Chapter 2
Nonlinear Time-Delay Systems

2.1 Time-Delay Continuous Systems

Definition 2.1 For $I \subseteq \mathscr{R}, \Omega \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$, consider a vector function $\mathbf{f}: \Omega \times \Omega \times I \times \Lambda \to \mathscr{R}^n$ which is $C^r(r \geq 1)$-continuous, and there is an ordinary differential equation with time-delay in the form of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, t, \mathbf{p}) \text{ for } t \in I; \mathbf{x}, \mathbf{x}^\tau \in \Omega \text{ and } \mathbf{p} \in \Lambda \tag{2.1}$$

where $\dot{\mathbf{x}} = d\mathbf{x}/dt$ is differentiation with respect to time t, which is simply called the velocity vector of the state variables $\mathbf{x}$. $\mathbf{x}^\tau = \mathbf{x}(t-\tau)$, and τ is time-delay. With an initial condition of $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_0 - \tau) = \mathbf{x}_0^\tau$, the solution of Eq. (2.1) is given by

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) \text{ with} \\ \mathbf{x}(t_0) &= \mathbf{\Phi}(\mathbf{x}_0, t_0 - t_0, \mathbf{p}) \text{ and } \mathbf{x}(t_0 - \tau) = \mathbf{x}_0^\tau = \mathbf{\Phi}(\mathbf{x}_0, -\tau, \mathbf{p}) \end{aligned} \tag{2.2}$$

(i) The ordinary differential equation with the initial condition is called a *time-delay dynamical system*.
(ii) The vector function $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, t, \mathbf{p})$ is called a *time-delay vector field* on domain Ω.
(iii) The solution $\mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$ is called the *flow* of time-delay dynamical systems.
(iv) The corresponding projection of the solution $\mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$ on domain Ω is called the trajectory, phase curve, or orbit of time-delay dynamical system, defined as follows:

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$$\begin{aligned}\Gamma &= \{\mathbf{x}(t) \in \Omega | \mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) \text{ for } t \in I\} \subset \Omega. \\ \Gamma^\tau &= \{\mathbf{x}(t-\tau) \in \Omega | \mathbf{x}(t-\tau) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0 - \tau, \mathbf{p}) \text{ for } t \in I\} \subset \Omega\end{aligned} \tag{2.3}$$

Definition 2.2 If the vector field of the time-delay dynamical system in Eq. (2.1) is independent of time, such a system is called an autonomous time-delay dynamical system. Thus, Eq. (2.1) becomes

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \text{ for } t \in I \subseteq \mathscr{R}, \mathbf{x} \in \Omega \subseteq \mathscr{R}^n \text{ and } \mathbf{p} \in \Lambda \subseteq \mathscr{R}^m \tag{2.4}$$

Otherwise, such a system is called non-autonomous time-delay dynamical systems if the vector field of the dynamical system in Eq. (2.1) is dependent on time and state variables.

Definition 2.3 For a vector function $\mathbf{f} \in \mathscr{R}^n$ with $\mathbf{x} \in \mathscr{R}^n$, the operator norm of $\mathbf{f}$ is defined by

$$||\mathbf{f}|| = \sum\nolimits_{i=1}^{n} \max_{||\mathbf{x}|| \le 1, ||\mathbf{x}^\tau|| \le 1, t \in I} |f_i(\mathbf{x}, \mathbf{x}^\tau, t)|. \tag{2.5}$$

For $\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x}^\tau$ with an $n \times n$ matrix $\mathbf{A} = (a_{ij})_{n \times n}$ and $\mathbf{B} = (b_{ij})_{n \times n}$, the corresponding norms are defined by

$$||\mathbf{A}|| = \sum\nolimits_{i,j=1}^{n} |a_{ij}| \text{ and } ||\mathbf{B}|| = \sum\nolimits_{i,j=1}^{n} |b_{ij}|. \tag{2.6}$$

Definition 2.4 For a vector function $\mathbf{x}(t) = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathscr{R}^n$, the derivative and integral of $\mathbf{x}(t)$ are defined by

$$\begin{aligned}\frac{d\mathbf{x}(t)}{dt} &= \left(\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}, \cdots, \frac{dx_n(t)}{dt}\right)^{\mathrm{T}}, \\ \int \mathbf{x}(t)dt &= \left(\int x_1(t)dt, \int x_2(t)dt, \cdots, \int x_n(t)dt\right)^{\mathrm{T}}.\end{aligned} \tag{2.7}$$

For an $n \times n$ matrix $\mathbf{A} = (a_{ij})_{n \times n}$, the corresponding derivative and integral are defined by

$$\frac{d\mathbf{A}(t)}{dt} = \left(\frac{da_{ij}(t)}{dt}\right)_{n \times n} \text{ and } \int \mathbf{A}(t)dt = \left(\int a_{ij}(t)dt\right)_{n \times n}. \tag{2.8}$$

Definition 2.5 For $I \subseteq \mathscr{R}, \Omega \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$, the vector function $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, t, \mathbf{p})$ with $\mathbf{f} : \Omega \times \Omega \times I \times \Lambda \to \mathscr{R}^n$ is differentiable at $\mathbf{x}_0 \in \Omega$ if

$$\begin{aligned}\left.\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, t, \mathbf{p})}{\partial \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{x}_0^\tau, t, \mathbf{p})} &= \lim_{\Delta \mathbf{x} \to \mathbf{0}} \frac{\mathbf{f}(\mathbf{x}_0 + \Delta \mathbf{x}, \mathbf{x}_0^\tau, t, \mathbf{p}) - \mathbf{f}(\mathbf{x}_0, \mathbf{x}_0^\tau, t, \mathbf{p})}{\Delta \mathbf{x}}; \\ \left.\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, t, \mathbf{p})}{\partial \mathbf{x}^\tau}\right|_{(\mathbf{x}_0, \mathbf{x}_0^\tau, t, \mathbf{p})} &= \lim_{\Delta \mathbf{x}^\tau \to \mathbf{0}} \frac{\mathbf{f}(\mathbf{x}_0, \mathbf{x}_0^\tau + \Delta \mathbf{x}^\tau, t, \mathbf{p}) - \mathbf{f}(\mathbf{x}_0, \mathbf{x}_0^\tau, t, \mathbf{p})}{\Delta \mathbf{x}^\tau}.\end{aligned} \tag{2.9}$$

$\partial\mathbf{f}/\partial\mathbf{x}$ and $\partial\mathbf{f}/\partial\mathbf{x}^{\tau}$ are called the spatial derivatives of $\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},t,\mathbf{p})$ at $(\mathbf{x}_0,\mathbf{x}_0^{\tau})$, and the derivatives are given by the non-time-delay and time-delay Jacobian matrices

$$
\begin{aligned}
D_{\mathbf{x}}\mathbf{f} &\equiv \frac{\partial\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},t,\mathbf{p})}{\partial\mathbf{x}} = (\partial f_i/\partial x_j)_{n\times n},\\
D_{\mathbf{x}^{\tau}}\mathbf{f} &= \frac{\partial\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},t,\mathbf{p})}{\partial\mathbf{x}^{\tau}} = (\partial f_i/\partial x_j^{\tau})_{n\times n}.
\end{aligned}
\tag{2.10}
$$

Definition 2.6 For $I\subseteq\mathscr{R}$, $\Omega\subseteq\mathscr{R}^n$ and $\Lambda\subseteq\mathscr{R}^m$, consider a vector function $\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},t,\mathbf{p})$ with $\mathbf{f}:\Omega\times\Omega\times I\times\Lambda\to\mathscr{R}^n$, $t\in I$ and $\mathbf{x},\mathbf{x}^{\tau}\in\Omega$ and $\mathbf{p}\in\Lambda$. The time-delay vector function $\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},t,\mathbf{p})$ is said to be satisfied the Lipschitz condition with respect to $\mathbf{x}$ if

$$
||\mathbf{f}(\mathbf{x}_2,\mathbf{x}_2^{\tau},t,\mathbf{p}) - \mathbf{f}(\mathbf{x}_1,\mathbf{x}_1^{\tau},t,\mathbf{p})|| \le L||\mathbf{x}_2-\mathbf{x}_1|| + L^{\tau}||\mathbf{x}_2^{\tau}-\mathbf{x}_1^{\tau}|| \tag{2.11}
$$

with $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_1^{\tau},\mathbf{x}_2^{\tau}\in\Omega$ and the constants L and L^{τ} are called the Lipschitz constants.

Theorem 2.1 *Consider a time-delay dynamical system as*

$$
\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},t,\mathbf{p}) \text{ with } \mathbf{x}(t_0)=\mathbf{x}_0, \mathbf{x}(t_0-\tau)=\mathbf{x}_0^{\tau} \tag{2.12}
$$

with $t_0,t\in I=[t_1,t_2]$, $\mathbf{x}\in\Omega=\{\mathbf{x}|||\mathbf{x}-\mathbf{x}_0||\le d\}$ *and* $\mathbf{p}\in\Lambda$. *If the vector function* $\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},t,\mathbf{p})$ *is* C^r*-continuous (* $r\ge 1$*) in* $G=\Omega\times I\times\Lambda$*, then the dynamical system in* Eq. (2.12) *has one and only one solution* $\boldsymbol{\Phi}(\mathbf{x}_0,t-t_0,\mathbf{p})$ *for*

$$
|t-t_0|\le \min(t_2-t_1,d/M) \text{ with } M=\max_G ||\mathbf{f}||. \tag{2.13}
$$

Proof The proof of this theorem can be similar to non-time-delay system in the book by Coddington and Levinson (1955). ■

Theorem 2.2 (Gronwall) *Suppose there are continuous real-valued function* $g(t)\ge 0,\delta(t)\ge 0$ *and* $\beta(t)$. *If* $\delta(t)$ *is non-decreasing and* $\beta(t)\ge 0$ *with*

$$
g(t)\le \delta(t) + \int_{t_0}^{t}\beta(\eta)g(\eta)d\eta \tag{2.14}
$$

then

$$
g(t)\le \delta(t)\exp\left(\int_{t_0}^{t}\beta(\eta)d\eta\right). \tag{2.15}
$$

Proof For $t\in[t_0,t_1]$, consider

$$
G(t)=\delta(t)+\int_{t_0}^{t}\beta(\eta)g(\eta)d\eta
$$

The derivative of the foregoing equation gives

$$\dot{G}(t) = \dot{\delta}(t) + \beta(t)g(t)$$

and with we have

$$\frac{\dot{G}(t)}{G(t)} = \frac{\dot{\delta} + g(t)\beta(t)}{G(t)} \leq \frac{\dot{\delta}}{G(t)} + \frac{g(t)}{G(t)}\beta(t) \leq \frac{\dot{\delta}}{\delta} + \beta(t).$$

Integration gives

$$\ln\, G(t)|_{t_0}^{t} \leq \ \ln\, \delta|_{t_0}^{t} + \int_{t_0}^{t} \beta(\eta)d\eta.$$

So for $\delta(t_0) > 0$ with $G(t_0) = \delta(t_0)$,

$$\ln\, G(t) - \ln\, \delta(t_0) \leq \ \ln\, \delta(t) - \ln\, \delta(t_0) + \int_{t_0}^{t} \beta(\eta)d\eta.$$

In other words, for all $t \in [t_0, t_1]$

$$G(t) \leq \delta(t) \exp(\int_{t_0}^{t} \beta(\eta)d\eta).$$

Therefore, for all $t \in [t_0, t_1]$ with $g(t) \leq G(t)$,

$$g(t) \leq \delta(t) \exp(\int_{t_0}^{t} \beta(\eta)d\eta).$$

For $\delta(t_0) = 0$, there is a positive $\varepsilon > 0$, and $\delta(t) = \lim_{\varepsilon \to 0}(\delta(t) + \varepsilon)$

$$g(t) \leq (\delta(t) + \varepsilon) \exp(\int_{t_0}^{t} \beta(\eta)d\eta).$$

As $\varepsilon \to 0$, the forgoing equation satisfies Eq. (2.15).
This theorem is proved. ■

Theorem 2.3 *Consider a time-delay system as* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, t, \mathbf{p})$ *with* $\mathbf{x}(t_0) = \mathbf{x}_0$ *and* $\mathbf{x}(t_0 - \tau) = \mathbf{x}_0^{\tau}$ *in* Eq. (2.12) *with* $t_0, t \in I = [t_1, t_2], \mathbf{x} \in \Omega = \{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\| \leq d\}$ *and* $\mathbf{p} \in \Lambda$. *The vector function* $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, t, \mathbf{p})$ *is* C^r*-continuous (* $r \geq 1$*) in* $G = \Omega \times I \times \Lambda$, *if the solution of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, t, \mathbf{p})$ *with* $\mathbf{x}(t_0) = \mathbf{x}_0$ *and* $\mathbf{x}(t_0 - \tau) = \mathbf{x}_0^{\tau}$ *is* $\mathbf{x}(t)$ *on G and the solution of* $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{y}^{\tau}, t, \mathbf{p})$ *with* $\mathbf{y}(t_0) = \mathbf{y}_0$ *and* $\mathbf{y}(t_0 - \tau) = \mathbf{y}_0^{\tau}$ *is* $\mathbf{y}(t)$ *on G. For given* $\varepsilon, \varepsilon^{\tau} > 0$, *if* $\|\mathbf{x}_0 - \mathbf{y}_0\| \leq \varepsilon$ *and* $\|\mathbf{x}_0^{\tau} - \mathbf{y}_0^{\tau}\| \leq \varepsilon^{\tau}$, *then*

$$||\mathbf{x}(t) - \mathbf{y}(t)|| \leq (\varepsilon + \int_{t_0}^{t} L^{\tau}(\eta)||\mathbf{x}^{\tau} - \mathbf{y}^{\tau}||d\eta) \exp(\int_{t_0}^{t} L(\eta)d\eta) \text{ on } I \times \Lambda. \quad (2.16)$$

Proof From the method of successive approximations, with the local Lipschitz condition, the two initial value problems become

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, t, \mathbf{p})d\tau \text{ and } \mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{y}, \mathbf{y}^{\tau}, t, \mathbf{p})d\tau.$$

Thus,

$$\mathbf{x}(t) - \mathbf{y}(t) = \mathbf{x}_0 - \mathbf{y}_0 + \int_{t_0}^{t} (\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \eta, \mathbf{p}) - \mathbf{f}(\mathbf{y}, \mathbf{y}^{\tau}, \eta, \mathbf{p}))d\eta,$$

$$||\mathbf{x}(t) - \mathbf{y}(t)|| \leq ||\mathbf{x}_0 - \mathbf{y}_0|| + \int_{t_0}^{t} ||\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \eta, \mathbf{p}) - \mathbf{f}(\mathbf{y}, \mathbf{y}^{\tau}, \eta, \mathbf{p})||d\eta.$$

Using the local Lipschitz condition of

$$||\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, t, \mathbf{p}) - \mathbf{f}(\mathbf{y}, \mathbf{y}^{\tau}, t, \mathbf{p})|| \leq L(t)||\mathbf{x} - \mathbf{y}|| + L^{\tau}(t)||\mathbf{x}^{\tau} - \mathbf{y}^{\tau}||$$

gives

$$||\mathbf{x}(t) - \mathbf{y}(t)|| \leq \varepsilon + \int_{t_0}^{t} L(\eta)||\mathbf{x} - \mathbf{y}||d\eta + \int_{t_0}^{t} L^{\tau}(\eta)||\mathbf{x}^{\tau} - \mathbf{y}^{\tau}||d\eta$$

where $||\mathbf{x}_0 - \mathbf{y}_0|| < \varepsilon$. So the Gronwall's inequality gives

$$||\mathbf{x}(t) - \mathbf{y}(t)|| \leq (\varepsilon + \int_{t_0}^{t} L^{\tau}(\eta)||\mathbf{x}^{\tau} - \mathbf{y}^{\tau}||d\eta) \exp(\int_{t_0}^{t} L(\eta)d\eta).$$

This theorem is proved. ■

2.2 Equilibriums and Stability

Definition 2.7 Consider a metric space Ω and $\Omega_{\alpha} \subseteq \Omega$ ($\alpha = 1, 2, \cdots$).

