces and moments including the drilling ones, see Eringen and Maugin (1990), Maugin (1988). In addition, this gives the possibility of description of the contact interaction of shells with distributed on its surface nano-objects Eremeyev (2005a), Eremeyev et al. (2015a) or sensors, actuators, absorbers, etc., see Koç et al. (2005), Akay et al. (2005), Carcaterra et al. (2012), Andreaus et al. (2004), Vidoli and dell'Isola (2001), dell'Isola and Vidoli (1998), dell'Isola et al. (2003), Maurini et al. (2004).

This chapter is almost based on recent works by Altenbach and Eremeyev (2013a, 2014a, b), Eremeyev et al. (2013, 2015b), Eremeyev and Zubov (2007). In what follows we use the direct tensor calculus as in (Lebedev et al. 2010; Eremeyev et al. 2013). Here vectors and tensors are denoted by semi-bold font shape.

2 On Rigid Body Dynamics

In this section we recall basic notions of rigid body dynamics such as the moments, the inertia tensors, the kinetic energy and others which are also used in continuum mechanics and mechanics of structures. For details we can refer to various textbooks, see e.g. Lurie (2001).

The *rigid body* $\mathcal{P}$ can be considered as a collection of mass points (material particles) and can be defined as follows.

Definition 2.1 A set of material points for which the mutual distances between the points remain unchanged in motion, is called rigid body.

The kinematics of the rigid body is determined by six parameters, by three translations of an arbitrary point of the rigid body and by three rotations. Let $o \in \mathcal{P}$ be a point of the body called the pole and $\mathbf{r}_0(t)$ is its position vector at instant t. This vector describes translations of the rigid body. For description of rotations we consider embedded the coordinate trihedron with unit vectors $\mathbf{d}_1(t)$, $\mathbf{d}_2(t)$, $\mathbf{d}_3(t)$, $\mathbf{d}_i \cdot \mathbf{d}_j = \delta_{ij}$, see Fig. 1, here δ_{ij} is the Kronecker symbol. Using $\mathbf{r}_0(t)$ and $\mathbf{d}_k(t)$ the position of any point $z \in \mathcal{P}$ is determined by

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \mathbf{z}(t), \quad \mathbf{z}(t) = z_i \mathbf{d}_i(t). \tag{1}$$

For the body, we fix an initial configuration $\varkappa$. For example, we can take the body position at instant $t = 0$ as the initial configuration. The position of the pole o, the point z and the embedded trihedron of the coordinate axes that are $\mathbf{R}_0$, $\mathbf{R} = \mathbf{R}_0 + \mathbf{Z}$, $\mathbf{D}_1, \mathbf{D}_3, \mathbf{D}_3$, respectively, in the initial configuration, define the body position uniquely at any instant. As the body is rigid, $\mathbf{Z} = z_i \mathbf{D}_i$.

To describe the body rotation, instead of vectors $\mathbf{d}_i$ we can introduce a proper orthogonal tensor $\mathbf{Q} = \mathbf{d}_i \otimes \mathbf{D}_i$, where $\otimes$ is the tensor product. Then Eq. (1) takes

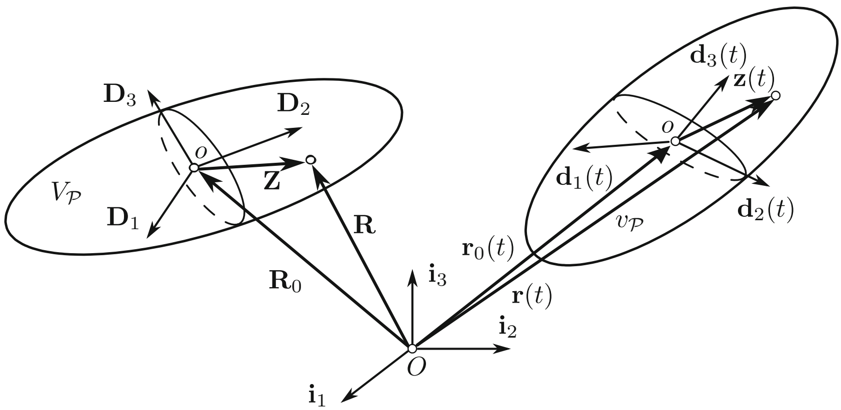

Fig. 1 Rigid body motion

the form

$$\mathbf{r}(t) = \mathbf{R}_0 + \mathbf{u}(t) + \mathbf{Q}(t) \cdot \mathbf{Z}. \tag{2}$$

Hence the rigid body motion is determined by two quantities, one of which is the translation vector of point o, i.e. $\mathbf{u}(t) = \mathbf{r}_0(t) - \mathbf{R}_0$, and another is the rotation tensor $\mathbf{Q}(t)$. To describe the motion, we also can use *Rodrigues's finite rotation vector* $\boldsymbol{\theta}$, cf. Lurie (2001) that we see in the representation of the proper orthogonal tensor

$$\mathbf{Q} = \frac{1}{(4+\theta^2)}\left[(4-\theta^2)\mathbf{I} + 2\boldsymbol{\theta}\otimes\boldsymbol{\theta} - 4\mathbf{I}\times\boldsymbol{\theta}\right], \quad \theta^2 = \boldsymbol{\theta}\boldsymbol{\cdot}\boldsymbol{\theta}. \tag{3}$$

Here $\times$ stands for the cross product while centered dot $\boldsymbol{\cdot}$ denotes the scalar (inner) product.

The other known vectorial parameterizations of an orthogonal tensor are presented by Pietraszkiewicz and Eremeyev (2009b), Bauchau and Trainelli (2003), Bauchau (2010), Wiśniewski (2010). By Eq. (3), vector $\boldsymbol{\theta}$ is determined by proper orthogonal tensor $\mathbf{Q}$ as follows

$$\boldsymbol{\theta} = 2(1 + \operatorname{tr}\mathbf{Q})^{-1}\mathbf{Q}_\times, \tag{4}$$

where $\operatorname{tr}\mathbf{Q}$ is the trace of the second-order tensor, and we introduced the vector invariant $\mathbf{Q}_\times$ by the formula

$$\mathbf{Q}_\times = (Q_{mn}\mathbf{e}^m \otimes \mathbf{e}^n)_\times \stackrel{\triangle}{=} Q_{mn}\mathbf{e}^m \times \mathbf{e}^n \tag{5}$$

for any base vectors $\mathbf{e}^k$. In particular, for a dyad $\mathbf{a}\otimes\mathbf{b}$ we have

$$(\mathbf{a}\otimes\mathbf{b})_\times = \mathbf{a}\times\mathbf{b}.$$

Differentiating (2) we find the velocity

$$\dot{\mathbf{r}}(t) = \dot{\mathbf{u}}(t) + \dot{\mathbf{Q}}(t)\cdot\mathbf{Z}. \tag{6}$$

Hereinafter, the overdot denotes the derivative with respect to time. $\mathbf{Q}$ is orthogonal so tensor $\dot{\mathbf{Q}}\cdot\mathbf{Q}^T$ is skew-symmetric. As any skew-symmetric tensor, it can be represented in the form

$$\dot{\mathbf{Q}}\cdot\mathbf{Q}^T = \boldsymbol{\omega}\times\mathbf{I}, \tag{7}$$

where $\boldsymbol{\omega}$ is called the angular velocity of $\mathcal{P}$. Vector $\boldsymbol{\omega}$ can be determined from (7) as follows

$$\boldsymbol{\omega} = -\frac{1}{2}(\dot{\mathbf{Q}}\cdot\mathbf{Q}^T)_\times, \tag{8}$$

Thus the velocity vector of a body point takes the form

$$\mathbf{v}(t) = \dot{\mathbf{u}}(t) + \boldsymbol{\omega}(t)\times\mathbf{Z}, \tag{9}$$

The rigid body can be considered as a system of mass-points and so we introduce the following definitions.

Definition 2.2 The momentum and the moment of momentum with respect to the pole o for a rigid body are the quantities

$$\mathfrak{P} = \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{v}\, dv, \qquad \mathfrak{M} = \iiint\limits_{v_{\mathcal{P}}} \rho(\mathbf{r} - \mathbf{r}_0) \times \mathbf{v}\, dv,$$

respectively.

Here ρ is the mass density of $\mathcal{P}$ so its mass m is given by the integral over the domain $v_{\mathcal{P}} \subset \mathbb{R}^3$ taken by the body in the space,

$$m(\mathcal{P}) = \iiint\limits_{v_{\mathcal{P}}} \rho\, dv.$$

Let us take as a pole the body mass center, that is the point whose radius vector $\mathbf{r}_0$ satisfies the relation

$$\iiint\limits_{v_{\mathcal{P}}} \rho(\mathbf{r} - \mathbf{r}_0)\, dv = \mathbf{0}.$$

Then the momentum and the moment of momentum of the rigid body take the form

$$\mathfrak{P} = m\mathbf{v}_0, \qquad \mathfrak{M} = \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{z} \times \dot{\mathbf{z}}\, dv = \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{z} \times (\boldsymbol{\omega} \times \mathbf{z})\, dv = \mathbf{J} \cdot \boldsymbol{\omega}, \tag{10}$$

where $\mathbf{v}_0 = \dot{\mathbf{u}}$ and $\mathbf{J}$ is the *inertia tensor*:

$$\mathbf{J} \triangleq \iiint\limits_{v_{\mathcal{P}}} \rho[(\mathbf{z} \cdot \mathbf{z})\mathbf{I} - \mathbf{z} \otimes \mathbf{z}]\, dv. \tag{11}$$

It is seen that $\mathbf{J}$ possesses the following property

$$\mathbf{J} = \mathbf{Q} \cdot \mathbf{J}_0 \cdot \mathbf{Q}^T, \qquad \mathbf{J}_0 \triangleq \iiint\limits_{V_{\mathcal{P}}} \rho[(\mathbf{Z} \cdot \mathbf{Z})\mathbf{I} - \mathbf{Z} \otimes \mathbf{Z}]\, dv, \tag{12}$$

where the volume integral is taken over $V_{\mathcal{P}}$ in the initial body configuration. The constant tensor $\mathbf{J}_0$ can be called the inertia tensor in the initial configuration. For example, for a solid homogeneous sphere of radius a, $\mathbf{J}$ is a spherical tensor

$$\mathbf{J} = \frac{2}{5} m a^2\, \mathbf{I} = \mathbf{J}_0.$$

If the directors $\mathbf{d}_k$ are unit vectors along the principle axes of the inertia tensor, we see that $\mathbf{J}$ and $\mathbf{J}_0$ are diagonal

$$\mathbf{J} = J_1\mathbf{d}_1 \otimes \mathbf{d}_1 + J_2\mathbf{d}_2 \otimes \mathbf{d}_2 + J_3\mathbf{d}_3 \otimes \mathbf{d}_3,$$

$$\mathbf{J}_0 = J_1\mathbf{D}_1 \otimes \mathbf{D}_1 + J_2\mathbf{D}_2 \otimes \mathbf{D}_2 + J_3\mathbf{D}_3 \otimes \mathbf{D}_3,$$

where J_1, J_2, J_3 are moments of inertia with respect to the principal axes.

With regard to (7) and (12) it can be shown that the derivative of $\mathbf{J}$ satisfies the relation

$$\dot{\mathbf{J}} = \boldsymbol{\omega} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\omega}. \tag{13}$$

Taking the mass center as a pole we can rewrite the *kinetic energy* of the rigid body as follows

$$K \triangleq \frac{1}{2} \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{v} \cdot \mathbf{v} \; dv = \frac{1}{2} m \mathbf{v}_0 \cdot \mathbf{v}_0 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega}. \tag{14}$$

The following identities are valid

$$\mathfrak{P} = \frac{\partial K}{\partial \mathbf{v}_0}, \qquad \mathfrak{M} = \frac{\partial K}{\partial \boldsymbol{\omega}}. \tag{15}$$

To a rigid body we can apply the forces and torques (couples or moments). The forces relates with the translation of the body whereas the torques involve body rotation.

The rigid body motion is described by two Euler's laws of motion.

1. *The time rate of the rigid body momentum is equal to the resultant vector of forces* $\mathfrak{F}$, *acting on the body:*

$$\frac{d}{dt}\mathfrak{P} = \mathfrak{F}, \qquad \mathfrak{F} \triangleq \iiint\limits_{v_{\mathcal{P}}} \rho \mathbf{f} \; dv. \tag{16}$$

2. *The time rate of the rigid body moment of momentum with respect to pole o is equal to the resultant moment of all forces with respect to the pole and the body moments:*

$$\frac{d}{dt}\mathfrak{M} = \mathfrak{C}, \qquad \mathfrak{C} \triangleq \iiint\limits_{v_{\mathcal{P}}} \rho[(\mathbf{r} - \mathbf{r}_0) \times \mathbf{f} + \mathbf{m}] \; dv. \tag{17}$$

Here $\mathbf{f}$ and $\mathbf{m}$ are the densities of the forces and the moments acting on the body, respectively.

In equilibrium, these laws reduce to the equality to zero of the resultant vector of the forces and the resultant moment:

$$\mathfrak{F} = \mathbf{0}, \qquad \mathfrak{C} = \mathbf{0}. \tag{18}$$

Substituting (10) to (17) and taking account (13), we get the motion equations of the rigid body

$$m\dot{\mathbf{v}}_0 = \mathfrak{F}, \quad \mathbf{J} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J} \cdot \boldsymbol{\omega} = \mathfrak{C}. \tag{19}$$

In mechanics, Eqs. (16) and (17) constitute the foundation of classic mechanics as well as of continuum mechanics. For example, in equilibrium state of a deformable media Eqs. (18) should be fulfilled for ant part of the media.

3 Kinematics of a Micropolar Shell

In what follows we consider a shell as a material surface and apply Euler's motion laws to an arbitrary part of the shell. Each point (particle) of the shell is considered as infinitesimal rigid body with six degrees of freedom. The deformation of the shell is described by a mapping from a fixed reference configuration into an actual configuration. In other words, we consider a mapping between two directed surfaces including rotations of their particles. Let Σ be a base surface of the micropolar shell in the reference configuration $\varkappa$, q^α ($\alpha = 1, 2$) Gaussian coordinates on Σ, and $\mathbf{P}(q^1, q^2)$ the position vector of the points of Σ, see Fig. 2. Usually but not necessary, one uses undeformed shell state as a initial configuration. In the actual, deformed, configuration χ the base surface is denoted by σ, and the position of its material points (infinitesimal point-bodies) is given by vector $\boldsymbol{\rho}(q^1, q^2, t)$. The point-body orientation is described by the *microrotation tensor* $\mathbf{Q}(q^1, q^2, t)$ that is a proper orthogonal tensor. Introducing three orthonormal vectors $\mathbf{D}_k$ ($k = 1, 2, 3$) describing the orientation in the reference configuration, and three orthonormal vectors $\mathbf{d}_k$ determining the orientation in the actual configuration, we get tensor $\mathbf{Q}$ in the form $\mathbf{Q} = \mathbf{d}_k \otimes \mathbf{D}_k$. Thus the micropolar shell is described by two kinematically independent fields

$$\boldsymbol{\rho} = \boldsymbol{\rho}(q^\alpha, t) \quad \text{and} \quad \mathbf{Q} = \mathbf{Q}(q^\alpha, t). \tag{20}$$

Instead of $\mathbf{Q}$ one can use vectorial representation (3) of $\mathbf{Q}$ or other vectorial representations of a rotation tensor, see Pietraszkiewicz and Eremeyev (2009b), Wiśniewski (2010).

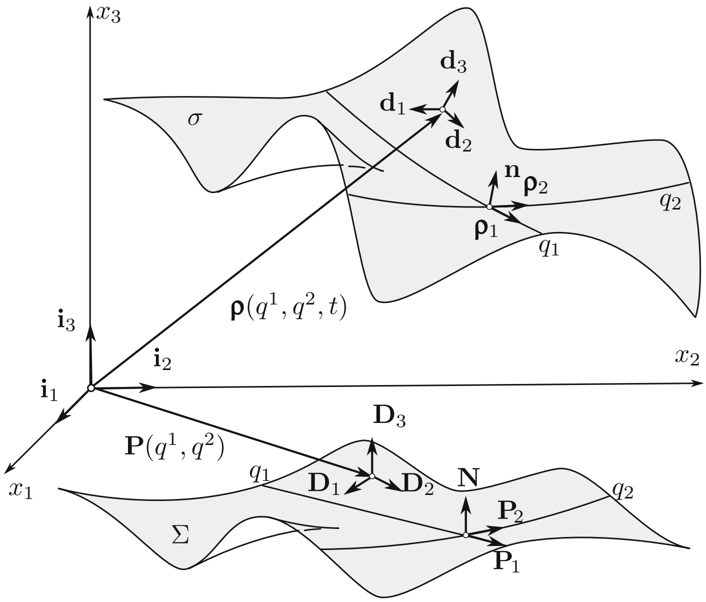

Fig. 2 Kinematics of a micropolar shell

4 Euler's Motion Laws of a Micropolar Shell

The shell motion equations can be introduced using two-dimensional analogues of Euler's motion laws. Here we use the referential (Lagrangian) description, so we introduce quantities using the reference configuration. Let us define the momentum $\mathfrak{P}$ and moment of momentum $\mathfrak{M}$ of an arbitrary shell part $\mathcal{P}$ as follows

$$\mathfrak{P}(\mathcal{P}) \triangleq \iint\limits_{\Sigma_{\mathcal{P}}} \rho \mathbf{K}_1 \, d\Sigma, \quad \mathfrak{M}(\mathcal{P}) \triangleq \iint\limits_{\Sigma_{\mathcal{P}}} \rho \left\{ (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \times \mathbf{K}_1 + \mathbf{K}_2 \right\} d\Sigma, \qquad (21)$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are defined by formulae

$$\mathbf{K}_1 \triangleq \frac{\partial K}{\partial \mathbf{v}} = \mathbf{v} + \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega}, \quad \mathbf{K}_2 \triangleq \frac{\partial K}{\partial \boldsymbol{\omega}} = \boldsymbol{\Theta}_1 \cdot \mathbf{v} + \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega}, \qquad (22)$$

$$K(\mathbf{v}, \boldsymbol{\omega}) = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \boldsymbol{\omega} \cdot \boldsymbol{\Theta}_1 \cdot \mathbf{v} + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega}, \qquad (23)$$

$\Sigma_{\mathcal{P}} \subset \Sigma$ is the part of Σ corresponding to $\mathcal{P}$ in the reference configuration, see Fig. 3.

Here

$$\mathbf{v} = \frac{d\boldsymbol{\rho}}{dt}, \qquad \boldsymbol{\omega} = \frac{1}{2} \left(\mathbf{Q}^T \cdot \frac{d\mathbf{Q}}{dt} \right)_{\times}$$

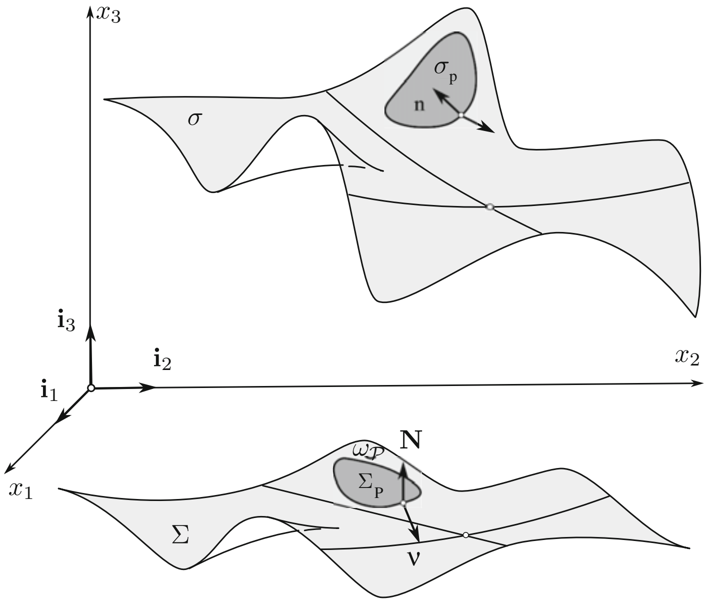

Fig. 3 Part $\mathcal{P}$ of the shell in reference and actual configurations

are the linear and angular velocities, respectively, ρ is the surface mass density in the reference configuration, ρK is the surface density of the *kinetic energy*, and $\rho\boldsymbol{\Theta}_1$, $\rho\boldsymbol{\Theta}_2$ are *the rotatory inertia tensors* ($\boldsymbol{\Theta}_2^T = \boldsymbol{\Theta}_2$). We assume that K is a quadratic form of velocities $\mathbf{v}$ and $\boldsymbol{\omega}$.

