Chapter 2
Linear Systems with State Measurement

1 Introduction

In this chapter we consider stabilizable LTI systems with state measurement. However, we also consider additive disturbances in the differential equations (which play the role of modeling errors) and additive disturbances in the measurement (which play the role of measurement errors).

First, we consider the basic case: undelayed continuous state measurement and continuously adjusted control input (Section 2). The three ways of implementing predictor feedback are analyzed in detail and it is shown that they result in different gains for the disturbance inputs. The gains as well as the asymptotic gains are estimated by explicit formulas. Two different things can be observed by the provided explicit formulas:

- the fact that the disturbance gains are significantly larger than the asymptotic gains, and
- the fact that the (asymptotic) gains cannot be assigned.

As already remarked in Chapter 1, the latter fact is not a technical consequence of the proof methodology: it is a fundamental feature of all systems with input delays. This is proved in Section 3 (Theorem 3.2) for general nonlinear systems.

Next we consider the construction of approximate predictors for LTI systems and their use in a predictor feedback control scheme (Section 4). We provide results which guarantee robustness with respect to perturbations of the sampling schedule and we consider different sampling and holding periods (Theorem 4.2).

Finally, the last section of the chapter is devoted to the study of the robustness properties of predictor feedback with respect to delay perturbations. Delay perturbations are vanishing perturbations (in the sense that they do not change the

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position of the equilibrium point) and there are only a few works in the literature which consider delay perturbations for LTI systems under predictor feedback. Specific results are provided for the cases of measurable time-varying delay perturbations and constant delay perturbations.

2 Basic Case: Undelayed Continuous State Measurement and Continuous Control

LTI systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau) \\ x(t) &\in \Re^n,\ u(t) \in \Re^m \end{aligned} \tag{2.1}$$

where $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$ are constant real matrices, are the topic of the present chapter. We assume that we have a linear stabilizing feedback law for the delay-free case, i.e., we can design a matrix $k \in \Re^{m \times n}$ such that $(A + Bk)$ is a Hurwitz matrix. Since the predictor mapping, i.e., the mapping that provides the state vector τ time units ahead is given by the variations of constants formula

$$x(t+\tau) = \exp(A\tau)x(t) + \int_0^\tau \exp(Aw)Bu(t-w)dw \tag{2.2}$$

the predictor feedback assumes the following simple form:

$$u(t) = k\exp(A\tau)x(t) + k\int_0^\tau \exp(Aw)Bu(t-w)dw, \quad \text{for } t \geq 0 \tag{2.3}$$

As already remarked in Chapter 1, we can view the closed-loop system (2.1) with (2.3) in three possible ways:

1) As the interconnection of a system of ODEs (namely, system (2.1)) with a system of IDEs (namely, system (2.3)). In this way, the state space of the closed-loop system is the space $\Re^n \times L^\infty\big([-\tau, 0); \Re^m\big)$.
2) As a system of RFDEs with distributed delays, namely the system

$$\dot{u}(t) = k(A + \mu I_n)\left(\exp(A\tau)x(t) + \int_0^\tau \exp(Aw)Bu(t-w)dw\right) + (kB - \mu I_m)u(t) \tag{2.4}$$

where $\mu > 0$ is an arbitrary constant, with system (2.1). In this way, the state space of the closed-loop system is the subspace

$$\left\{ (x,u) \in \Re^n \times C^0([-\tau,0];\Re^m) \colon u(0) = k\exp(A\tau)x + k\int_0^\tau \exp(Aw)Bu(-w)dw \right\}$$

3) As a hybrid system with delays, namely, system (2.1) with the following system:

$$u(t) = k\exp((A+Bk)(t-\tau_i))\left(\exp(A\tau)x(\tau_i) + \int_0^\tau \exp(Aw)Bu(\tau_i - w)dw\right), \quad \text{for all } t \in [\tau_i, \tau_{i+1}) \tag{2.5}$$

where $\{\tau_i\}_{i=0}^{\infty}$ is an arbitrary partition of $\Re_+$. In this way, the state space of the closed-loop system is the space $\Re^n \times L^\infty([-\tau,0);\Re^m)$.

The three ways of viewing the closed-loop system (2.1) with (2.3) can lead to three different implementations of the feedback law:

1) The direct implementation of (2.3): this implementation corresponds to the first viewpoint.
2) The dynamic implementation, i.e., the implementation of (2.4): this implementation corresponds to the second viewpoint. In this case, we can even consider as a state space the space $\Re^n \times C^0([-\tau,0];\Re^m)$.
3) The implementation of the following hybrid system:

$$\dot{z}(t) = (A+Bk)z(t), \text{ for all } t \in [\tau_i, \tau_{i+1}) \tag{2.6}$$

$$z(\tau_i) = \exp(A\tau)x(\tau_i) + \int_0^\tau \exp(Aw)Bu(\tau_i - w)dw, \text{ for all } i \in Z_+ \tag{2.7}$$

$$u(t) = kz(t), \text{ for all } t \geq 0 \tag{2.8}$$

This implementation corresponds to the third viewpoint.

The three different ways of implementing the closed-loop system (2.1) with (2.3) have important differences. The hybrid implementation does not require continuous measurement of $x(t)$: the value of $x(t)$ is only required at the discrete sampling times $\{\tau_i\}_{i=0}^{\infty}$.

More differences arise when disturbances are present. In this case, we have

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t-\tau) + Gw(t) \\ &x(t) \in \Re^n,\ u(t) \in \Re^m,\ w(t) \in \Re^q \end{aligned} \tag{2.9}$$

where $G \in \mathfrak{R}^{n \times q}$ is a constant real matrix and $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$ is the plant disturbance. The feedback law is modified in the presence of measurement errors $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^n)$:

1) In the case of direct implementation we have:

$$u(t) = k\exp(A\tau)(x(t)+\xi(t)) + k\int_0^{\tau}\exp(As)Bu(t-s)ds, \text{ for } t \geq 0 \tag{2.10}$$

2) In the case of the dynamic implementation we have:

$$\begin{aligned}\dot{u}(t) &= k(A+\mu I_n)\left(\exp(A\tau)(x(t)+\xi(t)) + \int_0^{\tau}\exp(As)Bu(t-s)ds\right)\\ &\quad + (kB-\mu I_m)u(t)\end{aligned} \tag{2.11}$$

where $\mu > 0$ is an arbitrary constant.

3) In the case of hybrid implementation we have:

$$\dot{z}(t) = (A+Bk)z(t), \text{ for all } t \in \left[\tau_i, \tau_{i+1}\right) \tag{2.12}$$

$$z(\tau_i) = \exp(A\tau)(x(\tau_i)+\xi(\tau_i)) + \int_0^{\tau}\exp(As)Bu(\tau_i - s)ds, \text{ for all } i \in Z_+ \tag{2.13}$$

$$u(t) = kz(t), \text{ for all } t \geq 0 \tag{2.14}$$

where $\{\tau_i\}_{i=0}^{\infty}$ is an arbitrary partition of $\mathfrak{R}_+$.

Depending on the implementation, we obtain different results. The following result deals with the direct implementation.

Theorem 2.1 (ISS w.r.t. disturbances under direct implementation): *For every $\left(x_0, \breve{u}_0\right) \in \mathfrak{R}^n \times L^{\infty}\left(\left[-\tau, 0\right); \mathfrak{R}^m\right)$, $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$, $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^n)$ the solution of (2.9), (2.10) with initial condition $x(0) = x_0$, $u(s) = \breve{u}_0(s)$ for $s \in \left[-\tau, 0\right)$ corresponding to inputs $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$, $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^n)$ exists for all $t \geq 0$ and satisfies the following estimates for all $t \geq 0$:*

$$\begin{aligned}|x(t)| &\leq M\exp(-2\sigma(t-\tau))\left(\max_{0\leq s\leq\tau}(\phi(s))|x_0| + |B|\|\breve{u}_0\|\int_0^{\tau}\phi(s)ds\right)\\ &+M\exp(-2\sigma(t-\tau))|G|\sup_{0\leq s\leq \min(t,\tau)}(|w(s)|)\int_0^{\tau}\phi(s)ds + \frac{M}{2\sigma}|Bk|\phi(\tau)\sup_{0\leq s\leq t}(|\xi(s)|)\\ &+\frac{M}{2\sigma}|G|\left(1+|Bk|\int_0^{\tau}\phi(s)ds\right)\sup_{0\leq s\leq t}(|w(s)|)\end{aligned} \tag{2.15}$$

$$
\begin{aligned}
\left\|\breve{u}_t\right\| \leq \exp(-2\sigma(t-\tau))\max\left(|k|M\left(\phi(\tau)|x_0| + |B|\left\|\breve{u}_0\right\|\int_0^{\tau}\phi(s)ds\right), \left\|\breve{u}_0\right\|\right) \\
+|k|\phi(\tau)\left(|Bk|\frac{M}{2\sigma}+1\right)\sup_{0\leq s\leq t}(|\xi(s)|) + |k||G|\frac{M}{2\sigma}\left(1+|Bk|\int_0^{\tau}\phi(s)ds\right)\sup_{0\leq s\leq t}(|w(s)|) \\
+|G||k|M\exp(-2\sigma(t-\tau))\sup_{0\leq s\leq \min(\tau,t)}(|w(s)|)\int_0^{\tau}\phi(s)ds
\end{aligned}
\tag{2.16}
$$

where $\sigma > 0$ *and* $M \geq 1$ *are constants satisfying* $|\exp((A+Bk)t)| \leq M\exp(-2\sigma t)$ *for all* $t \geq 0$ *and* $\phi \in C^0(\Re_+;\Re_+)$ *is a function that satisfies* $|\exp(At)| \leq \phi(t)$.

Proof: Local existence of solution is guaranteed by the results of Chapter 7. Global existence follows from the results of Chapter 7 and the estimates that are obtained next.

The variations of constants formula gives for all $t \geq 0$:

$$
x(t+\tau) = \exp(A\tau)x(t) + \int_0^{\tau}\exp(As)Bu(t-s)ds + \int_t^{t+\tau}\exp(A(t+\tau-s))Gw(s)ds
\tag{2.17}
$$

Using (2.10) and (2.17), we obtain for almost all $t \geq 0$:

$$
kx(t+\tau) = u(t) - k\exp(A\tau)\xi(t) + k\int_t^{t+\tau}\exp(A(t+\tau-s))Gw(s)ds
\tag{2.18}
$$

Consequently, the following differential equation holds for almost all $t \geq 0$:

$$
\begin{aligned}
\dot{x}(t+\tau) = (A+Bk)x(t+\tau) + Bk\exp(A\tau)\xi(t) \\
+ Gw(t+\tau) - Bk\int_t^{t+\tau}\exp(A(t+\tau-s))Gw(s)ds
\end{aligned}
\tag{2.19}
$$

The variations of constants formula in conjunction with (2.19) guarantees that the following formula holds for all $t \geq \tau$:

$$
x(t) = \exp((A+Bk)(t-\tau))x(\tau) + \int_{\tau}^{t}\exp((A+Bk)(t-s))v(s)ds
\tag{2.20}
$$

where

$$v(t) = Bk\exp(A\tau)\xi(t-\tau) + Gw(t) - Bk\int_{t-\tau}^{t}\exp(A(t-s))Gw(s)ds \quad \text{for almost all } t \geq \tau \tag{2.21}$$

Using the facts that $\sigma > 0$ and $M \geq 1$ are constants satisfying $|\exp((A+Bk)t)| \leq M\exp(-2\sigma t)$ for all $t \geq 0$ and $\phi \in C^0(\mathfrak{R}_+;\mathfrak{R}_+)$ is a function that satisfies $|\exp(At)| \leq \phi(t)$, we obtain from (2.20), (2.21) and the triangle inequality:

$$|x(t)| \leq M\exp(-2\sigma(t-\tau))|x(\tau)| + \frac{M}{2\sigma}\sup_{\tau \leq s \leq t}(|v(s)|) \quad \text{for all } t \geq \tau \tag{2.22}$$

$$|v(t)| \leq |Bk|\phi(\tau)|\xi(t-\tau)| + |G||w(t)| + |Bk||G|\int_0^{\tau}\phi(s)ds \sup_{t-\tau \leq s \leq t}(|w(s)|) \quad \text{for almost all } t \geq \tau \tag{2.23}$$

Using the variations of constants formula $x(t) = \exp(At)x(0) + \int_0^t \exp(A(t-s)) Bu(s-\tau)ds + \int_0^t \exp(A(t-s))Gw(s)ds$ for $t \in [0,\tau]$ we obtain:

$$|x(t)| \leq \phi(t)|x_0| + |B|\|\breve{u}_0\|\int_0^t \phi(s)ds + |G|\sup_{0 \leq s \leq t}(|w(s)|)\int_0^t \phi(s)ds \quad \text{for } t \in [0,\tau] \tag{2.24}$$

Combining (2.22), (2.23), and (2.24) we obtain estimate (2.15) for all $t \geq \tau$. Moreover, we notice that by virtue of (2.24), estimate (2.15) holds for $t \in [0,\tau]$ as well. Using (2.18) and (2.22) we obtain for almost all $t \geq 0$:

$$\begin{aligned} |u(t)| &\leq |k|M\exp(-2\sigma t)|x(\tau)| + |k|\frac{M}{2\sigma}\sup_{\tau \leq s \leq t+\tau}(|v(s)|) \\ &+ |k|\phi(\tau)\sup_{0 \leq s \leq t}(|\xi(s)|) + |k||G|\sup_{t \leq s \leq t+\tau}(|w(s)|)\int_0^{\tau}\phi(s)ds \end{aligned} \tag{2.25}$$

Combining (2.25), (2.24), and (2.23) we obtain for almost all $t \geq 0$:

$$
\begin{aligned}
|u(t)| \leq\ & |k| M \exp(-2\sigma t) \left(\phi(\tau)|x_0| + |B| \|\breve{u}_0\| \int_0^\tau \phi(s)ds \right) \\
& + |k|\phi(\tau)\left(|Bk|\frac{M}{2\sigma} + 1 \right) \sup_{0 \leq s \leq t} (|\xi(s)|) + |k||G| \sup_{t \leq s \leq t+\tau} (|w(s)|) \int_0^\tau \phi(s)ds \\
& + |k||Bk||G|\frac{M}{2\sigma}\int_0^\tau \phi(s)ds \sup_{0 \leq s \leq t+\tau} (|w(s)|) + |k||G|\frac{M}{2\sigma} \sup_{\tau \leq s \leq t+\tau} (|w(s)|) \\
& + |G||k| M \exp(-2\sigma t) \sup_{0 \leq s \leq \tau} (|w(s)|) \int_0^\tau \phi(s)ds
\end{aligned}
\tag{2.26}
$$

Using a standard causality argument, we conclude from (2.26) that estimate (2.16) holds for all $t \geq 0$. The proof is complete. ◁

Remark 2.2: Estimate (2.15) shows that the closed-loop system (2.9), (2.10) with output $Y(t) = x(t)$ satisfies the IOS property from the inputs $w \in L^\infty_{loc}(\Re_+;\Re^q)$, $\xi \in L^\infty_{loc}(\Re_+;\Re^n)$. Estimate (2.15) shows that the gain of the input $w \in L^\infty_{loc}(\Re_+;\Re^q)$ is equal to $\frac{M}{2\sigma}|G| \left(1 + (2\sigma + |Bk|) \int_0^\tau \phi(s)ds \right)$ and the asymptotic gain of the input $w \in L^\infty_{loc}(\Re_+;\Re^q)$ is equal to $\frac{M}{2\sigma}|G| \left(1 + |Bk| \int_0^\tau \phi(s)ds \right)$.

The following result deals with the dynamic implementation (2.11).