(i) A map $\mathbf{h}$ is called a homeomorphism of Ω_{α} onto Ω_{β} ($\alpha, \beta = 1, 2, \cdots$) if the map $\mathbf{h} : \Omega_{\alpha} \to \Omega_{\beta}$ is continuous and one to one, and $\mathbf{h}^{-1} : \Omega_{\beta} \to \Omega_{\alpha}$ is continuous.

(ii) Two set Ω_{α} and Ω_{β} are homeomorphic or topologically equivalent if there is a homeomorphism of Ω_{α} onto Ω_{β}.

Definition 2.8 A connected, metric space Ω with an open cover $\{\Omega_\alpha\}$ (i.e., $\Omega = \cup_\alpha \Omega_\alpha$) is called an n-dimensional, $C^r (r \geq 1)$ differentiable manifold if the following properties exist.

(i) There is an open unit ball $B = \{\mathbf{x} \in \mathscr{R}^n \| \|\mathbf{x}\| < 1\}$.
(ii) For all α, there is an homeomorphism $\mathbf{h}_\alpha : \Omega_\alpha \to B$.
(iii) If $\mathbf{h}_\alpha : \Omega_\alpha \to B$ and $\mathbf{h}_\beta : \Omega_\beta \to B$ are homeomorphisms for $\Omega_\alpha \cap \Omega_\beta \neq \varnothing$, then there is a C^r-differentiable map $\mathbf{h} = \mathbf{h}_\alpha \circ \mathbf{h}_\beta^{-1}$ for $\mathbf{h}_\alpha(\Omega_\alpha \cap \Omega_\beta) \subset \mathscr{R}^n$ and $\mathbf{h}_\beta(\Omega_\alpha \cap \Omega_\beta) \subset \mathscr{R}^n$ with

$$\mathbf{h} : \mathbf{h}_\beta(\Omega_\alpha \cap \Omega_\beta) \to \mathbf{h}_\alpha(\Omega_\alpha \cap \Omega_\beta), \tag{2.17}$$

and for all $\mathbf{x} \in \mathbf{h}_\beta(\Omega_\alpha \cap \Omega_\beta)$, the Jacobian determinant $\det D\mathbf{h}(\mathbf{x}) \neq 0$.
The manifold Ω is called to be analytic if the maps $\mathbf{h} = \mathbf{h}_\alpha \circ \mathbf{h}_\beta^{-1}$ are analytic.

Definition 2.9 Consider a nonlinear time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4). A point $\mathbf{x}^* = \mathbf{x}^{\tau *} \in \Omega$ is called an equilibrium point or critical point of a nonlinear time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ if

$$\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) = \mathbf{0} \text{ and } \mathbf{x}^* = \mathbf{x}^{\tau *} \tag{2.18}$$

The linearized system of the time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) at the equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is given by

$$\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})\mathbf{y}^\tau \tag{2.19}$$

where $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$.

Definition 2.10 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$. The linearized system of the nonlinear time-delay system at the equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is $\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})\mathbf{y}^\tau$ ($\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau *}$) in Eq. (2.19). The matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})$ possesses n eigenvalues λ_k $(k = 1, 2, \cdots n)$. Set $N = \{1, 2, \cdots, n\}, N_i = \{i_1, i_2, \cdots, i_{n_i}\} \cup \varnothing$ with $i_j \in N$ $(j = 1, 2, \cdots, n_i; i = 1, 2, 3)$ and $\Sigma_{i=1}^3 n_i = n$. $\cup_{i=1}^3 N_i = N$ and $N_i \cap N_l = \varnothing$ $(l \neq i)$. $N_i = \varnothing$ if $n_i = 0$. The corresponding vectors for the negative, positive, and zero eigenvalues of $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ are $\{\mathbf{u}_k\}$ $(k \in N_i, i = 1, 2, 3)$, respectively. The stable, unstable, and invariant subspaces of the linearized time-delay system in Eq. (2.19) are defined as follows:

$$\begin{aligned}
\mathcal{E}^{\mathrm{s}} &= span\left\{\mathbf{u}_k \left| \begin{array}{c} \lambda_k < 0, k_1 \subseteq N \cup \varnothing \\ (D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) - \lambda_k \mathbf{I})\mathbf{u}_k = \mathbf{0}, \end{array}\right.\right\}; \\
\mathcal{E}^{\mathrm{u}} &= span\left\{\mathbf{u}_k \left| \begin{array}{c} \lambda_k > 0, k \in N_2 \subseteq N \cup \varnothing \\ (D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) - \lambda_k \mathbf{I})\mathbf{u}_k = \mathbf{0}, \end{array}\right.\right\}; \\
\mathcal{E}^{\mathrm{i}} &= span\left\{\mathbf{u}_k \left| \begin{array}{c} \lambda_k = 0, k \in N_3 \subseteq N \cup \varnothing \\ (D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) - \lambda_k \mathbf{I})\mathbf{u}_k = \mathbf{0}, \end{array}\right.\right\}.
\end{aligned} \tag{2.20}$$

Definition 2.11 Consider a $2n$-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The linearized system of the time-delay system at the equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is $\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y}^\tau$ ($\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau*}$) in Eq. (2.19). The matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ possesses complex eigenvalues $\alpha_k \pm \mathbf{i}\beta_k$ with eigenvectors $\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k$ ($k \in \{1, 2, \cdots, n\}$), and the base of vector is

$$\mathbf{B} = \{\mathbf{u}_1, \mathbf{v}_1, \cdots, \mathbf{u}_k, \mathbf{v}_k, \cdots, \mathbf{u}_n, \mathbf{v}_n\}. \tag{2.21}$$

The stable, unstable, center subspaces of Eq. (2.19) are linear subspaces spanned by $\{\mathbf{u}_k, \mathbf{v}_k\}(k \in N_i, i = 1, 2, 3)$, respectively. $N_i = \{i_1, i_2, \cdots, i_{n_i}\} \cup \varnothing \subseteq N \cup \varnothing$ and $N = \{1, 2, \cdots, n\}$ with $i_j \in N$ ($j = 1, 2, \cdots, n_i$) and $\Sigma_{i=1}^3 n_i = n$. $\cup_{i=1}^3 N_i = N$ and $N_i \cap N_l = \varnothing (l \neq i)$. $N_i = \varnothing$ if $n_i = 0$. The stable, unstable, center subspaces of the linearized time-delay system in Eq. (2.19) are defined as follows:

$$\begin{aligned}
\mathcal{E}^{\mathrm{s}} &= span\left\{ (\mathbf{u}_k, \mathbf{v}_k) \left| \begin{array}{c} \alpha_k < 0, \beta_k \neq 0, k \in N_1 \subseteq \{1, 2, \cdots, n\} \cup \varnothing \\ \begin{pmatrix} D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) \\ + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) \\ -(\alpha_k \pm \mathbf{i}\beta_k)\mathbf{I} \end{pmatrix} (\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k) = \mathbf{0} \end{array} \right.\right\}; \\
\mathcal{E}^{\mathrm{u}} &= span\left\{ (\mathbf{u}_k, \mathbf{v}_k) \left| \begin{array}{c} \alpha_k > 0, \beta_k \neq 0, k \in N_2 \subseteq \{1, 2, \cdots, n\} \cup \varnothing \\ \begin{pmatrix} D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) \\ + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) \\ -(\alpha_k \pm \mathbf{i}\beta_k)\mathbf{I} \end{pmatrix} (\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k) = \mathbf{0} \end{array} \right.\right\}; \\
\mathcal{E}^{\mathrm{c}} &= span\left\{ (\mathbf{u}_k, \mathbf{v}_k) \left| \begin{array}{c} \alpha_k = 0, \beta_k \neq 0, k \in N_3 \subseteq \{1, 2, \cdots, n\} \cup \varnothing \\ \begin{pmatrix} D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) \\ + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}\mathbf{p}) \\ -(\alpha_k \pm \mathbf{i}\beta_k)\mathbf{I} \end{pmatrix} (\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k) = \mathbf{0} \end{array} \right.\right\}.
\end{aligned} \tag{2.22}$$

Theorem 2.4 *Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in* Eq. (2.4) *with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The linearized system of the nonlinear time-delay system at the equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$*

is $\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y}^\tau$ ($\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau*}$) in Eq. (2.19). *The eigenspace of* $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ *(i.e.,* $\mathscr{E} \subseteq \mathscr{R}^n$*) in the linearized time-delay system is expressed by direct sum of three subspaces*

$$\mathscr{E} = \mathscr{E}^s \oplus \mathscr{E}^u \oplus \mathscr{E}^c \tag{2.23}$$

where $\mathscr{E}^s, \mathscr{E}^u$ and $\mathscr{E}^c$ *are the stable, unstable, and center spaces* $\mathscr{E}^s, \mathscr{E}^u$ *and* $\mathscr{E}^c$*, respectively.*

Proof This proof is similar to the linear time-delay systems. ■

Definition 2.12 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ and $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) = \mathbf{\Phi}_t(\mathbf{x}_0)$. The linearized system of the time-delay system at the equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is $\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y}^\tau$ ($\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau*}$) in Eq. (2.19). Suppose there is a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ as $U(\mathbf{x}^*) \subset \Omega$, and in the neighborhood

$$\begin{aligned} &\lim_{||\mathbf{y}|| \to 0} \frac{||\mathbf{f}(\mathbf{x}^* + \mathbf{y}, \mathbf{x}^{*\tau}, \mathbf{p}) - D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{*\tau}, \mathbf{p})\mathbf{y}||}{||\mathbf{y}||} = 0, \\ &\lim_{||\mathbf{y}^\tau|| \to 0} \frac{||\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{*\tau} + \mathbf{y}^\tau, \mathbf{p}) - D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{*\tau}, \mathbf{p})\mathbf{y}^\tau||}{||\mathbf{y}^\tau||} = 0. \end{aligned} \tag{2.24}$$

(i) A C^rinvariant manifold

$$\mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*) = \left\{ \mathbf{x}, \mathbf{x}^\tau \in U(\mathbf{x}^*) \middle| \begin{array}{l} \lim\limits_{t \to \infty} \mathbf{x}(t) = \mathbf{x}^*, \mathbf{x}(t) \in U(\mathbf{x}^*) \\ \lim\limits_{t \to \infty} \mathbf{x}(t - \tau) = \mathbf{x}^*, \mathbf{x}(t - \tau) \in U(\mathbf{x}^*) \\ \text{for all } t \geq 0 \end{array} \right\}, \tag{2.25}$$

is called the local stable manifold of $\mathbf{x}^*$, and the corresponding global, stable manifold is defined as follows:

$$\mathscr{S}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*) = \cup_{t \leq 0} \mathbf{\Phi}_t(\mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)). \tag{2.26}$$

(ii) A C^r invariant manifold

$$\mathrm{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*) = \left\{ \mathbf{x}, \mathbf{x}^\tau \in U(\mathbf{x}^*) \middle| \begin{array}{l} \lim\limits_{t \to -\infty} \mathbf{x}(t) = \mathbf{x}^*, \mathbf{x}(t) \in U(\mathbf{x}^*) \\ \lim\limits_{t \to -\infty} \mathbf{x}(t - \tau) = \mathbf{x}^*, \mathbf{x}(t - \tau) \in U(\mathbf{x}^*) \\ \text{for all } t \leq 0 \end{array} \right\} \tag{2.27}$$

is called the unstable manifold of $\mathbf{x}^*$, and the corresponding global, unstable manifolds are defined as follows:

$$\mathscr{U}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*) = \cup_{t \geq 0} \mathbf{\Phi}_t(\mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)) \tag{2.28}$$

(iii) A C^{r-1} invariant manifold $\mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ is called the center manifolds of $\mathbf{x}^*$ if $\mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ possesses the same dimensions of $\mathscr{E}^c$, and the tangential spaces of $\mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ are identical to $\mathscr{E}^c$.

The stable and unstable manifolds are unique, but the center manifold is not unique. If the nonlinear time-delay vector field $\mathbf{f}$ is C^∞-continuous, then a C^r center manifold can be found for any $r < \infty$.

Theorem 2.5 *Consider an n-dimensional, autonomous, nonlinear, time-delay dynamical system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ *in* Eq. (2.4) *with a hyperbolic equilibrium point* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *and* $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ *is* C^r *(* $r \geq 1$*)-continuous in a neighborhood of the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$. *The corresponding solution is* $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) = \mathbf{\Phi}_t(\mathbf{x}_0)$. *The linearized system of the time-delay system at the equilibrium point* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *is* $\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y}^\tau$ $(\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau*})$ *in* Eq. (2.19)*. Suppose there is a neighborhood of the hyperbolic equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *as* $U(\mathbf{x}^*) \subset \Omega$. *If the homeomorphism between the local invariant subspace* $E(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*) \subset U(\mathbf{x}^*)$ *under the flow* $\mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$ *of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ *in* Eq. (2.4) *and the eigenspace* $\mathscr{E}$ *of the linearized system exists with the condition in* Eq. (2.24)*, the local invariant subspace is decomposed by*

$$E(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*) = \mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*) \oplus \mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*). \tag{2.29}$$

(a) *The local stable invariant manifolds* $\mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ *possess the following properties:*

   (i) *for* $\mathbf{x}^* \in \mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$, $\mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ *possesses the same dimension of* $\mathscr{E}^s$ *and the tangential space of* $\mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ *is identical to* $\mathscr{E}^s$;

   (ii) *for* $\mathbf{x}_0 \in \mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*), \mathbf{x}(t), \mathbf{x}(t - \tau) \in \mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ *for all time* $t \geq t_0$ *and* $\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}^*$ *and* $\lim_{t \to \infty} \mathbf{x}(t - \tau) = \mathbf{x}^*$;

   (iii) *for* $\mathbf{x}_0 \notin \mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*), ||\mathbf{x} - \mathbf{x}^*|| \geq \delta$ *for* $\delta > 0$ *with* $t \geq t_1 \geq t_0$ *and* $||\mathbf{x}(t - \tau) - \mathbf{x}^*|| \geq \delta$ *for* $\delta > 0$ *with* $t \geq t_2 \geq t_0$.