Equations (22) are called the *kinetic constitutive equations* of the micropolar shell. A more general form of the kinetic constitutive equations is discussed by Pietraszkiewicz (2011). Presented here definitions of momentum and moment of momentum for a part of the shell are straightforward generalizations of momentum and moment of momentum of a rigid body.

In a similar way, Euler's motion laws for the shell are analogues of Eqs. (16) and (17), they are formulated as follows:

1. Balance of momentum. First Euler's law of motion of the shell. *The time rate of change of the momentum of an arbitrary shell part $\mathcal{P}$ is equal to the total force acting on $\mathcal{P}$*:

$$\frac{d}{dt}\mathfrak{P}(\mathcal{P}) = \mathfrak{F}, \quad \mathfrak{F} \stackrel{\triangle}{=} \iint\limits_{\Sigma_{\mathcal{P}}} \mathbf{f}\, d\Sigma + \int\limits_{\omega_{\mathcal{P}}} \mathbf{t}\, d\omega. \tag{24}$$

Here $\mathbf{f}$ is the surface force density distributed on $\Sigma_{\mathcal{P}}$ and $\mathbf{t}$ is the linear density of forces distributed along corresponding parts of the contour $\omega_{\mathcal{P}}$, respectively.

2. Balance of moment of momentum. Second Euler's law of motion of the shell. *The time rate of change of the moment of momentum of an arbitrary shell part* $\mathcal{P}$ *about a fixed point* $\boldsymbol{\rho}_0$ *is equal to the total moment about* $\boldsymbol{\rho}_0$ *acting on* $\mathcal{P}$:

$$\frac{d}{dt}\mathfrak{M}(\mathcal{P}) = \mathfrak{C}, \tag{25}$$

$$\mathfrak{C} \triangleq \iint\limits_{\Sigma_{\mathcal{P}}} \{(\boldsymbol{\rho} - \boldsymbol{\rho}_0) \times \mathbf{f} + \mathbf{m}\}\, d\Sigma + \int\limits_{\omega_{\mathcal{P}}} \{(\boldsymbol{\rho} - \boldsymbol{\rho}_0) \times \mathbf{t} + \boldsymbol{\mu}\}\, d\omega.$$

The quantities $\mathbf{m}$ and $\boldsymbol{\mu}$ introduced here are the surface and linear densities of couples distributed along corresponding parts of $\Sigma_{\mathcal{P}}$ and $\omega_{\mathcal{P}}$, respectively.

So, here as in the case of rigid body dynamics we have forces and couples as basic loading parameters. Unlike a rigid body for a shell we have surface and linear densities of forces and couples. In other words, we assume that the interaction between shell and its environment or between shell parts is described only by forces and couples (moments).

As for 3D Cosserat continuum (Eremeyev et al. 2013), using (24) and (25) we can prove two-dimensional analogues to the Cauchy lemma and Cauchy theorem and afterwards introduce the surface stress measures and derive the motion equations of a micropolar shell. As the result we introduce the nonsymmetric second-order tensors $\mathbf{D}$ and $\mathbf{G}$ which relate to $\mathbf{t}$ and $\boldsymbol{\mu}$ by formulas

$$\mathbf{t} = \boldsymbol{\nu}\cdot\mathbf{D}, \quad \boldsymbol{\mu} = \boldsymbol{\nu}\cdot\mathbf{G},$$

where $\boldsymbol{\nu}$ is the external unit normal to the boundary curve $\omega_{\mathcal{P}}$ such that $\boldsymbol{\nu}\cdot\mathbf{N} = 0$. $\mathbf{D}$ and $\mathbf{G}$ are *the surface stress and couple stress tensors* of the 1st Piola-Kirchhoff type.

The following relations are valid:

$$\mathbf{N}\cdot\mathbf{D} = \mathbf{0} = \mathbf{N}\cdot\mathbf{G}. \tag{26}$$

In what follows we use the *divergence theorem* on the surface

$$\iint_{\Sigma} (\nabla_s \cdot \mathbf{T} + 2H\mathbf{N} \cdot \mathbf{T})\, d\Sigma = \int_{\partial\Sigma} \boldsymbol{\nu} \cdot \mathbf{T}\, ds, \tag{27}$$

where $\mathbf{T}$ is an arbitrary tensor field, ∇_s is the *surface nabla-operator* on Σ defined by the formula

$$\nabla_s = \mathbf{P}^{\alpha}\frac{\partial}{\partial q^{\alpha}}, \quad \mathbf{P}_1 = \frac{\partial \mathbf{P}}{\partial q^1}, \quad \mathbf{P}_2 = \frac{\partial \mathbf{P}}{\partial q^2}, \quad \mathbf{P}_{\alpha} \cdot \mathbf{P}^{\beta} = \delta_{\alpha}^{\beta} \quad (\alpha, \beta = 1, 2).$$

$\boldsymbol{\nu}$ is the unit external normal to contour $\partial\Sigma$, lying in the tangent plane to Σ, that is $\boldsymbol{\nu} \cdot \mathbf{N} = 0$, $\mathbf{N}$ is the normal to the surface Σ, and H is the mean curvature of Σ.

From (24) and (25) we obtain the Lagrangian *motion equations* of the micropolar shell

$$\nabla_s \cdot \mathbf{D} + \mathbf{f} = \rho \frac{d\mathbf{K}_1}{dt}, \qquad (28)$$
$$\nabla_s \cdot \mathbf{G} + \left[\mathbf{F}^T \cdot \mathbf{D}\right]_\times + \mathbf{m} = \rho \left(\frac{d\mathbf{K}_2}{dt} + \mathbf{v} \times \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega} \right).$$

Here $\mathbf{F} = \nabla_s \boldsymbol{\rho}$ is the *surface deformation gradient*. These equations are presented also by Chróścielewski et al. (2004b), Eremeyev and Zubov (2008), Libai and Simmonds (1998).

5 Strain Energy Density and Strain Measures

For a micropolar hyper-elastic shell we can introduce a strain energy density W. With regard for the local action principle by Truesdell and Noll (1965), W takes the form

$$W = W(\boldsymbol{\rho}, \nabla_s \boldsymbol{\rho}, \mathbf{Q}, \nabla_s \mathbf{Q}).$$

Here we recall that

$$\nabla_s \boldsymbol{\psi} \triangleq \mathbf{P}^\alpha \otimes \frac{\partial \boldsymbol{\psi}}{\partial q^\alpha} \quad (\alpha, \beta = 1, 2), \quad \mathbf{P}^\alpha \cdot \mathbf{P}_\beta = \delta_\beta^\alpha, \quad \mathbf{P}^\alpha \cdot \mathbf{N} = 0, \quad \mathbf{P}_\beta = \frac{\partial \mathbf{P}}{\partial q^\beta}.$$

Here vectors $\mathbf{P}_\beta$ and $\mathbf{P}^\alpha$ denote the natural and reciprocal bases on Σ respectively, $\mathbf{N}$ is the unit normal to Σ, ∇_s is the surface nabla operator on Σ, and $\boldsymbol{\psi}$ is an arbitrary differentiable tensor field given on Σ.

From the principle of material frame-indifference by Truesdell and Noll (1965) we can deduce that W depends on two surface strain measures $\mathbf{E}$ and $\mathbf{K}$ of Cosserat type:

$$W = W(\mathbf{E}, \mathbf{K}),$$

where

$$\mathbf{E} = \mathbf{F} \cdot \mathbf{Q}^T - \mathbf{A}, \quad \mathbf{K} = \frac{1}{2} \mathbf{P}^\alpha \otimes \left(\frac{\partial \mathbf{Q}}{\partial q^\alpha} \cdot \mathbf{Q}^T \right)_\times, \quad \mathbf{F} = \nabla_s \boldsymbol{\rho}. \qquad (29)$$

Here $\mathbf{F}$ is the surface deformation gradient, $\mathbf{A} \triangleq \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$, and $\mathbf{I}$ is the 3D unit tensor.

The proper orthogonal tensor describing the rotation about axis $\mathbf{e}$ for angle φ can be represented with use of the Gibbs's formula

$$\mathbf{Q} = (\mathbf{I} - \mathbf{e} \otimes \mathbf{e}) \cos \varphi + \mathbf{e} \otimes \mathbf{e} - \mathbf{e} \times \mathbf{I} \sin \varphi, \qquad (30)$$

where φ is the rotation angle about the axis with the unit vector $\mathbf{e}$.

Introducing the finite rotation vector $\boldsymbol{\theta} = 2\mathbf{e}\tan\varphi/2$ we get a representation of $\mathbf{Q}$ in the form (3) that does not contain trigonometric functions. By Eq. (3), a proper orthogonal tensor $\mathbf{Q}$ defines uniquely vector $\boldsymbol{\theta}$

$$\boldsymbol{\theta} = 2(1 + \operatorname{tr}\mathbf{Q})^{-1}\mathbf{Q}_\times. \tag{31}$$

Using the *finite rotation vector* $\boldsymbol{\theta}$ we can express $\mathbf{K}$ as follows

$$\mathbf{K} = \mathbf{P}^\alpha \otimes \mathbf{L}_\alpha = \frac{4}{4+\theta^2}\nabla_s\boldsymbol{\theta}\cdot\left(\mathbf{I} + \frac{1}{2}\mathbf{I}\times\boldsymbol{\theta}\right). \tag{32}$$

The strain measures $\mathbf{E}$ and $\mathbf{K}$ are two-dimensional analogues of the strain measures used in 3D Cosserat continuum, see Pietraszkiewicz and Eremeyev (2009a, b).

6 Constitutive Equations of an Elastic Isotropic Shell

For an elastic shell, the constitutive equations are defined by the surface strain energy density as the function of two strain measures. An example we present the model of a *physically linear isotropic shell*, see Chróścielewski et al. (2004b), Eremeyev and Pietraszkiewicz (2006), Eremeyev and Zubov (2008), whose energy is given by the quadratic form

$$\begin{aligned}2W = \ &\alpha_1\operatorname{tr}^2\mathbf{E}_\| + \alpha_2\operatorname{tr}\mathbf{E}_\|^2 + \alpha_3\operatorname{tr}\left(\mathbf{E}_\|\cdot\mathbf{E}_\|^T\right) + \alpha_4\mathbf{N}\cdot\mathbf{E}^T\cdot\mathbf{E}\cdot\mathbf{N}\\ &+ \beta_1\operatorname{tr}^2\mathbf{K}_\| + \beta_2\operatorname{tr}\mathbf{K}_\|^2 + \beta_3\operatorname{tr}\left(\mathbf{K}_\|\cdot\mathbf{K}_\|^T\right) + \beta_4\mathbf{N}\cdot\mathbf{K}^T\cdot\mathbf{K}\cdot\mathbf{N},\end{aligned} \tag{33}$$

where $\mathbf{E}_\| \triangleq \mathbf{E}\cdot\mathbf{A}$, $\mathbf{K}_\| \triangleq \mathbf{K}\cdot\mathbf{A}$. In Eq. (33) there is no term that is bilinear in $\mathbf{E}$ and $\mathbf{K}$, it is a consequence of the fact that the surface wryness tensor $\mathbf{K}$ is a axial tensor that changes the sign on a space mirror reflection. Discussion on axial and polar tensors can be found for example in Eremeyev et al. (2013). Note the constitutive equations contain 8 parameters, α_k, β_k $k = 1, 2, 3, 4$.

With respect to Eq. (33) $\mathbf{P}_1$ and $\mathbf{P}_2$ have the form

$$\mathbf{P}_1 = \alpha_1(\operatorname{tr}\mathbf{E}_\|)\mathbf{A} + \alpha_2\mathbf{E}_\|^T + \alpha_3\mathbf{E}_\| + \alpha_4(\mathbf{E}\cdot\mathbf{N})\otimes\mathbf{N}, \tag{34}$$

$$\mathbf{P}_2 = \beta_1(\operatorname{tr}\mathbf{K}_\|)\mathbf{A} + \beta_2\mathbf{K}_\|^T + \beta_3\mathbf{K}_\| + \beta_4(\mathbf{K}\cdot\mathbf{N})\otimes\mathbf{N}. \tag{35}$$

Introducing the fourth-order tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ by the formulae

$$\begin{aligned}\mathbf{C}_1 &= \alpha_1\mathbf{A}\otimes\mathbf{A} + \alpha_2\mathbf{P}_\alpha\otimes\mathbf{A}\otimes\mathbf{P}^\alpha + \alpha_4\mathbf{P}_\alpha\otimes\mathbf{N}\otimes\mathbf{P}^\alpha\otimes\mathbf{N},\\ \mathbf{C}_2 &= \beta_1\mathbf{A}\otimes\mathbf{A} + \beta_2\mathbf{P}_\alpha\otimes\mathbf{A}\otimes\mathbf{P}^\alpha\mathrm{P}^\beta + \beta_4\mathbf{P}_\alpha\otimes\mathbf{N}\otimes\mathbf{P}^\alpha\otimes\mathbf{N}\end{aligned}$$

we re-write (34) and (35) in a more compact form

$$\mathbf{P}_1 = \mathbf{C}_1 : \mathbf{E}, \quad \mathbf{P}_2 = \mathbf{C}_2 : \mathbf{K}, \tag{36}$$

where ":" denotes the inner product in the space of second-order tensors, for example

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) : (\mathbf{x} \otimes \mathbf{y}) = (\mathbf{c}\cdot\mathbf{x})(\mathbf{d}\cdot\mathbf{y})\mathbf{a} \otimes \mathbf{b}.$$

For the elastic moduli in Eq. (33) Chróścielewski et al. (2004b) proposed the relations:

$$\begin{aligned}
&\alpha_1 = C\nu, \quad \alpha_2 = 0, \quad \alpha_3 = C(1-\nu), \quad \alpha_4 = \alpha_s C(1-\nu),\\
&C = \frac{Eh}{1-\nu^2},\\
&\beta_1 = D\nu, \quad \beta_2 = 0, \quad \beta_3 = D(1-\nu), \quad \beta_4 = \alpha_t D(1-\nu),\\
&D = \frac{Eh^3}{12(1-\nu^2)},
\end{aligned}$$

where E is the Young modulus, ν is the Poisson ratio of the bulk material, and h is the shell thickness. Parameter α_s is the dimensionless shear correction factor. Reissner (1944) used $\alpha_s = 5/6$ in his plate theory, by Mindlin (1951) $\alpha_s = \pi^2/12$. For the couple stresses parameter α_t plays a role similar to α_s for the stresses. The value $\alpha_t = 0.7$ was proposed by Pietraszkiewicz (1979a, b), also see Chróścielewski et al. (2010). In Chróścielewski et al. (2004b), Chróścielewski and Witkowski (2010), Chróścielewski et al. (2010) the influence of α_s and α_t on the solution is investigated numerically for several boundary value problems.

For some types of anisotropy, other representations of shell energy density W were constructed by Eremeyev and Pietraszkiewicz (2006) using material symmetry groups. Let us note that the definition of the material symmetry group for shells is more complex than in the case of simple materials (Truesdell 1984) and even for micropolar elastic materials (Eremeyev and Pietraszkiewicz 2012, 2016).

7 The Virtual Work Principle and Formulation of Boundary Value Problems

Another way of derivation of motion and equilibrium equation is based on the virtual work principle. For formulations of the principle of virtual power for media with microstructure we refer to the landmark papers by Sedov (1968) and by Germain (1973a, b) see also Berdichevsky (2009). Lagrangian equilibrium equations for a micropolar shell can be derived from *the virtual work principle*

$$\delta \iint\limits_{\Sigma} W \, d\Sigma = \delta' A, \tag{37}$$

where

$$\delta' A = \iint\limits_{\Sigma} \left(\mathbf{f}\cdot\delta\boldsymbol{\rho} + \mathbf{m}\cdot\delta'\boldsymbol{\psi}\right)\, d\Sigma + \int\limits_{\omega_2} \mathbf{t}\cdot\delta\boldsymbol{\rho}\, ds + \int\limits_{\omega_4} \boldsymbol{\mu}\cdot\delta'\boldsymbol{\psi}\, ds,$$

$$\mathbf{I} \times \delta'\boldsymbol{\psi} = -\mathbf{Q}^T\cdot\delta\mathbf{Q}.$$

In Eq. (37), δ is the symbol of variation, $\delta'\boldsymbol{\psi}$ the virtual rotation vector, $\mathbf{f}$ and $\mathbf{m}$ the surface force density and the surface couple density distributed on Σ, respectively, $\mathbf{t}$ force distributed along $\omega_2 \subset \partial\Sigma$, and $\boldsymbol{\mu}$ the couples distributed on $\omega_4 \subset \partial\Sigma$. Here we used the symbol δ' to underline that $\delta' A$ and $\delta'\boldsymbol{\psi}$ are not variations, in general.

Using the formulae suggested in Eremeyev and Zubov (2008),

$$\delta W = \frac{\partial W}{\partial \mathbf{E}} : \delta\mathbf{E} + \frac{\partial W}{\partial \mathbf{K}} : \delta\mathbf{K},$$

$$\delta\mathbf{E} = (\nabla_s\delta\boldsymbol{\rho})\cdot\mathbf{Q}^T + \mathbf{F}\cdot\delta\mathbf{Q}^T, \quad \delta\mathbf{K} = (\nabla_s\delta'\boldsymbol{\psi})\cdot\mathbf{Q}^T,$$

$$\delta'\boldsymbol{\psi} = \frac{4}{4+\theta^2}\left(\delta\boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\theta}\times\delta\boldsymbol{\theta}\right)$$

and Eq. (37), we obtain the *Lagrangian shell equations*:

$$\nabla_s\cdot\mathbf{D} + \mathbf{f} = \mathbf{0}, \quad \nabla_s\cdot\mathbf{G} + \left[\mathbf{F}^T\cdot\mathbf{D}\right]_\times + \mathbf{m} = \mathbf{0}, \tag{38}$$

$$\mathbf{D} = \mathbf{P}_1\cdot\mathbf{Q}, \quad \mathbf{G} = \mathbf{P}_2\cdot\mathbf{Q}, \qquad \mathbf{P}_1 = \frac{\partial W}{\partial \mathbf{E}}, \quad \mathbf{P}_2 = \frac{\partial W}{\partial \mathbf{K}}. \tag{39}$$

They are supplemented by the boundary conditions:

$$\begin{array}{ll} \text{on } \omega_1 : \boldsymbol{\rho} = \boldsymbol{\rho}_0(s), & \text{on } \omega_2 : \boldsymbol{\nu}\cdot\mathbf{D} = \mathbf{t}(s), \\ \text{on } \omega_3 : \mathbf{Q} = \mathbf{h}(s), \quad \mathbf{h}\cdot\mathbf{h}^T = \mathbf{I}, & \text{on } \omega_4 : \boldsymbol{\nu}\cdot\mathbf{G} = \boldsymbol{\mu}(s). \end{array} \tag{40}$$

Here $\boldsymbol{\rho}_0(s)$, $\mathbf{h}(s)$ are given vector and tensor functions, and $\boldsymbol{\nu}$ is the external unit normal to the boundary curve ω ($\boldsymbol{\nu}\cdot\mathbf{N} = 0$). Equations (38) are the equilibrium equations for the linear momentum and angular momentum at any shell point. The stress measures $\mathbf{P}_1$ and $\mathbf{P}_2$ in Eqs. (38) are the referential stress and couple stress tensors, respectively, $\mathbf{N}\cdot\mathbf{P}_1 = \mathbf{N}\cdot\mathbf{P}_2 = \mathbf{0}$. The strain measures $\mathbf{E}$ and $\mathbf{K}$ are work-conjugate to the 1st Piola-Kirchhoff stress measures $\mathbf{D}$ and $\mathbf{G}$. The boundary ω of Σ is divided into two parts in such a way that $\omega = \omega_1 \cup \omega_2 = \omega_3 \cup \omega_4$.