Theorem 2.3 (ISS w.r.t. disturbances under dynamic implementation): *Let $\sigma > 0$ and $M \geq 1$ be constants satisfying $|\exp((A+Bk)t)| \leq M\exp(-2\sigma t)$ for all $t \geq 0$ and let $\phi \in C^0(\Re_+;\Re_+)$ be a function that satisfies $|\exp(At)| \leq \phi(t)$. For every $\mu > 2\sigma$, $(x_0, u_0) \in \Re^n \times C^0([-\tau, 0];\Re^m)$, $w \in L^\infty_{loc}(\Re_+;\Re^q)$, $\xi \in L^\infty_{loc}(\Re_+;\Re^n)$ the solution of (2.9), (2.11) with initial condition $x(0) = x_0$, $u(s) = u_0(s)$ for $s \in [-\tau, 0]$ corresponding to inputs $w \in L^\infty_{loc}(\Re_+;\Re^q)$, $\xi \in L^\infty_{loc}(\Re_+;\Re^n)$ exists for all $t \geq 0$ and satisfies the following estimates for all $t \geq 0$:*

$$
\begin{aligned}
|x(t)| \le\; & M\left(1+\frac{|B||k|}{\mu-2\sigma}\right)\exp(-2\sigma(t-\tau))\max_{0\le s\le\tau}(\phi(s))|x_0| \\
& +\frac{M|B|}{\mu-2\sigma}\left(1+\frac{|B||k|}{\mu-2\sigma}\right)\exp(-2\sigma(t-\tau))\left((\mu-2\sigma)\int_0^\tau \phi(s)ds+1\right)\|u_0\| \\
& +\frac{M}{2\sigma}\left(|G|+\frac{|B|}{\mu}|kG|+\frac{|B|}{\mu}|k(A+\mu I_n)||G|\int_0^\tau \phi(s)ds\right)\sup_{0\le s\le t}(|w(s)|) \\
& +M\exp(-2\sigma(t-\tau))|G|\left(1+\frac{|B||k|}{\mu-2\sigma}\right)\int_0^\tau \phi(s)ds\sup_{0\le s\le \min(\tau,t)}(|w(s)|) \\
& +\frac{M|B|}{2\mu\sigma}|k(A+\mu I_n)|\phi(\tau)\sup_{0\le s\le t}(|\xi(s)|)
\end{aligned}
\tag{2.27}
$$

$$
\begin{aligned}
\|u_t\| \le\; & |k|\left(1+M+M\frac{|B||k|}{\mu-2\sigma}\right)\exp(-2\sigma(t-\tau))\max_{0\le s\le\tau}(\phi(s))|x_0| \\
& +\left(1+M+M\frac{|B||k|}{\mu-2\sigma}\right)\left(1+|B||k|\left(\int_0^\tau \phi(s)ds+\frac{1}{\mu-2\sigma}\right)\right)\exp(-2\sigma(t-\tau))\|u_0\| \\
& +\left(\frac{1}{\mu}\left(|kG|+|k(A+\mu I_n)||G|\int_0^\tau \phi(s)ds\right)\left(1+|k||B|\frac{M}{2\sigma}\right)+|k||G|\frac{M}{2\sigma}\right)\sup_{0\le s\le t}(|w(s)|) \\
& +|k|\exp(-2\sigma(t-\tau))|G|\left(M\left(1+\frac{|B||k|}{\mu-2\sigma}\right)+1\right)\int_0^\tau \phi(s)ds\sup_{0\le s\le\min(\tau,t)}(|w(s)|) \\
& +\left(|k|\frac{M|B|}{2\sigma}+1\right)\frac{|k(A+\mu I_n)|}{\mu}\phi(\tau)\sup_{0\le s\le t}(|\xi(s)|)
\end{aligned}
\tag{2.28}
$$

Proof: Global existence of solutions follows from standard theory of RFDEs and the fact that the system (2.9), (2.11) is a linear system. Using the variations of constants formula (2.17) and (2.9), (2.11), we conclude that the following differential equation holds for almost all $t \ge 0$:

$$
\frac{d}{dt}(u(t)-kx(t+\tau)) = -\mu(u(t)-kx(t+\tau))+v(t) \tag{2.29}
$$

where

$$
\begin{aligned}
v(t) =\; & -k(A+\mu I_n)\int_t^{t+\tau}\exp(A(t+\tau-s))Gw(s)ds \\
& +k(A+\mu I_n)\exp(A\tau)\xi(t)-kGw(t+\tau)
\end{aligned}
\tag{2.30}
$$

Consequently, we obtain from (2.29) for all $t \geq 0$:

$$u(t) - kx(t+\tau) = \exp(-\mu t)(u(0) - kx(\tau)) + \int_0^t \exp(-\mu(t-s))v(s)ds \tag{2.31}$$

It follows from (2.9) and (2.31) that the following differential equation holds for almost all $t \geq 0$:

$$\begin{aligned} \dot{x}(t+\tau) &= (A+Bk)x(t+\tau) + B\exp(-\mu t)(u(0) - kx(\tau)) \\ &+ Gw(t+\tau) + B\int_0^t \exp(-\mu(t-s))v(s)ds \end{aligned} \tag{2.32}$$

Using the variations of constants formula for (2.32), the triangle inequality and the facts that $\sigma > 0$ and $M \geq 1$ are constants satisfying $|\exp((A+Bk)t)| \leq M\exp(-2\sigma t)$ for all $t \geq 0$ and $\phi \in C^0(\Re_+;\Re_+)$ is a function that satisfies $|\exp(At)| \leq \phi(t)$, we obtain for all $t \geq 0$:

$$\begin{aligned} |x(t+\tau)| &\leq M\exp(-2\sigma t)\left(|x(\tau)| + \frac{|B|}{\mu - 2\sigma}|u(0) - kx(\tau)|\right) \\ &+ \frac{M}{2\sigma}|G| \sup_{\tau \leq s \leq t+\tau}(|w(s)|) + \frac{M|B|}{2\mu\sigma} \sup_{0 \leq s \leq t}(|v(s)|) \end{aligned} \tag{2.33}$$

Using (2.24), (2.33), (2.30), the triangle inequality and the fact that $\phi \in C^0(\Re_+;\Re_+)$ is a function that satisfies $|\exp(At)| \leq \phi(t)$, we obtain estimate (2.27). Using (2.31), (2.24), (2.30) we obtain for all $t \geq 0$:

$$\begin{aligned} |u(t)| &\leq \exp(-2\sigma t)\left(1 + M + M\frac{|B||k|}{\mu - 2\sigma}\right)|k| \max_{0 \leq s \leq \tau}(\phi(s))|x_0| \\ &+ \exp(-2\sigma t)\left(1 + M + M\frac{|B||k|}{\mu - 2\sigma}\right)\left(1 + |B||k|\left(\int_0^\tau \phi(s)ds + \frac{1}{\mu - 2\sigma}\right)\right)\|u_0\| \\ &+ \left(\frac{1}{\mu}\left(|kG| + |k(A+\mu I_n)||G|\int_0^\tau \phi(s)ds\right)\left(1 + |k||B|\frac{M}{2\sigma}\right) + |k||G|\frac{M}{2\sigma}\right) \sup_{0 \leq s \leq t+\tau}(|w(s)|) \\ &+ |k|\exp(-2\sigma t)|G|\left(M\left(1 + \frac{|B||k|}{\mu - 2\sigma}\right) + 1\right)\int_0^\tau \phi(s)ds \sup_{0 \leq s \leq \tau}(|w(s)|) \\ &+ \left(|k|\frac{M|B|}{2\sigma} + 1\right)\frac{|k(A+\mu I_n)|}{\mu}\phi(\tau) \sup_{0 \leq s \leq t}(|\xi(s)|) \end{aligned} \tag{2.34}$$

Using estimate (2.34) and a standard causality argument, we obtain estimate (2.28). The proof is complete. ◁

Remark 2.4: Estimate (2.27) shows that the closed-loop system (2.9), (2.11) with output $Y(t) = x(t)$ satisfies the IOS property from the inputs $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$, $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^n)$. Estimate (2.15) shows that the gain of the input $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$ tends to $\frac{M}{2\sigma}|G|\left(1 + (2\sigma + |B||k|)\int_0^{\tau}\phi(s)ds\right)$ as $\mu \to +\infty$ and the asymptotic gain of the input $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$ tends to $\frac{M}{2\sigma}|G|\left(1 + |B||k|\int_0^{\tau}\phi(s)ds\right)$ as $\mu \to +\infty$. These gains are very similar to the gains obtained by the direct implementation of the predictor feedback.

Next, we consider the hybrid implementation (2.12), (2.13), and (2.14).

Theorem 2.5 (ISS w.r.t. disturbances under hybrid implementation): *Let $\sigma > 0$ and $M \geq 1$ be constants satisfying $|\exp((A+Bk)t)| \leq M\exp(-2\sigma t)$ for all $t \geq 0$ and let $\phi \in C^0(\mathfrak{R}_+;\mathfrak{R}_+)$ be a function that satisfies $|\exp(At)| \leq \phi(t)$. For every partition $\{\tau_i\}_{i=0}^{\infty}$ of $\mathfrak{R}_+$ with finite upper diameter (i.e., $\sup_{i\geq 0}(\tau_{i+1}-\tau_i) < +\infty$), for every $(x_0,\breve{u}_0) \in \mathfrak{R}^n \times L^{\infty}([-\tau,0);\mathfrak{R}^m)$, $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$, $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^n)$ the solution of (2.9), (2.12), (2.13), (2.14) with initial condition $x(0) = x_0$, $u(s) = \breve{u}_0(s)$ for $s \in [-\tau,0)$ corresponding to inputs $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$, $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^n)$ exists for all $t \geq 0$ and satisfies the following estimates for all $t \geq 0$:*

$$
\begin{aligned}
|x(t)| \leq{} & M\exp(-2\sigma(t-\tau))\left(\max_{0\leq s\leq\tau}(\phi(s))|x_0| + |B|\int_0^{\tau}\phi(s)ds\|\breve{u}_0\|\right) \\
&+\frac{M}{2\sigma}|G|\left(1 + |Bk|\int_0^{T_s+\tau}\phi(s)ds\right)\sup_{0\leq s\leq t}(|w(s)|) \\
&+M\exp(-2\sigma(t-\tau))|G|\int_0^{\tau}\phi(s)ds \sup_{0\leq s\leq \min(\tau,t)}(|w(s)|) \\
&+\frac{M}{2\sigma}|Bk|\max_{\tau\leq s\leq T_s+\tau}(\phi(s))\sup_{0\leq s\leq t}(|\xi(s)|)
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
\|\breve{u}_t\| \le {} & \exp(-2\sigma(t-\tau))\max\left(|k|M\phi(\tau)|x_0| + |k|M|B|\int_0^\tau \phi(s)ds\|\breve{u}_0\|,\ \|\breve{u}_0\|\right)\\
& +|k|\max_{\tau\le s\le T_s+\tau}(\phi(s))\left(\frac{M}{2\sigma}|Bk|+1\right)\sup_{0\le s\le t}(|\xi(s)|)\\
& +|k||G|\left(\left(1+\frac{M}{2\sigma}|Bk|\right)\int_0^{T_s+\tau}\phi(s)ds+\frac{M}{2\sigma}\right)\sup_{0\le s\le t}(|w(s)|)\\
& +|k|M\exp(-2\sigma(t-\tau))|G|\int_0^\tau \phi(s)ds \sup_{0\le s\le \min(\tau,t)}(|w(s)|)
\end{aligned}
\tag{2.36}
$$

where $T_s := \sup_{i\ge 0}(\tau_{i+1}-\tau_i)$.

Proof: It is straightforward to show that if $x(\tau_i)$ and $\breve{u}_{\tau_i}$ are defined for some $i \in Z_+$ then $x(t)$ and $\breve{u}_t$ can be uniquely determined by equations (2.9), (2.12), (2.13), (2.14) for all $t \in (\tau_i, \tau_{i+1}]$. Therefore, the solution of (2.9), (2.12), (2.13), (2.14) exists for all $t \ge 0$.

Using the variations of constants formula (2.17) we get from (2.12), (2.13), and (2.14) for all $i \in Z_+$:

$$z(\tau_i) - x(\tau_i+\tau) = \exp(A\tau)\xi(\tau_i) - \int_{\tau_i}^{\tau_i+\tau} \exp(A(\tau_i+\tau-s))Gw(s)ds \tag{2.37}$$

$$\frac{d}{dt}(z(t)-x(t+\tau)) = A(z(t)-x(t+\tau)) - Gw(t+\tau),\ \text{for } t\in[\tau_i,\tau_{i+1})\ \text{a.e.} \tag{2.38}$$

Using the variations of constants formula for (2.38) and (2.37), we obtain for all $t \in [\tau_i, \tau_{i+1})$ and $i \in Z_+$:

$$z(t) = x(t+\tau) + \exp(A(t+\tau-\tau_i))\xi(\tau_i) - \int_{\tau_i}^{t+\tau} \exp(A(t+\tau-s))Gw(s)ds \tag{2.39}$$

$$\dot{x}(t+\tau) = (A+Bk)x(t+\tau) + v(t) \tag{2.40}$$

where

$$v(t) = Gw(t+\tau) + Bk\exp(A(t+\tau-\tau_i))\xi(\tau_i) - Bk\int_{\tau_i}^{t+\tau} \exp(A(t+\tau-s))Gw(s)ds,\quad \text{for almost all } t\in[\tau_i,\tau_{i+1}) \tag{2.41}$$

Using the variations of constants formula for (2.40), the fact that $T_s := \sup_{i \geq 0} (\tau_{i+1} - \tau_i)$, the triangle inequality and the facts that $\sigma > 0$ and $M \geq 1$ are constants satisfying $|\exp((A+Bk)t)| \leq M\exp(-2\sigma t)$ for all $t \geq 0$ and $\phi \in C^0(\Re_+;\Re_+)$ is a function that satisfies $|\exp(At)| \leq \phi(t)$, we obtain for all $t \geq 0$:

$$|x(t+\tau)| \leq M\exp(-2\sigma t)|x(\tau)| + \frac{M}{2\sigma}|G|\left(1+|Bk|\int_0^{T_s+\tau}\phi(s)ds\right)\sup_{0\leq s\leq t+\tau}(|w(s)|)$$
$$+\frac{M}{2\sigma}|Bk|\max_{\tau\leq s\leq T_s+\tau}(\phi(s))\sup_{0\leq s\leq t}(|\xi(s)|) \tag{2.42}$$

Using (2.24) in conjunction with (2.42) we obtain estimate (2.35). Exploiting (2.14), (2.39), and (2.42) we get for all $t \geq 0$:

$$|u(t)| \leq |k|M\exp(-2\sigma t)|x(\tau)| + |k|\max_{\tau\leq s\leq T_s+\tau}(\phi(s))\left(\frac{M}{2\sigma}|Bk|+1\right)\sup_{0\leq s\leq t}(|\xi(s)|)$$
$$+|k||G|\left(\left(1+\frac{M}{2\sigma}|Bk|\right)\int_0^{T_s+\tau}\phi(s)ds+\frac{M}{2\sigma}\right)\sup_{0\leq s\leq t+\tau}(|w(s)|) \tag{2.43}$$

Estimate (2.36) is obtained by using estimate (2.43) in conjunction with (2.24) and a standard causality argument. The proof is complete. $\lhd$

Remark 2.6: Estimate (2.15) shows that the closed-loop system (2.9), (2.12), (2.13), (2.14) with output $Y(t) = x(t)$ satisfies the IOS property from the inputs $w \in L^\infty_{loc}(\Re_+;\Re^q)$, $\xi \in L^\infty_{loc}(\Re_+;\Re^n)$. Estimate (2.35) shows that the gain of the input $w \in L^\infty_{loc}(\Re_+;\Re^q)$ is equal to $\frac{M}{2\sigma}|G|\left(1+|Bk|\int_0^{T_s+\tau}\phi(s)ds + 2\sigma\int_0^{\tau}\phi(s)ds\right)$ and the asymptotic gain of the input $w \in L^\infty_{loc}(\Re_+;\Re^q)$ is equal to $\frac{M}{2\sigma}|G|\left(1+|Bk|\int_0^{T_s+\tau}\phi(s)ds\right)$. Both gains are higher than the corresponding gains obtained by the direct implementation and become equal to the gains obtained by the direct implementation as $T_s \to 0^+$, i.e., when we have continuous measurement.