(b) *The local unstable invariant manifold* $\mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ *possesses the following properties:*

   (i) *for* $\mathbf{x}^* \in \mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$, $\mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ *possesses the same dimension of* $\mathscr{E}^u$ *and the tangential space of* $\mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ *is identical to* $\mathscr{E}^u$;

   (ii) *for* $\mathbf{x}_0 \in \mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*), \mathbf{x}(t), \mathbf{x}(t - \tau) \in \mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{x}^*)$ *for all time* $t \leq t_0$ *and* $\lim_{t \to -\infty} \mathbf{x}(t) = \mathbf{x}^*$ *and* $\lim_{t \to -\infty} \mathbf{x}(t - \tau) = \mathbf{x}^*$;

   (iii) *for* $\mathbf{x}_0 \notin \mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^*), ||\mathbf{x} - \mathbf{x}^*|| \geq \delta$ *for* $\delta > 0$ *with* $t \leq t_1 \leq t_0$ *and* $||\mathbf{x}(t - \tau) - \mathbf{x}^*|| \geq \delta$ *for* $\delta > 0$ *with* $t \leq t_2 \leq t_0$.

Proof The proof for stable and unstable manifolds is similar to the non-time-delay system in Hartman (1964). ■

Theorem 2.6 *Consider an n-dimensional, autonomous, nonlinear, time-delay system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ *in* Eq. (2.4) *with a hyperbolic equilibrium point* $\mathbf{x}^* = \mathbf{x}^{\tau *}$ *and* $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ *is* C^r *(* $r \geq 1$*)-continuous in a neighborhood of the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau *}$. *The corresponding solution is* $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) = \mathbf{\Phi}_t(\mathbf{x}_0)$. *The linearized system of the nonlinear time-delay system at the equilibrium point* $\mathbf{x}^* = \mathbf{x}^{\tau *}$ *is* $\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})\mathbf{y}^{\tau}$ $(\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^{\tau} = \mathbf{x}^{\tau} - \mathbf{x}^{\tau *})$ *in* Eq. (2.19). *If the homeomorphism between the local invariant subspace* $E(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{x}^*) \subset U(\mathbf{x}^*)$ *under the flow* $\mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$ *of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ *in* Eq. (2.4) *and the eigenspace* $\mathscr{E}$ *of the linearized system exists with the condition in* Eq. (2.24)*, in addition to the local stable and unstable invariant manifolds, there is a* C^{r-1} *center manifold* $\mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{x}^*)$. *The center manifold possesses the same dimension of* $\mathscr{E}^c$ *for* $\mathbf{x}^* \in \mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{x}^*)$*, and the tangential space of* $\mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{x}^*)$ *is identical to* $\mathscr{E}^c$. *Thus, the local invariant subspace is decomposed by*

$$E(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{x}^*) = \mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{x}^*) \oplus \mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{x}^*) \oplus \mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{x}^*). \tag{2.30}$$

Proof The proof for stable and unstable manifolds is similar to the non-time-delay system in Hartman (1964). The proof for center manifold is similar to non-time-delay systems in Marsden and McCracken (1976) or Carr (1981). ■

Definition 2.13 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ and $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$.

(i) The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$is stable if all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\mathbf{x}_0 \in U_{\delta}(\mathbf{x}^*)$ where $U_{\delta}(\mathbf{x}^*) = \{\mathbf{x} | ||\mathbf{x} - \mathbf{x}^*|| < \delta\}$ and $t \geq t_0$,

$$\mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) \in U_{\varepsilon}(\mathbf{x}^*) \tag{2.31}$$

(ii) The equilibrium $\mathbf{x}^*$ is unstable if it is not stable or if all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\mathbf{x}_0 \in U_{\delta}(\mathbf{x}^*)$ where $U_{\delta}(\mathbf{x}^*) = \{\mathbf{x} | ||\mathbf{x} - \mathbf{x}^*|| < \delta\}$ and $t \geq t_1 > t_0$,

$$\mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) \notin U_{\varepsilon}(\mathbf{x}^*) \tag{2.32}$$

(iii) The equilibrium $\mathbf{x}^*$ is asymptotically stable if all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\mathbf{x}_0 \in U_{\delta}(\mathbf{x}^*)$ where $U_{\delta}(\mathbf{x}^*) = \{\mathbf{x} | ||\mathbf{x} - \mathbf{x}^*|| < \delta\}$ and $t \geq t_0$,

$$\lim_{t \to \infty} \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) = \mathbf{x}^* \tag{2.33}$$

(iv) The equilibrium $\mathbf{x}^*$ is asymptotically unstable if all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\mathbf{x}_0 \in U_{\delta}(\mathbf{x}^*)$ where $U_{\delta}(\mathbf{x}^*) = \{\mathbf{x} | ||\mathbf{x} - \mathbf{x}^*|| < \delta\}$ and $t \leq t_0$,

$$\lim_{t\to-\infty} \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p}) = \mathbf{x}^* \tag{2.34}$$

Definition 2.14 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$. Suppose there is a neighborhood of the equilibrium $\mathbf{x}^*$ as $U(\mathbf{x}^*) \subset \Omega$, then $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in a neighborhood of $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. For a linearized time-delay system in Eq. (2.19), consider a real eigenvalue λ_k of matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ $(k \in N = \{1, 2, \cdots, n\})$ with an eigenvector $\mathbf{v}_k$. For $\mathbf{y}^{(k)} = c^{(k)}\mathbf{v}_k$, $\dot{\mathbf{y}}^{(k)} = \dot{c}^{(k)}\mathbf{v}_k = \lambda_k c^{(k)}\mathbf{v}_k$, thus $\dot{c}^{(k)} = \lambda_k c^{(k)}$.

(i) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is stable if

$$\lim_{t\to\infty} c^{(k)} = \lim_{t\to\infty} c_0^{(k)} e^{\lambda_k t} = 0 \text{ for } \lambda_k < 0. \tag{2.35}$$

(ii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is unstable if

$$\lim_{t\to\infty} |c^{(k)}| = \lim_{t\to\infty} |c_0^{(k)} e^{\lambda_k t}| = \infty \text{ for } \lambda_k > 0. \tag{2.36}$$

(iii) $\mathbf{x}^{(i)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is uncertain (critical) if

$$\lim_{t\to\infty} c^{(k)} = \lim_{t\to\infty} e^{\lambda_k t} c_0^{(k)} = c_0^{(k)} \text{ for } \lambda_k = 0. \tag{2.37}$$

Definition 2.15 Consider a $2n$-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$. Suppose there is a neighborhood of the equilibrium $\mathbf{x}^*$ as $U(\mathbf{x}^*) \subset \Omega$, then $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The corresponding solution in $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. For a linearized time-delay system in Eq. (2.19), consider a pair of complex eigenvalues $\alpha_k \pm \mathbf{i}\beta_k$ $(k \in N = \{1, 2, \cdots, n\}$, $\mathbf{i} = \sqrt{-1})$ of matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k$. On the invariant plane of $(\mathbf{u}_k, \mathbf{v}_k)$, consider $\mathbf{y}^{(k)} = \mathbf{y}_+^{(k)} + \mathbf{y}_-^{(k)}$ with

$$\mathbf{y}^{(k)} = c^{(k)}\mathbf{u}_k + d^{(k)}\mathbf{v}_k, \dot{\mathbf{y}}^{(k)} = \dot{c}^{(k)}\mathbf{u}_k + \dot{d}^{(k)}\mathbf{v}_k \tag{2.38}$$

Thus, $\mathbf{c}^{(k)} = (c^{(k)}, d^{(k)})^{\mathrm{T}}$ with

$$\dot{\mathbf{c}}^{(k)} = \mathbf{E}_k \mathbf{c}^{(k)} \Rightarrow \mathbf{c}^{(k)} = e^{\alpha_k t}\mathbf{B}_k \mathbf{c}_0^{(k)} \tag{2.39}$$

where

$$\mathbf{E}_k = \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix} \text{ and } \mathbf{B}_k = \begin{bmatrix} \cos\beta_k t & \sin\beta_k t \\ -\sin\beta_k t & \cos\beta_k t \end{bmatrix}. \tag{2.40}$$

(i) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally stable if

$$\lim_{t\to\infty} ||\mathbf{c}^{(k)}|| = \lim_{t\to\infty} e^{\alpha_k t}||\mathbf{B}_k|| \times ||\mathbf{c}_0^{(k)}|| = 0 \text{ for } \mathrm{Re}\,\lambda_k = \alpha_k < 0. \tag{2.41}$$

(ii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally unstable if

$$\lim_{t\to\infty} ||\mathbf{c}^{(k)}|| = \lim_{t\to\infty} e^{\alpha_k t}||\mathbf{B}_k|| \times ||\mathbf{c}_0^{(k)}|| = \infty \text{ for } \mathrm{Re}\,\lambda_k = \alpha_k > 0. \tag{2.42}$$

(iii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is on the invariant circle if

$$\lim_{t\to\infty} ||\mathbf{c}^{(k)}|| = \lim_{t\to\infty} e^{\alpha_k t}||\mathbf{B}_k|| \times ||\mathbf{c}_0^{(k)}|| = ||\mathbf{c}_0^{(k)}|| \text{ for } \mathrm{Re}\,\lambda_k = \alpha_k = 0. \tag{2.43}$$

(iv) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is degenerate in the direction of $\mathbf{u}_k$ if $\mathrm{Im}\,\lambda_k = 0$.

Definition 2.16 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$. Suppose there is a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ as $U(\mathbf{x}^*) \subset \Omega$, then $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is $C^r(r \geq 1)$-continuous in a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. The linearized time-delay dynamical system at the equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is $\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y}^\tau$ ($\mathbf{y} = \mathbf{x} - \mathbf{x}^*$, and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau*}$) in Eq. (2.19).

(i) The equilibrium $\mathbf{x}^*$ is said *a hyperbolic equilibrium* if none of the eigenvalues of $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k\tau}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ is zero real part (i.e., $\mathrm{Re}\,\lambda_k \neq 0, k = 1, 2, \cdots, n$).

(ii) The equilibrium $\mathbf{x}^*$ is said *a sink* if all of the eigenvalues of $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k\tau}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ have negative real parts (i.e., $\mathrm{Re}\lambda_k < 0,\ k = 1, 2, \cdots, n$).

(iii) The equilibrium $\mathbf{x}^*$ is said *a source* if all of the eigenvalues of $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k\tau}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ have positive real parts (i.e., $\mathrm{Re}\lambda_k > 0, k = 1, 2, \cdots, n$).

(iv) The equilibrium $\mathbf{x}^*$ is said *a saddle* if it is a hyperbolic equilibrium and $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k\tau}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ have at least one eigenvalue with a positive real part [i.e., $\mathrm{Re}\lambda_j > 0$ ($j \in \{1, 2, \cdots, n\}$)] and one with a negative real part [i.e., $\mathrm{Re}\lambda_k < 0$ ($k \in \{1, 2, \cdots, n\}$)].

(v) The equilibrium $\mathbf{x}^*$ is called a center if all of the eigenvalues of $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ have zero real parts [i.e., $\mathrm{Re}\lambda_j = 0$ $(j = 1, 2, \cdots, n)$] with distinct eigenvalues.

(vi) The equilibrium $\mathbf{x}^*$ is called a stable node if all of the eigenvalues of $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ are real [i.e., $\lambda_k < 0$ $(k = 1, 2, \cdots n)$].

(vii) The equilibrium $\mathbf{x}^*$ is called an unstable node if all of the eigenvalues of $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ are real [i.e., $\lambda_k > 0$ $(k = 1, 2, \cdots n)$].

(viii) The equilibrium $\mathbf{x}^*$ is called a degenerate case if all of the eigenvalues of $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ are zero [i.e., $\lambda_k = 0$ $(k = 1, 2, \cdots n)$].

As in Luo (2012), the generalized stability and bifurcation of flows in linearized, nonlinear, time-delay systems in Eq. (2.4) will be discussed as follows.

Definition 2.17 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$. Suppose there is a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ as $U(\mathbf{x}^*) \subset \Omega$, and in the neighborhood $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is C^r $(r \geq 1)$-continuous and Eq. (2.24) holds. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. From Eq. (2.19), the matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ possesses n eigenvalues λ_k $(k = 1, 2, \cdots, n)$. Set $N = \{1, 2, \cdots, m, m+1, \cdots, (n+m)/2\}$, $N_i = \{i_1, i_2, \cdots, i_{n_i}\} \cup \varnothing$ with $i_j \in N$ $(j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, 6)$, $\Sigma_{i=1}^{3} n_i = m$ and $2\Sigma_{i=4}^{6} n_i = n - m$. $\cup_{i=1}^{6} N_i = N$ with $N_i \cap N_l = \varnothing$ $(l \neq i)$. $N_i = \varnothing$ if $n_i = 0$. The matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ possesses n_1-stable, n_2-unstable, and n_3-invariant real eigenvectors plus n_4-stable, n_5-unstable, and n_6-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $\mathrm{Re}\,\lambda_k = 0$ $(k \in N_3 \cup N_6)$, the flow $\mathbf{\Phi}(t)$ of the time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is an $(n_1 : n_2 : [n_3; m_3] | n_4 : n_5 : n_6)$ flow in the neighborhood of $\mathbf{x}^* = \mathbf{x}^{\tau*}$. However, with repeated complex eigenvalues of $\mathrm{Re}\,\lambda_k = 0$ $(k \in N_3 \cup N_6)$, the flow $\mathbf{\Phi}(t)$ of the time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is an $(n_1 : n_2 : [n_3; m_3] | n_4 : n_5 : [n_6, \mathbf{l}; \mathbf{m}_6])$ flow in the neighborhood of $\mathbf{x}^* = \mathbf{x}^{\tau*}$ where $\mathbf{m}_6 = (m_{61}, m_{62}, \cdots, m_{6s})$, and $\boldsymbol{l} = (l_1, l_2, \cdots, l_s)$ with $(s \in \{1, 2, \cdots, n/2\})$. The meanings of notations in the aforementioned structures are defined as follows:

(i) n_1 represents exponential sinks on n_1-directions of $\mathbf{v}_k$ if $\lambda_k < 0$ $(k \in N_1$ and $1 \leq n_1 \leq n)$ with distinct or repeated eigenvalues.

(ii) n_2 represents exponential sources on n_2-directions of $\mathbf{v}_k$ if $\lambda_k > 0$ $(k \in N_2$ and $1 \leq n_2 \leq n)$ with distinct or repeated eigenvalues.

(iii) $n_3 = 1$ represents an invariant center on 1-direction of $\mathbf{v}_k$ if $\lambda_k = 0$ $(k \in N_3$ and $n_3 = 1)$.