The *equilibrium equations* (38) may be transformed to the *Eulerian form* using the surface analogue of the Piola transformation

$$\tilde{\nabla}_s\cdot\mathbf{T} + J^{-1}\mathbf{f} = \mathbf{0}, \quad \tilde{\nabla}_s\cdot\mathbf{M} + \mathbf{T}_\times + J^{-1}\mathbf{m} = \mathbf{0}, \tag{41}$$

where

$$\tilde{\nabla}_s\cdot\boldsymbol{\psi} \triangleq \boldsymbol{\rho}^\alpha\cdot\frac{\partial\boldsymbol{\psi}}{\partial q^\alpha}, \quad \boldsymbol{\rho}^\alpha\cdot\boldsymbol{\rho}_\beta = \delta^\alpha_\beta, \quad \boldsymbol{\rho}^\alpha\cdot\mathbf{n} = 0, \quad \boldsymbol{\rho}_\beta = \frac{\partial\boldsymbol{\rho}}{\partial q^\beta},$$

$$\mathbf{T} = J^{-1}\mathbf{F}^T \cdot \mathbf{D}, \quad \mathbf{M} = J^{-1}\mathbf{F}^T \cdot \mathbf{G}, \tag{42}$$

$$J = \sqrt{\frac{1}{2}\left\{\left[\mathrm{tr}\,\left(\mathbf{F}\cdot\mathbf{F}^T\right)\right]^2 - \mathrm{tr}\,\left[\left(\mathbf{F}\cdot\mathbf{F}^T\right)^2\right]\right\}}.$$

Here $\mathbf{T}$ and $\mathbf{M}$ are Cauchy-type surface stress and couple stress tensors, $\tilde{\nabla}_s$ is the surface nabla operator on σ related with ∇_s by the formula

$$\nabla_s = \mathbf{F}\cdot\tilde{\nabla}_s,$$

and $\mathbf{n}$ is the unit normal to σ.

Under some natural restrictions, the equilibrium problem for a micropolar shell can be transformed to the system with respect to the strain measures:

$$\nabla_s \cdot \mathbf{P}_1 - \left(\mathbf{P}_1^T \cdot \mathbf{K}\right)_\times + \mathbf{f}^* = \mathbf{0}; \tag{43}$$

$$\nabla_s \cdot \mathbf{P}_2 - \left(\mathbf{P}_2^T \cdot \mathbf{K} + \mathbf{P}_1^T \cdot \mathbf{E}\right)_\times + \mathbf{m}^* = \mathbf{0}, \tag{44}$$

$$\omega_2 \ : \ \boldsymbol{\nu} \cdot \mathbf{P}_1 = \mathbf{t}^*, \qquad \omega_4 \ : \ \boldsymbol{\nu} \cdot \mathbf{P}_2 = \boldsymbol{\mu}^*, \tag{45}$$

$$\mathbf{f}^* \triangleq \mathbf{f} \cdot \mathbf{Q}^T, \quad \mathbf{m}^* \triangleq \mathbf{m} \cdot \mathbf{Q}^T, \quad \mathbf{t}^* \triangleq \mathbf{t} \cdot \mathbf{Q}^T, \quad \boldsymbol{\mu}^* \triangleq \boldsymbol{\mu} \cdot \mathbf{Q}^T.$$

Let the vectors $\mathbf{f}^*$, $\mathbf{m}^*$, and $\mathbf{t}^*$, $\boldsymbol{\mu}^*$ be given as some functions of coordinates q^1, q^2, and s. From the physical point of view, it means that the shell is loaded by tracking forces and couples. Then Eqs. (43)–(45) depend on $\mathbf{E}$, $\mathbf{K}$ that are the only independent fields.

For the dynamic problem (28), the initial conditions are

$$\boldsymbol{\rho}\big|_{t=0} = \boldsymbol{\rho}^\circ, \quad \mathbf{v}\big|_{t=0} = \mathbf{v}^\circ, \quad \mathbf{Q}\big|_{t=0} = \mathbf{Q}^\circ, \quad \boldsymbol{\omega}\big|_{t=0} = \boldsymbol{\omega}^\circ,$$

with given initial values $\boldsymbol{\rho}^\circ$, $\mathbf{v}^\circ$, $\mathbf{Q}^\circ$, $\boldsymbol{\omega}^\circ$.

8 Compatibility Conditions

Let us consider how to determine the position-vector $\boldsymbol{\rho}(q^1, q^2)$ of σ from the surface stretch tensor $\mathbf{E}$ and micro-rotation tensor $\mathbf{Q}$, which are assumed to be given as continuously differentiable functions on Σ. By using the equation

$$\mathbf{F} = (\mathbf{E} + \mathbf{A}) \cdot \mathbf{Q} \tag{46}$$

the problem is reduced to

$$\nabla_s \boldsymbol{\rho} = \mathbf{F}. \tag{47}$$

The necessary and sufficient condition for solvability of Eq. (47) is given by the relation

$$\operatorname{div}(\mathbf{e}\cdot\mathbf{F}) = \mathbf{0}, \quad \mathbf{e} \stackrel{\triangle}{=} -\mathbf{I}\times\mathbf{N}, \tag{48}$$

which we call the compatibility condition for the distortion tensor $\mathbf{F}$. Here $\mathbf{e}$ is the skew-symmetric discriminant tensor on the surface Σ. For a simply-connected region Σ, if the condition (48) is satisfied, the vector field $\boldsymbol{\rho}$ may be deduced from Eq. (47) only up to an additive vector.

Let us consider a more complex problem of determination of both the translations and rotations of the micropolar shell from the given fields of $\mathbf{E}$ and $\mathbf{K}$. At first, let us deduce the field $\mathbf{Q}(q^1, q^2)$ by using the system of equations following from definition (29) of $\mathbf{K}$

$$\frac{\partial\mathbf{Q}}{\partial q^\alpha} = -\mathbf{K}_\alpha\times\mathbf{Q}, \quad \mathbf{K}_\alpha \stackrel{\triangle}{=} \mathbf{P}_\alpha\cdot\mathbf{K}. \tag{49}$$

The integrability conditions for the system (49) are given by the relation

$$\frac{\partial\mathbf{K}_\alpha}{\partial q^\beta} - \frac{\partial\mathbf{K}_\beta}{\partial q^\alpha} = \mathbf{K}_\alpha\times\mathbf{K}_\beta \quad (\alpha,\beta = 1, 2). \tag{50}$$

Equations (50) were obtained by Pietraszkiewicz (1979a, 1989), Libai and Simmonds (1983) as the conditions of existence of the rotation field of the shell. They may be written in the following coordinate-free form

$$\operatorname{div}(\mathbf{e}\cdot\mathbf{K}) + \mathbf{K}^\perp\cdot\mathbf{n} = \mathbf{0}, \tag{51}$$

where

$$\mathbf{K}^\perp \stackrel{\triangle}{=} \frac{1}{2}\left(\mathbf{K}_\alpha\times\mathbf{K}_\beta\right)\otimes\left(\mathbf{P}^\alpha\times\mathbf{P}^\beta\right) = \mathbf{K}^2 - \mathbf{K}\operatorname{tr}\mathbf{K} + \frac{1}{2}\left(\operatorname{tr}^2\mathbf{K} - \operatorname{tr}\mathbf{K}^2\right)\mathbf{I}.$$

Using Eqs. (46) and (29) we transform the compatibility condition (48) into the coordinate-free form

$$\operatorname{div}(\mathbf{e}\cdot\mathbf{E}) + \left[(\mathbf{E}+\mathbf{A})^T\cdot\mathbf{e}\cdot\mathbf{K}\right]_\times = \mathbf{0}. \tag{52}$$

Two coordinate-free vector equations (51) and (52) are the compatibility conditions for the nonlinear micropolar shell. These conditions and the system of equations (43)–(45) constitute the complete boundary-value problem of statics of micropolar shells expressed entirely in terms of the surface strain measures $\mathbf{E}$ and $\mathbf{K}$.

9 Variational Statements

The presented above static and dynamic boundary-value problems of the micropolar shell theory have corresponding variational statements. Some of them for statics and for dynamics are presented below.

9.1 Lagrange-Type Principle

Let us assume that the external forces and couples are conservative. In the Lagrange-type variational principle

$$\delta\mathcal{E}_1 = 0$$

we use the total energy functional

$$\mathcal{E}_1[\boldsymbol{\rho}, \mathbf{Q}] = \iint\limits_{\Sigma} W\, d\Sigma - \mathcal{A}[\boldsymbol{\rho}, \mathbf{Q}], \tag{53}$$

where $\mathcal{A}$ is the potential of the external loads.

Here the translations and the rotations have to satisfy the kinematic boundary conditions $(40)_1$ and $(40)_3$ on ω_1 and ω_3, respectively. The stationarity of $\mathcal{E}_1$ is equivalent to the equilibrium equations (38), (39) and the static boundary conditions $(40)_2$ and $(40)_4$ on ω_2 and ω_4.

9.2 Hu-Washizu-Type Principle

For this principle the functional is given by

$$\begin{aligned}
\mathcal{E}_2[\boldsymbol{\rho}, \mathbf{Q}, \mathbf{E}, \mathbf{K}, \mathbf{D}, \mathbf{P}_2] = &\iint\limits_{\Sigma} \Big\{ W(\mathbf{E}, \mathbf{K}) - \mathbf{D} : (\mathbf{E} \cdot \mathbf{Q} - \nabla_s \boldsymbol{\rho}) \\
&- \mathbf{P}_2 : \left[\mathbf{K} - \frac{1}{2} \mathbf{P}^\alpha \otimes \left(\frac{\partial \mathbf{Q}}{\partial q^\alpha} \cdot \mathbf{Q}^T \right)_\times \right] \Big\}\, d\Sigma \\
&- \int\limits_{\omega_1} \boldsymbol{\nu} \cdot \mathbf{D} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_0)\, ds - \mathcal{A}[\boldsymbol{\rho}, \mathbf{Q}].
\end{aligned}$$

From the condition $\delta\mathcal{E}_2 = 0$ the equilibrium equations (38) and (39), the constitutive equations, and the relations (29) can be deduced. For this principle the natural boundary conditions are given by the relations $(40)_1$, $(40)_2$ and $(40)_4$, respectively.

Several other variational statements are given in Eremeyev and Zubov (2008). Mixed variational functionals are constructed in Chróścielewski et al. (2004b). They are used for the development of a family of finite elements with six degrees of freedom in each node. A number of nonlinear simulations of complex multifold shell structures were performed on the base of these elements Chróścielewski et al. (2004b, 2010), Chróścielewski and Witkowski (2010).

9.3 Hamilton-Type Principle

The *kinetic energy of micropolar shells* can be expressed as

$$\mathcal{K} = \iint\limits_{\Sigma} \rho K(\mathbf{v}, \boldsymbol{\omega})\, d\Sigma. \tag{54}$$

It is obvious that we should assume the kinetic energy to be a positive definite function that imposes some restriction on the form of the inertia tensors.

The Hamilton principle is a variational principle of dynamics. In real motion, the functional

$$\mathcal{E}_3[\boldsymbol{\rho}, \mathbf{Q}] = \int_{t_0}^{t_1} (\mathcal{K} - \mathcal{E}_1)\ dt \tag{55}$$

takes a stationary value on the set of all possible shell motions that at the range t_0, t_1 take given values of the real motion values and satisfy the kinematic boundary values. In other words, its first variation on a real motion is zero. From condition $\delta\mathcal{E}_3 = 0$, Eqs. (28) can be established.

10 Linear Theory of Micropolar Shells

For small strains the shell equations can be simplified significantly. In geometrically linear version, Eulerian and Lagrangian shell descriptions do not differ as the difference between σ and Σ is considered to be infinitesimal. Here we do not distinguish the operators $\tilde{\nabla}_s$ and ∇_s as well as the types of stress and couple stress tensors in different configurations.

Let us introduce the *vector of infinitesimal translations* $\mathbf{u}$ and the *vector of infinitesimal rotations* $\boldsymbol{\vartheta}$ such that

$$\boldsymbol{\rho} \approx \mathbf{P} + \mathbf{u}, \quad \mathbf{Q} \approx \mathbf{I} - \mathbf{I} \times \boldsymbol{\vartheta}. \tag{56}$$

The formula for $\mathbf{Q}$ follows from the representation of a proper orthogonal tensor through the finite rotation vector (3) for $|\boldsymbol{\theta}| \ll 1$.

The stretch measure $\mathbf{E}$ and the wryness tensor $\mathbf{K}$ can be expressed in terms of the *linear stretch tensor* $\boldsymbol{\epsilon}$ and the *linear wryness tensor* $\boldsymbol{\kappa}$ up to a linear addendum:

$$\mathbf{E} \approx \mathbf{I} + \boldsymbol{\varepsilon}, \quad \mathbf{K} \approx \boldsymbol{\varkappa}, \quad \boldsymbol{\varepsilon} = \nabla_s \mathbf{u} + \mathbf{A} \times \boldsymbol{\vartheta}, \quad \boldsymbol{\varkappa} = \nabla_s \boldsymbol{\vartheta}. \tag{57}$$

$\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$ are used in the linear theory of micropolar shells, cf. Chróścielewski et al. (2004b), Eremeyev and Zubov (2008), Lebedev et al. (2010), Zhilin (1976), Zubov (1997). As a consequence of (57) in the linear shell theory, the stress tensors $\mathbf{D}$, $\mathbf{P}_1$, and $\mathbf{T}$ coincide, the couple tensors $\mathbf{G}$, $\mathbf{P}_2$, $\mathbf{M}$ do not differ as well. In what follows, we will denote the stress tensor by $\mathbf{T}$ and the couple stress tensor by $\mathbf{M}$.

For a linearly elastic shell, the constitutive equations can be represented through the strain energy density $W = W(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa})$ as it follows

$$\mathbf{T} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{M} = \frac{\partial W}{\partial \boldsymbol{\varkappa}}. \tag{58}$$

The equilibrium equations in the linear theory are

$$\nabla_s \boldsymbol{\cdot} \mathbf{T} + \mathbf{f} = \mathbf{0}, \quad \nabla_s \boldsymbol{\cdot} \mathbf{M} + \mathbf{T}_\times + \mathbf{m} = \mathbf{0}, \tag{59}$$

whereas the boundary conditions transform to

$$\begin{array}{llll} \text{on } \omega_1: & \mathbf{u} = \mathbf{u}_0(s), & \text{on } \omega_2: & \boldsymbol{\nu} \cdot \mathbf{T} = \mathbf{t}(s), \\ \text{on } \omega_3: & \boldsymbol{\vartheta} = \boldsymbol{\vartheta}_0(s), & \text{on } \omega_4: & \boldsymbol{\nu} \cdot \mathbf{M} = \boldsymbol{\mu}(s), \end{array} \tag{60}$$

where $\mathbf{u}_0(s)$ and $\boldsymbol{\vartheta}_0(s)$ are given functions of the arclength s; the conditions define the translations and rotations on contour parts ω_k.

For small strains, an example of constitutive equations is defined by the following quadratic form

$$\begin{aligned} 2W = {} & \alpha_1 \mathrm{tr}^{\,2} \boldsymbol{\varepsilon}_\parallel + \alpha_2 \mathrm{tr}\, \boldsymbol{\varepsilon}_\parallel^2 + \alpha_3 \mathrm{tr}\, \left(\boldsymbol{\varepsilon}_\parallel \boldsymbol{\cdot} \boldsymbol{\varepsilon}_\parallel^T\right) + \alpha_4 \mathbf{N} \boldsymbol{\cdot} \boldsymbol{\varepsilon}^T \boldsymbol{\cdot} \boldsymbol{\varepsilon} \boldsymbol{\cdot} \mathbf{N} \\ & + \beta_1 \mathrm{tr}^{\,2} \boldsymbol{\varkappa}_\parallel + \beta_2 \mathrm{tr}\, \boldsymbol{\varkappa}_\parallel^2 + \beta_3 \mathrm{tr}\, \left(\boldsymbol{\varkappa}_\parallel \boldsymbol{\cdot} \boldsymbol{\varkappa}_\parallel^T\right) + \beta_4 \mathbf{N} \boldsymbol{\cdot} \boldsymbol{\varkappa}^T \boldsymbol{\cdot} \boldsymbol{\varkappa} \boldsymbol{\cdot} \mathbf{N} \end{aligned} \tag{61}$$

that describes a *linear isotropic shell*. Here α_k and β_k, $k = 1, 2, 3, 4$, are elastic constants, and $\boldsymbol{\varepsilon}_\parallel \triangleq \boldsymbol{\varepsilon} \boldsymbol{\cdot} \mathbf{A}$, $\boldsymbol{\varkappa}_\parallel \triangleq \boldsymbol{\varkappa} \boldsymbol{\cdot} \mathbf{A}$.

By Eqs. (58) and (61), the stress tensor and the couple stress tensor are

$$\mathbf{T} = \alpha_1 \mathbf{A} \mathrm{tr}\, \boldsymbol{\varepsilon}_\parallel + \alpha_2 \boldsymbol{\varepsilon}_\parallel^T + \alpha_3 \boldsymbol{\varepsilon}_\parallel + \alpha_4 \boldsymbol{\varepsilon} \cdot \mathbf{N} \otimes \mathbf{N}, \tag{62}$$

$$\mathbf{M} = \beta_1 \mathbf{A} \mathrm{tr}\, \boldsymbol{\varkappa}_\parallel + \beta_2 \boldsymbol{\varkappa}_\parallel^T + \beta_3 \boldsymbol{\varkappa}_\parallel + \beta_4 \boldsymbol{\varkappa} \cdot \mathbf{N} \otimes \mathbf{N}. \tag{63}$$

Supplemented with Eqs. (59) and (60), the linear constitutive equations (62) and (63) constitute the setup of the linear boundary value problem with respect to the fields of translations and rotations. It describes micropolar shell equilibrium when the strains are infinitesimal.

11 Linearized Boundary-Value Problems

Let $\boldsymbol{\rho}_0$ and $\mathbf{Q}_0$ are the known static solution of (28) and (40). The corresponding state of the shell we will call the basic actual configuration and denote it by χ_0. In addition, let us consider the actual configuration χ_*, which differs from χ_0 by infinitesimal deformation, and derive the linearized boundary-value problem. Denoting quantities related to χ_* by the lower index $*$ we have

$$\boldsymbol{\rho}_* = \boldsymbol{\rho}_0 + \delta\boldsymbol{\rho}, \quad \mathbf{Q}_* = \mathbf{Q}_0 + \delta\mathbf{Q},$$

where we use the symbol δ for infinitesimal increments of corresponding quantities. Since $\mathbf{Q}$ is an orthogonal tensor, the tensor $\mathbf{Q}^T \cdot \delta\mathbf{Q}$ is a skew-symmetric tensor and can be represented as follows

$$\mathbf{Q}^T \boldsymbol{\cdot} \delta\mathbf{Q} = -\mathbf{I} \times \boldsymbol{\psi},$$

where $\boldsymbol{\psi}$ is the infinitesimal rotation vector. It can be expressed by the increment of the finite rotation vector as follows

$$\boldsymbol{\psi} = \frac{4}{4+\theta^2}\left(\delta\boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\theta} \times \delta\boldsymbol{\theta}\right).$$

The increments of the strain measures are given by the formulae (Eremeyev and Zubov 2008)

$$\delta\mathbf{E} = (\nabla_s \delta\boldsymbol{\rho})\boldsymbol{\cdot}\mathbf{Q}_0^T + \mathbf{F}_0\boldsymbol{\cdot}\delta\mathbf{Q}^T = \mathbf{F}_0 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{Q}_0^T, \tag{64}$$

$$\delta\mathbf{K} = (\nabla_s \boldsymbol{\psi})\boldsymbol{\cdot}\mathbf{Q}^T = \mathbf{F}_0 \cdot \boldsymbol{\varkappa} \cdot \mathbf{Q}_0^T, \tag{65}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$ are the linear strain measures given by

$$\boldsymbol{\varepsilon} = \nabla_\chi \mathbf{w} + \mathbf{A} \times \boldsymbol{\psi}, \quad \boldsymbol{\varkappa} = \nabla_\chi \boldsymbol{\psi}, \tag{66}$$

$\mathbf{w} = \delta\boldsymbol{\rho}$ and $\mathbf{F}_0 = \nabla_\varkappa \boldsymbol{\rho}_0$. Here ∇_χ is the surface nabla-operator in the basic actual configuration χ_0. Assuming that $\delta\mathbf{f} = \mathbf{0}$ and $\delta\mathbf{m} = \mathbf{0}$ the linearization leads to the Lagrangian linearized equations of motion

$$\nabla_s \boldsymbol{\cdot} \delta\mathbf{D} = \rho \frac{d^2\mathbf{w}}{dt^2}, \tag{67}$$

$$\nabla_s \boldsymbol{\cdot} \delta\mathbf{G} + \left[(\nabla_s \mathbf{w})^T \boldsymbol{\cdot} \mathbf{D} + \mathbf{F}_0^T \boldsymbol{\cdot} \delta\mathbf{D}\right]_\times = \rho\gamma \frac{d^2\boldsymbol{\psi}}{dt^2}. \tag{68}$$

Here for simplicity we assume that $\boldsymbol{\Theta}_1 = \mathbf{0}$ and $\boldsymbol{\Theta}_2 = \gamma\mathbf{I}$.