3 Disturbance Attenuation Limitations Due to Delays

The results provided by Theorems 2.1, 2.3, and 2.5 show that the gains of the inputs $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$, $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^n)$ to the output $Y(t) = x(t)$ for the corresponding closed-loop system (2.9) with the predictor feedback based on the nominal feedback $u = kx$ are significantly larger than the gains of the inputs $w \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^q)$, $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R}^n)$ that would be obtained for the closed-loop delay free system (2.9) with $\tau = 0$ and $u = kx$. The following example illustrates this fact.

Example 3.1 (IOS gains for a scalar unstable system): Consider the scalar system

$$\begin{aligned} &\dot{x}(t) = x(t) + u(t-1) + w(t) \\ &x(t) \in \mathfrak{R}, u(t) \in \mathfrak{R}, w(t) \in \mathfrak{R} \end{aligned} \tag{3.1}$$

The predictor feedback based on the linear nominal controller $k = -p$ with $p > 1$, under the presence of measurement error $\xi \in L^{\infty}_{loc}(\mathfrak{R}_+;\mathfrak{R})$, is given by:

1) In the case of direct implementation:

$$u(t) = -pe(x(t) + \xi(t)) - p\int_0^1 \exp(s)u(t-s)ds, \text{ for } t \geq 0 \tag{3.2}$$

2) In the case of the dynamic implementation:

$$\dot{u}(t) = -pe(1+\mu)\left(x(t) + \xi(t) + \int_0^1 \exp(s-1)u(t-s)ds\right) - (p+\mu)u(t) \tag{3.3}$$

where $\mu > 0$ is an arbitrary constant.

3) In the case of hybrid implementation:

$$u(t) = -p\exp(1-(p-1)(t-\tau_i))(x(\tau_i) + \xi(\tau_i)) - p\int_0^1 \exp(s-(p-1)(t-\tau_i))u(\tau_i - s)ds, \quad \text{for all } t \in [\tau_i, \tau_{i+1}) \tag{3.4}$$

where $\{\tau_i\}_{i=0}^{\infty}$ is an arbitrary partition of $\mathfrak{R}_+$ with finite upper diameter.

Theorem 2.1, Theorem 2.3, and Theorem 2.5 guarantee that the corresponding closed-loop systems with output $Y(t) = x(t)$ satisfy the IOS property with linear gain functions. The following table shows the gains with respect to plant disturbances and measurement errors as predicted by Theorem 2.1, Theorem 2.3, and Theorem 2.5 (Table 2.1).

The gains with respect to the plant disturbance cannot become arbitrarily small. This is in sharp contrast to the delay-free case: the delay free version of system (3.1)

Table 2.1 Input Gains for System (3.1) With Predictor Feedback.

IOS Property with Output $Y(t) = x(t)$	Direct Implementation	Dynamic Implementation with $\mu > p - 1$	Hybrid Implementation with Sampling Partition of Upper Diameter $T_s > 0$
Gain w.r.t. w	$\frac{1 + (2p-1)(e-1)}{p-1}$	$\frac{p + \mu + p(1+\mu)(e-1)}{\mu(p-1)} + \frac{\mu+1}{\mu+1-p}(e-1)$	$\frac{1 + p(e^{1+T_s} - 1)}{p-1} + e - 1$
Asymptotic gain w.r.t. w	$\frac{1 + p(e-1)}{p-1}$	$\frac{p + \mu + p(1+\mu)(e-1)}{\mu(p-1)}$	$\frac{1 + p(e^{1+T_s} - 1)}{p-1}$
Gain w.r.t. ξ	$\frac{pe}{p-1}$	$\frac{ep(1+\mu)}{\mu(p-1)}$	$\frac{pe^{1+T_s}}{p-1}$

under the linear nominal controller $u = -px$ with $p > 1$ satisfies the ISS property with respect to the plant disturbance with gain $\frac{1}{p-1}$ and can become arbitrarily small as $p \to +\infty$. The study of this example will be continued. $\lhd$

After reading the above example, the reader might think that a "smart modification" of the predictor feedback may be able to guarantee smaller values for the gains. The following result guarantees that this is not possible.

Theorem 3.2 (Lower Bound on Asymptotic Gain): *Consider the system*

$$\begin{aligned} &\dot{x}(t) = F(x(t), u(t-\tau), w(t)) \\ &x(t) \in \Re^n, u(t) \in \Re^m, w(t) \in \Re^q \end{aligned} \tag{3.5}$$

where $F : \Re^n \times \Re^m \times \Re^p \to \Re^n$ is a locally Lipschitz mapping with $F(0,0,0) = 0$ and $\tau > 0$ is a constant. Suppose that system (3.5) with $u \equiv 0$ is forward complete for all inputs $w \in L^\infty_{loc}(\Re_+; \Re^q)$. Moreover, suppose that there is a time-invariant feedback law (static or dynamic) such that the corresponding closed-loop system of (3.5) with $w \equiv 0$ and output $Y(t) = \left(x(t), \breve{u}_t\right)$ is Globally Asymptotically Output Stable and Globally Asymptotically Stable. Finally, suppose that the closed-loop system is robustly forward complete w.r.t. the input $w \in L^\infty_{loc}(\Re_+; \Re^q)$.

Define the function

$$\zeta(s) := \sup\left\{|x(t)| : 0 \le t \le \tau,\, w \in L^\infty(\Re_+; \Re^q) \ \textit{with} \ \|w\| \le s\right\}, \text{ for } s \ge 0 \tag{3.6}$$

where $x(t) \in \Re^n$ denotes the solution of $\dot{x}(t) = F(x(t), 0, w(t))$ with initial condition $x(0) = 0$. Then the following condition holds

$$\zeta(s) \le \sup\left\{\limsup_{t \to +\infty} |x(t)| \ : \ z \in X,\, w \in L^\infty(\Re_+; \Re^q) \ \textit{with} \ \|w\| \le s\right\}, \quad \text{for all } s \ge 0 \tag{3.7}$$

where X is the state space (a normed linear space) of the closed-loop system and $x(t) \in \Re^n$ denotes the component of the solution of the closed-loop system with initial condition $z \in X$ corresponding to input $w \in L^\infty_{loc}(\Re_+; \Re^q)$.

Proof: Let $s > 0$, $\varepsilon > 0$ be given. Let $\widetilde{w} \in L^{\infty}(\Re_{+};\Re^{q})$ be an input that satisfies $\|\widetilde{w}\| \leq s$ and $\zeta(s) - \varepsilon \leq \max\{|\varphi(t)| \,:\, 0 \leq t \leq \tau\}$, where $\varphi(t) \in \Re^{n}$ denotes the solution of $\dot{\varphi}(t) = F(\varphi(t), 0, \widetilde{w}(t))$ with initial condition $\varphi(0) = 0$ (such an input always exists by virtue of definition (3.6)).

Let $\delta > 0$ be a sufficiently small positive number with the following property:

"if $(q(0), u) \in \Re^{n} \times L^{\infty}(\Re_{+};\Re^{m})$ satisfy $|q(0)| + \sup_{0 \leq s < \tau}(|u(s)|) \leq \delta$ then $\max\{|q(t) - \varphi(t)| \colon 0 \leq t \leq \tau\} \leq \varepsilon$"

where $q(t) \in \Re^{n}$ denotes the solution of $\dot{q}(t) = F(q(t), u(t), \widetilde{w}(t))$. The existence of $\delta > 0$ is guaranteed by virtue of continuity of the solution mapping with respect to initial conditions and inputs.

Let $R \geq 0$ be sufficiently large such that the solution $z(t) \in X$ of the closed-loop system with initial condition $z(0) \in X$ satisfying $|x(0)| + \sup_{0 \leq s < \tau}(|u(s)|) \leq \delta$ and $\|z(0)\|_{X} \leq \delta$ (here $\|\;\|_{X}$ denotes the norm of the normed linear space X) corresponding to input $\widetilde{w} \in L^{\infty}(\Re_{+};\Re^{q})$ is contained for all $t \in [0,\tau]$ in a ball of X of radius $R \geq 0$ centered at zero, i.e., $\|z(t)\|_{X} \leq R$ for all $t \in [0,\tau]$. The existence of $R \geq 0$ is guaranteed by robust forward completeness of the closed-loop system w.r.t. the input $w \in L^{\infty}_{loc}(\Re_{+};\Re^{q})$ (see [2]).

Let $T > 0$ be a sufficiently large time such that the solution of the closed-loop system with $w \equiv 0$ and arbitrary initial condition contained in a ball of X of radius $R \geq 0$ centered at zero satisfies $|x(T)| + \sup_{-\tau \leq s < 0}(|u(T+s)|) \leq \delta$ and $\|z(T)\|_{X} \leq \delta$. The existence of $T > 0$ is guaranteed by Global Asymptotical Output Stability of the closed-loop system with $w \equiv 0$ and Global Asymptotical Stability of the closed-loop system with $w \equiv 0$ (see [2]).

Next consider the solution of the closed-loop system with zero initial condition corresponding to the $(T+\tau)$-periodic input $w \in L^{\infty}_{loc}(\Re_{+};\Re^{q})$ defined by:

$$w(t) = \begin{cases} \widetilde{w}(t) & for \quad t \in [0,\tau) \\ 0 & for \quad t \in [\tau, T+\tau) \end{cases}, \text{ for } t \in [0, T+\tau) \tag{3.8}$$

Notice that $\|w\| \leq s$. All the above properties (and induction) guarantee that the component $x(t)$ of the solution of the closed-loop system with zero initial condition corresponding to the $(T+\tau)$-periodic input $w \in L^{\infty}_{loc}(\Re_{+};\Re^{q})$ satisfies:

$$\zeta(s) - 2\varepsilon \leq \max\{|x(t)| \,:\, k(T+\tau) \leq t \leq k(T+\tau) + \tau\} \text{ for all } k \in Z_{+} \tag{3.9}$$

Inequality (3.9) implies that

$$\zeta(s) - 2\varepsilon \leq \sup\left\{ \limsup_{t \to +\infty} |x(t)| \,:\, z \in X,\, w \in L^{\infty}(\Re_{+};\Re^{q}) \;\; with \;\; \|w\| \leq s \right\}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (3.7) holds. The proof is complete. ◁

Theorem 3.2 shows that there is a lower bound for the asymptotic gain of the output $y(t) = x(t)$ of the closed-loop system, no matter what controller we are using. The same thing holds for gains of the IOS property with output $Y(t) = x(t)$ (since gains are always higher than the asymptotic gains). For linear systems of the form (2.9) we obtain the following corollary.

Corollary 3.3 (Lower Bound on Asymptotic Gain for LTI Systems): *Consider the system* (*2.9*) *and suppose that there is a time-invariant feedback law* (*static or dynamic*) *such that the corresponding closed-loop system of* (*2.9*) *with* $w \equiv 0$ *and output* $Y(t) = \left(x(t), \breve{u}_t\right)$ *is Globally Asymptotically Output Stable and Globally Asymptotically Stable. Moreover, suppose that the closed-loop system is robustly forward complete w.r.t. the input* $w \in L^\infty_{loc}(\Re_+;\Re^q)$*. Define*

$$K := \sup\left\{ \left| \int_0^t \exp(A(t-s))Gw(s)ds \right| : 0 \le t \le \tau,\ w \in L^\infty(\Re_+;\Re^q) \text{ with } \|w\| \le 1 \right\} \tag{3.10}$$

Then the following condition holds

$$Ks \le \sup\left\{ \limsup_{t\to+\infty} |x(t)| : z \in X,\ w \in L^\infty(\Re_+;\Re^q) \text{ with } \|w\| \le s \right\}, \text{ for all } s \ge 0 \tag{3.11}$$

where X *is the state space* (*a normed linear space*) *of the closed-loop system and* $x(t) \in \Re^n$ *denotes the component of the solution of the closed-loop system with initial condition* $z \in X$ *corresponding to input* $w \in L^\infty_{loc}(\Re_+;\Re^q)$.

Therefore, for the LTI system (2.9), inequality (3.11) implies that for every controller, the asymptotic gain of the output $y(t) = x(t)$ of the closed-loop system with respect to the input $w \in L^\infty_{loc}(\Re_+;\Re^q)$ cannot be less than K, where K is defined by (3.10).

Example 3.1 (continued): Consider the scalar system (3.1). Using (3.10) we can conclude that the asymptotic gain (and certainly the gain) with respect to the input $w \in L^\infty_{loc}(\Re_+;\Re)$ cannot be less than $e-1$, no matter what controller we are using. This is exactly the limit of the asymptotic gain obtained by the direct implementation of the predictor feedback as $p \to +\infty$.

Therefore, predictor feedback achieves as good a performance as possible for large values of $p > 1$. Of course, as we will see later in this chapter, by letting the controller gain to have a large value, we have other problems for the closed-loop system: sensitivity to the value of the delay increases as $p \to +\infty$ (see Example 5.4 in Section 5 of the present chapter). $\lhd$

4 Approximate Predictors

Roughly speaking, a static approximate predictor is a mapping $\Phi : \Re^n \times L^\infty\left([0,T);\Re^m\right) \to \Re^n$, which can provide an approximation of the future value of the state vector at time $T > 0$ of a forward complete system. The use of an approximate predictor may be important in linear (and nonlinear) systems. For example, in large scale systems, it may be difficult to compute the matrix exponential as well as the convolution integrals that involve the matrix exponential. On the other hand, the use of an approximate predictor may be computationally cheap or may give us simple formulae which can be used in a straightforward way.

Next, we describe the construction of static approximate predictors for linear systems. Consider the solution of

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t) \\ &x(t) \in \Re^n,\, u(t) \in \Re^m \end{aligned} \tag{4.1}$$

on $[0,T]$ with initial condition $x(0) = x_0 \in \Re^n$ corresponding to input $u \in L^\infty\left([0,T);\Re^m\right)$. The theory of differential equations allows us to construct approximations of the solution of (4.1) which do not require the knowledge of the exponential matrix $\exp(At)$.