(iv) n_4 represents spiral sinks on n_4-pairs of $(\mathbf{u}_k, \mathbf{v}_k)$ if $\mathrm{Re}\,\lambda_k < 0$ and $\mathrm{Im}\,\lambda_k \neq 0$ $(k \in N_4$ and $1 \leq n_4 \leq n)$ with distinct or repeated eigenvalues.

(v) n_5 represents spiral sources on n_5-pairs of $(\mathbf{u}_k, \mathbf{v}_k)$ if $\mathrm{Re}\lambda_k > 0$ and $\mathrm{Im}\lambda_k = 0$ $(k \in N_5$ and $1 \leq n_5 \leq n)$ with distinct or repeated eigenvalues.

(vi) n_6 represents invariant centers on n_6-pairs of $(\mathbf{u}_k, \mathbf{v}_k)$ if $\mathrm{Re}\lambda_k = 0$ and $\mathrm{Im}\lambda_k \neq 0$ $(k \in N_6$ and $1 \leq n_6 \leq n)$ with distinct eigenvalues.

(vii) $\varnothing$ represents empty or none if $n_i = 0$ $(i \in \{1, 2, \cdots, 6\})$.

(viii) $[n_3; m_3]$ represents invariant centers on $(n_3 - m_3)$-directions of $\mathbf{v}_{k_3}$ $(k_3 \in N_3)$ and sources in m_3-directions of $\mathbf{v}_{j_3}$ $(j_3 \in N_3$ and $j_3 \neq k_3)$ if $\lambda_k = 0$ $(k \in N_3$ and $n_3 \leq n)$ with the (m_3+1)th-order nilpotent matrix $\mathbf{N}_3^{m_3+1} = \mathbf{0}$ $(0 < m_3 \leq n_2 - 1)$.

(ix) $[n_3; \varnothing]$ represents invariant centers on n_3-directions of $\mathbf{v}_k$ if $\lambda_k = 0$ $(k \in N_3$ and $1 < n_3 \leq n)$ with a nilpotent matrix $\mathbf{N}_3 = \mathbf{0}$.

(x) $[n_6, \boldsymbol{l}; \mathbf{m_6}]$ represents invariant centers on $(n_6 - \Sigma_{i=1}^{s} m_{6i})$-pairs of $(\mathbf{u}_{k_{6i}}, \mathbf{v}_{k_{6i}})$ $(k_{6i} \in N_{6i})$ and sources in $\Sigma_{i=1}^{s} m_{6i}$-pairs of $(\mathbf{u}_{j_{6i}}, \mathbf{v}_{j_{6i}})$ $(j_{6i} \in N_{6i}$ and $j_{6i} \neq k_{6i})$ if $\mathrm{Re}\lambda_{k_i} = 0$ and $\mathrm{Im}\lambda_{k_i} \neq 0$ $(k \in N_{6i}$ and $n_6 \leq n)$ for $(\Sigma_{i=1}^{s} l_i + s + 1)$-pairs of repeated eigenvalues with the $(\Sigma_{i=1}^{s} m_{6i} + 1)$th-order nilpotent matrix $\mathbf{N}_{6i}^{m_{6i}+1} = \mathbf{0}$ $(0 < m_{6i} \leq l_i)$ $(i = 1, 2, \cdots, s)$.

(xi) $[n_6, l; \varnothing]$ represents invariant centers on n_6-pairs of $(\mathbf{u}_k, \mathbf{v}_k)$ if $\mathrm{Re}\lambda_k = 0$ and $\mathrm{Im}\lambda_k \neq 0$ $(k \in N_6$ and $1 \leq n_6 \leq n)$ for $(l+1)$-pairs of repeated eigenvalues with a nilpotent matrix $\mathbf{N}_6 = \mathbf{0}$.

Definition 2.18 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$. Suppose there is a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ as $U(\mathbf{x}^*) \subset \Omega$, and in the neighborhood, $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ is C^r $(r \geq 1)$-continuous and Eq. (2.24) holds. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. From Eq. (2.19), the matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ possesses n eigenvalues λ_k $(k = 1, 2, \cdots, n)$. Set $N = \{1, 2, \cdots, m, m+1, \cdots, (n+m)/2\}$, $N_i = \{i_1, i_2, \cdots, i_{n_i}\} \cup \varnothing$ with $i_j \in N$ $(j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, 6)$, $\Sigma_{i=1}^{3} n_i = m$ and $2\Sigma_{i=4}^{6} n_i = n - m$. $\cup_{i=1}^{6} N_i = N$ with $N_i \cap N_l = \varnothing$ $(l \neq i)$. $N_i = \varnothing$ if $n_i = 0$. The matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ possesses n_1-stable, n_2-unstable, and n_3-invariant real eigenvectors plus n_4-stable, n_5-unstable, and n_6-center pairs of complex eigenvectors.

I. *Non-degenerate cases*

(i) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is an $(n_1 : n_2 : \varnothing | n_4 : n_5 : \varnothing)$ hyperbolic point (or saddle) for the time-delay system.

(ii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is an $(n_1 : \varnothing : \varnothing | n_4 : \varnothing : \varnothing)$ sink for the time-delay system.

(iii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is an $(\varnothing : n_2 : \varnothing | \varnothing : n_5 : \varnothing)$ source for the time-delay system.

(iv) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is an $(\varnothing : \varnothing : \varnothing | \varnothing : \varnothing : n/2)$ center for the time-delay system.

(v) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is an $(\varnothing : \varnothing : \varnothing | \varnothing : \varnothing : [n/2, \mathbf{l}; \varnothing])$ center for the time-delay system.

(vi) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is an $(\varnothing : \varnothing : \varnothing | \varnothing : \varnothing : [n/2, \mathbf{l}; \mathbf{m}])$ point for the time-delay system.

(vii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(n_1 : \varnothing : \varnothing | n_4 : \varnothing : n_6)$ point for the time-delay system.
(viii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : n_2 : \varnothing | \varnothing : n_5 : n_6)$ point for the time-delay system.
(ix) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(n_1 : n_2 : \varnothing | n_4 : n_5 : n_6)$ point for the time-delay system.

II. *Simple degenerate cases*

(i) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : \varnothing : [n; \varnothing] | \varnothing : \varnothing : \varnothing)$-invariant (or static) center for the time-delay system.
(ii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : \varnothing : [n; m_3] | \varnothing : \varnothing : \varnothing)$ point for the time-delay system.
(iii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : \varnothing : [n_3; \varnothing] | \varnothing : \varnothing : n_6)$ point for the time-delay system.
(iv) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : \varnothing : [n_3; m_3] | \varnothing : \varnothing : n_6)$ point for the time-delay system.
(v) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : \varnothing : [n_3; \varnothing] | \varnothing : \varnothing : [n_6; \varnothing])$ point for the time-delay system.
(vi) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : \varnothing : [n_3; m_3] | \varnothing : \varnothing : [n_6; \varnothing])$ point for the time-delay system.
(vii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : \varnothing : [n_3; \varnothing] | \varnothing : \varnothing : [n_6, \mathbf{l}; \mathbf{m_6}])$ point for the time-delay system.
(viii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : \varnothing : [n_3; m_3] | \varnothing : \varnothing : [n_6, \mathbf{l}; \mathbf{m_6}])$ point for the time-delay system.

III. *Complex degenerate cases*

(i) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(n_1 : \varnothing : [n_3; \varnothing] | n_4 : \varnothing : \varnothing)$ point for the time-delay system.
(ii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(n_1 : \varnothing : [n_3; m_3] | n_4 : \varnothing : \varnothing)$ point for the time-delay system.
(iii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : n_2 : [n_3; \varnothing] | \varnothing : n_5 : \varnothing)$ point for the time-delay system.
(iv) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : n_2 : [n_3; m_3] | \varnothing : n_5 : \varnothing)$ point for the time-delay system.
(v) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(n_1 : \varnothing : [n_3; \varnothing] | n_4 : \varnothing : n_6)$ point for the time-delay system.
(vi) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(n_1 : \varnothing : [n_3; m_3] | n_4 : \varnothing : n_6)$ point for the time-delay system.
(vii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : n_2 : [n_3; \varnothing] | \varnothing : n_5 : n_6)$ point for the time-delay system.
(viii) The equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is an $(\varnothing : n_2 : [n_3; m_3] | \varnothing : n_5 : n_6)$ point for the time-delay system.

2.3 Bifurcation and Stability Switching

The dynamical characteristics of equilibriums in nonlinear time-delay systems in Eq. (2.4) are based on the given parameters. With varying parameters in the time-delay dynamical systems, the corresponding dynamical behaviors will change qualitatively. The qualitative switching of dynamical behaviors in the time-delay dynamical systems is called *bifurcation*, and the corresponding parameter values are called *bifurcation values*. To understand the qualitative changes of dynamical behaviors of nonlinear time-delay systems with parameters in the neighborhood of equilibriums, the bifurcation theory for equilibrium of nonlinear time-delay system in Eq. (2.4) will be investigated. $D_{\mathbf{x}}() = \partial()/\partial\mathbf{x}$,$D_{\mathbf{x}^\tau}() = \partial()/\partial\mathbf{x}^\tau$, and $D_{\mathbf{p}}() = \partial()/\partial\mathbf{p}$ will be adopted from now on.

Definition 2.21 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $(\mathbf{x}^*, \mathbf{p})$ with $\mathbf{x}^* = \mathbf{x}^{\tau*}$. Suppose there is a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ as $U(\mathbf{x}^*) \subset \Omega$, and in the neighborhood $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is C^r $(r \geq 1)$-continuous and Eq. (2.24) holds. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. The linearized time-delay system at equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is $\dot{\mathbf{y}} = D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y} + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})\mathbf{y}^\tau$ ($\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau*}$) in Eq. (2.19).

- (i) The equilibrium point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ with $\mathbf{x}_0^* = \mathbf{x}_0^{\tau*}$ is called *the switching point* of equilibrium solutions if $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ at $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ possesses at least one more real eigenvalue (or one more pair of complex eigenvalues) with zero real part.
- (ii) The value $\mathbf{p}_0$ in Eq. (2.4) is called *a switching value* of $\mathbf{p}$ if the dynamical characteristics at point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ change from one state into another state.
- (iii) The equilibrium point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ with $\mathbf{x}_0^* = \mathbf{x}_0^{\tau*}$ is called the *bifurcation point* of equilibrium solutions if $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ at $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ possesses at least one more real eigenvalue (or one more pair of complex eigenvalues) with zero real part, and more than one branch of equilibrium solutions appears or disappears.
- (iv) The value $\mathbf{p}_0$ in Eq. (2.4) is called a *bifurcation value* of $\mathbf{p}$ if the dynamical characteristics at point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ with $\mathbf{x}_0^* = \mathbf{x}_0^{\tau*}$ change from one stable state into another unstable state.

2.3.1 Stability and Switching

To extend the idea of Definitions 2.14 and 2.15, a new function will be defined to determine the stability and the stability state switching.

Definition 2.25 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ and

$\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in a neighborhood of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. Suppose $U(\mathbf{x}^*) \subset \Omega$ is a neighborhood of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$, and there are n linearly independent vectors $\mathbf{v}_k$ $(k = 1, 2, \cdots, n)$. For a perturbation of equilibrium $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau *}$, let $\mathbf{y}^{(k)} = c_k \mathbf{v}_k$ and $\mathbf{y}^{\tau(k)} = c_k^\tau \mathbf{v}_k$, and $\dot{\mathbf{y}}^{(k)} = \dot{c}_k \mathbf{v}_k$, $\dot{\mathbf{y}}^{\tau(k)} = \dot{c}_k^\tau \mathbf{v}_k$

$$\begin{aligned} s_k &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y} = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x} - \mathbf{x}^*), \\ s_k^\tau &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y}^\tau = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}^\tau - \mathbf{x}^{\tau *}) \end{aligned} \tag{2.44}$$

where $s_k = c_k ||\mathbf{v}_k||^2$ and $s_k^\tau = c_k^\tau ||\mathbf{v}_k||^2$. Define the following functions:

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \tag{2.45}$$

and

$$\begin{aligned} & G_k^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \\ & = \mathbf{v}_k^{\mathrm{T}} \cdot [D_{s_k}(\cdot) + D_{s_k^\tau}(\cdot) s_k^\tau / s_k] \mathbf{f}(\mathbf{x}(s_k), \mathbf{x}^\tau(s_k^\tau), \mathbf{p}) \\ & = \mathbf{v}_k^{\mathrm{T}} \cdot [D_{\mathbf{x}}(\cdot) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}(\cdot)] \mathbf{f}(\mathbf{x}(s_k), \mathbf{x}^\tau(s_k^\tau), \mathbf{p}) \mathbf{v}_k ||\mathbf{v}_k||^{-2} \end{aligned} \tag{2.46}$$

$$\begin{aligned} G_{s_k}^{(m)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}} \cdot [D_{s_k}(\cdot) + D_{s_k^\tau}(\cdot) s_k^\tau / s_k]^m \mathbf{f}(\mathbf{x}(s_k), \mathbf{x}^\tau(s_k^\tau), \mathbf{p}) \\ &= \mathbf{v}_k^{\mathrm{T}} \cdot [D_{s_k}(\cdot) + D_{s_k^\tau}(\cdot) s_k^\tau / s_k] G_k^{(m-1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \end{aligned} \tag{2.47}$$

where $D_{s_k}(\cdot) = \partial(\cdot)/\partial s_k$ and $D_{s_k^\tau}(\cdot) = \partial(\cdot)/\partial s_k^\tau . G_{s_k}^{(0)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ if $m = 0$.

Definition 2.26 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau *}$ and $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. Suppose $U(\mathbf{x}^*) \subset \Omega$ is a neighborhood of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$, and there are n linearly independent vectors $\mathbf{v}_k$ $(k = 1, 2, \cdots, n)$. For a perturbation of equilibrium $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau *}$, let $\mathbf{y}^{(k)} = c_k \mathbf{v}_k$ and $\mathbf{y}^{\tau(k)} = c_k^\tau \mathbf{v}_k$, and $\dot{\mathbf{y}}^{(k)} = \dot{c}_k \mathbf{v}_k, \dot{\mathbf{y}}^{\tau(k)} = \dot{c}_k^\tau \mathbf{v}_k$.

(i) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ on the direction $\mathbf{v}_k$ is stable if

$$\begin{aligned} &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)) < 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)) > 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0; \end{aligned} \tag{2.48}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ is called a sink (or stable node) on the direction $\mathbf{v}_k$.

(ii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is unstable if

$$\begin{aligned}&\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0;\\ &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0;\end{aligned} \tag{2.49}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is called a source (or unstable node) on the direction $\mathbf{v}_k$.

(iii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is increasingly unstable if

$$\begin{aligned}&\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0;\\ &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0;\end{aligned} \tag{2.50}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is called an increasing saddle on the direction $\mathbf{v}_k$.