The increments of the stress and couple stress tensors are calculated by the relations

$$\delta \mathbf{D} = \delta \mathbf{P}_1 \cdot \mathbf{Q}_0 + \mathbf{P}_1 \cdot \delta \mathbf{Q} = \delta \mathbf{P}_1 \cdot \mathbf{Q}_0 - \mathbf{D} \times \boldsymbol{\psi}, \tag{69}$$

$$\delta \mathbf{G} = \delta \mathbf{P}_2 \cdot \mathbf{Q}_0 + \mathbf{P}_2 \cdot \delta \mathbf{Q} = \delta \mathbf{P}_2 \cdot \mathbf{Q}_0 - \mathbf{G} \times \boldsymbol{\psi}, \tag{70}$$

$$\delta \mathbf{P}_1 = \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{E}} : \delta \mathbf{E} + \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{K}} : \delta \mathbf{K}, \tag{71}$$

$$\delta \mathbf{P}_2 = \frac{\partial^2 W}{\partial \mathbf{K} \partial \mathbf{E}} : \delta \mathbf{E} + \frac{\partial^2 W}{\partial \mathbf{K} \partial \mathbf{K}} : \delta \mathbf{K}. \tag{72}$$

Using (36) for the physically linear shell we have

$$\delta \mathbf{P}_1 = \mathbf{C}_1 : \delta \mathbf{E} = \mathbf{D}_1 : \boldsymbol{\varepsilon}, \quad \delta \mathbf{P}_2 = \mathbf{C}_2 : \delta \mathbf{K} = \mathbf{D}_2 : \boldsymbol{\varkappa},$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are fourth-order tensors given by

$$\begin{aligned}
\mathbf{D}_1 = {} & \alpha_1 \mathbf{A} \otimes \mathbf{F}_0^T \cdot \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\alpha + \alpha_2 \mathbf{P}_\alpha \otimes \mathbf{P}_\beta \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\beta \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\alpha \\
& + \alpha_3 \mathbf{P}_\alpha \otimes \mathbf{P}_\beta \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\beta + \alpha_4 \mathbf{P}_\alpha \otimes \mathbf{N} \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{N}, \\
\mathbf{D}_2 = {} & \beta_1 \mathbf{A} \otimes \mathbf{F}_0^T \cdot \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\alpha + \beta_2 \mathbf{P}_\alpha \otimes \mathbf{P}_\beta \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\beta \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\alpha \\
& + \beta_3 \mathbf{P}_\alpha \otimes \mathbf{P}_\beta \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\beta + \beta_4 \mathbf{P}_\alpha \otimes \mathbf{N} \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{N}.
\end{aligned}$$

Assuming that $\delta \mathbf{t} = \mathbf{0}$, $\delta \boldsymbol{\mu} = \mathbf{0}$, $\delta \mathbf{r}_0 = \mathbf{0}$, and $\delta \mathbf{h} = \mathbf{0}$, we obtain the linearized boundary conditions

$$\begin{aligned}
&\text{on } \omega_1 : \mathbf{w} = \mathbf{0}, \quad \text{on } \omega_2 : \boldsymbol{\nu} \cdot \delta \mathbf{D} = \mathbf{0}, \\
&\text{on } \omega_3 : \boldsymbol{\psi} = \mathbf{0}, \quad \text{on } \omega_4 : \boldsymbol{\nu} \cdot \delta \mathbf{G} = \mathbf{0}.
\end{aligned} \tag{73}$$

Introducing the tensors

$$\boldsymbol{\Phi}_1 = J_0^{-1} \mathbf{F}_0^T \cdot \delta \mathbf{D}, \quad \boldsymbol{\Phi}_2 = J_0^{-1} \mathbf{F}_0^T \cdot \delta \mathbf{G}, \tag{74}$$

where $J_0 = J(\mathbf{F}_0)$, we transform Eqs. (67) and (68) into the linearized equations of motion in the actual configuration χ_0

$$\nabla_\chi \cdot \boldsymbol{\Phi}_1 = \rho \frac{d^2 \mathbf{w}}{dt^2}, \tag{75}$$

$$\nabla_\chi \cdot \boldsymbol{\Phi}_2 + \left[(\nabla_\chi \mathbf{w})^T \cdot \mathbf{T} + \boldsymbol{\Phi}_1 \right]_\times = \rho \gamma \frac{d^2 \boldsymbol{\psi}}{dt^2}. \tag{76}$$

For the physically linear isotropic micropolar shell $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are given by relations

$$\begin{gathered}
\boldsymbol{\Phi}_1 = \mathbf{H}_1 : \boldsymbol{\varepsilon} - \mathbf{T} \times \boldsymbol{\psi}, \quad \boldsymbol{\Phi}_1 = \mathbf{H}_2 : \boldsymbol{\varkappa} - \mathbf{M} \times \boldsymbol{\psi}, \\
\mathbf{H}_1 = J_0^{-1} \mathbf{F}_0^T \cdot \widetilde{\mathbf{D}}_1, \quad \mathbf{H}_2 = J_0^{-1} \mathbf{F}_0^T \cdot \widetilde{\mathbf{D}}_2,
\end{gathered}$$

where

$$\begin{aligned}
\widetilde{\mathbf{D}}_1 &= \alpha_1 \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{F}_0^T \cdot \mathbf{P}_\beta \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\beta \\
&+ \alpha_2 \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}_\beta \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\beta \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\alpha \\
&+ \alpha_3 \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}_\beta \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\beta \\
&+ \alpha_4 \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{N} \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{N}, \\
\widetilde{\mathbf{D}}_2 &= \beta_1 \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{F}_0^T \cdot \mathbf{P}_\beta \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\beta \\
&+ \beta_2 \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}_\beta \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\beta \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\alpha \\
&+ \beta_3 \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}_\beta \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{P}^\beta \\
&+ \beta_4 \mathbf{P}_\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{N} \otimes \mathbf{F}_0^T \cdot \mathbf{P}^\alpha \otimes \mathbf{Q}_0^T \cdot \mathbf{N}.
\end{aligned}$$

The fourth-order tensors $\mathbf{H}_1$ and $\mathbf{H}_2$ are tangent stiffness tensors in the non-linear theory of shells which have the same properties as in the three-dimensional non-linear elasticity, see Fu and Ogden (1999), Ogden (1997), Lurie (1990), Altenbach and Eremeyev (2010). The components of $\mathbf{H}_1$ and $\mathbf{H}_2$ depend on initial deformations and, as a result, have symmetry properties which are different from ones of $\mathbf{C}_1$ and $\mathbf{C}_2$, in general.

The linearized Eulerian boundary conditions are

$$\begin{aligned}
&\text{on } \ell_1 : \mathbf{w} = \mathbf{0}, \quad \text{on } \ell_2 : \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_1 = \mathbf{0}, \\
&\text{on } \ell_3 : \boldsymbol{\psi} = \mathbf{0}, \quad \text{on } \ell_4 : \boldsymbol{\eta} \cdot \boldsymbol{\Phi}_2 = \mathbf{0}.
\end{aligned} \tag{77}$$

Here $\boldsymbol{\eta}$ is the unit vector normal to the shell contour $\ell = \partial\sigma$, $\boldsymbol{\eta} \cdot \mathbf{n} = 0$, $\ell = \ell_1 \cup \ell_2 = \ell_3 \cup \ell_4$, ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are the parts of the shell contour in the actual configuration corresponding to ω_1, ω_2, ω_3, and ω_4, respectively.

The boundary-value problems (67), (68), (73), and (75)–(77) describe the motion of the prestressed micropolar shell. For $\chi_0 = \varkappa$ we have

$$\mathbf{F}_0 = \mathbf{A}, \quad \mathbf{Q}_0 = \mathbf{I}.$$

Assuming in addition the absence of initial stresses

$$\mathbf{T} = \mathbf{M} = \mathbf{0}$$

the linearized boundary-value problems coincide with the equations of motion of linear isotropic micropolar shells discussed in Chróścielewski et al. (2004b), Eremeyev and Zubov (2008), Lebedev et al. (2010), Eremeyev and Lebedev (2011), Eremeyev et al. (2015b).

12 Eigen-Vibrations of Prestressed Micropolar Shells

Let us consider eigen-vibrations of a prestressed shell. By linearity, eigen-solutions are proportional to $e^{i\Omega t}$:

$$\mathbf{w} = \mathbf{W}(q^1, q^2)e^{i\Omega t}, \quad \boldsymbol{\psi} = \boldsymbol{\Psi}(q^1, q^2)e^{i\Omega t}.$$

Substituting the latter relations into (75) and (77) we obtain the boundary-value problem for the physically linear isotropic prestressed micropolar shell

$$\nabla_\chi \boldsymbol{\cdot} \boldsymbol{\Phi}_1 = -\rho\Omega^2 \mathbf{W}, \tag{78}$$

$$\nabla_\chi \boldsymbol{\cdot} \boldsymbol{\Phi}_2 + \left[(\nabla_\chi \mathbf{w})^T \boldsymbol{\cdot} \mathbf{T} + \boldsymbol{\Phi}_1\right]_\times = -\rho\gamma\Omega^2 \boldsymbol{\Psi}, \tag{79}$$

$$\begin{aligned} &\text{on } \ell_1 : \mathbf{W} = \mathbf{0}, \quad \text{on } \ell_2 : \boldsymbol{\eta} \boldsymbol{\cdot} \boldsymbol{\Phi}_1 = \mathbf{0}, \\ &\text{on } \ell_3 : \boldsymbol{\Psi} = \mathbf{0}, \quad \text{on } \ell_4 : \boldsymbol{\eta} \boldsymbol{\cdot} \boldsymbol{\Phi}_2 = \mathbf{0}, \end{aligned} \tag{80}$$

where

$$\begin{gathered} \boldsymbol{\Phi}_1 = \mathbf{H}_1 : \boldsymbol{\varepsilon} - \mathbf{T} \times \boldsymbol{\Psi}, \quad \boldsymbol{\Phi}_1 = \mathbf{H}_2 : \boldsymbol{\varkappa} - \mathbf{M} \times \boldsymbol{\Psi}, \\ \boldsymbol{\varepsilon} = \nabla_\chi \mathbf{W} + \mathbf{A} \times \boldsymbol{\Psi}, \quad \boldsymbol{\varkappa} = \nabla_\chi \boldsymbol{\Psi}. \end{gathered} \tag{81}$$

Additionally we consider the linear boundary-value problem of the micropolar shell without initial deformation, that is when $\chi_0 = \varkappa$, which is given by

$$\nabla_\chi \boldsymbol{\cdot} \boldsymbol{\Phi}_1^0 = -\rho\Omega^2 \mathbf{W}, \quad \nabla_\chi \boldsymbol{\cdot} \boldsymbol{\Phi}_2^0 + \boldsymbol{\Phi}^0_{1\times} = -\rho\gamma\Omega^2 \boldsymbol{\Psi}, \tag{82}$$

$$\begin{aligned} &\text{on } \ell_1 : \mathbf{W} = \mathbf{0}, \quad \text{on } \ell_2 : \boldsymbol{\eta} \boldsymbol{\cdot} \boldsymbol{\Phi}_1^0 = \mathbf{0}, \\ &\text{on } \ell_3 : \boldsymbol{\Psi} = \mathbf{0}, \quad \text{on } \ell_4 : \boldsymbol{\eta} \boldsymbol{\cdot} \boldsymbol{\Phi}_2^0 = \mathbf{0}, \end{aligned} \tag{83}$$

$$\boldsymbol{\Phi}_1^0 = \mathbf{C}_1 : \boldsymbol{\varepsilon}, \quad \boldsymbol{\Phi}_1^0 = \mathbf{C}_2 : \boldsymbol{\varkappa}. \tag{84}$$

The comparison of $\boldsymbol{\Phi}_1^0$ and $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2^0$ and $\boldsymbol{\Phi}_2$ shows that difference between these boundary-value problems consists of

1. the difference between the elastic moduli tensors $\mathbf{C}_\alpha$ and $\mathbf{H}_\alpha$, $\alpha = 1, 2$, and
2. the existence of initial stress tensors $\mathbf{T}$ and $\mathbf{M}$ in $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$.

In what follows we show the influence on eigen-frequencies of the prestressed shell using the variational approach.

12.1 *Rayleigh Principle*

In the linear and linearized shell theories presented above there is a variational principle for eigen-vibrations called the Rayleigh variational principle. To formulate it we consider the second variation of the functional of the total energy of the micropolar shell. Suppose that $\mathbf{m} = \boldsymbol{\mu} = \mathbf{0}$ and the external forces are "dead". This means that $\mathbf{f}$ and $\mathbf{t}$ do not depend on $\mathbf{u}$ and $\mathbf{Q}$. Thus the functional of the total potential energy of the shell is

$$\Pi = \iint\limits_{\Sigma} W \, d\Sigma - \iint\limits_{\Sigma} \mathbf{f}\cdot\mathbf{u} \, d\Sigma - \int\limits_{\omega_2} \mathbf{t}\cdot\mathbf{u} \, ds.$$

The first variation of Π is given by

$$\delta\Pi = \iint\limits_{\Sigma} \left[\mathrm{tr}\,\left(\mathbf{D}^T\cdot\nabla_s\mathbf{w}\right) + \mathrm{tr}\,\left(\mathbf{D}^T\cdot\mathbf{F}_0\times\boldsymbol{\psi}\right) + \mathrm{tr}\,\left(\mathbf{G}^T\cdot\nabla_s\boldsymbol{\psi}\right)\right] d\Sigma \tag{85}$$
$$- \iint\limits_{\Sigma} \mathbf{f}\cdot\mathbf{w} \, d\Sigma - \int\limits_{\omega_2} \mathbf{t}\cdot\mathbf{w} \, ds.$$

Since $\boldsymbol{\rho}_0$ and $\mathbf{Q}_0$ are assumed to satisfy equilibrium equations and boundary conditions (40), the first variation of the energy vanishes

$$\delta\Pi = 0.$$

The second variation of the energy takes the form

$$\delta^2\Pi = \iint\limits_{\Sigma} \left\{\mathrm{tr}\,\left(\delta\mathbf{D}^T\cdot\nabla_s\mathbf{w}\right) + \mathrm{tr}\,\left(\delta\mathbf{D}^T\cdot\mathbf{F}_0\times\boldsymbol{\psi}\right) + \mathrm{tr}\,\left[\mathbf{D}^T\cdot(\nabla_s\mathbf{w})\times\boldsymbol{\psi}\right]\right.$$
$$\left. + \,\mathrm{tr}\,\left(\delta\mathbf{G}^T\cdot\nabla_s\boldsymbol{\psi}\right)\right\} d\Sigma.$$

Using identities $\nabla_\chi = \mathbf{F}\cdot\nabla_s$, $d\sigma = J\,d\Sigma$, and (74), we transform $\delta^2\Pi$ to

$$\delta^2\Pi = \iint\limits_{\sigma} \left\{\boldsymbol{\Phi}_1 : \left(\nabla_\chi\mathbf{w} + \mathbf{A}\times\boldsymbol{\psi}\right) + \boldsymbol{\Phi}_1 : \nabla_\chi\boldsymbol{\psi}\right.$$
$$\left. + \,\mathrm{tr}\,\left[\mathbf{T}^T\cdot(\nabla_\chi\mathbf{w})\times\boldsymbol{\psi}\right]\right\} d\sigma$$
$$= \iint\limits_{\sigma} \{\boldsymbol{\Phi}_1 : \boldsymbol{\varepsilon} + \boldsymbol{\Phi}_2 : \boldsymbol{\varkappa}$$
$$+ \,\mathrm{tr}\,\left[\mathbf{T}^T\cdot(\nabla_\chi\mathbf{w})\times\boldsymbol{\psi}\right]\} \, d\sigma.$$

Finally, with Eqs. (81), the second energy variation takes the form

$$\delta^2\Pi = 2\iint\limits_{\sigma} w \, d\sigma, \quad w = w_1 + w_2, \tag{86}$$

where

$$
\begin{aligned}
w_1(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) \quad &= \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{H}_1 : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varkappa} : \mathbf{H}_2 : \boldsymbol{\varkappa}, \\
w_2(\boldsymbol{\psi}, \boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) &= \operatorname{tr}\left(\boldsymbol{\psi} \times \mathbf{T}^T \cdot \boldsymbol{\varepsilon}\right) - \frac{1}{2}\operatorname{tr}\left(\boldsymbol{\psi} \times \mathbf{T}^T \times \boldsymbol{\psi}\right) + \frac{1}{2}\operatorname{tr}\left(\boldsymbol{\psi} \times \mathbf{M}^T \cdot \boldsymbol{\varkappa}\right).
\end{aligned}
\tag{87}
$$

Let us note that w is the increment of the elastic energy density of the initially prestressed shell under additional infinitesimal deformations. By Eqs. (86) and (87), w splits into two terms. The first term, w_1, is similar to the strain energy density of the linear shell. w_1 is the quadratic form of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$ with the elastic moduli tensors $\mathbf{H}_1$ and $\mathbf{H}_2$. w_2 is also a quadratic form but depending on $\boldsymbol{\psi}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$. The coefficients in the quadratic form w_2 are expressed in terms of the initial stress and couple stress tensors only, they do not depend on the properties of shell material.

If $\chi_0 = \varkappa$, that is $\mathbf{T} = \mathbf{M} = \mathbf{0}$, then the energy density w is a quadratic form of tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$ having the form

$$
w = w_0 \equiv \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{C}_1 : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varkappa} : \mathbf{C}_2 : \boldsymbol{\varkappa}.
$$

Here w_0 is the strain energy density of an isotropic linear micropolar shell under infinitesimal deformations, see Chróścielewski et al. (2004b), Eremeyev and Zubov (2008), Eremeyev et al. (2013), Eremeyev and Lebedev (2011).

Now the Rayleigh variational principle can be formulated as follows. The modes of shell eigen-oscillations are stationary points of the energy functional

$$
\mathcal{E}[\mathbf{W}, \boldsymbol{\Psi}] = \iint\limits_{\sigma} \left[w_1(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) + w_2(\boldsymbol{\Psi}, \boldsymbol{\varepsilon}, \boldsymbol{\varkappa})\right] d\sigma,
\tag{88}
$$

where

$$
\boldsymbol{\varepsilon} = \nabla_\chi \mathbf{W} + \mathbf{A} \times \boldsymbol{\Psi}, \qquad \boldsymbol{\varkappa} = \nabla_\chi \boldsymbol{\Psi},
$$

on the set of functions that satisfy the kinematic boundary conditions

$$
\text{on } \ell_1 : \quad \mathbf{W} = \mathbf{0} \quad \text{ and on } \ell_3 : \quad \boldsymbol{\Psi} = \mathbf{0}
\tag{89}
$$

and the restriction

$$
\mathcal{K}(\mathbf{W}, \boldsymbol{\Psi}) \equiv \frac{1}{2} \iint\limits_{\sigma} \rho \left(\mathbf{W} \cdot \mathbf{W} + \gamma \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}\right) \, d\sigma = 1.
\tag{90}
$$

Here the functions $\mathbf{W}$, $\boldsymbol{\Psi}$ are the oscillation amplitudes for the translations and rotations, respectively.

The Rayleigh variational principle is equivalent to the stationary principle for the Rayleigh quotient

$$\mathcal{R}[\mathbf{W}, \boldsymbol{\Psi}] = \frac{\mathcal{E}[\mathbf{W}, \boldsymbol{\Psi}]}{\mathcal{K}(\mathbf{W}, \boldsymbol{\Psi})}, \tag{91}$$

that is defined on kinematically admissible functions $\mathbf{W}$, $\boldsymbol{\Psi}$.

The proof of the principle in the case of a prestressed shell is standard and mimics one which can be found for example in Berdichevsky (2009) or in the case of the micropolar shell theory in Eremeyev and Lebedev (2011). For comparison purposes we introduce the Rayleigh quotient of the shell without initial stresses

$$\mathcal{R}_0[\mathbf{W}, \boldsymbol{\Psi}] = \frac{\mathcal{E}_0[\mathbf{W}, \boldsymbol{\Psi}]}{\mathcal{K}(\mathbf{W}, \boldsymbol{\Psi})}, \qquad \mathcal{E}_0[\mathbf{W}, \boldsymbol{\Psi}] = \iint\limits_{\sigma} w_0(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa})\, d\sigma. \tag{92}$$

Note that the least squared eigenfrequencies of the shell correspond to the minimal values of $\mathcal{R}$ and $\mathcal{R}_0$

$$\Omega^2_{\min} = \inf \mathcal{R}[\mathbf{W}, \boldsymbol{\Psi}], \quad {\Omega_0}^2_{\min} = \inf \mathcal{R}_0[\mathbf{W}, \boldsymbol{\Psi}]$$

on $\mathbf{W}$, $\boldsymbol{\Psi}$ that satisfy (89). By the Courant minimax principle, see Courant and Hilbert (1991), Berdichevsky (2009), the Rayleigh quotient (91) allows us to estimate the values of higher eigen-frequencies. For this we should consider $\mathcal{R}$ on the set of functions that are orthogonal to the previous modes of eigen-oscillations in some functional energy space.