1st step: Successive approximations

Starting with the constant approximation $y_0(t,u,x_0) \equiv x_0$ on $[0,T]$, we generate approximations by means of the recursive formula:

$$y_{k+1}(t,u,x_0) = x_0 + \int_0^t Ay_k(s,u)ds + \int_0^t Bu(s)ds, \text{ for } k \geq 0 \quad \text{and } t \in [0,T]$$

It can be shown in a straightforward way that

$$\begin{aligned} y_l(t,u,x_0) = \left(\sum_{k=0}^{l} \frac{t^k}{k!}A^k\right)x_0 + \int_0^t \left(\sum_{k=0}^{l-1} \frac{(t-s)^k}{k!}A^k\right)Bu(s)ds, \\ \text{for } l \geq 1 \text{ and } t \in [0,T] \end{aligned} \tag{4.2}$$

The error of the l-th approximation is given by:

$$\begin{aligned} x(t) - y_l(t,u,x_0) = \left(\sum_{k=l+1}^{\infty} \frac{t^k}{k!}A^k\right)x_0 + \int_0^t \left(\sum_{k=l}^{\infty} \frac{(t-s)^k}{k!}A^k\right)Bu(s)ds, \\ \text{for } l \geq 1 \text{ and } t \in [0,T] \end{aligned} \tag{4.3}$$

Since $\left|\sum_{k=l+1}^{\infty} \frac{t^k}{k!} A^k\right| \leq \sum_{k=l+1}^{\infty} \frac{t^k}{k!} |A|^k = p_l(|A|t)$, where $p_l(t) := \exp(t) - \sum_{k=0}^{l} \frac{t^k}{k!}$ and since $p_l(t) \leq \frac{t^{l+1}}{(l+1)!}\exp(t)$ for all $t \geq 0$, we obtain from (4.3):

$$|x(t) - y_l(t,u,x_0)| \leq \frac{T^{l+1}|A|^l}{(l+1)!}\exp(|A|T)\left(|A||x_0| + |B| \sup_{0\leq s<T}(|u(s)|)\right), \quad \text{for } l \geq 1 \text{ and } t \in [0,T] \tag{4.4}$$

2nd step: Combination of successive and numerical approximations

First, we divide the interval $[0,T]$ in N subintervals of equal length $h = T/N$ and we approximate the solution at the time instants ih ($i = 1, \ldots, N$) in the following way for certain integer $l \geq 1$:

$$z_{i+1} = \left(\sum_{k=0}^{l} \frac{h^k}{k!} A^k\right) z_i + \int_0^h \left(\sum_{k=0}^{l-1} \frac{(h-s)^k}{k!} A^k\right) Bu(ih+s)ds, \quad i = 0, \ldots, N-1 \text{ with } z_0 = x_0 \tag{4.5}$$

The value of the state vector is approximated at all points by means of the formula (4.2). This approximation scheme generalizes the successive approximation scheme (described above) as well as classical numerical schemes as the explicit Euler scheme, which corresponds to the special case $l = 1$. For this approximation scheme, we can prove the following technical result.

Lemma 4.1: *For every integer $N \geq 1, x(0) = x_0 \in \Re^n$ and $u \in L^\infty\left([0,T);\Re^m\right)$, the solution $x(t)$ of (4.1) satisfies*

$$|z_N - x(T)| \leq T\frac{h^l|A|^l}{(l+1)!}\exp(|A|T)\left(|A||x_0| + |B| \sup_{0\leq s<T}(|u(s)|)\right) \tag{4.6}$$

Proof: If we define $e_j = z_j - x(jh)$ for $j = 0, \ldots, N$ then it follows that $e_{j+1} = \left(\sum_{k=0}^{l} \frac{h^k}{k!} A^k\right) e_j - \int_0^h \left(\sum_{k=l}^{\infty} \frac{(h-s)^k}{k!} A^k\right) Bu(jh+s)ds - \left(\sum_{k=l+1}^{\infty} \frac{h^k}{k!} A^k\right) x(jh)$ for $j = 0, \ldots, N-1$. Therefore, proceeding as above, we get:

$$|e_{j+1}| \leq \left(\sum_{k=0}^{l} \frac{h^k}{k!}|A|^k\right)|e_j| + \frac{h^{l+1}|A|^l}{(l+1)!}\exp(|A|h)\left(|A||x(jh)| + |B| \sup_{0\leq s<T}(|u(s)|)\right), \quad \text{for } j = 0, \ldots, N-1 \tag{4.7}$$

Using the fact that $|\exp(At)| \leq \exp(|A|t)$, we get $|A||x(jh)| \leq \left(|A|\exp(|A|jh)|x_0| + |B|(\exp(|A|jh) - 1)\sup_{0\leq s<T}(|u(s)|)\right)$. Combining with (4.7) we obtain:

$$\left|e_{j+1}\right| \leq \exp(|A|h)\left|e_j\right| + \frac{h^{l+1}|A|^l}{(l+1)!}\exp(|A|(j+1)h)\left(|A||x_0| + |B|\sup_{0\leq s<T}(|u(s)|)\right),$$
$$\text{for } j = 0, \ldots, N-1 \tag{4.8}$$

Using (4.8) and the facts that $e_0 = 0$ and $hN = T$, we obtain $|e_N| \leq T\frac{h^l|A|^l}{(l+1)!}\exp(|A|T)\left(|A||x_0| + |B|\sup_{0\leq s<T}(|u(s)|)\right)$. The previous inequality in conjunction with definition $e_N = z_N - x(T)$ directly implies inequality (4.6). $\triangleleft$

Formula (4.5) can be written in a different way:

$$z_N = \left(\sum_{k=0}^{l}\frac{h^k}{k!}A^k\right)^N x_0 + \sum_{i=0}^{N-1}\int_{ih}^{(i+1)h}\left(\sum_{k=0}^{l}\frac{h^k}{k!}A^k\right)^{N-1-i}\left(\sum_{k=0}^{l-1}\frac{((i+1)h-s)^k}{k!}A^k\right)Bu(s)ds \tag{4.9}$$

where $h = T/N$, $l, N \geq 1$.

When $r \geq 0$ is the measurement delay (i.e., when the measurement is $y(t) = x(t-r)$) then we need a predictor with time horizon equal to $r + \tau$. Every choice of integers $l, N \geq 1$ gives us an *approximate predictor*, i.e., a mapping $\Phi\colon \Re^n \times L^\infty\left([0, r+\tau); \Re^m\right) \to \Re^n$ for which there exist constants $a_1, a_2 > 0$ such that:

$$\left|\Phi(x_0,u) - \exp(A(r+\tau))x_0 - \int_0^{r+\tau}\exp(A(r+\tau-s))Bu(s)ds\right| \leq a_1|x_0| + a_2\sup_{0\leq s<r+\tau}(|u(s)|),$$
$$\text{for all } (x_0,u) \in \Re^n \times L^\infty\left([0, r+\tau); \Re^m\right) \tag{4.10}$$

Clearly, the linear mapping

$$\begin{aligned}\Phi(x_0,u) &= \left(\sum_{k=0}^{l}\frac{h^k}{k!}A^k\right)^N x_0 \\ &+ \sum_{i=0}^{N-1}\int_{ih}^{(i+1)h}\left(\sum_{k=0}^{l}\frac{h^k}{k!}A^k\right)^{N-1-i}\left(\sum_{k=0}^{l-1}\frac{((i+1)h-s)^k}{k!}A^k\right)Bu(s)ds\end{aligned} \tag{4.11}$$

where $h = (r+\tau)/N$, $l, N \geq 1$, is an approximate predictor since (4.6) and (4.9) imply that (4.10) holds with $a_1 := \frac{(r+\tau)^{l+1}|A|^{l+1}}{(l+1)!N^l}\exp(|A|(r+\tau))$, $a_2 := |B|\frac{(r+\tau)^{l+1}|A|^l}{(l+1)!N^l}\exp(|A|(r+\tau))$. It should be noted that for every $r, \tau > 0$, we are in a position to select sufficiently large values for $l, N \geq 1$ so that the constants $a_1, a_2 > 0$ are sufficiently small.

Approximate predictors can be used in the case of state measurement. In general, as formula (4.11) shows, an approximate predictor for a linear control system will involve a matrix $Q \in \mathfrak{R}^{n\times n}$ and a linear operator $\widetilde{\Phi} : L^{\infty}\left(\left[-\tau - r, 0\right); \mathfrak{R}^m\right) \to \mathfrak{R}^n$ for which there exist constants $a_1, a_2 > 0$ such that (4.10) holds for $\Phi(x,u) = Qx + \widetilde{\Phi}(u)$, i.e.,

$$\left|(Q - \exp(A(\tau + r)))x + \widetilde{\Phi}(u) - \int_0^{r+\tau} \exp(As)Bu(-s)ds\right| \leq a_1|x| + a_2\|u\|,$$
$$\forall (x,u) \in \mathfrak{R}^n \times L^{\infty}\left(\left[-\tau - r, 0\right); \mathfrak{R}^m\right) \tag{4.12}$$

The approximate predictor (P) will be given by:

$$\widetilde{x}(t) = Q\hat{x}(t) + \widetilde{\Phi}\left(\breve{u}_t\right) \tag{4.13}$$

where $\hat{x}(t)$ is the measurement. Notice that the predictor is not necessarily equal to the mapping $\Phi(\hat{x}, u) = \exp(A(\tau + r))\hat{x} + \int_0^{r+\tau} \exp(As)Bu(-s)ds$ (exact predictor mapping) but may provide a predicted value of the future value of the state vector which differs from the nominal one, i.e., $\exp(A(\tau + r))\hat{x}(t) + \int_0^{r+\tau} \exp(As)Bu(t-s)ds$. The error of the prediction is bounded by the right hand side of inequality (4.12).

We also assume that we have a linear stabilizing feedback law for the delay-free case, i.e., we can design a matrix $k \in \mathfrak{R}^{m\times n}$ such that $(A + Bk)$ is a Hurwitz matrix. Next, we consider the control scheme

$$\dot{z}(t) = Az(t) + Bu(t), \text{ for all } t \in \left[\tau_i, \tau_{i+1}\right) \tag{4.14}$$

$$z(\tau_i) = Q(x(\tau_i - r) + \xi(\tau_i)) + \widetilde{\Phi}\left(\breve{u}_{\tau_i}\right), \text{ for all } i \in Z_+ \tag{4.15}$$

$$u(t) = kz(jT_H), \text{ for all } t \in \left[jT_H, (j+1)T_H\right), \; j \in Z_+ \tag{4.16}$$

where $r \geq 0$ is the measurement delay $T_H > 0$ is the holding period, ξ is the measurement error (i.e., the measurement at the sampling time τ_i is $y(\tau_i) = x(\tau_i - r) + \xi(\tau_i)$).

The reader can realize that the above control scheme is the hybrid implementation scheme with an approximate predictor and input applied with Zero-Order-Hold (ZOH). Notice that the inter-sample predictor for the state (given by (4.14)) provides an estimation of the future value of the state vector: this is in sharp contrast with the control schemes employed in the following chapter. For this particular control scheme we can prove the following result.

Theorem 4.2: *Let $\sigma > 0$ and $M \geq 1$ be constants satisfying $|\exp((A+Bk)t)| \leq M\exp(-2\sigma t)$ for all $t \geq 0$ and let $\phi \in C^0(\Re_+;\Re_+)$ be a function that satisfies $|\exp(At)| \leq \phi(t)$ for all $t \geq 0$. Let $T_s > 0$ be a given constant and assume that:*

$$\frac{M|B|}{\sigma}|k|\exp(\sigma T_H)\left((a_1+a_2|k|)\max_{0\leq s\leq T_s}(\phi(s))\exp(\sigma(r+\tau+T_s))+\max_{0\leq s\leq T_H}(|\exp(As)-I|)\right)$$
$$+|k|\left(a_2\max_{0\leq s\leq T_s}(\phi(s))\exp(\sigma(r+\tau+T_H+T_s))+\frac{M|B|}{\sigma}(1-\exp(-\sigma T_H))\right)<1 \tag{4.17}$$

Then there exists a constant $K > 0$ such that for every $x_0 \in C^0([-r,0];\Re^n)$, $\breve{u}_0 \in L^\infty\big([-r-\tau,0);\Re^m\big)$, $(\xi,w) \in L^\infty_{loc}(\Re_+;\Re^n\times\Re^q)$ and for every partition $\{\tau_i\}_{i=0}^\infty$ of $\Re_+$ with $\sup_{i\geq 0}(\tau_{i+1}-\tau_i) \leq T_s$, the solution $\big(x_t,\breve{u}_t\big) \in C^0([-r,0];\Re^n)\times L^\infty\big([-r-\tau,0);\Re^m\big)$ of the closed-loop system (2.9), (4.14), (4.15), (4.16) with initial condition $u(s)=\breve{u}_0(s)$ for $s\in[-r-\tau,0)$, $x(s)=x_0(s)$ for $s\in[-r,0]$ corresponding to inputs $(\xi,w)\in L^\infty_{loc}(\Re_+;\Re^n\times\Re^q)$ satisfies the following inequality for all $t \geq 0$:

$$\|x_t\|+\|\breve{u}_t\| \leq K\exp(-\sigma t)\big(\|x_0\|+\|\breve{u}_0\|\big)+K\left(\sup_{0\leq s\leq t}(|w(s)|)+\sup_{0\leq s\leq t}(|\xi(s)|)\right) \tag{4.18}$$

Moreover, the following inequality holds for all $t \geq 0$:

$$|x(t)| \leq K\exp(-\sigma t)\left(\sup_{-r\leq s\leq r+T_s+T_H+\tau}(|x(s)|)+\sup_{-\tau-r\leq s<r+T_s+T_H}(|u(s)-kx(\tau+s)|)\right)$$
$$+2\Xi_1\Xi_2\Xi_3|Q||B|\sup_{0\leq s\leq t}(|\xi(s)|)$$
$$+\Xi_1|G|\left(2\Xi_2|B|\left(\frac{M}{2\sigma}(1-\exp(-2\sigma T_H))+\int_0^{T_s}\phi(s)ds+\Xi_3\int_0^{r+\tau}\phi(s)ds\right)+1\right)\sup_{0\leq s\leq t}(|w(s)|) \tag{4.19}$$

where $\Xi_1 := (1-L)^{-1}\frac{M}{2\sigma}$, $\Xi_2 := |k|(1-\Lambda)^{-1}$, $\Xi_3 := \max_{0\le s\le T_s}(\phi(s))$, $\Xi_4 := \max_{0\le s\le T_H}(|\exp(As)-I|)$, $L := \frac{M|B|}{\sigma}\Xi_2\exp(\sigma T_H)((a_1+a_2|k|)\Xi_3\exp(\sigma(r+\tau+T_s))+\Xi_4)$ *and* $\Lambda := |k|\left(a_2\Xi_3\exp(\sigma(r+T_H+\tau+T_s))+\frac{M|B|}{\sigma}(1-\exp(-\sigma T_H))\right)$.

It should be emphasized at this point that the result of Theorem 4.2 includes the case where the input is adjusted in a continuous way, i.e., when (4.16) is replaced by the equation $u(t)=kz(t)$, for all $t\ge 0$. Indeed, in this case we can simply let $T_H\to 0^+$ and obtain the corresponding estimates. Moreover, the result of Theorem 4.2 includes the case where the exact predictor is used, i.e., when (4.15) is replaced by the equation $z(\tau_i)=\exp(A(\tau+r))(x(\tau_i)+\xi(\tau_i))+\int_0^{\tau}\exp(Aw)Bu(\tau_i-w)dw$. Indeed, in this case we can set $a_1=a_2=0$. Finally, notice that when the measurement delay is absent (i.e., $r=0$), when the exact predictor is used (i.e., $a_1=a_2=0$) and when the input is adjusted in a continuous way (i.e., $T_H=0$) then the result of Theorem 4.2 is directly comparable to the result of Theorem 2.5. However, in this particular case the estimates of the asymptotic gains of the disturbance inputs to the output $Y(t)=x(t)$ provided by Theorem 4.2 are more conservative compared to the estimates provided by Theorem 2.5 (because Theorem 4.2 is more general; this is the "price" you pay for generality).

It is also important to notice that the state space for the component $x(t)$ of the solution is not $\mathfrak{R}^n$ but $C^0([-r,0];\mathfrak{R}^n)$: the reason for this change is that state delays are present. For obvious reasons we use the convention $C^0([-r,0];\mathfrak{R}^n)\equiv\mathfrak{R}^n$ for the case $r=0$.