(iv) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is decreasingly unstable if

$$\begin{aligned}&\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0;\\ &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0;\end{aligned} \tag{2.51}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is called a decreasing saddle on the direction $\mathbf{v}_k$.

(v) $\mathbf{x}^{(i)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is invariant if

$$\begin{aligned}&\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) = 0\\ &\text{for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) \neq 0;\end{aligned} \tag{2.52}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is called to be degenerate on the direction $\mathbf{v}_k$.

Theorem 2.7 *Consider an n-dimensional, autonomous, nonlinear, time-delay system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ *in Eq. (2.4) with an equilibrium point* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *and* $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ *is* C^r($r \geq 1$)*-continuous in a neighborhood of the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *(i.e.,* $U(\mathbf{x}^*) \subset \Omega$*). The corresponding solution is* $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. *Suppose Eq. (2.24) holds in* $U(\mathbf{x}^*) \subset \Omega$. *For a linearized time-delay system in Eq. (2.19), consider a real eigenvalue* λ_k *of matrix* $D\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ *(* $k \in N = \{1, 2, \cdots, n\}$*) with an eigenvector* $\mathbf{v}_k$. *Let* $\mathbf{y}^{(k)} = c_k \mathbf{v}_k$ *and* $\mathbf{y}^{\tau(k)} = c_k^\tau \mathbf{v}_k$, *and* $\dot{\mathbf{y}}^{(k)} = \dot{c}_k \mathbf{v}_k$, $\dot{\mathbf{y}}^{\tau(k)} = \dot{c}_k^\tau \mathbf{v}_k$. $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y} = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x} - \mathbf{x}^*)$ *with* $s_k = c_k ||\mathbf{v}_k||^2$. *Define*

$$\dot{s}_k = \mathbf{v}_k^{\mathrm{T}} \cdot \dot{\mathbf{y}} = \mathbf{v}_k^{\mathrm{T}} \cdot \dot{\mathbf{x}} = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}). \tag{2.53}$$

(i) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is stable if and only if*

$$\begin{aligned} G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) < 0 \text{ for } s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) > 0 \text{ for } s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0 \end{aligned} \tag{2.54}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

(ii) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is unstable if and only if*

$$\begin{aligned} G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) > 0 \text{ for } s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) < 0 \text{ for } s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0 \end{aligned} \tag{2.55}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

(iii) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is increasingly unstable if and only if*

$$\begin{aligned} G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) > 0 \text{ for } s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) > 0 \text{ for } s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0 \end{aligned} \tag{2.56}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

(iv) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is decreasingly unstable if and only if*

$$\begin{aligned} G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) < 0 \text{ for } s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) < 0 \text{ for } s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0 \end{aligned} \tag{2.57}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

(v) $\mathbf{x}^{(i)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is invariant if*

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = 0 \tag{2.58}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

Proof Because

$$\begin{aligned} \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) &= \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) + \dot{\mathbf{x}}(t)\varepsilon + o(\varepsilon) - \mathbf{x}(t)) \\ &= \mathbf{v}_k^{\mathrm{T}} \cdot \dot{\mathbf{x}}(t)\varepsilon + o(\varepsilon) \end{aligned}$$

and $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$, we have

$$\begin{aligned} \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\varepsilon + o(\varepsilon) \\ &= G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\varepsilon + o(\varepsilon) \end{aligned}$$

(i) Due to any selection of $\varepsilon > 0$, for $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) < 0$$

vice versa, and for $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) > 0$$

vice versa.

(ii) For $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) > 0$$

vice versa, and for $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) < 0$$

vice versa.

(iii) For $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) > 0$$

vice versa, and for $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) > 0$$

vice versa.

(iv) For $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) < 0$$

vice versa, and for $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) < 0$$

vice versa.

(v) For $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) = 0 \text{ if } G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = 0$$

vice versa. Similarly, for $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0$

$$\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) = 0 \,\text{if}\, G_k(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) = 0$$

vice versa. The theorem is proved. ■

Theorem 2.8 *Consider an n-dimensional, autonomous, nonlinear, time-delay system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ *in Eq. (2.4) with an equilibrium point* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *and* $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ *is* $C^r(\, r \geq 1)$*-continuous in a neighborhood of the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *(i.e.,* $U(\mathbf{x}^*) \subset \Omega$*). The corresponding solution is* $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$*. Suppose Eq. (2.24) holds in* $U(\mathbf{x}^*) \subset \Omega$*. For a linearized time-delay system in Eq. (2.19), consider a real eigenvalue* λ_k *of matrix* $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^*}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ *(*$k \in N = \{1, 2, \cdots, n\}$*) with an eigenvector* $\mathbf{v}_k$*. Let* $\mathbf{y}^{(k)} = c_k \mathbf{v}_k$ *and* $\mathbf{y}^{\tau(k)} = c_k^{\tau} \mathbf{v}_k$*, and* $\dot{\mathbf{y}}^{(k)} = \dot{c}_k \mathbf{v}_k$*,* $\dot{\mathbf{y}}^{\tau(k)} = \dot{c}_k^{\tau} \mathbf{v}_k$*.* $s_k = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y} = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x} - \mathbf{x}^*)$ *with* $s_k = c_k ||\mathbf{v}_k||^2$*. Define* $\dot{s}_k = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ *in Eq. (2.53) with* $||G_k^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})|| < \infty$*.*

(i) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^*$ *on the direction* $\mathbf{v}_k$ *is stable if and only if*

$$G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = \lambda_k < 0 \tag{2.59}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$*.*

(ii) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^*$ *on the direction* $\mathbf{v}_k$ *is unstable if and only if*

$$G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = \lambda_k > 0 \tag{2.60}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$*.*

(iii) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^*$ *on the direction* $\mathbf{v}_k$ *is increasingly unstable if and only if*

$$G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = \lambda_k = 0, \text{and}\, G_k^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) > 0 \tag{2.61}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$*.*

(iv) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^*$ *on the direction* $\mathbf{v}_k$ *is decreasingly unstable if and only if*

$$G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = \lambda_k = 0, \text{and}\, G_k^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) < 0 \tag{2.62}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$*.*

(v) $\mathbf{x}^{(i)}$ *at the equilibrium* $\mathbf{x}^*$ *on the direction* $\mathbf{v}_k$ *is invariant if and only if*

$$G_{s_s}^{(m)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0 \quad (m = 0, 1, 2, \cdots) \tag{2.63}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$*.*

Proof For $\mathbf{x} = \mathbf{x}^* = \mathbf{x}^{\tau *}$, $s_k = s_k^\tau = 0$. Using Taylor series expansion gives

$$\begin{aligned}
\dot{s}_k &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \\
&= \mathbf{v}_k^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) + D_{s_k}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k \\
&\quad + D_{s_k^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^\tau] + o(\max(s_k, s_k^\tau)) \\
&= \mathbf{v}_k^{\mathrm{T}} \cdot [D_{s_k}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) + e^{-\lambda_k \tau} D_{s_k^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})] s_k + o(max(s_k, s_k^\tau)) \\
&= G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k + o(\max(s_k, s_k^\tau))
\end{aligned}$$

and

$$\begin{aligned}
G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}} \cdot [D_{\mathbf{x}}\mathbf{f}(\mathbf{x}(s_k), \mathbf{x}^\tau(s_k), \mathbf{p}) \partial_{c_k}\mathbf{x} \partial_{s_k} c_k \\
&\quad + D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}(s_k), \mathbf{x}^\tau(s_k^\tau), \mathbf{p}) \partial_{c_k^\tau}\mathbf{x}^\tau \partial_{s_k^\tau} c_k^\tau] \\
&= \mathbf{v}_k^{\mathrm{T}} \cdot [D_{\mathbf{x}}\mathbf{f}(\mathbf{x}(s_k), \mathbf{p}) + e^{-\lambda_k \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}(s_k), \mathbf{x}^\tau(s_k^\tau), \mathbf{p})] \mathbf{v}_k ||\mathbf{v}_k||^{-2} \\
&= \lambda_k.
\end{aligned}$$

Thus,

$$\dot{s}_k = G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k + o(\max(s_k, s_k^\tau)) = \lambda_k s_k + o(\max(s_k, s_k^\tau)).$$

(i) For $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \lambda_k s_k < 0$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \lambda_k s_k > 0.$$

Thus, $G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) = \lambda_k < 0$.

(ii) For $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \lambda_k s_k > 0$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \lambda_k s_k < 0.$$

Thus, $G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) = \lambda_k > 0$.

(iii) For $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \lambda_k s_k > 0$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \lambda_k s_k > 0.$$

Thus, $G_{s_k}^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = \lambda_k = 0$ and the higher order should be considered. With $s = max(s_k, s_k^\tau)$, the higher-order Taylor series expansion gives

$$\begin{aligned}
\dot{s}_k &= \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \\
&= \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + [D_{s_k}(\cdot) + D_{s_k^\tau}(\cdot) s_k^\tau / s_k] \mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k \\
&\quad + \frac{1}{2!} [D_{s_k}(\cdot) + D_{s_k^\tau}(\cdot) s_k^\tau / s_k]^2 \mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^2) + o(s^2) \\
&= \frac{1}{2!} [\mathbf{v}_k^{\mathrm{T}} \cdot [D_{s_k}(\cdot) + D_{s_k^\tau}(\cdot) s_k^\tau / s_k]^2 \mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})] s_k^2 + o(s^2) \\
&= \frac{1}{2!} G_{s_k}^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^2 + o(s^2).
\end{aligned}$$

For $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{2!} G_{s_k}^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^2 > 0$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{2!} G_{s_k}^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^2 > 0.$$

So we have

$$G_{s_k}^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) > 0.$$

(iv) Similar to (iii), we have $G_k^{(1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = \lambda_k = 0$. For $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{2!} G_{s_k}^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^2 < 0$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{2!} G_{s_k}^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^2 < 0.$$

So

$$G_{s_k}^{(2)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) < 0.$$

(v) with $s = max(s_x, s_k^{\tau})$, using Taylor series expansion yields

$$\dot{s}_k = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) = \sum\nolimits_{m=1}^{N} \frac{1}{m!} G_{s_k}^{(m)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^m + o(s^N) = 0$$
$$(N = 1, 2, \cdots)$$

for any selected values of s_k. Thus, only if

$$G_{s_k}^{(m)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0 \, (m = 1, 2, \cdots),$$

the above equation holds, vice versa. The theorem is proved. ■

Definition 2.27 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ in Eq. (2.4) with an equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ and $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ is $C^r(r \geq 1)$-continuous in a neighborhood of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. Suppose $U(\mathbf{x}^*) \subset \Omega$ is a neighborhood of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$, and there are n linearly independent vectors $\mathbf{v}_k$ $(k = 1, 2, \cdots, n)$. For a perturbation of equilibrium $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{y}^{\tau} = \mathbf{x}^{\tau} - \mathbf{x}^{\tau*}$, let $\mathbf{y}^{(k)} = c_k \mathbf{v}_k$ and $\mathbf{y}^{\tau(k)} = c_k^{\tau} \mathbf{v}_k$, and $\dot{\mathbf{y}}^{(k)} = \dot{c}_k \mathbf{v}_k$, $\dot{\mathbf{y}}^{\tau(k)} = \dot{c}_k^{\tau} \mathbf{v}_k$.

(i) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is stable of the $(2m_k + 1)$th-order if

$$\begin{aligned} & G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0, r_k = 0, 1, 2, \cdots, 2m_k; \\ & \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)) < 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ & \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)) > 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0 \end{aligned} \tag{2.64}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is called a sink (or stable node) of the $(2m_k + 1)$th-order on the direction $\mathbf{v}_k$.

(ii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is unstable of the $(2m_k + 1)$th-order if

$$\begin{aligned} & G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0, r_k = 0, 1, 2, \cdots, 2m_k; \\ & \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)) > 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ & \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)) < 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0. \end{aligned} \tag{2.65}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is called a source (or unstable node) of the $(2m_k + 1)$th-order on the direction $\mathbf{v}_k$.

(iii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is increasingly unstable of the $(2m_k)$th-order if

$$\begin{aligned} &G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0, r_k = 0, 1, 2, \cdots, 2m_k - 1; \\ &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) > 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0 \end{aligned} \tag{2.66}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is called an increasing saddle of the $(2m_k)$th-order on the direction $\mathbf{v}_k$.

(iv) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the direction $\mathbf{v}_k$ is decreasingly unstable of the $(2m_k)$th-order if

$$\begin{aligned} &G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0, r_k = 0, 1, 2, \cdots, 2m_k - 1; \\ &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) > 0; \\ &\mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t+\varepsilon) - \mathbf{x}(t)) < 0 \text{ for } \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}(t) - \mathbf{x}^*) < 0 \end{aligned} \tag{2.67}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ and all $t \in [t_0, \infty)$. The equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ is called a decreasing saddle of the $(2m_k)$th-order on the direction $\mathbf{v}_k$.

Theorem 2.9 *Consider an n-dimensional, autonomous, nonlinear, time-delay system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ *in Eq. (2.4) with an equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *and* $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ *is* C^r *(* $r \geq 1$*)-continuous in a neighborhood of equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$*. The corresponding solution is* $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$*. Suppose* $U(\mathbf{x}^*) \subset \Omega$ *is a neighborhood of equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$*, and there are n linearly independent vectors* $\mathbf{v}_k$ *(* $k = 1, 2, \cdots, n$*). For a perturbation of equilibrium* $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ *and* $\mathbf{y}^\tau = \mathbf{x}^\tau - \mathbf{x}^{\tau*}$*.*

(i) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is stable of the* $(2m_k + 1)$th-*order if and only if*

$$\begin{aligned} &G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0, r_k = 0, 1, 2, \cdots, 2m_k; \\ &G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) < 0 \end{aligned} \tag{2.68}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

(ii) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is unstable of the* $(2m_k + 1)$th-*order if and only if*

$$\begin{aligned} &G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0, r_k = 0, 1, 2, \cdots, 2m_k; \\ &G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) > 0 \end{aligned} \tag{2.69}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

(iii) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is increasingly unstable of the* $(2m_k)$th-*order if and only if*

$$\begin{aligned} &G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0, r_k = 0, 1, 2, \cdots, 2m_k - 1; \\ &G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) > 0 \end{aligned} \tag{2.70}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

(iv) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau*}$ *on the direction* $\mathbf{v}_k$ *is decreasingly unstable of the* $(2m_k)$th-*order if and only if*

$$\begin{aligned} &G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0, r_k = 0, 1, 2, \cdots, 2m_k - 1; \\ &G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) < 0 \end{aligned} \tag{2.71}$$

for all $\mathbf{x} \in U(\mathbf{x}^*) \subset \Omega$ *and all* $t \in [t_0, \infty)$.