12.2 Influence of Initial (Residual) Stresses

To analyze the influence of initial (residual) stresses we compare the functionals $\mathcal{R}$ and $\mathcal{R}_0$ that is equivalent to comparison of $\mathcal{E}$ and $\mathcal{E}_0$. It is obvious that the difference between $\mathcal{E}$ and $\mathcal{E}_0$ consist of two terms: the difference in elastic moduli, that is the difference between $\mathbf{C}_1$ and $\mathbf{H}_1$, $\mathbf{C}_2$ and $\mathbf{H}_2$, and the term w_2 depending on initial stress and couple stress tensors.

Let us consider first w_1 and w_0. In the linear theory of shell it is assumed that w_0 is a positive definite quadratic form of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$. We also assume that $w_1(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa})$ is a positive definite quadratic form. This means that w_1 satisfies the following inequality

$$w_1(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) \geq c_1\|\boldsymbol{\varepsilon}\|^2 + c_2\|\boldsymbol{\varkappa}\|^2$$

with positive constants c_1 and c_2 depending on the shell geometry. This restriction plays the same role as the generalized Coleman-Noll inequality used in the non-linear elasticity, see Eremeyev and Zubov (2007). This case is similar to the dependence of the eigen-frequency of a spring on its stiffness: the increase of stiffness leads to the increase of eigen-frequency.

To analyze the influence of w_2 let us assume that $\mathbf{C}_1 = \mathbf{H}_1$, $\mathbf{C}_2 = \mathbf{H}_2$. This means that we neglect in influence of initial strains on the elastic moduli of the shell. Here we have $w - w_0 = w_2$. It is obvious that w_2 is not a positive definite function, in general. Indeed, let us consider as an example the uniform stretching of the shell with $\mathbf{T} = T\mathbf{A}$, $\mathbf{M} = \mathbf{0}$, T is the uniform tension. We have

$$\begin{aligned} w_2(\boldsymbol{\Psi}, \boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) &= T \operatorname{tr}\, (\boldsymbol{\Psi} \times \mathbf{A} \cdot \boldsymbol{\varepsilon}) - \frac{T}{2} \operatorname{tr}\, (\boldsymbol{\Psi} \times \mathbf{A} \times \boldsymbol{\Psi}) \\ &= T \operatorname{tr}\, \left(\boldsymbol{\Psi} \times \nabla_\chi \mathbf{W}\right) + \frac{T}{2} \operatorname{tr}\, (\boldsymbol{\Psi} \times \mathbf{A} \times \boldsymbol{\Psi}) \\ &= T \operatorname{tr}\, \left(\boldsymbol{\Psi} \times \nabla_\chi \mathbf{W}\right) + \frac{T}{2} [\boldsymbol{\Psi} \cdot \boldsymbol{\Psi} + (\boldsymbol{\Psi} \cdot \mathbf{N})^2]. \end{aligned}$$

Assuming $\nabla_\chi \mathbf{W} = \mathbf{0}$ we obtain

$$w_2 = \frac{T}{2} [\boldsymbol{\Psi} \cdot \boldsymbol{\Psi} + (\boldsymbol{\Psi} \cdot \mathbf{N})^2].$$

Thus, the sign of w_2 coincides with the sign of T. As a result we have

$$\mathcal{E}[\mathbf{0}, \boldsymbol{\Psi}] - \mathcal{E}_0[\mathbf{0}, \boldsymbol{\Psi}] = \frac{T}{2} \iint_\sigma [\boldsymbol{\Psi} \cdot \boldsymbol{\Psi} + (\boldsymbol{\Psi} \cdot \mathbf{N})^2]\, d\sigma.$$

Positive values of T leads to an increase of Ω. This case is similar to the dependence of eigen-frequency of a string on tension (Courant and Hilbert 1991): stretching ($T > 0$) leads to the increase while compression ($T < 0$) leads to the decrease of the eigen-frequencies in comparison with the unstressed shell. Moreover, since initial stresses and couple stresses may lead to instability of the shell that is when $\delta^2 \Pi$ becomes non-positive their influence on eigen-oscillations is more important than the change of elastic moduli tensors.

A few examples showing the influence of initial stresses on the least eigen-frequencies of a prestressed six-parameter shell are given by Altenbach and Eremeyev (2014a). Eremeyev et al. (2015b) presented extension of the eigen-frequencies analysis for higher eigen-frequencies using the Courant variational principle.

13 Constitutive Restrictions for Micropolar Shells

As in 3D elasticity, in the theory of micropolar shells we should supplement the equilibrium/motion equations with constitutive restrictions. We will do that in the frame of general nonlinear shell theory similarly to what was done in 3D elasticity. Following Eremeyev and Zubov (2007) we will formulate the generalized Coleman-Noll inequality (GCN-condition), the strong ellipticity condition for the equilibrium equations and the Hadamard inequality. The inequalities impose some restrictions on constitutive equations of elastic shells under finite deformation. We also will prove that the Coleman-Noll inequality implies strong ellipticity of shell equilibrium equations. We begin with the linear theory.

13.1 Linear Theory of Micropolar Shell

Suppose that specific strain energy $W(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa})$ is positive definite. W is a quadratic form depending on the linear strain tensor and linear bending strain tensor. For an isotropic shell, W takes the form (61). Positivity of the quadratic form (61) with respect to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$ is equivalent to the following set of inequalities

$$2\alpha_1 + \alpha_2 + \alpha_3 > 0, \quad \alpha_2 + \alpha_3 > 0, \quad \alpha_3 - \alpha_2 > 0, \quad \alpha_4 > 0, \tag{93}$$

$$2\beta_1 + \beta_2 + \beta_3 > 0, \quad \beta_2 + \beta_3 > 0, \quad \beta_3 - \beta_2 > 0, \quad \beta_4 > 0.$$

The inequality

$$W(\boldsymbol{\varepsilon}, \boldsymbol{\varkappa}) > 0, \quad \forall\ \boldsymbol{\varepsilon}, \boldsymbol{\varkappa} \neq \mathbf{0}$$

and the inequalities for elastic constants of isotropic material (93) that are its consequences, present the simplest example of additional inequalities in the shell theory. If the inequalities fail this leads to a number of pathological consequences. For example boundary value problems of linear shell theory can have few solutions or can have no solution for some loads. Next, the propagation of waves in some directions becomes impossible that is not natural from the physical point of view. Note that for finite strains, the positive definiteness of specific energy $W(\mathbf{E}, \mathbf{K})$ is not a warranty that the desired properties of boundary value problems or wave propagation hold, here we should introduce some additional restrictions.

13.2 Coleman-Noll Inequality for Elastic Shells

Suppose a solution of equilibrium problem for a nonlinear elastic shell of Cosserat type is known. Let us call it the initial or basic stressed state. The state is defined by vector field $\boldsymbol{\rho}(q^\alpha)$ and tensor field $\mathbf{Q}(q^\alpha)$. Now we consider some equilibrium shell state that perturbs the basic state. If the difference between the state is infinitesimal we can linearize the equations with respect to the quantities characterizing the difference between the states. Let us denote the small increment of various quantities characterizing the perturbed equilibrium with the dot superscript like $\mathbf{D}^{\cdot}$. This quantity can be calculated by the formula:

$$\mathbf{D}^{\cdot} = \frac{d}{d\tau}\mathbf{D}\left[\nabla_s(\boldsymbol{\rho} + \tau\mathbf{u}, \mathbf{Q} - \tau\mathbf{Q}\times\boldsymbol{\theta}, \nabla_s(\mathbf{Q} - \tau\mathbf{Q}\times\boldsymbol{\theta}))\right]\Big|_{\tau=0}, \tag{94}$$

where $\mathbf{u}$ is the vector of additional infinitesimal translation and $\boldsymbol{\theta}$ is the vector of additional infinitesimal rotation characterizing the small rotation with respect to the initial stressed state. The following relations are valid

$$\boldsymbol{\rho}^{\cdot} = \mathbf{u}, \quad \mathbf{Q}^{\cdot} = -\mathbf{Q} \times \boldsymbol{\theta}, \quad \mathbf{E}^{\cdot} = \mathbf{F}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{Q}^{T}, \quad \mathbf{K}^{\cdot} = \mathbf{F}\cdot\boldsymbol{\kappa}\cdot\mathbf{Q}^{T}, \tag{95}$$

$$\boldsymbol{\varepsilon} = \nabla_s \mathbf{u} + \mathbf{A} \times \boldsymbol{\theta}, \quad \boldsymbol{\kappa} = \nabla_s \boldsymbol{\theta}, \tag{96}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ are the linear stretch tensor and linear wryness tensor introduced in (57).

As a reference configuration it can be chosen any stressed shell state. To avoid awkward expressions and to simplify the calculations, we assume the reference configuration to be the initial (basic) stressed state of the shell. This means that in the reference configuration $\mathbf{F} = \mathbf{E} = \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$, $\mathbf{Q} = \mathbf{I}$, $\mathbf{K} = \mathbf{0}$. Under this choice, using Eqs. (39), (42), (94)–(96) we have

$$\begin{aligned} \mathbf{D}^{\cdot} &= \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{E}} : \boldsymbol{\varepsilon} + \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{K}} : \boldsymbol{\kappa} - \mathbf{T} \times \boldsymbol{\theta}, \\ \mathbf{G}^{\cdot} &= \frac{\partial^2 W}{\partial \mathbf{K} \partial \mathbf{E}} : \boldsymbol{\varepsilon} + \frac{\partial^2 W}{\partial \mathbf{K} \partial \mathbf{K}} : \boldsymbol{\kappa} - \mathbf{M} \times \boldsymbol{\theta}. \end{aligned} \tag{97}$$

Suppose that in the initial and perturbed stressed shell states the external couples are zero $\mathbf{m} = \boldsymbol{\mu} = \mathbf{0}$ and the external forces are "dead". Then the total potential energy of the shell is

$$\Pi = \iint\limits_{\Sigma} W \, d\Sigma - \iint\limits_{\Sigma} \mathbf{f}\cdot(\boldsymbol{\rho} - \mathbf{P}) \, d\Sigma - \int\limits_{\omega_2} \mathbf{t}\cdot(\boldsymbol{\rho} - \mathbf{P}) \, ds.$$

Let us consider the energy increment for the perturbed equilibrium state with respect to the initial energy taking into account the members of the second order of smallness

$$\Pi - \Pi_0 = \tau \left(\frac{d\Pi}{d\tau} \right)_{\tau=0} + \frac{1}{2}\tau^2 \left(\frac{d^2\Pi}{d\tau^2} \right)_{\tau=0} + \ldots$$

By the constitutive relations (39) and Eqs. (95), (96), we get

$$\begin{aligned} \frac{d\Pi}{d\tau} = & \iint\limits_{\Sigma} \left[\operatorname{tr} \left(\mathbf{D}^{T} \cdot \nabla_s \mathbf{u} \right) + \operatorname{tr} \left(\mathbf{D}^{T} \cdot \mathbf{F} \times \boldsymbol{\theta} \right) + \operatorname{tr} \left(\mathbf{G}^{T} \cdot \nabla_s \boldsymbol{\theta} \right) \right] d\Sigma \\ & - \iint\limits_{\Sigma} \mathbf{f}\cdot\mathbf{u} \, d\Sigma - \int\limits_{\omega_2} \boldsymbol{\varphi}\cdot\mathbf{u} \, ds. \end{aligned} \tag{98}$$

We recall that the basic stressed shell state is the reference configuration of the shell. Differentiating Eq. (98) with respect to parameter τ and using Eqs. (95) we obtain

$$\left.\frac{d^2\Pi}{d\tau^2}\right|_{\tau=0} = \iint\limits_{\Sigma} \left[\mathrm{tr}\ \left(\mathbf{D}^{\cdot T} \cdot \nabla_s \mathbf{u}\right) + \mathrm{tr}\ \left(\mathbf{D}^{\cdot T} \times \boldsymbol{\theta}\right) + \mathrm{tr}\ \left(\mathbf{T}^T \cdot (\nabla_s \mathbf{u}) \times \boldsymbol{\theta}\right) + \mathrm{tr}\ \left(\mathbf{G}^{\cdot T} \cdot \boldsymbol{\kappa}\right)\right] d\Sigma.$$

As we have chosen equilibrium of shell as the basic state, with use of Eqs. (38) and (40) we get that the first variation of the energy vanishes

$$\left.\frac{d\Pi}{d\tau}\right|_{\tau=0} = 0.$$

By Eqs. (96) and (97), the second energy variation takes the form

$$\left.\frac{d^2\Pi}{d\tau^2}\right|_{\tau=0} = 2\iint\limits_{\Sigma} w\ d\Sigma, \quad w = w' + w'', \tag{99}$$

where

$$\begin{aligned} w' &= \frac{1}{2}\boldsymbol{\varepsilon} : \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{E}} : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{K}} : \boldsymbol{\kappa} + \frac{1}{2}\boldsymbol{\kappa} : \frac{\partial^2 W}{\partial \mathbf{K} \partial \mathbf{K}} : \boldsymbol{\kappa}, \\ w'' &= \mathrm{tr}\ \left(\boldsymbol{\theta} \times \mathbf{T}^T \cdot \boldsymbol{\varepsilon}\right) - \frac{1}{2}\mathrm{tr}\ \left(\boldsymbol{\theta} \times \mathbf{T}^T \times \boldsymbol{\theta}\right) + \frac{1}{2}\mathrm{tr}\ \left(\boldsymbol{\theta} \times \mathbf{M}^T \cdot \boldsymbol{\kappa}\right). \end{aligned} \tag{100}$$

w is the increment of the elastic energy of the initially prestressed shell under additional infinitesimal deformations. By Eqs. (99) and (100), this incremental energy splits into two parts: the pure strain energy, w', and the energy of rotations w''. The coefficients in the quadratic form w'' are expressed in terms of the stress and couple stress tensors of the initially prestressed state, they do not depend on the properties of shell material. If the basic stressed state of the shell is natural, that is $\mathbf{T} = \mathbf{M} = \mathbf{0}$, then $w = w'$ and the energy is a quadratic form of tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$. It is easily seen that the decomposition (86) and Eqs. (87) coincide with the corresponding quantities for the increment of the strain energy density of 3D micropolar body (Eremeyev and Zubov 1994) up to the notation.

The Coleman-Noll constitutive inequality is one of well-known in nonlinear elasticity (Truesdell and Noll 1965; Truesdell 1977, 1984). Its differential form, a so-called GCN-condition, expresses the property that for any reference configuration, the increment of the elastic energy density for arbitrary infinitesimal non-zero strains should be positive. Note that the Coleman–Noll inequality in 3D elasticity does not restrict the constitutive equations with respect to the rotations.

Taking into account the decomposition (86) of the energy we obtain an analogue of the *Coleman-Noll inequality* for micropolar elastic shells

$$w'(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) > 0 \quad \forall\ \boldsymbol{\varepsilon} \neq \mathbf{0}, \quad \boldsymbol{\kappa} \neq \mathbf{0}. \tag{101}$$

Using Eqs. (87) we rewrite (101) in the equivalent form

$$\left.\frac{d^2}{d\tau^2} W\left(\mathbf{E}+\tau\boldsymbol{\varepsilon}, \mathbf{K}+\tau\boldsymbol{\kappa}\right)\right|_{\tau=0} > 0 \quad \forall\, \boldsymbol{\varepsilon} \neq \mathbf{0}, \quad \boldsymbol{\kappa} \neq \mathbf{0}. \tag{102}$$

Condition (102) satisfies the principle of material frame-indifference, it can serve as a constitutive inequality for elastic shells.

13.3 Strong Ellipticity and Hadamard Inequality

In nonlinear elasticity, the strong ellipticity condition and its weak form, the Hadamard inequality, are other known constitutive restrictions. Following the partial differential equations theory (PDE) (Lions and Magenes 1968; Fichera 1972; Hörmander 1976) we formulate the strong ellipticity condition of the equilibrium equations (38). For dead loads, the linearized equilibrium equations are

$$\nabla_s\cdot\mathbf{D}^{\boldsymbol{\cdot}} = \mathbf{0}, \quad \nabla_s\cdot\mathbf{G}^{\boldsymbol{\cdot}} + \left[\mathbf{F}^T\cdot\mathbf{D}^{\boldsymbol{\cdot}} + (\nabla_s\mathbf{u})^T\cdot\mathbf{D}\right]_\times = \mathbf{0}, \tag{103}$$

where $\mathbf{D}^{\boldsymbol{\cdot}}$ and $\mathbf{G}^{\boldsymbol{\cdot}}$ are defined by the formulae similar to (94). Equations (103) constitute a system of linear PDE of second order with respect to $\mathbf{u}$ and $\boldsymbol{\theta}$. The second order parts of the differential operators in Eqs. (103) are

$$\nabla_s\cdot\left\{\left[\frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{E}} : \left((\nabla_s\mathbf{u})\cdot\mathbf{Q}^T\right) + \frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{K}} : \left((\nabla_s\boldsymbol{\theta})\cdot\mathbf{Q}^T\right)\right]\cdot\mathbf{Q}\right\},$$
$$\nabla_s\cdot\left\{\left[\frac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{E}} : \left((\nabla_s\mathbf{u})\cdot\mathbf{Q}^T\right) + \frac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{K}} : \left((\nabla_s\boldsymbol{\theta})\cdot\mathbf{Q}^T\right)\right]\cdot\mathbf{Q}\right\}.$$

Now we can formulate the condition of strong ellipticity for system (103). Following a formal procedure from Fichera (1972), we replace the differential operators ∇_s by the unit vector $\boldsymbol{\nu}$ tangential to surface Σ and vector fields $\mathbf{u}$ and $\boldsymbol{\theta}$ by vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. Thus, we get the algebraic expressions

$$\boldsymbol{\nu}\cdot\left\{\left[\frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{E}} : \left(\boldsymbol{\nu}\otimes\mathbf{a}\cdot\mathbf{Q}^T\right) + \frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{K}} : \left(\boldsymbol{\nu}\otimes\mathbf{b}\cdot\mathbf{Q}^T\right)\right]\cdot\mathbf{Q}\right\},$$
$$\boldsymbol{\nu}\cdot\left\{\left[\frac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{E}} : \left(\boldsymbol{\nu}\otimes\mathbf{a}\cdot\mathbf{Q}^T\right) + \frac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{K}} : \left(\boldsymbol{\nu}\otimes\mathbf{b}\cdot\mathbf{Q}^T\right)\right]\cdot\mathbf{Q}\right\}.$$

Multiply the first equation by vector $\mathbf{a}$, the second equation by $\mathbf{b}$ and add the results. Then we get the strong ellipticity condition of Eqs. (103):

$$\mathbf{v}\cdot\left\{\left[\frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{E}}:\left(\mathbf{v}\otimes\mathbf{a}\cdot\mathbf{Q}^T\right)+\frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{K}}:\left(\mathbf{v}\otimes\mathbf{b}\cdot\mathbf{Q}^T\right)\right]\cdot\mathbf{Q}\right\}\cdot\mathbf{a}$$
$$+\mathbf{v}\cdot\left\{\left[\frac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{E}}:\left(\mathbf{v}\otimes\mathbf{a}\cdot\mathbf{Q}^T\right)+\frac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{K}}:\left(\mathbf{v}\otimes\mathbf{b}\cdot\mathbf{Q}^T\right)\right]\cdot\mathbf{Q}\right\}\cdot\mathbf{b}>0,$$
$$\forall\,\mathbf{a},\mathbf{b}\neq\mathbf{0}.$$

Replacing dot product by the operation ":", we transform the inequality into a symmetric form

$$\left(\mathbf{v}\otimes\mathbf{a}\cdot\mathbf{Q}^T\right):\frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{E}}:\left(\mathbf{v}\otimes\mathbf{a}\cdot\mathbf{Q}^T\right)+2\left(\mathbf{v}\otimes\mathbf{a}\cdot\mathbf{Q}^T\right):\frac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{K}}:\left(\mathbf{v}\otimes\mathbf{b}\cdot\mathbf{Q}^T\right)$$
$$+\left(\mathbf{v}\otimes\mathbf{b}\cdot\mathbf{Q}^T\right):\frac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{K}}:\left(\mathbf{v}\otimes\mathbf{b}\cdot\mathbf{Q}^T\right)>0,\qquad\forall\,\mathbf{a},\mathbf{b}\neq\mathbf{0}.$$

In matrix notations, we rewrite this in a compact form

$$\boldsymbol{\xi}\cdot\mathbb{A}(\mathbf{v})\cdot\boldsymbol{\xi}>0,\quad\forall\;\mathbf{v}\neq\mathbf{0},\quad\mathbf{v}\cdot\mathbf{N}=0,\quad\forall\;\boldsymbol{\xi}\in\mathbb{R}^6,\quad\boldsymbol{\xi}\neq\mathbf{0},\tag{104}$$

where $\boldsymbol{\xi}=(\mathbf{a}',\mathbf{b}')\in\mathbb{R}^6$, $\mathbf{a}'=\mathbf{a}\cdot\mathbf{Q}^T$, $\mathbf{b}'=\mathbf{b}\cdot\mathbf{Q}^T$, and matrix $\mathbb{A}(\mathbf{v})$ is

$$\mathbb{A}(\mathbf{v})\triangleq\begin{bmatrix}\dfrac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{E}}\{\mathbf{v}\} & \dfrac{\partial^2 W}{\partial\mathbf{E}\partial\mathbf{K}}\{\mathbf{v}\}\\[2ex] \dfrac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{E}}\{\mathbf{v}\} & \dfrac{\partial^2 W}{\partial\mathbf{K}\partial\mathbf{K}}\{\mathbf{v}\}\end{bmatrix},$$

where for any fourth-order tensor $\mathbf{K}$ and vector $\mathbf{v}$ we denote

$$\mathbf{K}\{\mathbf{v}\}\triangleq K_{klmn}\nu_k\nu_m\mathbf{i}_l\otimes\mathbf{i}_n.$$

Inequality (104) is the *strong ellipticity condition* of the equilibrium equations (38) for the elastic shell. A weak form of inequality (104) is an analogue of the *Hadamard inequality*. These inequalities are examples of possible restrictions of the constitutive equations of elastic shells under finite deformations. As for the theory of simple materials, a failure in inequality (104) can lead to the existence of non-smooth solutions to equilibrium equations (38), see Lurie (1990).