Proof: Existence and uniqueness of the closed-loop system (2.9), (4.14), (4.15), (4.16) is straightforward. The following equation is a direct consequence of (2.9) and (4.14):

$$z(t)-x(t+\tau)=\exp(A(t-\tau_i))(z(\tau_i)-x(\tau_i+\tau))-\int_{\tau_i}^{t}\exp(A(t-s))Gw(s+\tau)ds$$
$$\text{for all } i\in Z_+ \text{ and } t\in[\tau_i,\tau_{i+1}) \tag{4.20}$$

Using the fact that $|\exp(At)|\le\phi(t)$ for all $t\ge 0$ and the fact that $\sup_{i\ge 0}(\tau_{i+1}-\tau_i)\le T_s$, we obtain from (4.20) and (4.15):

$$\begin{aligned}|z(t)-x(t+\tau)| &\le |G|\int_0^{T_s}\phi(s)ds\sup_{0\le s\le t+\tau}(|w(s)|)\\ &+\max_{0\le s\le T_s}(\phi(s))\left|Qx(\tau_i-r)+Q\xi(\tau_i)+\widetilde{\Phi}(\breve{u}_{\tau_i})-x(\tau_i+\tau)\right|\\ &\text{for all } i\in Z_+ \text{ and } t\in[\tau_i,\tau_{i+1})\end{aligned} \tag{4.21}$$

Using (2.9) and (4.21) we get:

$$
\begin{aligned}
&|z(t)-x(t+\tau)| \leq \max_{0\leq s\leq T_s}(\phi(s))\left|\int_{\tau_i-r}^{\tau_i+\tau}\exp(A(\tau_i+\tau-s))Gw(s)ds\right| \\
&+\max_{0\leq s\leq T_s}(\phi(s))\left|Qx(\tau_i-r)+\widetilde{\Phi}\left(\breve{u}_{\tau_i}\right)-\exp(A(r+\tau))x(\tau_i-r)-\int_0^{r+\tau}\exp(Aw)Bu(\tau_i-w)dw\right| \\
&+\max_{0\leq s\leq T_s}(\phi(s))|Q|\sup_{0\leq s\leq t}(|\xi(s)|)+|G|\int_0^{T_s}\phi(s)ds\sup_{0\leq s\leq t+\tau}(|w(s)|) \\
&\qquad \text{for all } i\in Z_+ \text{ with } \tau_i\geq r \text{ and for all } t\in\left[\tau_i,\tau_{i+1}\right)
\end{aligned}
\tag{4.22}
$$

Using the fact that $|\exp(At)|\leq\phi(t)$ for all $t\geq 0$ and (4.12), we obtain from (4.22):

$$
\begin{aligned}
&|z(t)-x(t+\tau)| \leq a_1\max_{0\leq s\leq T_s}(\phi(s))|x(\tau_i-r)|+a_2\max_{0\leq s\leq T_s}(\phi(s))\left\|\breve{u}_{\tau_i}\right\| \\
&+\max_{0\leq s\leq T_s}(\phi(s))|Q|\sup_{0\leq s\leq t}(|\xi(s)|)+|G|\left(\int_0^{T_s}\phi(s)ds+\max_{0\leq s\leq T_s}(\phi(s))\int_0^{r+\tau}\phi(s)ds\right)\sup_{0\leq s\leq t+\tau}(|w(s)|) \\
&\qquad \text{for all } i\in Z_+ \text{ with } \tau_i\geq r \text{ and for all } t\in\left[\tau_i,\tau_{i+1}\right)
\end{aligned}
\tag{4.23}
$$

Using (4.23), the triangle inequality and the fact that $\sup_{i\geq 0}(\tau_{i+1}-\tau_i)\leq T_s$, we get:

$$
\begin{aligned}
&|z(t)-x(t+\tau)|\exp(\sigma t)\leq(a_1+a_2|k|)\max_{0\leq s\leq T_s}(\phi(s))\exp(\sigma(r+T_s))\sup_{0\leq s<t+\tau}(|x(s)|\exp(\sigma s)) \\
&+a_2\max_{0\leq s\leq T_s}(\phi(s))\exp(\sigma(r+\tau+T_s))\sup_{-\tau\leq s<t}(|u(s)-kx(\tau+s)|\exp(\sigma s)) \\
&+\max_{0\leq s\leq T_s}(\phi(s))|Q|\exp(\sigma t)\sup_{0\leq s\leq t}(|\xi(s)|) \\
&+|G|\left(\int_0^{T_s}\phi(s)ds+\max_{0\leq s\leq T_s}(\phi(s))\int_0^{r+\tau}\phi(s)ds\right)\exp(\sigma t)\sup_{0\leq s\leq t+\tau}(|w(s)|) \\
&\qquad \text{for all } i\in Z_+ \text{ with } \tau_i\geq r \text{ and for all } t\geq\tau_i
\end{aligned}
\tag{4.24}
$$

Using (2.9), (4.24) and the following inequality:

$$
\begin{aligned}
&|u(t)-kx(t+\tau)|=|kz(jT_H)-kx(t+\tau)| \\
&\leq|k||z(jT_H)-x(jT_H+\tau)|+|k||x(jT_H+\tau)-x(t+\tau)|
\end{aligned}
$$

which holds for all $j \in Z_+$ and $t \in \left[jT_H, (j+1)T_H\right)$, we get for all $i, j \in Z_+$ with $\tau_i \geq r$, $jT_H \geq \tau_i$ and $t \in \left[jT_H, (j+1)T_H\right)$:

$$
\begin{aligned}
&|u(t) - x(t+\tau)|\exp(\sigma t) \leq |k|(a_1 + a_2|k|)\Xi_3 \exp(\sigma(r + T_H + T_s)) \sup_{0 \leq s < jT_H + \tau} (|x(s)|\exp(\sigma s)) \\
&+ a_2|k|\Xi_3 \exp(\sigma(r + T_H + \tau + T_s)) \sup_{-\tau \leq s < jT_H} (|u(s) - kx(\tau + s)|\exp(\sigma s)) \\
&+ \Xi_3 |Q||k| \exp(\sigma t) \sup_{0 \leq s \leq jT_H} (|\xi(s)|) + |k||x(t+\tau) - x(jT_H + \tau)|\exp(\sigma t) \\
&+ |G||k| \left(\int_0^{T_s} \phi(s)ds + \max_{0 \leq s \leq T_s} (\phi(s)) \int_0^{r+\tau} \phi(s)ds \right) \exp(\sigma t) \sup_{0 \leq s \leq jT_H + \tau} (|w(s)|)
\end{aligned}
\tag{4.25}
$$

where $\Xi_3 := \max_{0 \leq s \leq T_s} (\phi(s))$. Using the fact that $|\exp((A + Bk)t)| \leq M \exp(-2\sigma t)$ for all $t \geq 0$, we obtain:

$$
\begin{aligned}
&|x(t)|\exp(\sigma t) \leq M \exp(\sigma t_0)|x(t_0)| + \frac{M|G|}{2\sigma} \exp(\sigma t) \sup_{t_0 \leq s \leq t} (|w(s)|) \\
&+ \frac{M|B|}{\sigma} \sup_{t_0 \leq s \leq t} (|u(s-\tau) - kx(s)|\exp(\sigma s)), \\
&\qquad \text{for all } t \geq t_0 \geq 0
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
&|x(t+\tau) - x(jT_H + \tau)|\exp(\sigma t) \leq \frac{M|G|(1 - \exp(-2\sigma T_H))}{2\sigma} \exp(\sigma t) \sup_{jT_H + \tau \leq s \leq t + \tau} (|w(s)|) \\
&+ \exp(\sigma(T_H - \tau)) \max_{0 \leq s \leq T_H} (|\exp((A + Bk)s) - I|)|x(jT_H + \tau)|\exp(\sigma(jT_H + \tau)) \\
&+ \frac{M|B|(1 - \exp(-\sigma T_H))}{\sigma} \sup_{jT_H \leq s \leq t} (|u(s) - kx(s+\tau)|\exp(\sigma s)) \\
&\qquad \text{for all } j \in Z_+ \text{ and } t \in \left[jT_H, (j+1)T_H\right)
\end{aligned}
\tag{4.27}
$$

Combining (4.25) and (4.27) we obtain for all $i, j \in Z_+$ with $\tau_i \geq r$, $t \geq jT_H \geq \tau_i$:

$$\begin{aligned}
&\sup_{jT_H\le s\le t}\left(|u(s)-kx(\tau+s)|\exp(\sigma s)\right)\le \Xi_2\Xi_3|Q|\exp(\sigma t)\sup_{0\le s\le t}\left(|\xi(s)|\right)\\
&\Xi_2\exp(\sigma(T_H-\tau))((a_1+a_2|k|)\Xi_3\exp(\sigma(r+\tau+T_s))+\Xi_4)\sup_{0\le s\le t+\tau}\left(|x(s)|\exp(\sigma s)\right)\\
&+\Lambda(1-\Lambda)^{-1}\sup_{-\tau\le s<jT_H}\left(|u(s)-kx(\tau+s)|\exp(\sigma s)\right)\\
&+|G|\Xi_2\left(\frac{M}{2\sigma}(1-\exp(-2\sigma T_H))+\int_0^{T_s}\phi(s)ds+\Xi_3\int_0^{r+\tau}\phi(s)ds\right)\exp(\sigma t)\sup_{0\le s\le t+\tau}\left(|w(s)|\right)
\end{aligned}\tag{4.28}$$

where $\Lambda:=|k|\left(a_2\Xi_3\exp(\sigma(r+T_H+\tau+T_s))+\frac{M|B|}{\sigma}(1-\exp(-\sigma T_H))\right)$, $\Xi_4:=\max_{0\le s\le T_H}\left(|\exp(As)-I|\right)$ and $\Xi_2:=|k|(1-\Lambda)^{-1}$. Inequality (4.19) is a direct consequence of (4.26), (4.28) and the fact that the smallest τ_i with $\tau_i\ge r$ satisfies $\tau_i\in[r,r+T_s]$ (a consequence of $\sup_{i\ge 0}(\tau_{i+1}-\tau_i)\le T_s$) and the selection $j=\left[\frac{r+T_s}{T_H}\right]+1$ satisfies $r+T_s+T_H\ge jT_H\ge\tau_i$. Using (4.19), (4.28) and a standard causality argument, we conclude that there exists a constant $K_1>0$ such that for every $x_0\in C^0([-r,0];\Re^n)$, $\breve{u}_0\in L^\infty\left([-r-\tau,0);\Re^m\right)$, $(\xi,w)\in L^\infty_{loc}(\Re_+;\Re^n\times\Re^q)$ and for every partition $\{\tau_i\}_{i=0}^\infty$ of $\Re_+$ with $\sup_{i\ge 0}(\tau_{i+1}-\tau_i)\le T_s$, the solution $\left(x_t,\breve{u}_t\right)\in C^0([-r,0];\Re^n)\times L^\infty\left([-r-\tau,0);\Re^m\right)$ of the closed-loop system (2.9), (4.14), (4.15), (4.16) with initial condition $u(s)=\breve{u}_0(s)$ for $s\in[-r-\tau,0)$, $x(s)=x_0(s)$ for $s\in[-r,0]$ corresponding to inputs $(\xi,w)\in L^\infty_{loc}(\Re_+;\Re^n\times\Re^q)$ satisfies the following inequality for all $t\ge 0$:

$$\begin{aligned}
&\|x_t\|+\left\|\breve{u}_t\right\|\le K_1\left(\sup_{0\le s\le t}\left(|w(s)|\right)+\sup_{0\le s\le t}\left(|\xi(s)|\right)\right)\\
&+K_1\exp(-\sigma t)\left(\sup_{-r\le s\le r+\tau+T_s+T_H}\left(|x(s)|\right)+\sup_{-\tau-r\le s\le r+T_s+T_H}\left(|u(s)|\right)\right)
\end{aligned}\tag{4.29}$$

Inequality (4.12) in conjunction with the fact that $|\exp(At)|\le\phi(t)$ for all $t\ge 0$ gives:

$$\left|\widetilde{\Phi}(u)\right|\le\left(a_2+|B|\int_0^{r+\tau}\phi(s)ds\right)\|u\|,\quad\forall u\in L^\infty\left([-\tau-r,0);\Re^m\right)\tag{4.30}$$

Inequality (4.30) in conjunction with (4.21) gives for all $t\ge 0$:

$$
\begin{aligned}
|z(t)| \le\ & (1+|Q|(1+\Xi_3)) \sup_{-r\le s\le t+\tau}(|x(s)|) + |G|\int_0^{T_s}\phi(s)ds \sup_{0\le s\le t+\tau}(|w(s)|) \\
&+\Xi_3|Q|\sup_{0\le s\le t}(|\xi(s)|) + \Xi_3\left(a_2+|B|\int_0^{r+\tau}\phi(s)ds\right)\sup_{-r-\tau\le s<t}(|u(s)|)
\end{aligned}
\tag{4.31}
$$

where $\Xi_3 := \max_{0\le s\le T_s}(\phi(s))$. Using the fact that $|\exp(At)|\le\phi(t)$ for all $t\ge 0$, we obtain the inequality

$$
|x(t+\tau)| \le \phi(\tau)|x(t)| + |B|\int_0^{\tau}\phi(s)ds \sup_{-r-\tau\le s<t}(|u(s)|) + |G|\int_0^{\tau}\phi(s)ds \sup_{0\le s\le t+\tau}(|w(s)|)
$$

for all $t\ge 0$, which combined with (4.31) and a standard causality argument gives the following inequality for all $t\ge 0$:

$$
\begin{aligned}
|z(t)| \le\ & \max_{0\le s\le T_s+\tau}(\phi(s))(3+|Q|)\sup_{-r\le s\le t}(|x(s)|) \\
&+2|G|\int_0^{T_s+\tau}\phi(s)ds \sup_{0\le s\le t}(|w(s)|) + \Xi_3|Q|\sup_{0\le s\le t}(|\xi(s)|) \\
&+\Xi_3\left(a_2+2|B|\int_0^{r+\tau}\phi(s)ds\right)\sup_{-r-\tau\le s<t}(|u(s)|)
\end{aligned}
\tag{4.32}
$$

We next make the following claim:

(Claim): *For every $j\in Z_+$ there exists a constant $\Gamma_j\ge 0$ such that for every $x_0\in C^0([-r,0];\mathfrak{R}^n)$, $\breve{u}_0\in L^\infty([-r-\tau,0);\mathfrak{R}^m)$, $(\xi,w)\in L^\infty_{loc}(\mathfrak{R}_+;\mathfrak{R}^n\times\mathfrak{R}^q)$ and for every partition $\{\tau_i\}_{i=0}^\infty$ of $\mathfrak{R}_+$ with $\sup_{i\ge 0}(\tau_{i+1}-\tau_i)\le T_s$, the solution $(x_t,\breve{u}_t)\in C^0([-r,0];\mathfrak{R}^n)\times L^\infty([-r-\tau,0);\mathfrak{R}^m)$ of the closed-loop system (2.9), (4.14), (4.15), (4.16) with initial condition $u(s)=\breve{u}_0(s)$ for $s\in[-r-\tau,0)$, $x(s)=x_0(s)$ for $s\in[-r,0]$ corresponding to inputs $(\xi,w)\in L^\infty_{loc}(\mathfrak{R}_+;\mathfrak{R}^n\times\mathfrak{R}^q)$ satisfies the following inequality for all $t\in[0,jT_H]$:*

$$
\|x_t\| + \|\breve{u}_t\| \le \Gamma_j\left(\sup_{0\le s\le t}(|w(s)|) + \sup_{0\le s\le t}(|\xi(s)|) + \|x_0\| + \|u_0\|\right) \tag{4.33}
$$

Indeed, the claim holds for $j=0$ with $\Gamma_0=1$. Next suppose that the claim holds for certain $j\in Z_+$. Using (4.32) and (4.33) we obtain

$|z(jT_H)| \leq K_2(1+\Gamma_j)\left(\|x_0\| + \|u_0\| + \sup_{0\leq s\leq t}(|w(s)|) + \sup_{0\leq s\leq t}(|\xi(s)|)\right)$ for certain constant $K_2 > 0$. Consequently, (4.16) implies the inequality $\|\breve{u}_t\| \leq \max(|k|K_2(1+\Gamma_j), \Gamma_j)\left(\|x_0\| + \|u_0\| + \sup_{0\leq s\leq t}(|w(s)|) + \sup_{0\leq s\leq t}(|\xi(s)|)\right)$, for all $t\in[0,(j+1)T_H]$. The previous inequality in conjunction with (2.9) implies that (4.33) holds for all $t\in[0,(j+1)T_H]$ with $\Gamma_{j+1} := \max_{0\leq s\leq T_H}(\phi(s))\Gamma_j + \left(1 + \int_0^{T_H}\phi(s)ds\right)(1+\Gamma_j)\max(|k|K_2, 1)$.