Proof For $\mathbf{x} = \mathbf{x}^*$, $s_k = 0$. Using Taylor series expansion gives

$$\begin{aligned} \dot{s}_k &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) \\ &= \sum\nolimits_{r_k=1}^{2m_k} \frac{1}{r_k!} G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^{r_k} \\ &\quad + \frac{1}{(2m_k+1)!} G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^{2m_k+1} + o(s_k^{2m_k+1}) \end{aligned}$$

and

$$\begin{aligned} &G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) = 0 \text{ for } r_k = 0, 1, 2, \cdots, 2m_k; \\ &\dot{s}_k = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) = \frac{1}{(2m_k+1)!} G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^{2m_k+1}. \end{aligned}$$

(i) For $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) = \dot{s}_k = \frac{1}{(2m_k+1)!} G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^{2m_k+1} < 0,$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) = \dot{s}_k = \frac{1}{(2m_k+1)!} G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^{2m_k+1} > 0.$$

Thus, $G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) < 0$.

(ii) For $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) = \dot{s}_k = \frac{1}{(2m_k+1)!} G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) s_k^{2m_k+1} > 0,$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{(2m_k+1)!} G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^{2m_k+1} < 0.$$

Thus, $G_{s_k}^{(2m_k+1)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) > 0$.

(iii) For $\mathbf{x} = \mathbf{x}^*$, $s_k = 0$. Using Taylor series expansion gives

$$\begin{aligned} \dot{s}_k &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \\ &= \sum\nolimits_{r_k=1}^{2m_k-1} \frac{1}{r_k!} G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^{r_k} + \frac{1}{(2m_k)!} G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^{2m_k} + o(s_k^{2m_k}) \end{aligned}$$

and

$$\begin{aligned} &G_{s_k}^{(r_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) = 0 \text{ for } r_k = 0, 1, \cdots, 2m_k - 1; \\ &\dot{s}_k = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \frac{1}{(2m_k)!} G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^{2m_k}. \end{aligned}$$

For $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{(2m_k)!} G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^{2m_k} > 0,$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{(2m_k)!} G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^{2m_k} > 0.$$

So we have

$$G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) > 0.$$

(iv) Similar to (iii), for $s_k > 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{(2m_k)!} G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^{2m_k} < 0,$$

and for $s_k < 0$

$$G_k(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) = \dot{s}_k = \frac{1}{(2m_k)!} G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) s_k^{2m_k} < 0.$$

So

$$G_{s_k}^{(2m_k)}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) < 0.$$

The theorem is proved. ■

Definition 2.28 Consider an n-dimensional, autonomous, nonlinear, time-delay dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ in Eq. (2.4) with an equilibrium point $\mathbf{x}^* = \mathbf{x}^{\tau*}$ and $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in a neighborhood of the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ (i.e.,$U(\mathbf{x}^*) \subset \mathbf{\Omega}$). The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. Suppose Eq. (2.24) holds in $U(\mathbf{x}^*) \subset \Omega$. For a linearized time-delay system in Eq. (2.19), consider a pair of complex eigenvalues $\alpha_k \pm \mathbf{i}\beta_k$ $(k \in N = \{1, 2, \cdots, n\}, \mathbf{i} = \sqrt{-1})$ of matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k$. On the invariant plane of $(\mathbf{u}_k, \mathbf{v}_k)$, consider $\mathbf{r}_k = \mathbf{y}_k = \mathbf{y}_+^{(k)} + \mathbf{y}_-^{(k)}$ with

$$\begin{aligned}
\mathbf{r}_k &= c_k\mathbf{u}_k + d_k\mathbf{v}_k = r_k\mathbf{e}_{r_k}, \mathbf{r}_k^{\tau} = c_k^{\tau}\mathbf{u}_k + d_k^{\tau}\mathbf{v}_k = r_k^{\tau}\mathbf{e}_{r_k};\\
\dot{\mathbf{r}}_k &= \dot{c}_k\mathbf{u}_k + \dot{d}_k\mathbf{v}_k = \dot{r}_k\mathbf{e}_{r_k} + r_k\dot{\mathbf{e}}_{r_k},\\
\dot{\mathbf{r}}_k^{\tau} &= \dot{c}_k^{\tau}\mathbf{u}_k + \dot{d}_k^{\tau}\mathbf{v}_k = \dot{r}_k^{\tau}\mathbf{e}_{r_k} + r_k^{\tau}\dot{\mathbf{e}}_{r_k}
\end{aligned} \tag{2.72}$$

and

$$\begin{aligned}
c_k &= \frac{1}{\Delta}[\Delta_2(\mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{y}) - \Delta_{12}(\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y})],\\
d_k &= \frac{1}{\Delta}[\Delta_1(\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y}) - \Delta_{12}(\mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{y})];
\end{aligned}$$

$$\begin{aligned}
c_k^{\tau} &= \frac{1}{\Delta}[\Delta_2(\mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{y}^{\tau}) - \Delta_{12}(\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y}^{\tau})],\\
d_k^{\tau} &= \frac{1}{\Delta}[\Delta_1(\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y}^{\tau}) - \Delta_{12}(\mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{y}^{\tau})];\\
\Delta_1 &= ||\mathbf{u}_k||^2, \Delta_2 = ||\mathbf{v}_k||^2, \Delta_{12} = \mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{v}_k,\\
\Delta &= \Delta_1\Delta_2 - \Delta_{12}^2.
\end{aligned} \tag{2.73}$$

Consider a polar coordinate of (r_k, θ_k) defined by

$$\begin{aligned}
& c_k = r_k \cos\theta_k, \text{and}\, d_k = r_k \sin\,\theta_k;\\
& r_k = \sqrt{c_k^2 + d_k^2}, \text{and}\, \theta_k = \arctan d_k/c_k;\\
& \begin{bmatrix} c_k^{\tau} \\ d_k^{\tau} \end{bmatrix} = e^{-\alpha_k\tau}\begin{bmatrix} \cos\beta_k\tau & \sin\beta_k\tau \\ -\sin\beta_k\tau & \cos\beta_k\tau \end{bmatrix}\begin{bmatrix} c_k \\ d_k \end{bmatrix};\\
& c_k^{\tau} = r_k^{\tau}\cos(\theta_k - \beta\tau), \text{and}\, d_k^{\tau} = r_k^{\tau}\sin(\theta_k - \beta\tau);\\
& r_k^{\tau} = \sqrt{(c_k^{\tau})^2 + (d_k^{\tau})^2}, \text{and}\, \theta_k^{\tau} = \theta_k - \beta\tau = \arctan d_k^{\tau}/c_k^{\tau};
\end{aligned} \tag{2.74}$$

$$\begin{aligned}
& \mathbf{e}_{r_k} = \cos\,\theta_k\mathbf{u}_k + \sin\,\theta_k\mathbf{v}_k \text{ and } \mathbf{e}_{\theta_k} = -\cos\,\theta_k\mathbf{u}_k^{\perp}\Delta_3 + \sin\,\theta_k\mathbf{v}_k^{\perp}\Delta_4\\
& \Delta_3 = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{u}_k^{\perp} \text{ and } \Delta_4 = \mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{v}_k^{\perp}
\end{aligned} \tag{2.75}$$

where $\mathbf{u}_k^{\perp}$ and $\mathbf{v}_k^{\perp}$ are the normal vectors of $\mathbf{u}_k$ and $\mathbf{v}_k$, respectively.

$$\begin{aligned}\dot{c}_k &= \frac{1}{\Delta}[\Delta_2 G_{c_k}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) - \Delta_{12} G_{d_k}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})]\\ \dot{d}_k &= \frac{1}{\Delta}[\Delta_1 G_{d_k}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) - \Delta_{12} G_{c_k}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})]\end{aligned}\tag{2.76}$$

where

$$\begin{aligned}G_{c_k}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) = \sum\nolimits_{m=1}^{\infty} G_{c_k}^{(m)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}) r_k^m,\\ G_{d_k}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) = \sum\nolimits_{m=1}^{\infty} G_{d_k}^{(m)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}) r_k^m;\end{aligned}\tag{2.77}$$

$$\begin{aligned}G_{d_k}^{(1)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}} \cdot D_{(\mathbf{x},\mathbf{x}^\tau)}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})|_{(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p})},\\ G_{d_k}^{(1)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}} \cdot D_{(\mathbf{x},\mathbf{x}^\tau)}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})|_{(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p})};\\ G_{c_k}^{(m)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}} \cdot D^{(m)}_{(\mathbf{x},\mathbf{x}^\tau)}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})|_{(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p})},\\ G_{d_k}^{(m)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}} \cdot D^{(m)}_{(\mathbf{x},\mathbf{x}^\tau)}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})|_{(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p})};\end{aligned}$$

$$\begin{aligned}D_{(\mathbf{x},\mathbf{x}^\tau)}(\cdot) &= \{(\partial_{\mathbf{x}}(\cdot)[\mathbf{u}_k\cos\theta_k + \mathbf{v}_k\sin\theta_k]\\ &\quad + e^{-\alpha_k\tau}\partial_{\mathbf{x}^\tau}(\cdot)[\mathbf{u}_k\cos(\theta_k-\beta\tau) + \mathbf{v}_k\sin(\theta_k-\beta\tau)]\},\\ D^{(m)}_{(\mathbf{x},\mathbf{x}^\tau)}(\cdot) &= \{(\partial_{\mathbf{x}}(\cdot)[\mathbf{u}_k\cos\theta_k + \mathbf{v}_k\sin\theta_k]\\ &\quad + e^{-\alpha_k\tau}\partial_{\mathbf{x}^\tau}(\cdot)[\mathbf{u}_k\cos(\theta_k-\beta\tau) + \mathbf{v}_k\sin(\theta_k-\beta\tau)]\}^m.\end{aligned}\tag{2.78}$$

Thus,

$$\begin{aligned}\dot{r}_k &= \dot{c}_k\cos\theta_k + \dot{d}_k\sin\theta_k = \sum\nolimits_{m=1}^{\infty} G_{r_k}^{(m)}(\theta_k) r_k^m\\ \dot{\theta}_k &= r_k^{-1}(\dot{d}_k\cos\theta_k - \dot{c}_k\sin\theta_k) = r_k^{-1}\sum\nolimits_{m=1}^{\infty} G_{\theta_k}^{(m)}(\theta_k) r_k^{m-1}\end{aligned}\tag{2.79}$$

where

$$\begin{aligned}G_{r_k}^{(m)}(\theta_k) &= \frac{1}{\Delta}[(\Delta_2\cos\theta_k - \Delta_{12}\sin\theta_k)\mathbf{u}_k^{\mathrm{T}}\\ &\quad + (\Delta_2\sin\theta_k - \Delta_{12}\cos\theta_k)\mathbf{v}_k^{\mathrm{T}}] \cdot D^{(m)}_{(\mathbf{x},\mathbf{x}^\tau)}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})|_{(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p})},\\ G_{\theta_k}^{(m)}(\theta_k) &= -\frac{1}{\Delta}[(\Delta_2\sin\theta_k + \Delta_{12}\cos\theta_k)\mathbf{u}_k^{\mathrm{T}}\\ &\quad - (\Delta_1\cos\theta_k - \Delta_{12}\sin\theta_k)\mathbf{v}_k^{\mathrm{T}}] \cdot D^{(m)}_{(\mathbf{x},\mathbf{x}^\tau)}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})|_{(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p})}.\end{aligned}\tag{2.80}$$

From the foregoing definition, consider the first-order terms of G-function

$$\begin{aligned}G_{c_k}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) &= G_{c_k1}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) + G_{c_k2}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\\ G_{d_k}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) &= G_{d_k1}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) + G_{d_k2}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\end{aligned}\tag{2.81}$$

where

$$
\begin{aligned}
G_{c_k1}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}}\cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{c_k}\mathbf{x} + \frac{\partial c_k^\tau}{\partial c_k}\mathbf{u}_k^{\mathrm{T}}\cdot D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{c_k^\tau}\mathbf{x}^\tau\\
&\quad + \frac{\partial d_k^\tau}{\partial c_k}\mathbf{u}_k^{\mathrm{T}}\cdot D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{d_k^\tau}\mathbf{x}^\tau\\
&= \mathbf{u}_k^{\mathrm{T}}\cdot[D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) + \frac{\partial c_k^\tau}{\partial c_k}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})]\mathbf{u}_k\\
&\quad + \mathbf{u}_k^{\mathrm{T}}\cdot\frac{\partial d_k^\tau}{\partial c_k}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\mathbf{v}_k\\
&= \mathbf{u}_k^{\mathrm{T}}\cdot(\alpha_k\mathbf{u}_k - \beta_k\mathbf{v}_k)\\
&= \alpha_k\Delta_1 - \beta_k\Delta_{12}
\end{aligned}
$$

$$
\begin{aligned}
G_{c_k2}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}}\cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{d_k}\mathbf{x} + \frac{\partial d_k^\tau}{\partial d_k}\mathbf{u}_k^{\mathrm{T}}\cdot D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{d_k^\tau}\mathbf{x}\\
&\quad + \frac{\partial c_k^\tau}{\partial d_k}\mathbf{u}_k^{\mathrm{T}}\cdot D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{c_k^\tau}\mathbf{x}\\
&= \mathbf{u}_k^{\mathrm{T}}\cdot[D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) + \frac{\partial d_k^\tau}{\partial d_k}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})]\mathbf{v}_k\\
&\quad + \mathbf{u}_k^{\mathrm{T}}\cdot\frac{\partial c_k^\tau}{\partial d_k}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\mathbf{u}_k\\
&= \mathbf{u}_k^{\mathrm{T}}\cdot(\beta_k\mathbf{u}_k + \alpha_k\mathbf{v}_k)\\
&= \alpha_k\Delta_{12} + \beta_k\Delta_1;
\end{aligned}
\tag{2.82}
$$

and

$$
\begin{aligned}
G_{d_k1}^{(1)}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}}\cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{c_k}\mathbf{x} + \frac{\partial c_k^\tau}{\partial d_k}\mathbf{v}_k^{\mathrm{T}}\cdot D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{c_k^\tau}\mathbf{x}^\tau\\
&\quad + \frac{\partial d_k^\tau}{\partial d_k}\mathbf{v}_k^{\mathrm{T}}\cdot D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\partial_{d_k^\tau}\mathbf{x}^\tau\\
&= \mathbf{v}_k^{\mathrm{T}}\cdot[D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p}) + \frac{\partial c_k^\tau}{\partial d_k}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})]\mathbf{u}_k\\
&\quad + \mathbf{v}_k^{\mathrm{T}}\cdot\frac{\partial d_k^\tau}{\partial d_k}D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x},\mathbf{x}^\tau,\mathbf{p})\mathbf{v}_k\\
&= \mathbf{v}_k^{\mathrm{T}}\cdot(\alpha_k\mathbf{u}_k - \beta_k\mathbf{v}_k)\\
&= \alpha_k\Delta_{12} - \beta_k\Delta_2,
\end{aligned}
$$

$$
\begin{aligned}
G_{d_k2}^{(1)}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}}\cdot D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\partial_{d_k}\mathbf{x} + \frac{\partial d_k^{\tau}}{\partial d_k}\mathbf{u}_k^{\mathrm{T}}\cdot D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\partial_{d_k^{\tau}}\mathbf{x} \\
&\quad + \frac{\partial c_k^{\tau}}{\partial d_k}\mathbf{u}_k^{\mathrm{T}}\cdot D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\partial_{c_k^{\tau}}\mathbf{x} \\
&= \mathbf{v}_k^{\mathrm{T}}\cdot[D_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p}) + \frac{\partial d_k^{\tau}}{\partial d_k}D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})]\mathbf{v}_k \\
&\quad + \mathbf{v}_k^{\mathrm{T}}\cdot\frac{\partial c_k^{\tau}}{\partial d_k}D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\mathbf{u}_k \\
&= \mathbf{v}_k^{\mathrm{T}}\cdot(\beta_k\mathbf{u}_k + \alpha_k\mathbf{v}_k) \\
&= \alpha_k\Delta_2 + \beta_k\Delta_{12}.
\end{aligned}
\tag{2.83}
$$