The strong ellipticity condition can be written in the equivalent form

$$\left.\frac{d^2}{d\tau^2}W\left(\mathbf{E}+\tau\mathbf{v}\otimes\mathbf{a}',\mathbf{K}+\tau\mathbf{v}\otimes\mathbf{b}'\right)\right|_{\tau=0}>0\quad\forall\;\mathbf{v},\;\mathbf{a}',\;\mathbf{b}'\neq\mathbf{0}.\tag{105}$$

Comparing the strong ellipticity condition (105) and the Coleman-Noll inequality (102) one can see that the latter implies the former. Indeed, inequality (102) holds for any tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$. Note that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ may be nonsymmetric tensors, in general.

Substituting relations $\boldsymbol{\varepsilon} = \mathbf{v} \otimes \mathbf{a}'$ and $\boldsymbol{\kappa} = \mathbf{v} \otimes \mathbf{b}'$ to inequality (102), we immediately obtain inequality (105). Thus, the strong ellipticity condition is a particular case of the Coleman–Noll inequality. We watch an essential difference between the micropolar shell theory and the theory of simple elastic materials (Truesdell and Noll 1965; Truesdell 1977): in the latter these two properties are independent in the sense that neither of them implies the other.

In the shell theory, the following particular constitutive relation is widely used

$$W(\mathbf{E}, \mathbf{K}) = W_1(\mathbf{E}) + W_2(\mathbf{K}). \tag{106}$$

For example, Eq. (33) has the form of (106). Now condition (104) is equivalent to two simpler inequalities

$$\mathbf{a} \cdot \frac{\partial^2 W_1}{\partial \mathbf{E} \partial \mathbf{E}} \{\mathbf{v}\} \cdot \mathbf{a} > 0, \quad \mathbf{b} \cdot \frac{\partial^2 W_2}{\partial \mathbf{K} \partial \mathbf{K}} \{\mathbf{v}\} \cdot \mathbf{b} > 0.$$

As an example, let us consider consequences of conditions (104) for constitutive equation (33). In this case we have

$$\begin{aligned} \frac{\partial^2 W_1}{\partial \mathbf{E} \partial \mathbf{E}} \{\mathbf{v}\} &= \alpha_3 \mathbf{A} + (\alpha_1 + \alpha_2) \mathbf{v} \otimes \mathbf{v} + \alpha_4 \mathbf{N} \otimes \mathbf{N}, \\ \frac{\partial^2 W_2}{\partial \mathbf{K} \partial \mathbf{K}} \{\mathbf{v}\} &= \beta_3 \mathbf{A} + (\beta_1 + \beta_2) \mathbf{v} \otimes \mathbf{v} + \beta_4 \mathbf{N} \otimes \mathbf{N}. \end{aligned} \tag{107}$$

Now inequality (104) is valid under the following conditions

$$\begin{aligned} &\alpha_3 > 0, \quad \alpha_1 + \alpha_2 + \alpha_3 > 0, \quad \alpha_4 > 0, \\ &\beta_3 > 0, \quad \beta_1 + \beta_2 + \beta_3 > 0, \quad \beta_4 > 0. \end{aligned} \tag{108}$$

For a linear isotropic shell, inequalities (108) provide the strong ellipticity of equilibrium equation (59), they are weaker than the conditions of positive definiteness (93). Considering the constitutive equations of an isotropic micropolar shell (33) we have reduced inequality (104) to the inequalities (108).

13.4 Strong Ellipticity Condition and Acceleration Waves

Using the approach of Eremeyev (2005b), Eremeyev and Zubov (2007), Altenbach et al. (2010b), we will show that inequality (104) coincides with the conditions of the propagation of acceleration waves in a shell. We consider a motion when on a smooth curve $\mathcal{C}(t) \subset \Sigma$ called *singular* (Fig. 4), continuous kinematic and dynamic quantities can jump. We assume that the limit values of these quantities exist on $\mathcal{C}$ being different from the opposite sides of $\mathcal{C}$ in general. The jump of quantity $\boldsymbol{\psi}$ on

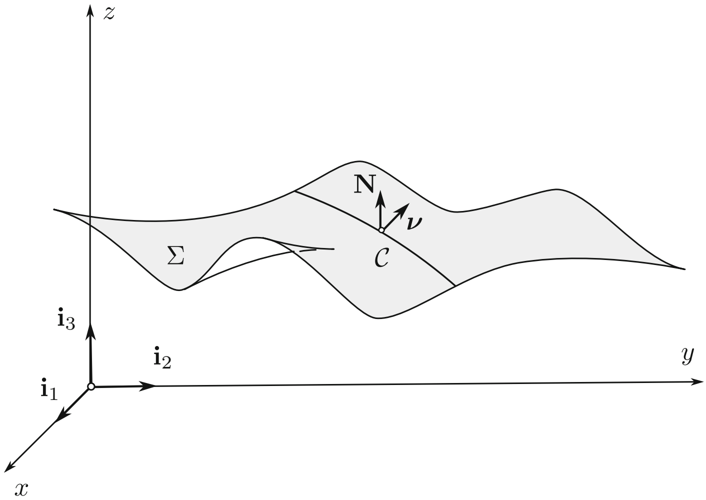

Fig. 4 Shell with a singular curve

$\mathcal{C}$ will be denoted by the double brackets: $[\![\boldsymbol{\psi}]\!] = \boldsymbol{\psi}^+ - \boldsymbol{\psi}^-$, where $\boldsymbol{\psi}^\pm$ are one-side limits of $\boldsymbol{\psi}$.

An *acceleration wave* (a *weak-discontinuity wave* or *second-order singular curve*) is a moving singular curve $\mathcal{C}$ on which the second derivatives of the radius-vector $\boldsymbol{\rho}$ and the microrotation tensor $\mathbf{Q}$ with respect to the spatial coordinates and time are discontinuous, while $\boldsymbol{\rho}$, $\mathbf{Q}$ and their first derivatives are continuous that means that on $\mathcal{C}$

$$[\![\mathbf{F}]\!] = \mathbf{0}, \quad [\![\nabla_s \mathbf{Q}]\!] = \mathbf{0}, \quad [\![\mathbf{v}]\!] = \mathbf{0}, \quad [\![\boldsymbol{\omega}]\!] = \mathbf{0}. \tag{109}$$

By Eqs. (29), the stretch measure $\mathbf{E}$ and the wryness tensor $\mathbf{K}$ are continuous on $\mathcal{C}$. By constitutive equations (39), the jumps of tensors $\mathbf{D}$ and $\mathbf{G}$ are absent. Applying the Maxwell theorem formulated by Truesdell (1977) to continuous fields of velocities $\mathbf{v}$ and $\boldsymbol{\omega}$, surface stress tensor $\mathbf{D}$, and the surface couple stress tensor $\mathbf{G}$, we deduce a system of equations that relates the jumps of the derivatives of these quantities with respect to the spatial coordinates and time

$$\left[\!\left[\frac{d\mathbf{v}}{dt}\right]\!\right] = -V\mathbf{a}, \quad [\![\nabla_s \mathbf{v}]\!] = \boldsymbol{\nu} \otimes \mathbf{a}, \quad \left[\!\left[\frac{d\boldsymbol{\omega}}{dt}\right]\!\right] = -V\mathbf{b}, \quad [\![\nabla_s \boldsymbol{\omega}]\!] = \boldsymbol{\nu} \otimes \mathbf{b}, \tag{110}$$
$$V\,[\![\nabla_s \cdot \mathbf{D}]\!] = -\boldsymbol{\nu} \cdot \left[\!\left[\frac{d\mathbf{D}}{dt}\right]\!\right], \quad V\,[\![\nabla_s \cdot \mathbf{G}]\!] = -\boldsymbol{\nu} \cdot \left[\!\left[\frac{d\mathbf{G}}{dt}\right]\!\right].$$

Here $\mathbf{a}$ and $\mathbf{b}$ are the vectorial amplitudes of the jumps of the linear and angular accelerations, respectively, $\boldsymbol{\nu}$ is the unit normal vector to $\mathcal{C}$ such that $\mathbf{N} \cdot \boldsymbol{\nu} = 0$, and V

is the velocity of the surface $\mathcal{C}$ in the direction $\mathbf{v}$. If the external forces and couples are continuous, the relations

$$[\![\nabla_s \cdot \mathbf{D}]\!] = \rho \left[\frac{d\mathbf{K}_1}{dt} \right], \quad [\![\nabla_s \cdot \mathbf{G}]\!] = \rho \left[\frac{d\mathbf{K}_2}{dt} \right]$$

follow immediately from the motion equations (28).

Differentiating constitutive equations (39) and using Eqs. (109) and (110), we express the last relations in terms of vector amplitudes $\mathbf{a}$ and $\mathbf{b}$

$$\begin{aligned}
\mathbf{v} \cdot \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{E}} &: \left(\mathbf{v} \otimes \mathbf{a} \cdot \mathbf{Q}^T\right) + \mathbf{v} \cdot \frac{\partial^2 W}{\partial \mathbf{E} \partial \mathbf{K}} : \left(\mathbf{v} \otimes \mathbf{b} \cdot \mathbf{Q}^T\right) \\
&= \rho V^2 \left[\mathbf{a} \cdot \mathbf{Q}^T + \left(\mathbf{Q} \cdot \boldsymbol{\Theta}_1^T \cdot \mathbf{Q}^T\right) \cdot \left(\mathbf{b} \cdot \mathbf{Q}^T\right)\right], \\
\mathbf{v} \cdot \frac{\partial^2 W}{\partial \mathbf{K} \partial \mathbf{E}} &: \left(\mathbf{v} \otimes \mathbf{a} \cdot \mathbf{Q}^T\right) + \mathbf{v} \cdot \frac{\partial^2 W}{\partial \mathbf{K} \partial \mathbf{K}} : \left(\mathbf{v} \otimes \mathbf{b} \cdot \mathbf{Q}^T\right) \\
&= \rho V^2 \left[\left(\mathbf{Q} \cdot \boldsymbol{\Theta}_1 \cdot \mathbf{Q}^T\right) \cdot \left(\mathbf{a} \cdot \mathbf{Q}^T\right) + \left(\mathbf{Q} \cdot \boldsymbol{\Theta}_2 \cdot \mathbf{Q}^T\right) \cdot \left(\mathbf{b} \cdot \mathbf{Q}^T\right)\right].
\end{aligned}$$

Hence the strong ellipticity condition can be written in a compact form

$$\mathbb{A}(\mathbf{v}) \cdot \boldsymbol{\xi} = \rho V^2 \mathbb{B} \cdot \boldsymbol{\xi}, \qquad \mathbb{B} = \begin{bmatrix} \mathbf{I} & \mathbf{Q} \cdot \boldsymbol{\Theta}_1^T \cdot \mathbf{Q}^T \\ \mathbf{Q} \cdot \boldsymbol{\Theta}_1 \cdot \mathbf{Q}^T & \mathbf{Q} \cdot \boldsymbol{\Theta}_2 \cdot \mathbf{Q}^T \end{bmatrix}. \tag{111}$$

Thus, the problem of propagation of an acceleration wave in a shell is reduced to the spectral problem given by algebraic Eqs. (111). Existence of potential-energy function W implies that $\mathbb{A}(\mathbf{v})$ is symmetric. Matrix $\mathbb{B}$ is also symmetric and positive definite. This enables us to formulate an analogue of the *Fresnel–Hadamard–Duhem theorem* for the elastic shell:

Theorem 13.1 *In an elastic shell, for any propagation direction specified by vector* $\mathbf{v}$*, the squared velocities of a second order singular curve (the acceleration wave) are real.*

Note that positive definiteness of $\mathbb{A}(\mathbf{v})$, which is necessary and sufficient for the wave velocity V to be real, coincides with the strong ellipticity inequality (104).

For a physically linear shell, we present an example of solution of the problem (111). Let $\boldsymbol{\Theta}_1$ be zero and $\boldsymbol{\Theta}_2$ be a spherical part of tensor (ball tensor), that is $\boldsymbol{\Theta}_2 = j\,\mathbf{I}$, where j is the rotatory inertia measure. Let the inequalities (108) hold. Then the solutions of Eq. (111) are

$$V_1 = \sqrt{\frac{\alpha_3}{\rho}}, \ \boldsymbol{\xi}_1 = (\boldsymbol{\tau}, \mathbf{0}), \ V_2 = \sqrt{\frac{\alpha_1 + \alpha_2 + \alpha_3}{\rho}}, \ \boldsymbol{\xi}_2 = (\mathbf{v}, \mathbf{0}), \tag{112}$$

$$V_3 = \sqrt{\frac{\alpha_4}{\rho}}, \qquad \boldsymbol{\xi}_3 = (\mathbf{N}, \mathbf{0}), \; V_4 = \sqrt{\frac{\beta_3}{\rho j}}, \; \boldsymbol{\xi}_4 = (\mathbf{0}, \boldsymbol{\tau}),$$
$$V_5 = \sqrt{\frac{\beta_1 + \beta_2 + \beta_3}{\rho j}}, \; \boldsymbol{\xi}_5 = (\mathbf{0}, \boldsymbol{\nu}), \; V_6 = \sqrt{\frac{\beta_4}{\rho j}}, \; \boldsymbol{\xi}_6 = (\mathbf{0}, \mathbf{N}).$$

The solutions (112) are similar to the 3D case Eremeyev (2005b), Altenbach et al. (2010b) and describe the *transverse and longitudinal acceleration waves* and *transverse and longitudinal acceleration waves of microrotation*, respectively.

13.5 Ordinary Ellipticity

If the equilibrium equations are not elliptic the continuity of solutions can fail. Let us consider this in more detail. We will assume the singular curves to be time-independent. Suppose on the shell surface Σ there exists a curve $\mathcal{C}$ on which there happen a jump in the values of second derivatives of position vector $\boldsymbol{\rho}$ or microrotation tensor $\mathbf{Q}$. Such a jump will be called the *weak discontinuity*. As the curvature of Σ is determined through second derivatives of $\boldsymbol{\rho}$, such discontinuity can be exhibited as wrinkling of the shell surface.

From the equilibrium equations it follows $[\![\nabla_s \cdot \mathbf{D}]\!] = \mathbf{0}, \quad [\![\nabla_s \cdot \mathbf{G}]\!] = \mathbf{0}$. Repeating the transformations of the previous section, we transform these to

$$\mathbb{A}(\boldsymbol{\nu}) \cdot \boldsymbol{\xi} = \mathbf{0}, \quad \boldsymbol{\xi} = (\mathbf{a}', \mathbf{b}') \in \mathbb{R}^6. \tag{113}$$

Existence of nontrivial solutions of Eq. (113) means that the weak discontinuities arise. The nontrivial solutions exist if the determinant of matrix $\mathbb{A}(\boldsymbol{\nu})$ is zero. If

$$\det \mathbb{A}(\boldsymbol{\nu}) \neq 0, \tag{114}$$

the weak discontinuities are impossible.

For the constitutive relation $W = W_1(\mathbf{E}) + W_2(\mathbf{K})$, condition (114) splits into two conditions

$$\det \frac{\partial^2 W_1}{\partial \mathbf{E} \partial \mathbf{E}} \{\boldsymbol{\nu}\} \neq 0, \quad \det \frac{\partial^2 W_2}{\partial \mathbf{K} \partial \mathbf{K}} \{\boldsymbol{\nu}\} \neq 0. \tag{115}$$

As an example, we consider conditions (115) for the constitutive relations of a physically linear shell (33). Using Eqs. (107) we can show that conditions (115) reduce to the inequalities

$$\alpha_3 \neq 0, \quad \alpha_1 + \alpha_2 + \alpha_3 \neq 0, \quad \alpha_4 \neq 0, \quad \beta_3 \neq 0, \quad \beta_1 + \beta_2 + \beta_3 \neq 0, \quad \beta_4 \neq 0.$$

Condition (114) is the *ellipticity condition* of the equilibrium equations of shell theory (ellipticity in the Petrovsky sense). The condition follows from the general definition of ellipticity in PDE theory (Agranovich 1997; Hörmander 1976; Nirenberg 2001). Condition (114) is also called the *ordinary ellipticity condition*, it is weaker than the strong ellipticity condition (104).

14 Applications

The recent progress in material technology of materials extends the field of application of classical and non-classical theories of plates and shells towards the new phenomena which should be taken into account. In this section we discuss some applications and extensions of the presented above theory.

14.1 Surface Stresses

One example of phenomena which are significant at the micro- and nanoscales is the surface effects. For example, nanomaterials have physical properties which are different from the bulk material. The classical elasticity can be extended to the nanoscale by taking into account the surface stresses, cf. Duan et al. (2008), Wang et al. (2010, 2011), Javili et al. (2012), Altenbach and Morozov (2013). In particular, the surface stresses are responsible for the size-effect, that means the apparent material properties of a specimen depend on its size. For example, Young's modulus of a rod-like specimen increases significantly, when the cross-section area becomes very small. The surface stresses are the generalization of the scalar surface tension which is a well-known phenomenon in the theory of capillarity. The investigations of the surface phenomena were initiated by Laplace, Young and Gibbs within the sharp interface model, see original papers Laplace (1805), Laplace (1806), Young (1805), Longley and Name (1928), and extended by van der Waals (1893), Korteweg (1901) using the second-gradient models, see also Rowlinson and Widom (2003), Finn (1986), de Gennes et al. (2004), Javili et al. (2013). Rational mechanics of nonlinear elastic solids with surface stresses is developed in Gurtin and Murdoch (1975), Steigmann and Ogden (1999). The theory of elasticity with surface stresses is applied to the modifications of the two-dimensional theories of nanosized plates, see, for example, Altenbach and Eremeyev (2011b), Altenbach et al. (2012b) and the reference therein.

Using six-parameter theory of shells the modification of the constitutive equations taking into account surface stresses is proposed in Altenbach and Eremeyev (2011b). It is shown that both the stress resultant and the couple stress tensors are represented as a sum of two terms as follows

$$\mathbf{T}^* = \mathbf{T} + \mathbf{T}_S \quad \mathbf{M}^* = \mathbf{M} + \mathbf{M}_S\,, \tag{116}$$

where $\mathbf{T}$ and $\mathbf{M}$ are the classical stress resultant tensors presented for example in Lebedev et al. (2010), Libai and Simmonds (1998), while $\mathbf{T}_S$ and $\mathbf{M}_S$ are the resultant tensors induced by the surface stresses

$$\mathbf{T}_S = G_+\mathbf{S}_+ + G_-\mathbf{S}_- , \tag{117}$$

$$\mathbf{M}_S = -h/2\left[G_+\mathbf{S}_+ \times \mathbf{z}_+ - G_-\mathbf{S}_- \times \mathbf{z}_-\right] . \tag{118}$$

Here $\mathbf{S}_\pm$ are the tensors of surface stresses acting on the shell faces, $\mathbf{z}_\pm$ and $G = \det \mathbf{G}$ are the deviation and the geometric scale factor defined in Libai and Simmonds (1998), and $(\ldots)_\pm = (\ldots)\big|_{\pm h/2}$.