Inequality (4.18) for certain constant $K > 0$ is a direct consequence of (4.29), (4.33) for sufficiently large $j\in Z_+$ and a standard causality argument. The proof is complete. ◁

Next we present an example, which shows how Theorem 4.2 can be applied.

Example 4.3: Consider the scalar system (3.1) again. We consider the case of zero measurement delay (i.e., $r = 0$). All assumptions of Theorem 4.2 hold with $A = B = M = \tau = 1$, $k = -p$, $\phi(t) = \exp(t)$ for all $t \geq 0$ and $\sigma\in\left(0, \frac{p-1}{2}\right]$, where $p > 1$. Here we consider the implementation of the hybrid scheme (4.14), (4.15), (4.16) with the approximate predictor formula (4.11) with $r = 0$, $\tau = 1$ and $l = 1$ (explicit Euler scheme):

$$
\begin{aligned}
z(t) &= \exp(t-\tau_i)\left(1+N^{-1}\right)^N(x(\tau_i)+\xi(\tau_i)) \\
&+ \exp(t-\tau_i)\sum_{j=0}^{N-1}\left(1+N^{-1}\right)^{N-1-j}\int_0^{N^{-1}} u\left(\tau_i + jN^{-1} + s - 1\right)ds \\
&+ \int_{\tau_i}^{t}\exp(t-s)u(s)ds,
\end{aligned}
$$

$$
\text{for all } t\in\left[\tau_i, \tau_{i+1}\right) \text{ and } i\in Z_+ \tag{4.34}
$$

$$
u(t) = -pz(jT_H), \text{ for all } t\in\left[jT_H, (j+1)T_H\right), \; j\in Z_+ \tag{4.35}
$$

where $\xi\in L_{loc}^{\infty}(\mathfrak{R}_+;\mathfrak{R})$ is the measurement error, $\{\tau_i\}_{i=0}^{\infty}$ is the sampling partition (a partition of $\mathfrak{R}_+$) with $\sup_{i\geq 0}(\tau_{i+1}-\tau_i) \leq T_s$, $T_H > 0$ is the holding period and $N > 0$ is the number of grid points used in the approximation of the solution mapping. Applying Lemma 4.1 with $T = r + \tau = 1$ we can guarantee that (4.12) holds with $a_1 = a_2 = \frac{\exp(1)}{2N}$. Inequality (4.17) is equivalent to the following inequality:

$$\frac{p\exp((\sigma+1)(1+T_s))(1+p+\sigma)\exp(\sigma T_H)}{\sigma-p(1-\exp(-\sigma T_H))-p\exp(\sigma T_H)(\exp(T_H)-1)}<2N \tag{4.36}$$

for $\sigma\in\left(0,\frac{p-1}{2}\right]$. It follows from (4.36) that the number of grid points $N>0$ is proportional to the quantity $\exp((\sigma+1)(1+T_s))$, i.e., $N>0$ must satisfy the inequality $N>K(T_H)\exp((\sigma+1)(1+T_s))$, where

$$K(T_H):=\frac{p(1+p+\sigma)\exp(\sigma T_H)}{\sigma-p(1-\exp(-\sigma T_H))-p\exp(\sigma T_H)(\exp(T_H)-1)}$$

In other words, the number of grid points $N>0$ used in the approximation of the solution mapping must increase exponentially with the upper diameter of the sampling period $T_s>0$. This feature is expected: the approximation $z(t)$ of the future value of the state $x(t+\tau)$ becomes less and less accurate when the measurements become sparser. In order to face this potential loss of accuracy, the approximation scheme requires more grid points.

We conclude from Theorem 4.2 that if inequality (4.36) holds, then there exists a constant $K>0$ such that for every $x_0\in\mathfrak{R}$, $\breve{u}_0\in L^\infty\left(\left[-1,0\right);\mathfrak{R}\right)$, $(\xi,w)\in L^\infty_{loc}(\mathfrak{R}_+;\mathfrak{R}\times\mathfrak{R})$ and for every partition $\{\tau_i\}_{i=0}^\infty$ of $\mathfrak{R}_+$ with $\sup\limits_{i\geq 0}(\tau_{i+1}-\tau_i)\leq T_s$, the solution $\left(x_t,\breve{u}_t\right)\in\mathfrak{R}\times L^\infty\left(\left[-1,0\right);\mathfrak{R}\right)$ of the closed-loop system (3.1), (4.34), (4.35) with initial condition $u(s)=\breve{u}_0(s)$ for $s\in\left[-1,0\right)$, $x(0)=x_0$ corresponding to inputs $(\xi,w)\in L^\infty_{loc}(\mathfrak{R}_+;\mathfrak{R}\times\mathfrak{R})$ satisfies inequality (4.18) for all $t\geq 0$. ◁

5 Delay-Robustness of Predictor Feedback

In this section we consider the system:

$$\begin{aligned}&\dot{x}(t)=Ax(t)+Bu(t-\tau-\varepsilon d(t))\\&x(t)\in\mathfrak{R}^n,\ u(t)\in\mathfrak{R}^m,\ d(t)\in[-1,1],\ \text{ for } t\geq 0,\ \text{a.e.}\end{aligned} \tag{5.1}$$

where $0<\varepsilon\leq\tau$ are constants. The linear predictor feedback is based on the constant nominal value of the delay $\tau>0$:

$$u(t)=k\exp(A\tau)x(t)+k\int_t^{t+\tau}\exp(A(t+\tau-s))Bu(s-\tau)ds,\ \text{ for } t\geq 0 \tag{5.2}$$

where $k\in\mathfrak{R}^{m\times n}$ is a constant matrix such that the matrix $(A+Bk)$ is Hurwitz. In this section we show that, provided $\varepsilon>0$ is sufficiently small, there exist constants $Q,\sigma>0$ such that for all $x_0\in\mathfrak{R}^n$, $u_0\in C^0([-\tau-\varepsilon,0];\mathfrak{R}^m)$ with

$u_0(0) = k\exp(A\tau)x_0 + k\int\limits_{-\tau}^{0}\exp(-As)Bu_0(s)ds$ the solution $(x(t),u(t))\in\Re^n\times\Re^m$ of (5.1), (5.2) with initial condition $x(0)=x_0, u(t)=u_0(t)$ for $t\in[-\tau-\varepsilon,0]$ satisfies the following exponential stability estimate in the supremum norm of the actuator state:

$$|x(t)| + \max_{t-\varepsilon-\tau\le s\le t}(|u(s)|) \le Q\exp(-\sigma t)\left(|x_0| + \max_{-\varepsilon-\tau\le s\le 0}(|u_0(s)|)\right),\ \forall t\ge 0 \quad (5.3)$$

for arbitrary disturbance $d:\Re_+\to[-1,1]$ that belongs to one of the following classes:

1) The perturbation $d:\Re_+\to[-1,1]$ is an arbitrary measurable function, i.e., $d\in L^\infty(\Re_+;[-1,1])$ (Theorem 5.1).
2) The perturbation $d:\Re_+\to[-1,1]$ is constant (Corollary 5.3).

Clearly, (5.3) shows robust global exponential stability for the closed-loop system (5.1), (5.2). The estimation of $\varepsilon>0$ is given by explicit inequalities, which are derived by small-gain arguments. The inequalities can be used easily by the control practitioner in order to guarantee the successful application of the linear predictor feedback control strategy.

Arbitrary measurable perturbations $d\in L^\infty(\Re_+;[-1,1])$ of the delay can be considered for system (5.1). Indeed, we notice that this fact follows from the consideration of system (5.1) with

$$\begin{aligned}\dot u(t) &= k\exp(A\tau)(Ax(t)+Bu(t-\tau-\varepsilon d(t))-Bu(t-\tau))\\ &+kA\int\limits_{-\tau}^{0}\exp(-As)Bu(t+s)ds + kBu(t)\end{aligned} \quad (5.4)$$

Differential equation (5.4) is obtained by formally differentiating (5.2) with respect to $t\ge 0$. System (5.1) with (5.4) is a linear autonomous system described by Retarded Functional Differential Equations with disturbance $d\in L^\infty(\Re_+;[-1,1])$ and state space $\Re^n\times C^0([-\tau-\varepsilon,0];\Re^m)$ and satisfies all hypotheses (S1), (S2), (S3), (S4) in [2] for existence and uniqueness of solutions, for robustness of the equilibrium point and for the "Boundedness-Implies-Continuation" property. If we define the subspace

$$S := \left\{(x,u)\in\Re^n\times C^0([-\tau-\varepsilon,0];\Re^m) : u(0) = k\exp(A\tau)x + k\int\limits_{-\tau}^{0}\exp(-As)Bu(s)ds\right\} \quad (5.5)$$

then we are in a position to guarantee that S is a positively invariant set for system (5.1) with (5.4) (the dynamic implementation discussed in Section 2 of the present chapter). Moreover, every solution of (5.1) with (5.4) and initial condition $(x_0, u_0) \in S$ is a solution of (5.1), (5.2) and every solution of (5.1), (5.2) with initial condition $(x_0, u_0) \in S$ is a solution of (5.1) with (5.4). Finally, we notice that there exist constants $M, L > 0$ such that for every $\varepsilon > 0, x_0 \in \Re^n, u_0 \in C^0([-\tau - \varepsilon, 0]; \Re^m)$, $d \in L^\infty(\Re_+; [-1,1])$ with $u_0(0) = k\exp(A\tau)x_0 + k\int_{-\tau}^{0} \exp(-As)Bu_0(s)ds$ the unique solution $x \in C^0(\Re_+; \Re^n)$, $u \in C^0\big([-\tau - \varepsilon, +\infty); \Re^m\big)$ of system (5.1), (5.2) with initial conditions $x(0) = x_0$, $u(t) = u_0(t)$ for $t \in [-\tau - \varepsilon, 0]$ satisfies the exponential growth estimate:

$$|x(t)| + |u(t)| \le M\exp(Lt)\left(|x_0| + \max_{-\tau-\varepsilon \le s \le 0} |u_0(s)|\right), \quad \forall t \ge 0 \tag{5.6}$$

The existence of constants $M, L > 0$ satisfying estimate (5.6) follows directly from the integral representation of the solution of (5.1) with (5.4) and the Gronwall–Bellman Lemma.

Discontinuities of $u(t)$ cannot be handled in this framework: the initial condition $u_0 \in C^0([-\tau - \varepsilon, 0]; \Re^m)$ must be continuous and must satisfy (5.2) for $t = 0$. The reason for this regularity requirement is that the right hand side of (5.1) and (5.4) must be measurable in $t \ge 0$. Since the disturbance $d \in L^\infty(\Re_+; [-1,1])$ is measurable, the only way to guarantee this regularity requirement is to demand continuity of $u(t)$ (the composition of a continuous function with a measurable one gives a measurable function).

Our main result is the following theorem, which provides an explicit inequality for the magnitude $\varepsilon > 0$ of the delay perturbation under which robust global exponential stability for the closed-loop system (5.1), (5.2) is guaranteed.

Theorem 5.1 (Robustness to Time-Varying Perturbations of Small Magnitude but Unlimited Rate): *Consider system (5.1), (5.2), where* $0 < \varepsilon \le \tau$ *are constants,* $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $k \in \Re^{m \times n}$ *and* $(A + Bk)$ *is Hurwitz. There exist constants* $Q, \sigma > 0$ *such that for all* $d \in L^\infty(\Re_+; [-1,1])$, $x_0 \in \Re^n$, $u_0 \in C^0([-\tau - \varepsilon, 0]; \Re^m)$ *with* $u_0(0) = k\exp(A\tau)x_0 + k\int_{-\tau}^{0} \exp(-As)Bu_0(s)ds$ *the solution* $(x(t), u(t)) \in \Re^n \times \Re^m$ *of (5.1), (5.2) with initial condition* $x(0) = x_0, u(t) = u_0(t)$ *for* $t \in [-\tau - \varepsilon, 0]$ *satisfies estimate (5.3), provided that the following inequality holds:*

$$\Theta|\exp(A\tau)Bk|\left(\exp(|A + Bk|\varepsilon) - \exp(-\lambda\varepsilon)\right) < \lambda \tag{5.7}$$

where $\Theta, \lambda > 0$ *are constants satisfying* $|\exp((A + Bk)t)| \le \Theta\exp(-\lambda t)$ *for all* $t \ge 0$. *Moreover, if* $n = 1$ *then inequality (5.7) can be replaced by the inequality*

$$2|Bk|\exp(A\tau)(1 - \exp(-|A + Bk|\varepsilon)) < |A + Bk| \tag{5.8}$$

Remark 5.2: Since the left hand-side of inequality (5.7) becomes zero for $\varepsilon = 0$, by continuity, there exists $\varepsilon > 0$ (sufficiently small) such that inequality (5.7) holds. The least upper bound value for $\varepsilon > 0$ can be determined numerically.

For the case of constant perturbations of the delay, we obtain the following result.

Corollary 5.3: *Consider the system*

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t - \tilde{\tau}) \\ &x(t) \in \Re^n,\ u(t) \in \Re^m \end{aligned} \tag{5.9}$$

with (5.2), where $\tau, \tilde{\tau} \geq 0$ are constants, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $k \in \Re^{m\times n}$ and $(A + Bk)$ is Hurwitz. The zero solution of the closed-loop system is Globally Exponentially Stable if and only if all roots of the following equation:

$$\det(sI - (A + Bk) + \exp(A\tau)Bk(\exp(-\tau s) - \exp(-\tilde{\tau} s))) = 0 \tag{5.10}$$

have negative real parts.

Let $F_\tau \subseteq \Re_+$ denote the set of all $\tilde{\tau} \geq 0$, for which the roots of equation (5.10) have negative real parts, for fixed $\tau \geq 0$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and $k \in \Re^{m\times n}$. Theorem 5.1 guarantees that there exists $\varepsilon \in (0, \tau]$ such that $(\tau - \varepsilon, \tau + \varepsilon) \subseteq F_\tau$. However, Corollary 5.3 does not guarantee that $F_\tau \subseteq \Re_+$ is a convex set of $\Re_+$, i.e., an interval. Indeed, one cannot exclude the possibility of having two delays $\tau_2 > \tau_1$ in F_τ and a delay value $r \in (\tau_1, \tau_2)$ with $r \notin F_\tau$. The determination of the topological properties of the set $F_\tau \subseteq \Re_+$ is an open problem.

The following example illustrates inequality (5.8) and Corollary 5.3.

Example 5.4: Consider the scalar system

$$\dot{x}(t) = x(t) + u(t - 1 - \varepsilon d(t)) \ \text{ with } x(t) \in \Re,\ u(t) \in \Re, d(t) \in [-1, 1] \tag{5.11}$$

where $\varepsilon > 0$. For this example $A = 1 = B = r$ and we may choose $k = -p$, where $p > 1$. Theorem 5.1 guarantees that the closed-loop system (5.11) with

$$u(t) = -pe\,x(t) - p\int_0^1 \exp(s)u(t - s)ds \tag{5.12}$$

and $d \in L^\infty(\Re_+; [-1, 1])$ is robustly globally exponentially stable provided that $\varepsilon > 0$ satisfies

$$\varepsilon < \frac{1}{p - 1}\ln\left(\frac{2pe}{2pe - p + 1}\right) \tag{5.13}$$

In other words, system (5.11) with (5.12) is robustly globally exponentially stable provided that $\tau(t) \in (\tau_{\min}, \tau_{\max})$, where $\tau(t) = 1 + \varepsilon d(t)$, $\tau_{\min} = 1 - \varepsilon$, $\tau_{\max} = 1 + \varepsilon$ and $\varepsilon = \frac{1}{p-1} \ln\left(\frac{2pe}{2pe-p+1}\right)$, $p > 1$.