Substitution of Eqs. (2.81)–(2.83) into Eq. (2.78) gives

$$
\begin{aligned}
G_{c_k}^{(1)}(\mathbf{x},\mathbf{p}) &= G_{c_k1}^{(1)}(\mathbf{x},\mathbf{p})\cos\theta_k + G_{c_k2}^{(1)}(\mathbf{x},\mathbf{p})\sin\theta_k \\
&= (\alpha_k\Delta_1 - \beta_k\Delta_{12})\cos\theta_k + (\alpha_k\Delta_{12} + \beta_k\Delta_1)\sin\theta_k, \\
G_{d_k}^{(1)}(\mathbf{x},\mathbf{p}) &= G_{d_k1}^{(1)}(\mathbf{x},\mathbf{p})\cos\theta_k + G_{d_k2}^{(1)}(\mathbf{x},\mathbf{p})\sin\theta_k \\
&= (-\beta_k\Delta_2 + \alpha_k\Delta_{12})\cos\theta_k + (\alpha_k\Delta_2 + \beta_k\Delta_{12})\sin\theta_k.
\end{aligned}
\tag{2.84}
$$

From Eq. (2.80), we have

$$
\begin{aligned}
G_{r_k}^{(1)}(\theta_k) &= \frac{1}{\Delta}[(G_{c_k}^{(1)}\Delta_2 - G_{d_k}^{(1)}\Delta_{12})\cos\theta_k + (G_{d_k}^{(1)}\Delta_1 - G_{c_k}^{(1)}\Delta_{12})\sin\theta_k] = \alpha_k; \\
G_{\theta_k}^{(1)}(\theta_k) &= \frac{1}{\Delta}[(G_{d_k}^{(1)}\Delta_1 - G_{c_k}^{(1)}\Delta_{12})\cos\theta_k - (G_{c_k}^{(1)}\Delta_2 - G_{d_k}^{(1)}\Delta_{12})\sin\theta_k] = -\beta_k.
\end{aligned}
\tag{2.85}
$$

Furthermore, Eq. (2.79) gives

$$\dot{r}_k = \alpha_k r_k + o(r_k) \text{ and } \dot{\theta}_k r_k = -\beta_k r_k + o(r_k). \tag{2.86}$$

As $r_k << 1$ and $r_k \to 0$, we have

$$\dot{r}_k = \alpha_k r_k \text{ and } \dot{\theta}_k = -\beta_k. \tag{2.87}$$

With an initial condition of $r_k = r_k^0$ and $\theta_k = \theta_k^0$, the corresponding solution of Eq. (2.87) is

$$r_k = r_k^0 e^{\alpha_k t} \text{ and } \theta_k = -\beta_k t + \theta_k^0. \tag{2.88}$$

and

$$\begin{aligned} c_k &= r_k^0 e^{\alpha_k t}\cos(-\beta_k t + \theta_k^0) = e^{\alpha_k t}[\cos(\beta_k t)c_k^0 + \sin(\beta_k t)d_k^0]; \\ d_k &= r_k^0 e^{\alpha_k t}\sin(-\beta_k t + \theta_k^0) = e^{\alpha_k t}[-\sin(\beta_k t)c_k^0 + \cos(\beta_k t)d_k^0]. \end{aligned} \tag{2.89}$$

Letting $\mathbf{c}^{(k)} = (c^{(k)}, d^{(k)})^{\mathbf{T}}$, we have

$$\dot{\mathbf{c}}^{(k)} = \mathbf{E}_k \mathbf{c}^{(k)} \Rightarrow \mathbf{c}^{(k)} = e^{\alpha_k t}\mathbf{B}_k \mathbf{c}_0^{(k)} \tag{2.90}$$

where

$$\mathbf{E}_i = \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix} \text{ and } \mathbf{B}_k = \begin{bmatrix} \cos\beta_k t & \sin\beta_k t \\ -\sin\beta_k t & \cos\beta_k t \end{bmatrix}. \tag{2.91}$$

If $G_{r_k}^{(m)}(\theta_k)$ and $G_{\theta_k}^{(m)}(\theta_k)$ are dependent on θ_k, Eq. (2.79) gives the dynamical systems based on the polar coordinates on the invariant plane of $(\mathbf{u}_k, \mathbf{v}_k)$ of matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k$. If $G_{r_k}^{(m)}(\theta_k)$ and $G_{\theta_k}^{(m)}(\theta_k)$ are independent of θ_k, the deformed dynamical system on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is dependent on r_k, then the G-functions can be used to determine the stability of $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$.

Definition 2.29 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ and $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is $C^r(r \geq 1)$-continuous in a neighborhood of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. Suppose $U(\mathbf{x}^*) \subset \Omega$ is a neighborhood of equilibrium $\mathbf{x}^*$. For a linearized time-delay system in Eq. (2.19), consider a pair of complex eigenvalues $\alpha_k \pm \mathbf{i}\beta_k$ $(k \in N = \{1, 2, \cdots, l\},\ l < n,\ \mathbf{i} = \sqrt{-1})$ of matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}) + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k$. On the invariant plane of $(\mathbf{u}_k, \mathbf{v}_k)$, consider $\mathbf{y}^{(k)} = \mathbf{y}_+^{(k)} + \mathbf{y}_-^{(k)}$ with Eqs. (2.72) and (2.74). For any arbitrarily small $\varepsilon > 0$, the stability of the equilibrium $\mathbf{x}^*$ on the invariant plane of $(\mathbf{u}_k, \mathbf{v}_k)$ can be determined.

(i) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally stable if

$$r_k(t + \varepsilon) - r_k(t) < 0. \tag{2.92}$$

(ii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally unstable if

$$r_k(t + \varepsilon) - r_k(t) > 0. \tag{2.93}$$

(iii) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is stable with the m_kth-order singularity if for $\theta_k \in [0, 2\pi]$

$$\begin{aligned} &G_{r_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 0, 1, 2, \cdots, m_k - 1 \\ &r_k(t + \varepsilon) - r_k(t) < 0. \end{aligned} \tag{2.94}$$

(iv) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally unstable with the m_kth-order singularity if for $\theta_k \in [0, 2\pi]$

$$\begin{aligned} &G_{r_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 0, 1, 2, \cdots, m_k - 1 \\ &r_k(t+\varepsilon) - r_k(t) > 0. \end{aligned} \tag{2.95}$$

(v) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is circular if for $\theta_k \in [0, 2\pi]$

$$r_k(t+\varepsilon) - r_k(t) = 0. \tag{2.96}$$

(vi) $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is degenerate in the direction of $\mathbf{u}_k$ if

$$\beta_k = 0 \text{ and } \theta_k(t+\varepsilon) - \theta_k(t) = 0. \tag{2.97}$$

Theorem 2.10 *Consider an n-dimensional, autonomous, nonlinear, time-delay system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ *in Eq.* (2.4) *with an equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau *}$ *and* $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ *is* C^r *(* $r \geq 1$*)-continuous in a neighborhood of equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau *}$*. The corresponding solution is* $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$*. Suppose* $U(\mathbf{x}^*) \subset \Omega$ *is a neighborhood of equilibrium* $\mathbf{x}^*$*. For a linearized time-delay system in Eq.* (2.19)*, consider a pair of complex eigenvalues* $\alpha_k \pm \mathbf{i}\beta_k$ *(* $k \in N = \{1, 2, \cdots, l\}$*,* $n/2 \leq l < n$*,* $\mathbf{i} = \sqrt{-1}$*) of matrix* $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}) + e^{-(\alpha_k \pm \mathbf{i}\beta_k)\tau} D_{\mathbf{x}^{\tau}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p})$ *with a pair of eigenvectors* $\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k$*. On the invariant plane of* $(\mathbf{u}_k, \mathbf{v}_k)$*, consider* $\mathbf{y}^{(k)} = \mathbf{y}_+^{(k)} + \mathbf{y}_-^{(k)}$ *with* Eqs. (2.72) *and* (2.74) *with* $G_{r_k}^{(s_k)}(\theta_k) = const$*. For any arbitrarily small* $\varepsilon > 0$*, the stability of the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau *}$ *on the invariant plane of* $(\mathbf{u}_k, \mathbf{v}_k)$ *can be determined.*

(i) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau *}$ *on the plane of* $(\mathbf{u}_k, \mathbf{v}_k)$ *is spirally stable if and only if*

$$G_{r_k}^{(1)}(\theta_k) = \alpha_k < 0. \tag{2.98}$$

(ii) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau *}$ *on the plane of* $(\mathbf{u}_k, \mathbf{v}_k)$ *is spirally unstable if and only if*

$$G_{r_k}^{(1)}(\theta_k) = \alpha_k > 0. \tag{2.99}$$

(iii) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau *}$ *on the plane of* $(\mathbf{u}_k, \mathbf{v}_k)$ *is spirally stable with the* m_kth*-order singularity if and only if for* $\theta_k \in [0, 2\pi]$

$$\begin{aligned} & G_{r_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 1, 2, \cdots, m_k - 1 \\ & \quad \text{and } G_{r_k}^{(m_k)}(\theta_k) < 0. \end{aligned} \tag{2.100}$$

(iv) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^* = \mathbf{x}^{\tau *}$ *on the plane of* $(\mathbf{u}_k, \mathbf{v}_k)$ *is spirally unstable with the* m_kth-*order singularity if and only if for* $\theta_k \in [0, 2\pi]$

$$\begin{aligned} & G_{r_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 1, 2, \cdots, m_k - 1 \\ & \quad \text{and } G_{r_k}^{(m_k)}(\theta_k) > 0. \end{aligned} \tag{2.101}$$

(v) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^*$ *on the plane of* $(\mathbf{u}_k, \mathbf{v}_k)$*is circular if and only if for* $\theta_k \in [0, 2\pi]$

$$G_{r_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 1, 2, \cdots. \tag{2.102}$$

(vi) $\mathbf{x}^{(k)}$ *at the equilibrium* $\mathbf{x}^*$ *on the plane of* $(\mathbf{u}_k, \mathbf{v}_k)$ *is degenerate in the direction of* $\mathbf{u}_k$ *if and only if*

$$\operatorname{Im} \lambda_k = \beta_k = 0 \text{ and } G_{\theta_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 2, 3, \cdots. \tag{2.103}$$

Proof The proof is similar to the non-time-delay systems as in Luo (2012). Consider the first-order approximation of $\dot{c}_k$ and $\dot{d}_k$ in Taylor series expansion gives

$$\begin{aligned} \dot{c}_k &= \frac{1}{\Delta}[\Delta_2 G_{c_k}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) - \Delta_{12} G_{d_k}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})] \\ \dot{d}_k &= \frac{1}{\Delta}[\Delta_1 G_{d_k}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) - \Delta_{12} G_{c_k}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})] \end{aligned}$$

where $r_k = \sqrt{c_k^2 + d_k^2}$ and

$$\begin{aligned} G_{c_k 1}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}} \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{c_k} \mathbf{x} + \frac{\partial c_k^\tau}{\partial c_k} \mathbf{u}_k^{\mathrm{T}} \cdot D_{\mathbf{x}^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{c_k^\tau} \mathbf{x}^\tau \\ &\quad + \frac{\partial d_k^\tau}{\partial c_k} \mathbf{u}_k^{\mathrm{T}} \cdot D_{\mathbf{x}^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{d_k^\tau} \mathbf{x}^\tau \\ &= \alpha_k \Delta_1 - \beta_k \Delta_{12}, \\ G_{c_k 2}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}} \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{c_k} \mathbf{x} + \frac{\partial c_k^\tau}{\partial c_k} \mathbf{u}_k^{\mathrm{T}} \cdot D_{\mathbf{x}^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{c_k^\tau} \mathbf{x}^\tau \\ &\quad + \frac{\partial d_k^\tau}{\partial c_k} \mathbf{u}_k^{\mathrm{T}} \cdot D_{\mathbf{x}^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{d_k^\tau} \mathbf{x}^\tau \\ &= \alpha_k \Delta_{12} + \beta_k \Delta_1; \end{aligned}$$

and

$$
\begin{aligned}
G_{d_k 1}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}} \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{c_k} \mathbf{x} + \frac{\partial c_k^\tau}{\partial d_k} \mathbf{v}_k^{\mathrm{T}} \cdot D_{\mathbf{x}^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{c_k^\tau} \mathbf{x}^\tau \\
&\quad + \frac{\partial d_k^\tau}{\partial d_k} \mathbf{v}_k^{\mathrm{T}} \cdot D_{\mathbf{x}^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{d_k^\tau} \mathbf{x}^\tau \\
&= -\beta_k \Delta_2 + \alpha_k \Delta_{12}, \\
G_{d_k 2}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}} \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{c_k} \mathbf{x} + \frac{\partial c_k^\tau}{\partial d_k} \mathbf{v}_k^{\mathrm{T}} \cdot D_{\mathbf{x}^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{c_k^\tau} \mathbf{x}^\tau \\
&\quad + \frac{\partial d_k^\tau}{\partial d_k} \mathbf{v}_k^{\mathrm{T}} \cdot D_{\mathbf{x}^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) \partial_{d_k^\tau} \mathbf{x}^\tau \\
&= \alpha_k \Delta_2 + \beta_k \Delta_{12}.
\end{aligned}
$$

Therefore, using

$$
\begin{aligned}
G_{c_k}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) &= G_{c_k 1}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) c_k + G_{c_k 2}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) d_k, \\
G_{d_k}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) &= G_{d_k 1}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) c_k + G_{d_k 2}^{(1)}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) d_k
\end{aligned}
$$

to the first-order approximation of $\dot{c}_k$ and $\dot{d}_k$ yields

$$
\dot{c}_k = \alpha_k c_k + \beta_k d_k \text{ and } \dot{d}_k = -\beta_k c_k + \alpha_k d_k
$$

or

$$
\begin{bmatrix} \dot{c}_k \\ \dot{d}_k \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix} \begin{bmatrix} c_k \\ d_k \end{bmatrix}.
$$