The first term in Eqs. (116) is the volume stress resultant while the second one determined by the surface stresses and the shell geometry. In the linear case this modification reduces to the addition of new terms to the elastic stiffness parameters, see Eremeyev et al. (2009), Altenbach et al. (2009, 2010a)

$$\alpha_1 = C\nu + 2\lambda_S , \quad \alpha_3 = C(1-\nu) + 4\mu_S ,$$

$$\alpha_4 = \alpha_s C(1-\nu) ,$$

$$\beta_1 = D\nu + h^2\lambda_S/2 , \quad \beta_3 = D(1-\nu) + h^2\mu_S ,$$

$$\beta_4 = \alpha_t D(1-\nu) ,$$

$$C^* = C + 4\mu_S + 2\lambda_S,$$

$$D^* = D + h^2\mu_S + h^2\lambda_S/2 .$$

Here λ_S and μ_S are the surface elastic moduli, C^* and D^* are the effective in-plane and bending stiffness of the plate with surface stresses. It is clear that $C^* > C$ and $D^* > D$, i.e. the plate with surface stresses is stiffer. The elastic moduli α_4 and β_4 do not depend on the surface stresses.

The model of plates and shells with the surface elasticity was extended for the case of surface viscoelasticity by Altenbach et al. (2012b).

14.2 Thin-Walled Structures Made of Micropolar Materials

The interest to the theory of thin-walled structures made of a micropolar material is based on prospective applications of this theory to mechanics of plates and shells made of materials with complex inner structure, such as, for example, cellular materials and foams, see Lakes (1986), Diebels and Steeb (2003), Bigoni and Drugan (2007), Goda et al. (2012), Reda et al. (2016), Eremeyev et al. (2013). In the literature theories of plates and shells and theories based on the reduction of the three-dimensional equations of the micropolar continuum are also known, see Eringen

(1967a), Eringen (1999), Reissner (1977), Altenbach and Eremeyev (2009c), Zubov (2009), Sargsyan (2011), Steinberg and Kvasov (2013) and the review Altenbach et al. (2010c), where various averaging procedures in the thickness direction together with the approximation of the displacements and rotations or the stresses and couple stresses in the thickness direction are discussed. As it is shown by Altenbach and Eremeyev (2009c) the 3D to 2D reduction procedure leads to Eqs. (58) and (59) with modified stiffness parameters α_k and β_k. The nonlinear case is considered by Zubov (2009). On the other hand there reduction procedure leading to the more complicated structure of governing equations, than the presented in this paper, see for example, Eringen (1967a), Sargsyan (2011). The model of micropolar shells can be used for modelling of thin structures made of certain composites, see dell'Isola et al. (2016a, b), Giorgio et al. (2015).

14.3 Thin-Walled Structures Made of Viscoelastic Materials

The two-dimensional constitutive equations for resultant force and couple stress tensors are derived from the constitutive equations of three-dimensional viscoelastic Cosserat continuum. For the linear theory of viscoelasticity given in Eringen (1967b) the application of the correspondence principle gives the possibility to derive the theory of viscoelasticity in the case of thin-walled structures such as plates and shells. The presented here results demonstrate how the viscoelastic properties of three-dimensional continuum inherit in the constitutive equations for plates and shells. Within framework of the linear micropolar viscoelasticity with the constitutive equations of differential type it is shown that 2D relaxation functions of shells have more complicated structures then the relaxation function of the bulk material. In particular, even for homogeneous shells the spectrum of relaxation time do not coincide with the spectrum of the bulk material. For nonhomogeneous shells the spectrum may depend also on the structure of the shell in the thickness direction and its curvature. The basics of such a theory considering general linear viscoelastic behavior are given by Altenbach and Eremeyev (2008, 2009a, b, 2011a) within the framework of five-parameter theory of shells and by Altenbach and Eremeyev (2015) for the six-parameter theory of shells. It is shown how the effective viscoelastic properties reflect the properties in the thickness direction.

14.4 Shells and Plates with Phase Transitions (PT)

The interest to thin-walled structures undergoing PT grows recently with perspective applications of martensite films in engineering, see e.g. Miyazaki et al. (2009). The major known theories of PT in deformable solids relate to the three-dimensional thermoelasticity, see Bhattacharya (2003), Abeyaratne and Knowles (2006), Berezovski et al. (2008) and references cited there.

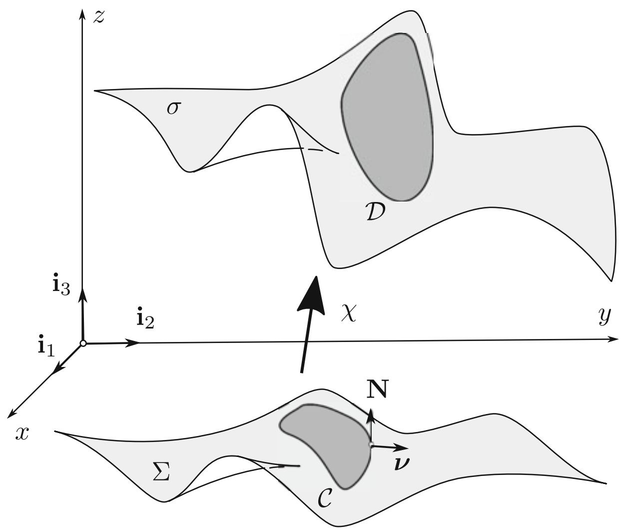

Fig. 5 Two-phase shell with phase interface in reference and actual configurations

The first two-dimensional (2D) mechanical models of PT in thin films are proposed by Bhattacharya and James (1999), James and Rizzoni (2000), see also Bhattacharya (2003), Miyazaki et al. (2009). Alternative 2D models of PT with applications to biomembrane modeling are suggested by Boulbitch (1999), Agrawal and Steigmann (2008), Elliott and Stinner (2010). The model of shell with PT discussed by Shkutin (2007) relates with phase-field models in the continuum mechanics.

The non-linear equilibrium conditions of elastic shells undergoing PT of martensitic type are formulated by Eremeyev and Pietraszkiewicz (2004) and extended in Pietraszkiewicz et al. (2007) taking into account the line tension energy. By analogy to the 3D case, the two-phase shell is regarded as a Cosserat surface consisting of two material phases divided by a sufficiently smooth surface singular curve (phase interface). The existence of such a curve is confirmed by several experiments on thin-walled samples, see e.g. He and Sun (2009, 2010) and the discission in Eremeyev and Pietraszkiewicz (2011). The quasistatic behavior of two-phase shells is analyzed by Eremeyev and Pietraszkiewicz (2009, 2010, 2011). It is assumed that in the actual configuration of the shell consists of different material phases occupying different complementary subregions separated by the curvilinear phase interface $\mathcal{D} \in \sigma$ (Fig. 5). For a piecewise differentiable mapping $\chi : \Sigma \to \sigma$ one can introduce on Σ a singular image curve $\mathcal{C} = \chi^{-1}(\mathcal{D})$. The a priori unknown curves $\mathcal{D}$ and $\mathcal{C}$ are called phase interfaces in the reference and actual configurations, respectively.

The two-dimensional local laws of shell thermomechanics can be derived by direct and exact through-the-thickness integration of global three-dimensional balances of forces, moments, energy and the entropy inequality, see Eremeyev and Pietraszkiewicz (2009, 2011) for details. After appropriate transformations the resulting 2D local Lagrangian laws contain also the surface temperature deviation, the extra surface heat flux, and dual entropy-type quantities in addition to the mean surface temperature and entropy fields. Other versions of the thermodynamics of shells and two-dimensional structures can be found in Green and Naghdi (1970), Green and Naghdi (1979), Murdoch (1976a, b), Zhilin (1976), Simmonds (1984, 2005, 2011), Makowski and Pietraszkiewicz (2002), Steinmann and Häsner (2005).

There are two types of phase interfaces: the coherent in rotations phase interface and the incoherent in rotations one (Eremeyev and Pietraszkiewicz 2004). Using the integral balance laws, the local balance equations along the coherent and incoherent phase interfaces $\mathcal{C}$, i.e. Lagrangian dynamic compatibility conditions, the local energy balance equation, and the local entropy inequality, the kinetic equation describing motion of the phase interface for all quasistatic processes, is formulated in the form

$$V = -\mathcal{F}\left(\boldsymbol{\nu} \cdot [\![\mathbf{C}]\!] \cdot \boldsymbol{\nu}\right), \tag{119}$$

where V is the velocity of the phase interface, the double brackets stand for the jump of $\mathbf{C}$ across $\mathcal{C}$, $\mathcal{F}$ is the non-negative definite kinetic function depending on the jump of $\mathbf{C}$ at $\mathcal{C}$, i.e. $\mathcal{F}(x) \geq 0$ for $x > 0$, and $\mathbf{C}$ is the Eshelby tensor in the non-linear shell theory introduced by Eremeyev and Pietraszkiewicz (2004). For the coherent phase interface $\mathbf{C}$ is given by the formula

$$\mathbf{C} = \mathbf{C}_c \equiv W\mathbf{A} - \mathbf{T} \cdot \mathbf{F}^T - \mathbf{M} \cdot \mathbf{K}^T,$$

and for the phase interface incoherent in rotations by

$$\mathbf{C} = \mathbf{C}_i \equiv W\mathbf{A} - \mathbf{T} \cdot \mathbf{F}^T.$$

For the sake of simplicity these formulas are restricted by pure mechanical theory.

After Berezovski et al. (2008) $\mathcal{F}(x)$ is assumed in the form

$$\mathcal{F}(x) = \begin{cases} \dfrac{k(x - \varsigma_0)}{1 + a(x - \varsigma_0)} & x \geq \varsigma_0, \\ 0 & -\varsigma_0 < x < \varsigma_0, \\ \dfrac{k(x + \varsigma_0)}{1 - a(x + \varsigma_0)} & x \leq -\varsigma_0. \end{cases} \tag{120}$$

Here ς_0 describes the effects associated with nucleation of the new phase, a is a parameter describing the limit value of PT, and k is a positive kinetic factor.

Equation (120) with the appropriate boundary conditions and constitutive equations constitute the non-linear boundary-value problem for a shell with PT with respect to unknown surface fields, as well as the position of the phase interface $\mathcal{C}$. Considering the model s one observes the existence of hysteresis loop characteristic to the behaviour of phase transitions in martensitic materials. The size of the loop depends upon the values of several loading and material parameters.

14.5 Beams and Rods

The presented here direct approach based on Cosserat models can be easily transformed for more technically simple cases of beams and rods. For this purposes we refer to Altenbach et al. (2012a, 2013), Bîrsan et al. (2012) and the reference therein.

15 Conclusions

We presented here the basic equations of the micropolar shell theory using the concept of deformable directed surfaces as a model of a shell. The model coincides kinematically with the general six-parameter resultant shell theory. The presented theory is full analogues to the three-dimensional Cosserat or micropolar theory of elastic solids. The main peculiarity of the model that the interaction between the parts of the shell is determined only by the force and moment tensors including drilling moment. As the consequence the translations and the rotations of the material points of the deformable surface are kinematically independent.

References

Abeyaratne, R., & Knowles, J. K. (2006). *Evolution of phase transitions: A continuum theory*. Cambridge: Cambridge University Press.

Agranovich, M. (1997). Elliptic boundary problems. In M. Agranovich, Y. Egorov, & M. Shubin (Eds.), *Partial differential equations IX: Elliptic boundary problems* (pp. 1–144)., Encyclopaedia of Mathematical Sciences, volume 79 Berlin: Springer.

Agrawal, A., & Steigmann, D. J. (2008). Coexistent fluid-phase equilibria in biomembranes with bending elasticity. *The Journal of Elasticity*, *93*(1), 63–80.

Akay, A., Xu, Z., Carcaterra, A., & Koç, I. M. (2005). Experiments on vibration absorption using energy sinks. *The Journal of the Acoustical Society of America*, *118*(5), 3043–3049.

Alijani, F., & Amabili, M. (2014). Non-linear vibrations of shells: A literature review from 2003 to 2013. *International Journal of Non-Linear Mechanics*, *58*, 233–257.

Altenbach, H., & Eremeyev, V. A. (2010). On the effective stiffness of plates made of hyperelastic materials with initial stresses. *International Journal of Non-Linear Mechanics*, *45*(10), 976–981.

Altenbach, H., & Eremeyev, V. A. (2011a) Mechanics of viscoelastic plates made of FGMs. In J. Murín, V. Kompiš, & V. Kutiš (Eds.), *Computational modelling and advanced simulations* (pp. 33–48), volume 24 of *Computational Methods in Applied Sciences*. Springer.

Altenbach, H., & Eremeyev, V. A. (Eds.). (2013a) *Generalized continua from the theory to engineering applications*, volume 541 of *CISM International Centre for Mechanical Sciences*. Springer Vienna.

Altenbach, H., & Eremeyev, V. A. (2014a). Vibration analysis of non-linear 6-parameter prestressed shells. *Meccanica*, *49*, 1751–1761.

Altenbach, H., & Eremeyev, V. A. (2008). On the analysis of viscoelastic plates made of functionally graded materials. *ZAMM*, *88*(5), 332–341.

Altenbach, H., & Eremeyev, V. A. (2009a). On the bending of viscoelastic plates made of polymer foams. *Acta Mechanica*, *204*(3–4), 137–154.

Altenbach, H., & Eremeyev, V. A. (2009b). On the time-dependent behavior of FGM plates. *Key Engineering Materials*, *399*, 63–70.

Altenbach, H., & Eremeyev, V. A. (2011b). On the shell theory on the nanoscale with surface stresses. *The International Journal of Engineering Science*, *49*, 1294–1301.

Altenbach, H., & Eremeyev, V. A. (2009c). On the linear theory of micropolar plates. *ZAMM*, *89*(4), 242–256.

Altenbach, H., & Eremeyev, V. A. (2011c). *Shell-like Structures: Non-classical Theories and Applications*, volume 15 of *Advanced Structured Materials*. Springer.

Altenbach, H., & Eremeyev, V. A. (2013b). Cosserat-type shells. In H. Altenbach & V. A. Eremeyev (Eds.), *Generalized continua from the theory to engineering applications* (Vol. 541, pp. 131–178). CISM Courses and Lectures Wien: Springer.

Altenbach, H., & Eremeyev, V. A. (2014b). Actual developments in the nonlinear shell theory—state of the art and new applications of the six-parameter shell theory. In W. Pietraszkiewicz & J. Górski (Eds.), *Shell structures: Theory and applications* (Vol. 3, pp. 3–12). Taylor & Francis.

Altenbach, H., & Eremeyev, V. A. (2015). On the constitutive equations of viscoelastic micropolar plates and shells of differential type. *Mathematics and Mechanics of Complex Systems*, *3*(3), 273–283.

Altenbach, H., & Mikhasev, G. (Eds.). (2014). *Shell and Membrane Theories in Mechanics and Biology: From Macro-to Nanoscale Structures*, volume 45 of *Advanced Structured Materials*. Springer.

Altenbach, H., & Morozov, N. F. (Eds.). (2013). *Surface Effects in Solid Mechanics—Models, Simulations and Applications*. Heidelberg: Springer.

Altenbach, H., Eremeyev, V. A., & Morozov, N. F. (2009). Linear theory of shells taking into account surface stresses. *Doklady Physics*, *54*(12), 531–535.

Altenbach, H., Eremeev, V. A., & Morozov, N. F. (2010a). On equations of the linear theory of shells with surface stresses taken into account. *Mechanics of Solids*, *45*(3), 331–342.

Altenbach, H., Eremeyev, V. A., Lebedev, L. P., & Rendón, L. A. (2010b). Acceleration waves and ellipticity in thermoelastic micropolar media. *Archive of Applied Mechanics*, *80*(3), 217–227.

Altenbach, H., Bîrsan, M., & Eremeyev, V. A. (2012a). On a thermodynamic theory of rods with two temperature fields. *Acta Mechanica*, *223*(8), 1583–1596.

Altenbach, H., Eremeyev, V. A., & Morozov, N. F. (2012b). Surface viscoelasticity and effective properties of thin-walled structures at the nanoscale. *The International Journal of Engineering Science*, *59*, 83–89.

Altenbach, H., Bîrsan, M., & Eremeyev, V. A. (2013). Cosserat-type rods. In H. Altenbach & V. A. Eremeyev (Eds.), *Generalized Continua from the Theory to Engineering Applications* (Vol. 541, pp. 179–248). CISM Courses and Lectures Wien: Springer.

Altenbach, J., Altenbach, H., & Eremeyev, V. A. (2010c). On generalized Cosserat-type theories of plates and shells: A short review and bibliography. *Archive of Applied Mechanics*, *80*, 73–92.

Amabili, M. (2008). *Nonlinear vibrations and stability of shells and plates*. Cambridge: Cambridge University Press.

Ambartsumyan, S. A. (1970). *Theory of anisotropic plates: Strength, stability, vibration*. Stamford: Technomic.

Andreaus, U., dell'Isola, F., & Porfiri, M. (2004). Piezoelectric passive distributed controllers for beam flexural vibrations. *Journal of Vibration and Control*, *10*(5), 625–659.

Bauchau, O. A. (2010). *Flexible Multibody Dynamics* (Vol. 176). Solid Mechanics and its Applications Dordrecht: Springer.

Bauchau, O. A., & Trainelli, L. (2003). The vectorial parameterization of rotation. *Nonlinear Dynamics*, *32*(1), 71–92.

Berdichevsky, V. L. (2009). *Variational principles of continuum mechanics. I. Fundamentals*. Heidelberg: Springer.

Berezovski, A., Engelbrecht, J., & Maugin, G. A. (2008). *Numerical simulation of waves and fronts in inhomogeneous solids*. New Jersey: World Scientific.

Bhattacharya, K. (2003). *Microstructure of martensite: Why it forms and how it gives rise to the shape-memory effect*. Oxford: Oxford University Press.
Bhattacharya, K., & James, R. D. (1999). A theory of thin films of martensitic materials with applications to microactuators. *Journal of the Mechanics and Physics of Solids*, *36*(3), 531–576.
Bigoni, D., & Drugan, W. J. (2007). Analytical derivation of Cosserat moduli via homogenization of heterogeneous elastic materials. Transactions of ASME. *Journal of Applied Mechanics*, *74*(4), 741–753.
Bîrsan, M., Altenbach, H., Sadowski, T., Eremeyev, V. A., & Pietras, D. (2012). Deformation analysis of functionally graded beams by the direct approach. *Composites B: Engineering*, *43*(3), 1315–1328.
Boulbitch, A. A. (1999). Equations of heterophase equilibrium of a biomembrane. *Archive of Applied Mechanics*, *69*(2), 83–93.
Carcaterra, A., Akay, A., & Bernardini, C. (2012). Trapping of vibration energy into a set of resonators: Theory and application to aerospace structures. *Mechanical Systems and Signal Processing*, *26*, 1–14.
Carrera, E., Brischetto, S., & Nali, P. (2011). *Plates and shells for smart structures: Classical and advanced theories for modeling and analysis*. Chichester: Wiley.
Chróścielewski, J., & Witkowski, W. (2010). On some constitutive equations for micropolar plates. *ZAMM*, *90*(1), 53–64.
Chróścielewski, J., Makowski, J., & Pietraszkiewicz, W. (2004a). *Statics and dynamics of multifolded shells: Nonlinear theory and finite element method*. Warsaw: IPPT PAN.
Chróścielewski, J., Makowski, J., & Pietraszkiewicz, W. (2004b). *Statics and dynamics of multyfolded shells: Nonlinear theory and finite elelement method (in Polish)*. Warszawa: Wydawnictwo IPPT PAN.
Chróścielewski, J., Pietraszkiewicz, W., & Witkowski, W. (2010). On shear correction factors in the non-linear theory of elastic shells. *International Journal of Solids and Structures*, *47*(25–26), 3537–3545.
Ciarlet, Ph. (1997). *Mathematical elasticity, Volume II: Theory of plates*. Amsterdam: Elsevier.
Ciarlet, Ph. (2000). *Mathematical elasticity, Volume III: Theory of shells*. Amsterdam: Elsevier.
Courant, R., & Hilbert, D. (1991). *Methods of mathematical physics* (Vol. 1). New York: Wiley.
de Gennes, P. G., Brochard-Wyart, F., & Quéré, D. (2004). *Capillarity and wetting phenomena: Drops, bubbles, pearls, waves*. New York: Springer.
dell'Isola, F., & Vidoli, S. (1998). Damping of bending waves in truss beams by electrical transmission lines with PZT actuators. *Archive of Applied Mechanics, 68*(9), 626–636.
dell'Isola, F., Porfiri, M., & Vidoli, S. (2003). Piezo-ElectroMechanical (PEM) structures: passive vibration control using distributed piezoelectric transducers. *Comptes Rendus Mécanique, 331*(1), 69–76.
dell'Isola, F., Della Corte, A., Greco, L., & Luongo, A. (2016a). Plane bias extension test for a continuum with two inextensible families of fibers: A variational treatment with lagrange multipliers and a perturbation solution. *International Journal of Solids and Structures, 81*, 1–12.
dell'Isola, F., Giorgio, I., Pawlikowski, M., & Rizzi, N. L. (2016b). Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. *Proceedings of the Royal Society of London. Series A, 472*(2185), 20150790.
Diebels, S., & Steeb, H. (2003). Stress and couple stress in foams. *Computational Materials Science*, *28*(3), 714–722.
Duan, H. L., Wang, J., & Karihaloo, B. L. (2008). Theory of elasticity at the nanoscale. In *Advances in applied mechanics* (Vol. 42, pp. 1–68). Elsevier.
Elliott, C. M., & Stinner, B. (2010). A surface phase field model for two-phase biological membranes. *SIAM Journal on Applied Mathematics*, *70*(8), 2904–2928.