On the other hand, if constant delay perturbations are considered, then the roots of the equation $s + (p-1) + p\exp(1-\tau s) - p\exp(1-s) = 0$ must have negative real parts. For every value of $p > 1$ there exist delay values $0 < \tau_{\min} < 1 < \tau_{\max}$ such that if $\tau \in (\tau_{\min}, \tau_{\max})$ then all roots of the equation $s + (p-1) + p\exp(1-\tau s) - p\exp(1-s) = 0$ have negative real parts. In order to determine the range of values of τ for which the roots of the equation $s + (p-1) + p\exp(1-\tau s) - p\exp(1-s) = 0$ have negative real parts, we determine the curves in the parameter plane (the (p,τ) plane) composed of points for which there exists $\omega \in \Re$ such that $\omega j + (p-1) + p\exp(1-\tau\omega j) - p\exp(1-\omega j) = 0$, where j is the imaginary unit. The procedure that we follow for every $p > 1$, is:

(i) first we find numerically all solutions $\omega \in (0, 2pe)$ of the equation $(p-1)\cos(\omega) - \omega\sin(\omega) = \frac{(p-1)^2 + \omega^2}{2pe}$ (which is obtained from the equations $\cos(\omega\tau) - \cos(\omega) = -\frac{p-1}{pe}$ and $\sin(\omega\tau) - \sin(\omega) = \frac{\omega}{pe}$) ,

(ii) for every $\omega \in (0, 2pe)$ found from the previous step, we determine the unique solution $\phi \in \Re$ of the equations $\cos(\phi) = \cos(\omega) - \frac{p-1}{pe}$ and $\sin(\phi) = \sin(\omega) + \frac{\omega}{pe}$,

(iii) we find the positive solutions of $\tau = \frac{\phi + 2k\pi}{\omega}$, where k is an arbitrary integer, and

(iv) finally, we collect all positive values of $\tau = \frac{\phi + 2k\pi}{\omega}$ from the previous step and we find the highest value of τ that is less than 1 (this is $\tau_{\min}$) and the lowest value of τ that is higher than 1 (this is $\tau_{\max}$).

The results are shown in Figure 2.1 both for time-varying delay perturbations which are measurable (where $\tau_{\min} = 1 - \varepsilon$, $\tau_{\max} = 1 + \varepsilon$ and $\varepsilon = \frac{1}{p-1}\ln\left(\frac{2pe}{2pe-p+1}\right)$) and for constant delay perturbations.

The bounds for the magnitude of the delay perturbation obtained from (5.13) are about 50 % of the bounds obtained for constant perturbations. However, this is expected since (5.13) applies for time-varying delay perturbations which are measurable. Moreover, notice that the curves of $\tau_{\min}$ and $\tau_{\max}$ obtained for constant perturbations are not perfectly symmetric around 1. $\lhd$

The proof of Theorem 5.1 relies on the following technical theorem.

Theorem 5.5: *Consider the system*

$$\begin{array}{l} \dot{x}(t) = Ax(t) + q(t)C(x(t - \tau - \varepsilon d(t)) - x(t - \tau)) \\ x(t) \in \Re^n,\ d(t) \in [-1, 1],\ q(t) \in [-1, 1] \end{array}, \quad \text{for } t \geq 0, \text{ a.e.} \tag{5.14}$$

where $d \in L^\infty(\Re_+; [-1,1])$, $q \in L^\infty(\Re_+; [-1,1])$, $A, C \in \Re^{n \times n}$ *are constant matrices,* $\tau \geq \varepsilon \geq 0$ *are constants and* $A \in \Re^{n\times n}$ *is Hurwitz. Suppose that*

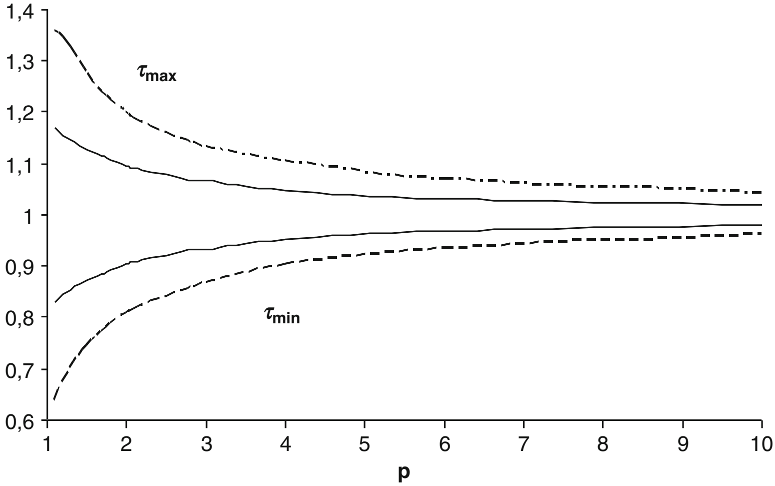

Figure 2.1 $\tau_{\min}$ and $\tau_{\max}$ for the closed-loop system (5.11) with (5.12). The solid line is for measurable delay perturbations as calculated by (5.13) and the dashed line is for constant delay perturbations.

$$\Theta|C|(\exp(|A|\,\varepsilon) - \exp(-\lambda\varepsilon)) < \lambda \tag{5.15}$$

where $\Theta, \lambda > 0$ *are constants satisfying* $|\exp(At)| \leq \Theta\exp(-\lambda t)$ *for all* $t \geq 0$. *Then there exist constants* $Q, \sigma > 0$ *such that for all* $d \in L^{\infty}(\Re_+; [-1,1])$, $q \in L^{\infty}(\Re_+; [-1,1])$, $x_0 \in C^0([-\tau-\varepsilon, 0]; \Re^n)$ *the solution* $x(t) \in \Re^n$ *of* (*5.14*) *with initial condition* $x(t) = x_0(t)$ *for* $t \in [-r-\varepsilon, 0]$ *that corresponds to inputs* $d \in L^{\infty}(\Re_+; [-1,1])$, $q \in L^{\infty}(\Re_+; [-1,1])$, *satisfies the following estimate*

$$|x(t)| \leq Q\exp(-\sigma t)\|x_0\|, \quad \forall t \geq 0 \tag{5.16}$$

Moreover, if $n = 1$ *then inequality* (*5.15*) *can be replaced by the inequality*

$$2|C|(1 - \exp(-|A|\,\varepsilon)) < |A| \tag{5.17}$$

The proof of Theorem 5.5 is based on a small-gain argument. The small-gain argument for the proof of Theorem 5.5 was inspired by the results contained in [11], but the methodology of the proof is essentially different from that followed in [11].

Finally, the proofs of Theorem 5.1 and Corollary 5.3 are based on the following result, which has its own interest.

Proposition 5.6: *Consider system* (*5.1*), (*5.2*), *where* $0 < \varepsilon \leq \tau$ *are constants,* $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $k \in \Re^{m\times n}$ *and* $(A + Bk)$ *is Hurwitz. Let* $\Omega \subseteq L^{\infty}(\Re_+; [-1,1])$ *be a set of time-varying inputs which is invariant under time translation, i.e., if* $d \in \Omega$ *then for every* $s > 0$ *the input* $\widetilde{d} : \Re_+ \to [-1,1]$ *defined by* $\widetilde{d}(t) = d(t+s)$ *for all* $t \geq 0$ *is in* $\Omega \subseteq L^{\infty}(\Re_+; [-1,1])$. *There exist constants* $Q, \sigma > 0$

such that for all $d \in \Omega$, $x_0 \in \Re^n$, $u_0 \in C^0([-\tau-\varepsilon,0];\Re^m)$ *with* $u_0(0) = k\exp(A\tau)x_0 + k\int\limits_{-\tau}^{0}\exp(-As)Bu_0(s)ds$ *the solution* $(x(t),u(t)) \in \Re^n \times \Re^m$ *of* (*5.1*), (*5.2*) *with initial condition* $x(0) = x_0, u(t) = u_0(t)$ *for* $t \in [-\tau-\varepsilon,0]$ *satisfies estimate* (*5.3*), *if and only if there exist constants* $\widetilde{Q}, \widetilde{\sigma} > 0$ *such that for all* $d \in \Omega$, $p_0 \in C^0([-\tau-\varepsilon,0];\Re^n)$, *the solution* $p(t) \in \Re^n$ *of*

$$\dot{p}(t) = (A+Bk)p(t) + \exp(A\tau)Bk(p(t-\tau-\varepsilon d(t)) - p(t-\tau)) \tag{5.18}$$

with initial condition $p(t) = p_0(t)$ *for* $t \in [-\tau-\varepsilon,0]$ *corresponding to input* $d \in \Omega$ *satisfies the following estimate*

$$|p(t)| \le \widetilde{Q}\exp(-\widetilde{\sigma}t)\max_{-\varepsilon-\tau \le s \le 0}(|p_0(s)|), \ \ \forall t \ge 0 \tag{5.19}$$

Remark 5.7: The proof of Theorem 5.1 relies on showing the exponential stability properties of the system (5.18), where

$$p(t) = \exp(A\tau)x(t) + \int\limits_{t}^{t+\tau}\exp(A(t+\tau-s))Bu(s-\tau)ds$$

is the "predictor state." The exponential stability properties of system (5.18) are guaranteed by means of Theorem 5.5. On the other hand Example 5.4 showed that the allowable magnitude for time-varying delay perturbations which are measurable is less than the magnitude obtained for constant perturbations from Corollary 5.3. We do not know if the conservatism is due to the small-gain approach (which is used for the proof of Theorem 5.5) or if the conservatism is due to the possibility that the stability analysis for delay perturbations depends not only on the magnitude of the perturbation but also on the rate of change of the perturbation. The latter implies that the rate of change of the perturbation may be important in stability analysis. Indeed, the recent work [35] has provided the construction of a Lyapunov functional for delay perturbations with constrained rate and results in [35] have showed that time-varying delays are more demanding than constant (uncertain) delays. Moreover, for time-varying delay perturbations with sufficiently small rate of change, there exists a function $\phi : \Re_+ \to [0,\tau+\varepsilon]$, which satisfies $\phi(t) = \tau + \varepsilon d(t+\phi(t))$ for all $t \ge 0$: these are exactly the class of delays considered in [36] for which the following linear time-varying predictor feedback can be applied for the stabilization of (5.1):

$$u(t)= k\exp(A\phi(t))x(t)+k\int_{t}^{t+\phi(t)}\exp(A(t+\phi(t)-s))Bu(s-\tau-\varepsilon d(s))ds,\ \text{for } t\geq 0 \tag{5.20}$$

provided that the function $d\colon \mathfrak{R}_+ \to [-1,1]$ is known.

We next provide the proof of Theorem 5.5.

Proof of Theorem 5.5: If (5.15) holds, then (by continuity) there exists $\sigma\in(0,\lambda)$ such that:

$$\exp(\sigma(\tau+\varepsilon))\frac{\Theta|C|}{\lambda-\sigma}(1-\exp(-(\lambda-\sigma)\varepsilon)+(\exp(|A|\varepsilon)-1))<1 \tag{5.21}$$

Let $d\in L^{\infty}(\mathfrak{R}_+;[-1,1])$, $q\in L^{\infty}(\mathfrak{R}_+;[-1,1])$, $x_0\in C^0([-\tau-\varepsilon,0];\mathfrak{R}^n)$ be arbitrary and consider the solution $x(t)\in\mathfrak{R}^n$ of (5.14) with initial condition $x(t)=x_0(t)$ for $t\in[-\tau-\varepsilon,0]$ that corresponds to inputs $d\in L^{\infty}(\mathfrak{R}_+;[-1,1])$, $q\in L^{\infty}(\mathfrak{R}_+;[-1,1])$. We define:

$$v(t)=x(t-\tau)-x(t-\tau-\varepsilon d(t)) \tag{5.22}$$

$$\|x\|_{[t_1,t_2]} := \max_{t_1\leq s\leq t_2}(\exp(\sigma s)|x(s)|),\quad \|v\|_{[t_1,t_2]} := \sup_{t_1\leq s\leq t_2}(\exp(\sigma s)|v(s)|) \tag{5.23}$$

for every $t_1\leq t_2$ and we distinguish the following cases:

Case 1: $d(t)\leq 0$. In this case the following formula holds for the solution of system (5.14) for almost all $t\geq\tau$:

$$-v(t)=(\exp(A\varepsilon|d(t)|)-I)x(t-\tau)-\int_{t-\tau}^{t-\tau-\varepsilon d(t)}\exp(A(t-\tau-\varepsilon d(t)-s))q(s)Cv(s)ds \tag{5.24}$$

Using the fact that $|\exp(At)|\leq\Theta\exp(-\lambda t)$ for all $t\geq 0$ and the fact that $|\exp(At)-I|\leq\exp(|A||t|)-1$, for all $t\in\mathfrak{R}$, we obtain from (5.24) for almost all $t\geq\tau$:

$$\begin{aligned}|v(t)|\exp(\sigma t)&\leq\exp(\sigma\tau)(\exp(|A|\varepsilon)-1)|x(t-\tau)|\exp(\sigma(t-\tau))\\&+\Theta\exp(\sigma\tau)\frac{1-\exp(-(\lambda-\sigma)\varepsilon)}{\lambda-\sigma}|C|\sup_{t-\tau\leq s\leq t-\tau+\varepsilon}(\exp(\sigma s)|v(s)|)\end{aligned} \tag{5.25}$$

Indeed, using the fact that $|\exp(At)|\leq\Theta\exp(-\lambda t)$ for all $t\geq 0$, we get:

$$\left|\int_{t-\tau}^{t-\tau-\varepsilon d(t)} \exp(A(t-\tau-\varepsilon d(t)-s))q(s)Cv(s)ds\right|$$

$$\leq \int_{t-\tau}^{t-\tau-\varepsilon d(t)} |\exp(A(t-\tau-\varepsilon d(t)-s))||q(s)||C||v(s)|ds$$

$$\leq |C| \int_{t-\tau}^{t-\tau-\varepsilon d(t)} |\exp(A(t-\tau-\varepsilon d(t)-s))||v(s)|ds$$

$$\leq |C|\Theta \exp(-\lambda(t-\tau-\varepsilon d(t))) \int_{t-\tau}^{t-\tau-\varepsilon d(t)} \exp((\lambda-\sigma)s)\exp(\sigma s)|v(s)|ds$$

$$\leq |C|\Theta \exp(-\lambda(t-\tau-\varepsilon d(t))) \int_{t-\tau}^{t-\tau-\varepsilon d(t)} \exp((\lambda-\sigma)s)ds\|v\|$$

$$\leq |C|\Theta \exp(-\lambda(t-\tau-\varepsilon d(t))) \frac{\exp\big((\lambda-\sigma)\big(t-\tau-\varepsilon d(t)\big)-\exp((\lambda-\sigma)(t-\tau))}{\lambda-\sigma}\|v\|$$

$$\leq |C|\Theta \exp(-\sigma(t-\tau))\exp(\sigma\varepsilon d(t)) \frac{1-\exp\big(-(\lambda-\sigma)\varepsilon|d(t)|\big)}{\lambda-\sigma}\|v\|$$

$$\leq |C|\Theta \exp(-\sigma(t-\tau)) \frac{1-\exp\big(-(\lambda-\sigma)\varepsilon\big)}{\lambda-\sigma}\|v\|$$

where $\|v\| = \sup_{t-\tau\leq s\leq t-\tau+\varepsilon} (\exp(\sigma s)|v(s)|)$. The above inequality in conjunction with (5.24) and the fact that $|\exp(At)-I| \leq \exp(|A||t|)-1$, for all $t\in\Re$, implies (5.25).