Eremeyev, V. A. (2005a). Nonlinear micropolar shells: Theory and applications. In W. Pietraszkiewicz & C. Szymczak (Eds.), *Shell structures: Theory and applications* (pp. 11–18). London: Taylor & Francis.
Eremeyev, V. A. (2005b). Acceleration waves in micropolar elastic media. *Doklady Physics*, *50*(4), 204–206.
Eremeyev, V. A., & Lebedev, L. P. (2011). Existence theorems in the linear theory of micropolar shells. *ZAMM*, *91*(6), 468–476.
Eremeyev, V. A., & Pietraszkiewicz, W. (2006). Local symmetry group in the general theory of elastic shells. *Journal of Elasticity*, *85*(2), 125–152.
Eremeyev, V. A., & Pietraszkiewicz, W. (2011). Thermomechanics of shells undergoing phase transition. *Journal of the Mechanics and Physics of Solids*, *59*(7), 1395–1412.
Eremeyev, V. A., & Pietraszkiewicz, W. (2004). The non-linear theory of elastic shells with phase transitions. *The Journal of Elasticity*, *74*(1), 67–86.
Eremeyev, V. A., & Pietraszkiewicz, W. (2010). On tension of a two-phase elastic tube. In W. Pietraszkiewicz & I. Kreja (Eds.), *Shell structures: Theory and applications* (Vol. 2, pp. 63–66). Boca Raton: CRC Press.
Eremeyev, V. A., & Pietraszkiewicz, W. (2009). Phase transitions in thermoelastic and thermoviscoelastic shells. *Archives of Mechanics*, *61*(1), 41–67.
Eremeyev, V. A., & Pietraszkiewicz, W. (2012). Material symmetry group of the non-linear polar-elastic continuum. *International Journal of Solids and Structures*, *49*(14), 1993–2005.
Eremeyev, V. A., & Pietraszkiewicz, W. (2014). Editorial: Refined theories of plates and shells. *ZAMM*, *94*(1–2), 5–6.
Eremeyev, V. A., & Pietraszkiewicz, W. (2016). Material symmetry group and constitutive equations of micropolar anisotropic elastic solids. *Mathematics and Mechanics of Solids*, *21*(2), 210–221.
Eremeyev, V. A., & Zubov, L. M. (1994). On the stability of elastic bodies with couple stresses. *Mechanics of Solids*, *29*(3), 172–181.
Eremeyev, V. A., & Zubov, L. M. (2007). On constitutive inequalities in nonlinear theory of elastic shells. *ZAMM*, *87*(2), 94–101.
Eremeyev, V. A., & Zubov, L. M. (2008). *Mechanics of elastic shells (in Russian)*. Moscow: Nauka.
Eremeyev, V. A., Altenbach, H., & Morozov, N. F. (2009). The influence of surface tension on the effective stiffness of nanosize plates. *Doklady Physics*, *54*(2), 98–100.
Eremeyev, V. A., Lebedev, L. P., & Altenbach, H. (2013). *Foundations of microplar mechanics*. Heidelberg: Springer.
Eremeyev, V. A., Ivanova, E. A., & Morozov, N. F. (2015a). On free oscillations of an elastic solids with ordered arrays of nano-sized objects. *Continuum Mechanics and Thermodynamics*, *27*(4–5), 583–607.
Eremeyev, V. A., Lebedev, L. P., & Cloud, M. J. (2015b). The Rayleigh and Courant variational principles in the six-parameter shell theory. *Mathematics and Mechanics of Solids*, *20*(7), 806–822.
Eringen, A. C. (1967a). Theory of micropolar plates. *ZAMP*, *18*(1), 12–30.
Eringen, A. C. (1967b). Linear theory of micropolar viscoelasticity. *International Journal of Engineering Science*, *5*(2), 191–204.
Eringen, A. C. (1999). *Microcontinuum field theory. I: Foundations and solids*. New York: Springer.
Eringen, A. C., & Maugin, G. A. (1990). *Electrodynamics of continua*. New York: Springer.
Fichera, G. (1972). Existence theorems in elasticity. In S. Flügge (Ed.), *Handbuch der Physik* (pp. 347–389), volume VIa/2 Berlin: Springer.
Finn, R. (1986). *Equilibrium capillary surfaces*. New York: Springer.
Fu, Y. B., & Ogden, R. W. (1999). Nonlinear stability analysis of pre-stressed elastic bodies. *Continuum Mechanics and Thermodynamics*, *11*, 141–172.
Germain, P. (1973a). La méthode des puissances virtuelles en mécanique des milieux continus - première partie, théorie du second gradient. *Journal de Mécanique*, *12*, 235–274.
Germain, P. (1973b). The method of virtual power in continuum mechanics. part 2: Microstructure. *SIAM Journal on Applied Mathematics*, *25*(3), 556–575.

Giorgio, I., Grygoruk, R., dell'Isola, F., & Steigmann, D. J. (2015). Pattern formation in the three-dimensional deformations of fibered sheets. *Mechanics Research Communications*, *69*, 164–171.
Goda, I., Assidi, M., Belouettar, S., & Ganghoffer, J. F. (2012). A micropolar anisotropic constitutive model of cancellous bone from discrete homogenization. *Journal of the Mechanical Behavior of Biomedical Materials*, *16*, 87–108.
Goldenveizer, A. L. (1976). *Theory of thin elastic shells (in Russ.)*. Moscow: Nauka.
Green, A. E., & Naghdi, P. M. (1979). On thermal effects in the theory of shells. *Proceedings of the Royal Society of London Series A*, *365*(1721), 161–190.
Green, A. E., & Naghdi, P. M. (1970). Non-isothermal theory of rods, plates and shells. *International Journal of Solids and Structures*, *6*, 209–244.
Gurtin, M. E., & Murdoch, A. I. (1975). A continuum theory of elastic material surfaces. *The Archive for Rational Mechanics and Analysis*, *57*(4), 291–323.
He, Y. J., & Sun, Q. P. (2009). Effects of structural and material length scales on stress-induced martensite macro-domain patterns in tube configurations. *The International Journal of Solids and Structures*, *46*(16), 3045–3060.
He, Y. J., & Sun, Q. P. (2010). Macroscopic equilibrium domain structure and geometric compatibility in elastic phase transition of thin plates. *The International Journal of Mechanical Sciences*, *52*(2), 198–211.
Hörmander, L. (1976). *Linear partial differential equations* (4th ed., Vol. 116). A Series of Comprehensive Studies in Mathematics Berlin: Springer.
Jaiani, G. (2011). *Cusped shell-like structures*. Heidelberg: Springer.
Jaiani, G., & Podio-Guidugli, P. (2008). *IUTAM Symposium on Relations of Shell, Plate, Beam and 3D Models: Proceedings of the IUTAM Symposium on the Relations of Shell, Plate, Beam, and 3D Models Dedicated to the Centenary of Ilia Vekua's Birth, held Tbilisi*, Georgia, April 23–27, 2007, Vol. 9. Springer.
James, R. D., & Rizzoni, R. (2000). Pressurized shape memory thin films. *The Journal of Elasticity*, *59*(1–3), 399–436.
Javili, A., McBride, A., & Steinmann, P. (2012). Thermomechanics of solids with lower-dimensional energetics: On the importance of surface, interface, and curve structures at the nanoscale. A unifying review. *Applied Mechanics Reviews, 65*, 010802–1–31.
Javili, A., dell'Isola, F., & Steinmann, P. (2013). Geometrically nonlinear higher-gradient elasticity with energetic boundaries. *Journal of the Mechanics and Physics of Solids*, *61*(12), 2381–2401.
Jemielita, G. (2001). Meandry teorii płyt i powłok. In Cz. Woźniak (Ed.), *Mechanics of elastic plates and shells (in Polish)*, volume VIII of *Mechanika Techniczna*. PWN, Warszawa.
Kabrits, S. A., Mikhailovskiy, E. I., Tovstik, P. E., Chernykh, K. F., & Shamina, V. A. (2002). *General nonlinear theory of elastic shells (in Russ.)*. St. Petersburg State University, St. Petersburg.
Koç, I. M., Carcaterra, A., Xu, Z., & Akay, A. (2005). Energy sinks: Vibration absorption by an optimal set of undamped oscillators. *The Journal of the Acoustical Society of America*, *118*(5), 3031–3042.
Konopińska, V., & Pietraszkiewicz, W. (2007). Exact resultant equilibrium conditions in the non-linear theory of branching and self-intersecting shells. *The International Journal of Solids and Structures*, *44*(1), 352–369.
Korteweg, D. J. (1901). Sur la forme que prennent les équations des mouvements des fluides si l'on tient compte des forces capillaires par des variations de densité. *Archives Néderlandaises des sciences exactes et naturelles, Sér.*, *II*(6), 1–24.
Kreja, I. (2007). *Geometrically non-linear analysis of layered composite plates and shells*. Gdańsk: Gdańsk University of Technology.
Lakes, R. S. (1986). Experimental microelasticity of two porous solids. *The International Journal of Solids and Structures*, *22*(1), 55–63.
Laplace, P. S. (1805). Sur l'action capillaire. supplément à la théorie de l'action capillaire. *Traité de mécanique céleste* (Vol. 4, pp. 771–777). Supplement 1, Livre X Paris: Gauthier-Villars et fils.

Laplace, P. S. (1806). À la théorie de l'action capillaire. supplément à la théorie de l'action capillaire. *Traité de mécanique céleste* (Vol. 4, pp. 909–945). Supplement 2, Livre X Paris: Gauthier-Villars et fils.
Lebedev, L. P., Cloud, M. J., & Eremeyev, V. A. (2010). *Tensor analysis with applications in mechanics*. New Jersey: World Scientific.
Libai, A., & Simmonds, J. G. (1983). Nonlinear elastic shell theory. *Advances in Applied Mechanics*, *23*, 271–371.
Libai, A., & Simmonds, J. G. (1998). *The nonlinear theory of elastic shells* (2nd ed.). Cambridge: Cambridge University Press.
Lions, J.-L., & Magenes, E. (1968). *Problèmes aux limites non homogènes et applications*. Paris: Dunod.
Longley, W. R. & Van Name, R. G. (Eds.). (1928). *The collected works of J. Willard Gibbs, PHD., LL.D. Vol. I Thermodynamics*. Longmans, New York.
Lurie, A. I. (1990). *Nonlinear theory of elasticity*. Amsterdam: North-Holland.
Lurie, A. I. (2001). *Analytical mechanics*. Berlin: Springer.
Makowski, J., & Pietraszkiewicz, W. (2002). *Thermomechanics of shells with singular curves*. Zesz. Nauk. No 528/1487/2002, IMP PAN, Gdańsk.
Maugin, G. A. (1988). *Continuum mechanics of electromagnetic solids*. Oxford: Elsevier.
Maurini, C., dell'Isola, F., & Del Vescovo, D. (2004). Comparison of piezoelectronic networks acting as distributed vibration absorbers. *Mechanical Systems and Signal Processing*, *18*(5), 1243–1271.
Mindlin, R. D. (1951). Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. Transactions of ASME. *Journal of Applied Mechanics*, *18*, 31–38.
Miyazaki, S., Fu, Y. Q., & Huang, W. M. (Eds.). (2009). *Thin film shape memory alloys: Fundamentals and device applications*. Cambridge: Cambridge University Press.
Murdoch, A. I. (1976a). A thermodynamical theory of elastic material interfaces. *The Quarterly Journal of Mechanics and Applied Mathematics*, *29*(3), 245–274.
Murdoch, A. I. (1976b). On the entropy inequality for material interfaces. *ZAMP*, *27*(5), 599–605.
Naghdi, P. M. (1972). The theory of plates and shells. In S. Flügge (Ed.), *Handbuch der Physik* (pp. 425–640), volume VIa/2 Heidelberg: Springer.
Nirenberg, L. (2001). *Topics in nonlinear functional analysis*. New York: American Mathematical Society.
Novozhilov, V. V., Chernykh, K. F., & Mikhailovskiy, E. I. (1991). *Linear theory of thin shells (in Russ.)*. Politekhnika, Leningrad.
Ogden, R. W. (1997). *Non-linear elastic deformations*. Mineola: Dover.
Pietraszkiewicz, W. (2011). Refined resultant thermomechanics of shells. *International Journal of Engineering Science*, *49*(10), 1112–1124.
Pietraszkiewicz, W. (1979a). *Finite rotations and langrangian description in the non-linear theory of shells*. Warszawa-Poznań: Polish Sci. Publ.
Pietraszkiewicz, W. (1979b). Consistent second approximation to the elastic strain energy of a shell. *ZAMM*, *59*, 206–208.
Pietraszkiewicz, W. (1989). Geometrically nonlinear theories of thin elastic shells. *Uspekhi Mechaniki (Advances in Mechanics)*, *12*(1), 51–130.
Pietraszkiewicz, W. (2015). The resultant linear six-field theory of elastic shells: What it brings to the classical linear shell models? *ZAMM*, 10.1002/zamm.201500184.
Pietraszkiewicz, W., & Eremeyev, V. A. (2009a). On natural strain measures of the non-linear micropolar continuum. *International Journal of Solids and Structures*, *46*(3–4), 774–787.
Pietraszkiewicz, W., & Eremeyev, V. A. (2009b). On vectorially parameterized natural strain measures of the non-linear Cosserat continuum. *International Journal of Solids and Structures*, *46* (11–12), 2477–2480.
Pietraszkiewicz, W., & Górski, J. (Eds.). (2014). *Shell structures: Theory and applications* (Vol. 3). Boca Raton: CRC Press.

Pietraszkiewicz, W., & Konopińska, V. (2015). Junctions in shell structures: A review. *Thin-Walled Structures*, *95*, 310–334.
Pietraszkiewicz, W., & Konopińska, V. (2011). On unique kinematics for the branching shells. *The International Journal of Solids and Structures*, *48*(14), 2238–2244.
Pietraszkiewicz, W., & Kreja, I. (Eds.). (2010). *Shell structures: Theory and applications* (Vol. 2). Boca Raton: CRC Press.
Pietraszkiewicz, W., & Szymczak, C. (Eds.). (2005). *Shell structures: Theory and applications*. London: Taylor & Francis.
Pietraszkiewicz, W., Eremeyev, V. A., & Konopińska, V. (2007). Extended non-linear relations of elastic shells undergoing phase transitions. *ZAMM*, *87*(2), 150–159.
Reda, H., Rahali, Y., Ganghoffer, J. F., & Lakiss, H. (2016). Wave propagation in 3d viscoelastic auxetic and textile materials by homogenized continuum micropolar models. *Composite Structures*, *141*, 328–345.
Reddy, J. N. (2003). *Mechanics of laminated composite plates and shells: Theory and analysis* (2nd ed.). Boca Raton: CRC Press.
Reissner, E. (1944). On the theory of bending of elastic plates. *Journal of Mathematical Physics*, *23*, 184–194.
Reissner, E. (1985). Reflection on the theory of elastic plates. *Applied Mechanics Reviews*, *38*(11), 1453–1464.
Reissner, E. (1977). A note on generating generalized two-dimensional plate and shell theories. *ZAMP*, *28*(4), 633–642.
Rowlinson, J. S., & Widom, B. (2003). *Molecular theory of capillarity*. New York: Dover.
Sargsyan, S. H. (2011). The general dynamic theory of micropolar elastic thin shells. *Doklady Physics*, *56*(1), 39–42.
Sedov, L. I. (1968). Models of continuous media with internal degrees of freedom. *Journal of Applied Mathematics and Mechanics*, *32*(5), 803–819.
Shkutin, L. I. (2007). Analysis of axisymmetric phase strains in plates and shells. *Journal of Applied Mechanics and Technical Physics*, *48*(2), 285–291.
Simmonds, J. G. (2005). A simple nonlinear thermodynamic theory of arbitrary elastic beams. *The Journal of Elasticity*, *81*(1), 51–62.
Simmonds, J. G. (2011). A classical, nonlinear thermodynamic theory of elastic shells based on a single constitutive assumption. *The Journal of Elasticity*, *105*(1–2), 305–312.
Simmonds, J. G. (1984). The thermodynamical theory of shells: Descent from 3-dimensions without thickness expansions. In E. L. Axelrad & F. A. Emmerling (Eds.), *Flexible shells* (pp. 1–11). Theory and Applications Berlin: Springer.
Steigmann, D. J., & Ogden, R. W. (1999). Elastic surface-substrate interactions. Proceedings of the Royal Society of London Series A: Mathematical, Physical and Engineering *Science*, *455*(1982), 437–474.
Steinberg, L., & Kvasov, R. (2013). Enhanced mathematical model for Cosserat plate bending. *Thin-Walled Structures*, *63*, 51–62.
Steinmann, P., & Häsner, O. (2005). On material interfaces in thermomechanical solids. *Archive of Applied Mechanics*, *75*(1), 31–41.
Tovstik, P. E., & Smirnov, A. L. (2001). *Asymptotic methods in the buckling theory of elastic shells*. Singapore: World Scientific.
Truesdell, C. (1977). *A first course in rational continuum mechanics*. New York: Academic Press.
Truesdell, C. (1984). *Rational thermodynamics* (2nd ed.). New York: Springer.
Truesdell, C., & Noll, W. (1965). The nonlinear field theories of mechanics. In S. Flügge (Ed.), *Handbuch der Physik* (pp. 1–602). III(3) Berlin: Springer.
van der Waals, J. D. (1893). The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density (Engl. transl. by J. S. Rowlinson). *Journal of Statistical Physics*, *20*, 200–244.
Vidoli, S., & dell'Isola, F. (2001). Vibration control in plates by uniformly distributed PZT actuators interconnected via electric networks. *European Journal of Mechanics: A/Solids*, *20*(3), 435–456.

Wang, C. M., Reddy, J. N., & Lee, K. H. (2000). *Shear deformable beams and shells*. Amsterdam: Elsevier.

Wang, J., Huang, Z., Duan, H., Yu, S., Feng, X., Wang, G., et al. (2011). Surface stress effect in mechanics of nanostructured materials. *Acta Mechanica Solida Sinica*, *24*, 52–82.

Wang, Z. Q., Zhao, Y.-P., & Huang, Z.-P. (2010). The effects of surface tension on the elastic properties of nano structures. *International Journal of Engineering Science*, *48*(2), 140–150.

Wiśniewski, K. (2010). *Finite rotation shells: Basic equations and finite elements for Reissner kinematics*. Berlin: Springer.

Young, T. (1805). An essay on the cohesion of fluids. *Philosophical Transactions of the Royal Society of London*, *95*, 65–87.

Zhilin, P. A. (1976). Mechanics of deformable directed surfaces. *International Journals of Solids and Structures*, *12*(9–10), 635–648.

Zubov, L. M. (1997). *Nonlinear theory of dislocations and disclinations in elastic bodies*. Berlin: Springer.

Zubov, L. M. (2009). Micropolar shell equilibrium equations. *Doklady Physics*, *54*(6), 290–293.

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