A direct consequence of definition (5.23) and inequality (5.25) is the following inequality which holds for all $t \geq \tau$:

$$\|v\|_{[\tau,t]} \leq \exp(\sigma\tau)\left((\exp(|A|\varepsilon)-1)\|x\|_{[0,t-\tau]} + \Theta\frac{1-\exp(-(\lambda-\sigma)\varepsilon)}{\lambda-\sigma}|C|\,\|v\|_{[0,t-\tau+\varepsilon]}\right) \tag{5.26}$$

Case 2: $d(t) \geq 0$. In this case the following formula holds for the solution of system (5.14) for almost all $t \geq \tau+\varepsilon$:

$$v(t) = (\exp(A\varepsilon|d(t)|)-I)x(t-\tau-\varepsilon d(t)) - \int_{t-\tau-\varepsilon d(t)}^{t-\tau} \exp(A(t-\tau-s))q(s)Cv(s)ds \tag{5.27}$$

Similarly as in the previous case, using (5.27), we show that the following inequality holds for all $t \geq \tau + \varepsilon$:

$$\begin{aligned} \|v\|_{[\tau+\varepsilon,t]} &\leq \exp(\sigma(\tau+\varepsilon))(\exp(|A|\varepsilon)-1)\|x\|_{[0,t-\tau]} \\ &+\Theta \exp(\sigma(\tau+\varepsilon))\frac{1-\exp(-(\lambda-\sigma)\varepsilon)}{\lambda-\sigma}|C|\,\|v\|_{[0,t-\tau]} \end{aligned} \tag{5.28}$$

Consequently, we conclude from (5.26) and (5.28) that the following inequality holds for all $t \geq \tau + \varepsilon$:

$$\begin{aligned} \|v\|_{[\tau+\varepsilon,t]} &\leq \exp(\sigma(\tau+\varepsilon))(\exp(|A|\varepsilon)-1)\|x\|_{[0,t-\tau]} \\ &+\Theta \exp(\sigma(\tau+\varepsilon))\frac{1-\exp(-(\lambda-\sigma)\varepsilon)}{\lambda-\sigma}|C|\,\|v\|_{[0,t-\tau+\varepsilon]} \end{aligned} \tag{5.29}$$

Using the fact that $|\exp(At)| \leq \Theta\exp(-\lambda t)$ for all $t \geq 0$ and the variations of constants formula $x(t) = \exp(At)x(0) - \int\limits_0^t \exp(A(t-s))q(s)Cv(s)ds$ for all $t \geq 0$, we obtain the estimate:

$$\begin{aligned} |x(t)|\exp(\sigma t) &\leq \Theta\exp(-(\lambda-\sigma)t)|x(0)|, \text{ for all } t \geq 0 \\ &+\Theta\frac{1-\exp(-(\lambda-\sigma)t)}{\lambda-\sigma}|C| \sup_{0\leq s\leq t}(\exp(\sigma s)|v(s)|) \end{aligned} \tag{5.30}$$

Definition (5.23) and inequality (5.30) in conjunction with the fact that $\sigma \in (0, \lambda)$ imply the following inequality:

$$\|x\|_{[0,t]} \leq \Theta|x(0)| + \frac{\Theta|C|}{\lambda-\sigma}\|v\|_{[0,t]}, \text{ for all } t \geq 0 \tag{5.31}$$

Combining (5.29) and (5.31), we obtain for all $t \geq \tau + \varepsilon$:

$$\begin{aligned} \|v\|_{[\tau+\varepsilon,t]} &\leq \exp(\sigma(\tau+\varepsilon))(\exp(|A|\varepsilon)-1)\Theta|x(0)| \\ &+\exp(\sigma(\tau+\varepsilon))\frac{\Theta|C|}{\lambda-\sigma}(1-\exp(-(\lambda-\sigma)\varepsilon)+(\exp(|A|\varepsilon)-1))\|v\|_{[0,t]} \end{aligned} \tag{5.32}$$

Inequality (5.21) in conjunction with (5.32), implies the following inequality for all $t \geq 0$:

$$\|v\|_{[0,t]} \leq \exp(\sigma(\tau+\varepsilon))\frac{\exp(|A|\varepsilon)-1}{1-\delta}\Theta|x(0)| + \|v\|_{[0,\tau+\varepsilon]} \tag{5.33}$$

where $\delta := \exp(\sigma(\tau+\varepsilon))\frac{\Theta|C|}{\lambda-\sigma}(1-\exp(-(\lambda-\sigma)\varepsilon)+(\exp(|A|\varepsilon)-1)) < 1$. Indeed, the equality $\|v\|_{[0,t]} = \max\left(\|v\|_{[0,\tau+\varepsilon]}, \|v\|_{[\tau+\varepsilon,t]}\right)$ allows us to consider two cases:

- Case 1: $\|v\|_{[0,t]} = \|v\|_{[0,\tau+\varepsilon]}$. In this case (5.32), in conjunction with the fact that $\delta := \exp(\sigma(\tau+\varepsilon))\frac{\Theta|C|}{\lambda-\sigma}(1-\exp(-(\lambda-\sigma)\varepsilon)+(\exp(|A|\varepsilon)-1))<1$, implies (5.33).
- Case 2: $\|v\|_{[0,t]} = \|v\|_{[\tau+\varepsilon,t]}$. In this case (5.32) implies $\|v\|_{[\tau+\varepsilon,t]} \leq \exp(\sigma(\tau+\varepsilon))\frac{\exp(|A|\varepsilon)-1}{1-\delta}\Theta|x(0)|$ and consequently (5.33) holds.

Inequality (5.33) in conjunction with (5.31) and the fact that there exist constants $L,M>0$ such that all solutions of (5.14) satisfy the estimate $|x(t)| \leq M\exp(Lt)\max_{-\tau-\varepsilon\leq s\leq 0}|x(s)|$ and in conjunction with the fact that $\|v\|_{[0,\tau+\varepsilon]} \leq 2\exp(\sigma(\tau+\varepsilon))\left(\|x\|_{[0,\tau+\varepsilon]} + \max_{-\tau-\varepsilon\leq s\leq 0}|x(s)|\right)$ (a direct consequence of definition (5.22)) imply that there exists a constant $Q>0$ such that estimate (5.16) holds.

If $n=1$ then $\Theta=1$ and $\lambda=|A|$. If (5.17) holds then (by continuity) there exists $\sigma\in(0,|A|)$ such that $\delta := \frac{|C|\exp(\sigma(\tau+\varepsilon))}{|A|-\sigma}(2-\exp(-(|A|-\sigma)\varepsilon)-\exp(-|A|\varepsilon))<1$. Moreover, inequalities (5.26) and (5.28) are replaced by the following inequalities:

$$\|v\|_{[\tau,t]} \leq \exp(\sigma\tau)(1-\exp(-|A|\varepsilon))\|x\|_{[0,t-\tau]} + \exp(\sigma\tau)\frac{1-\exp(-(|A|-\sigma)\varepsilon)}{|A|-\sigma}|C|\,\|v\|_{[0,t-\tau+\varepsilon]}$$

$$\|v\|_{[\tau+\varepsilon,t]} \leq \exp(\sigma(\tau+\varepsilon))(1-\exp(-|A|\varepsilon))\|x\|_{[0,t-\tau]} + \exp(\sigma(\tau+\varepsilon))\frac{1-\exp(-(|A|-\sigma)\varepsilon)}{|A|-\sigma}|C|\,\|v\|_{[0,t-\tau]}$$

It follows that inequality (5.29) is replaced by

$$\|v\|_{[\tau+\varepsilon,t]} \leq \exp(\sigma(\tau+\varepsilon))(1-\exp(-|A|\varepsilon))\|x\|_{[0,t-\tau]} + \exp(\sigma(\tau+\varepsilon))\frac{1-\exp(-(|A|-\sigma)\varepsilon)}{|A|-\sigma}|C|\,\|v\|_{[0,t-\tau+\varepsilon]} \tag{5.34}$$

Combining (5.34) with (5.31) and $\Theta=1$, $\lambda=|A|$, we obtain the estimate:

$$\|v\|_{[\tau+\varepsilon,t]} \leq \exp(\sigma(\tau+\varepsilon))(1-\exp(-|A|\varepsilon))|x(0)| + \frac{|C|\exp(\sigma(\tau+\varepsilon))}{|A|-\sigma}(2-\exp(-(|A|-\sigma)\varepsilon)-\exp(-|A|\varepsilon))\|v\|_{[0,t]}$$

Since $\delta := \frac{|C|\exp(\sigma(\tau+\varepsilon))}{|A|-\sigma}(2-\exp(-(|A|-\sigma)\varepsilon)-\exp(-|A|\varepsilon))<1$, the above inequality implies the inequality $\|v\|_{[0,t]} \leq \exp(\sigma(\tau+\varepsilon))\frac{1-\exp(-|A|\varepsilon)}{1-\delta}|x(0)| + \|v\|_{[0,\tau+\varepsilon]}$. The previous inequality in conjunction with (5.31) and the fact that there exist constants $L, M > 0$ such that all solutions of (5.14) satisfy the estimate $|x(t)| \leq M\exp(Lt) \max\limits_{-\tau-\varepsilon\leq s\leq 0}|x(s)|$ and in conjunction with the fact that $\|v\|_{[0,\tau+\varepsilon]} \leq 2\exp(\sigma(\tau+\varepsilon))\left(\|x\|_{[0,\tau+\varepsilon]} + \max\limits_{-\tau-\varepsilon\leq s\leq 0}|x(s)|\right)$ (a direct consequence of definition (5.22)) imply that there exists a constant $Q > 0$ such that estimate (5.16) holds. The proof is complete. ◁

We are now ready to provide the proof of Theorem 5.1.

Proof of Theorem 5.1: Proposition 5.6 with $\Omega = L^\infty(\mathfrak{R}_+;[-1,1])$ guarantees the conclusion of the theorem provided that system (5.18) is robustly globally exponentially stable. Theorem 5.5 with $A \in \mathfrak{R}^{n\times n}$ replaced by $(A+Bk)$ and $C = \exp(A\tau)Bk$ guarantees the robust global exponential stability of system (5.18) provided that (5.7) or (5.8) hold. The proof is complete. ◁

Next, we provide the proof of Corollary 5.3.

Proof of Corollary 5.3: Classical theory on linear delay systems guarantees that all roots of equation (5.10) have negative real parts if and only if the zero solution is Globally Exponentially Stable for the system:

$$\dot{p}(t) = (A+Bk)p(t) + \exp(A\tau)Bk(p(t-\widetilde{\tau}) - p(t-\tau)) \tag{5.35}$$

The rest of proof is a direct consequence of Proposition 5.6 with $\Omega \subset L^\infty(\mathfrak{R}_+;[-1,1])$ being the set of constant functions which are identically equal to 1 or -1 and $\widetilde{\tau} = \tau \pm \varepsilon$.

The proof is complete. ◁

Proof of Proposition 5.6: Let arbitrary $(x_0, u_0) \in S$ (where S is defined by (5.5)), $d \in \Omega$ and consider the solution $(x(t), u(t)) \in \mathfrak{R}^n \times \mathfrak{R}^m$ of (5.1), (5.2) with initial conditions $x(0) = x_0$, $u(t) = u_0(t)$ for $t \in [-\tau-\varepsilon, 0]$ corresponding to $d \in \Omega$. Define for all $t \geq 0$:

$$p(t) = \exp(A\tau)x(t) + \int_t^{t+\tau} \exp(A(t+\tau-s))Bu(s-\tau)ds \tag{5.36}$$

Notice that (5.2) and definition (5.36) implies that the following equality holds for all $t \geq 0$:

$$u(t) = kp(t), \text{ for all } t \geq 0 \tag{5.37}$$

By using (5.1) and definition (5.36), it follows that the following differential equation holds for almost all $t \geq 0$:

$$\begin{aligned}\dot{p}(t) &= \exp(A\tau)Ax(t) + \exp(A\tau)Bu(t-\tau-\varepsilon d(t)) \\ &+A(p(t)-\exp(A\tau)x(t)) + Bu(t) - \exp(A\tau)Bu(t-\tau)\end{aligned} \tag{5.38}$$

Using the identity $A\exp(A\tau) = \exp(A\tau)A$ and (5.37) it follows that the following differential equation holds for almost all $t \geq \tau + \varepsilon$:

$$\dot{p}(t) = (A+Bk)p(t) + \exp(A\tau)Bk(p(t-\tau-\varepsilon d(t)) - p(t-\tau)) \tag{5.39}$$

Since $\Omega \subseteq L^{\infty}(\mathfrak{R}_+;[-1,1])$ is a set of time-varying inputs which is invariant under time translation (which implies that the input defined by $\tilde{d}(t) = d(t+\tau+\varepsilon)$ is in $\Omega \subseteq L^{\infty}(\mathfrak{R}_+;[-1,1])$) and since (5.19) holds for certain constants $\tilde{Q}, \tilde{\sigma} > 0$, it follows that the following inequality holds:

$$|p(t)| \leq \tilde{Q}\exp(-\tilde{\sigma}(t-\tau-\varepsilon)) \max_{0 \leq s \leq \tau+\varepsilon} |p(s)|, \quad \forall t \geq \tau + \varepsilon \tag{5.40}$$

Using (5.40) in conjunction with (5.36), (5.37), (5.6) and the following equality:

$$x(t) = \exp(-A\tau)p(t) - \int_{t}^{t+\tau} \exp(A(t-s))Bkp(s-\tau)ds \tag{5.41}$$

which holds for all $t \geq \tau$ and is a direct consequence of (5.36) and (5.37), we obtain (5.3) with $\sigma := \tilde{\sigma}$ and

$$Q := M\exp(2L(\tau+\varepsilon) + 2|A|\tau)\exp(2\tilde{\sigma}(\tau+\varepsilon))\Big((1+\tau|Bk|+|k|)\tilde{Q}(1+\tau|B|)+1\Big).$$

Conversely, let arbitrary $p_0 \in C^0([-\tau-\varepsilon,0];\mathfrak{R}^n)$, $d \in \Omega$ and consider the solution $p(t) \in \mathfrak{R}^n$ of (5.18), (5.2) with initial condition $p(t) = p_0(t)$ for $t \in [-\tau-\varepsilon, 0]$ corresponding to $d \in \Omega$. Define $u_0(t) = kp_0(t)$ for $t \in [-\tau-\varepsilon,0]$ and $x_0 = \exp(-A\tau)\left(p(0) - \int_{-\tau}^{0} \exp(-As)Bu_0(s)ds\right)$. Notice that $(x_0,u_0) \in S$ (where S is defined by (5.5)). Therefore, the solution $(x(t),u(t)) \in \mathfrak{R}^n \times \mathfrak{R}^m$ of (5.1), (5.2) with initial condition $x(0) = x_0, u(t) = u_0(t)$ for $t \in [-\tau-\varepsilon,0]$ satisfies estimate (5.3) for certain constants $Q, \sigma > 0$. Notice that the solution $(x(t),u(t)) \in \mathfrak{R}^n \times \mathfrak{R}^m$ of (5.1), (5.2) with initial condition $x(0) = x_0, u(t) = u_0(t)$ for $t \in [-\tau-\varepsilon,0]$ satisfies (5.37) and (5.41) for all $t \geq 0$. Consequently, (5.36) holds for all $t \geq 0$. Estimate (5.19) with $\tilde{Q} := Q\exp(2|A|\tau)(1+\tau|B|)(1+\tau|Bk|\exp(|A|\tau)+|k|)$ and $\tilde{\sigma} := \sigma$ is a direct consequence of (5.3), (5.36) and the definitions $x_0 = \exp(-A\tau)\left(p(0) - \int_{-\tau}^{0} \exp(-As)Bu_0(s)ds\right)$ and $u_0(t) = kp_0(t)$ for $t \in [-\tau-\varepsilon, 0]$.

The proof is complete. $\lhd$

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