Introduce the rotation coordinates $(\mathbf{e}_{r_k}, \mathbf{e}_{\theta_k})$

$$
\mathbf{r}_k = c_k \mathbf{u}_k + d_k \mathbf{v}_k = r_k \mathbf{e}_{r_k},
$$

where

$$
\begin{aligned}
c_k &= r_k \cos\theta_k, d_k = r_k \sin\theta_k; \\
\mathbf{e}_{r_k} &= \cos\theta_k \mathbf{u}_k + \sin\theta_k \mathbf{v}_k, \\
\mathbf{e}_{\theta_k} &= -\cos\theta_k \mathbf{u}_k^{\perp} \Delta_3 + \sin\theta_k \mathbf{v}_k^{\perp} \Delta_4
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\mathbf{r}}_k &= \dot{c}_k \mathbf{u}_k + \dot{d}_k \mathbf{v}_k = \dot{r}_k \mathbf{e}_{r_k} + r_k \dot{\mathbf{e}}_{r_k}, \\
\dot{\mathbf{e}}_{r_k} &= -\dot{\theta}_k \mathbf{u}_k \sin\theta_k + \dot{\theta}_k \mathbf{v}_k \cos\theta_k.
\end{aligned}
$$

Thus,

$$\dot{r}_k = \dot{c}_k \cos\theta_k + \dot{d}_k \sin\theta_k,$$
$$\dot{\theta}_k = r_k^{-1}(\dot{d}_k \cos\theta_k - \dot{c}_k \sin\theta_k).$$

For the first approximation of the relative change rate in the $\mathbf{e}_{r_k}$-direction, we obtain

$$\begin{aligned}\dot{r}_k &= (\alpha_k c_k + \beta_k d_k)\cos\theta_k + (-\beta_k c_k + \alpha_k d_k)\sin\theta_k \\ &= \alpha_k r_k.\end{aligned}$$

Further

$$\dot{r}_k = \alpha_k r_k.$$

Similarly, the first approximation of rotation speed in the hoop direction is

$$\begin{aligned}\dot{\theta}_k r_k &= (-\beta_k c_k + \alpha_k d_k)\cos\theta_k + (\alpha_k c_k + \beta_k d_k)\sin\theta_k \\ &= -\beta_k r_k,\end{aligned}$$

so

$$\dot{\theta}_k r_k = -\beta_k r_k \Rightarrow \dot{\theta}_k = -\beta_k.$$

Therefore,

$$G_{r_k}^{(1)}(\theta_k) = \alpha_k \text{ and } G_{\theta_k}^{(1)}(\theta_k) = -\beta_k.$$

In fact, the relative change rate in the $\mathbf{e}_{r_k}$-direction is of interest. The corresponding higher-order expression is given by

$$\dot{r}_k = \sum\nolimits_{s_k=1}^{m_k-1} \frac{1}{s_k!} G_{r_k}^{(s_k)}(\theta_k) r_k^{s_k} + \frac{1}{m_k!} G_{r_k}^{(m_k)}(\theta_k) r_k^{m_k} + o(r_k^{m_k}).$$

Because for $\varepsilon > 0$ and $\varepsilon \to 0$,

$$\begin{aligned}r_k(t+\varepsilon) - r_k(t) &= \dot{r}_k \varepsilon \\ &= \varepsilon \sum\nolimits_{s_k=1}^{m_k-1} \frac{1}{s_k!} G_{r_k}^{(s_k)}(\theta_k) r_k^{s_k} + \varepsilon \frac{1}{m_k!} G_{r_k}^{(m_k)}(\theta_k) r_k^{m_k} + o(\varepsilon r_k^{m_k}).\end{aligned}$$

(i) For equilibrium stability, $r_k > 0$ and $r_k \to 0$. If $G_{r_k}^{(1)}(\theta_k) = \alpha_k \neq 0$, we have

$$\dot{r}_k = G_{r_k}^{(1)}(\theta_k) r_k = \alpha_k r_k.$$

Due to $r_k > 0$, if $\alpha_k < 0$, then $\dot{r}_k < 0$. Therefore,

$$r_k(t+\varepsilon) - r_k(t) = \dot{r}_k \varepsilon < 0$$

which implies $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally stable, vice versa.

(ii) Due to $r_k > 0$, if $\alpha_k > 0$, then $\dot{r}_k > 0$. Thus,

$$r_k(t+\varepsilon) - r_k(t) = \dot{r}_k \varepsilon > 0,$$

which implies $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally unstable, vice versa.

(iii) If for $\theta_k \in [0, 2\pi]$ the following conditions exist:

$$\begin{aligned} &G_{r_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 1, 2, \cdots, m_k - 1; \\ &G_{r_k}^{(m_k)}(\theta_k) \neq 0, \text{and} |G_{r_k}^{(s_k)}(\theta_k)| < \infty \text{ for } s_k = m_k + 1, m_k + 2, \cdots, \end{aligned}$$

then the higher-order terms can be ignored, i.e.,

$$\dot{r}_k = \frac{1}{m_k!} G_{r_k}^{(m_k)}(\theta_k) r_k^{m_k}.$$

If $G_{r_k}^{(m_k)}(\theta_k)$ is independent of θ_k (i.e., $G_{r_k}^{(m_k)}(\theta_k) = \text{const}$), it can be used to determine the equilibrium stability. Due to $r_k > 0$, if $G_{r_k}^{(m_k)}(\theta_k) < 0$, then $\dot{r}_k < 0$. Therefore,

$$r_k(t+\varepsilon) - r_k(t) = \dot{r}_k \varepsilon < 0.$$

In other words, $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally stable with the m_kth-order singularity, vice versa.

(iv) Due to $r_k > 0$, if $G_{r_k}^{(m_k)}(\theta_k) > 0$, then $\dot{r}_k > 0$. Therefore,

$$r_k(t+\varepsilon) - r_k(t) = \dot{r}_k \varepsilon > 0.$$

In other words, $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is spirally unstable with the $(m_k - 1)$th-order singularity, vice versa.

(v) If for $\theta_k \in [0, 2\pi]$ the following conditions exist:

$$G_{r_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 1, 2, \cdots,$$

then

$$r_k(t+\varepsilon) - r_k(t) = \dot{r}_k \varepsilon = 0,$$

vice versa. Therefore, $r_k(t)$ is constant. $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is circular.

(vi) Consider

$$\begin{aligned} &\theta_k(t+\varepsilon) - \theta_k(t) = \dot{\theta}_k \varepsilon \\ &= \varepsilon[-\beta_k + \sum\nolimits_{s_k=2}^{m_k-1} \frac{1}{s_k!} G_{\theta_k}^{(s_k)}(\theta_k) r_k^{s_k-1} + \frac{1}{m_k!} G_{\theta_k}^{(m_k)}(\theta_k) r_k^{m_k-1} + o(r_k^{m_k-1})]. \end{aligned}$$

If for $\theta_k \in [0, 2\pi]$ the following conditions exist:

$$\beta_k = 0 \text{ and } G_{\theta_k}^{(s_k)}(\theta_k) = 0 \text{ for } s_k = 2, 3, \cdots$$

Then,

$$\theta_k(t+\varepsilon) - \theta_k(t) = \dot{\theta}_k \varepsilon = 0.$$

Therefore, $\mathbf{x}^{(k)}$ at the equilibrium $\mathbf{x}^*$ on the plane of $(\mathbf{u}_k, \mathbf{v}_k)$ is degenerate in the direction of $\mathbf{u}_k$. This theorem is proved. ■

Note that $G_{r_k}^{(s_k)}(\theta_k) = const$ requires $s_k = 2m_k - 1$ and one obtains $G_{r_k}^{(s_k)}(\theta_k) = 0$ for $s_k = 2m_k$.

2.3.2 *Bifurcations*

Definition 2.30 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ in Eq. (2.4) with an equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ and $\mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is $C^r(r \geq 1)$-continuous in a neighborhood of $\mathbf{x}^* = \mathbf{x}^{\tau *}$ (i.e., $U(\mathbf{x}^*) \subset \Omega$). The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. Suppose Eq. (2.24) holds in $U(\mathbf{x}^*) \subset \Omega$. For a linearized time-delay system in Eq. (2.19), consider a real eigenvalue λ_k of matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}^*) + e^{-\lambda \tau} D_{\mathbf{x}^\tau}\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau *}, \mathbf{p}^*)$ $(k \in N = \{1, 2, \cdots, n\})$ with an eigenvector $\mathbf{v}_k$. Suppose one of n independent solutions $\mathbf{y} = c_k \mathbf{v}_k$ and $\dot{\mathbf{y}} = \dot{c}_k \mathbf{v}_k$,

$$\begin{aligned} s_k &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y} = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x} - \mathbf{x}^*), \\ s_k^\tau &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y}^\tau = \mathbf{v}_k^{\mathrm{T}} \cdot (\mathbf{x}^\tau - \mathbf{x}^{\tau *}) \end{aligned} \tag{2.104}$$

where $s_k = c_k ||\mathbf{v}_k||^2$.

$$\dot{s}_k = \mathbf{v}_k^{\mathrm{T}} \cdot \dot{\mathbf{y}} = \mathbf{v}_k^{\mathrm{T}} \cdot \dot{\mathbf{x}} = \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}). \tag{2.105}$$

In the vicinity of point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau *}, \mathbf{p}_0)$, $\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})$ is expended for $(0<\theta<1)$ as follows:

$$\begin{aligned}
\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p}) &= a_k(s_k - s_{k0}^*) + b_k(s_k^\tau - s_{k0}^{\tau *}) + \mathbf{b}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\
&\quad + \sum_{r=2}^{m} \sum_{\substack{r_1, r_2, r_3 = 0 \\ (r_1 + r_2 + r_3 = r)}}^{r} \begin{pmatrix} r \\ r_1, r_2, r_3 \end{pmatrix} \mathbf{a}_k^{(r_1, r_2, r_3)} (s_k - s_{k0}^*)^{r_1} (s_k^\tau - s_{k0}^{\tau *})^{r_2} (\mathbf{p} - \mathbf{p}_0)^{r_3} \\
&\quad + [(s_k - s_{k0}^*)\partial_{s_k} + (s_k^\tau - s_{k0}^{\tau *})\partial_{s_k^\tau} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
&\quad \times (\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_0^* + \theta \Delta \mathbf{x}, \mathbf{x}_0^{\tau *} + \theta \Delta \mathbf{x}^\tau, \mathbf{p}_0 + \theta \Delta \mathbf{p}))
\end{aligned} \tag{2.106}$$

where

$$\begin{aligned}
& a_k = \mathbf{v}_k^{\mathrm{T}} \cdot \partial_{s_k} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau *}, \mathbf{p}_0)}, b_k = \mathbf{v}_k^{\mathrm{T}} \cdot \partial_{s_k^\tau} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau *}, \mathbf{p}_0)} \\
& \mathbf{b}_k^{\mathrm{T}} = \mathbf{v}_k^{\mathrm{T}} \cdot \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\big|_{(\mathbf{x}_0^*, \mathbf{p}_0)}, \mathbf{a}_k^{(r_1, r_2, r_3)} = \mathbf{v}_k^{\mathrm{T}} \cdot \partial_{s_k}^{(r_1)} \partial_{s_k^\tau}^{(r_2)} \partial_{\mathbf{p}}^{(r_3)} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{p}_0)}, \\
& \begin{pmatrix} l \\ r_1, r_2, r_3 \end{pmatrix} = \frac{l!}{r_1! r_2! r_3!} = C_{r_1}^{l} C_{r_2}^{l - r_1} C_{r_3}^{l - r_1 - r_2}.
\end{aligned} \tag{2.107}$$

If $a_k + e^{-\lambda_k \tau} b_k = 0$ with $\lambda_k = 0$ at $\mathbf{p} = \mathbf{p}_0$, the stability of current equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ on an eigenvector $\mathbf{v}_k$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_k$ is determined by

$$\begin{aligned}
& \sum_{l=2}^{m} \sum_{\substack{r_1, r_2, r_3 = 0 \\ (r_1 + r_2 + r_3 = l)}}^{l} \begin{pmatrix} l \\ r_1, r_2, r_3 \end{pmatrix} \mathbf{a}_k^{(r_1, r_2, r_3)} (s_k - s_{k0}^*)^{r_1} (s_k^\tau - s_{k0}^{\tau *})^{r_2} (\mathbf{p} - \mathbf{p}_0)^{r_3} \\
& \quad + \mathbf{b}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0.
\end{aligned} \tag{2.108}$$

In the neighborhood of $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau *}, \mathbf{p}_0)$, when other components of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ on the eigenvector of $\mathbf{v}_j$ for all $j \neq k$, $(j, k \in N)$ do not change their stability states, Eq. (2.108) possesses l-branch solutions of equilibrium $s_k^* = s_k^{\tau *}$ $(0<l\leq m)$ with l_1-stable and l_2-unstable solutions $(l_1, l_2 \in \{0, 1, 2, \cdots, l\})$. Such l-branch solutions are called the bifurcation solutions of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau *}$ on the eigenvector of $\mathbf{v}_k$ in the neighborhood of $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau *}, \mathbf{p}_0)$. Such a bifurcation at point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau *}, \mathbf{p}_0)$ is called the hyperbolic bifurcation of mth-order on the eigenvector of $\mathbf{v}_k$. Three special cases are defined as follows:

(i) If

$$\begin{aligned}
&\mathbf{a}_k^{(0,0,2)} = \mathbf{0}, \mathbf{a}_k^{(0,1,1)} = \mathbf{0}, \mathbf{a}_k^{(1,0,1)} = \mathbf{0}, \\
&\mathbf{b}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + \frac{1}{2!}[a_k^{(2,0,0)} + 2a_k^{(1,1,0)} + a_k^{(0,2,0)}](s_k^* - s_{k0}^*)^2 = 0, \\
&\text{or} \\
&\mathbf{b}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + \frac{1}{2!}G_k^{(2,0)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)(s_k^* - s_{k0}^*)^2 = 0
\end{aligned} \tag{2.109}$$

where

$$\begin{aligned}
G_k^{(2,0)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) &= \mathbf{v}_k^{\mathrm{T}} \cdot [\partial_{s_k}^{(1)}(\cdot) + \partial_{s_k^\tau}^{(1)}(\cdot)]^2 \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} \neq 0 \\
\mathbf{G}_k^{(1,1)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) &= \mathbf{v}_k^{\mathrm{T}} \cdot [\partial_{s_k}^{(1)}(\cdot) + \partial_{s_k^\tau}^{(1)}(\cdot)]\partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} = \mathbf{0} \\
\mathbf{G}_k^{(0,2)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) &= \mathbf{v}_k^{\mathrm{T}} \cdot \partial_{\mathbf{p}}^{(2)} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} = \mathbf{0} \\
G_k^{(1,0)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) &= \mathbf{v}_k^{\mathrm{T}} \cdot [\partial_{s_k}^{(1)}(\cdot) + \partial_{s_k^\tau}^{(1)}(\cdot)] \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} = 0 \\
\mathbf{b}_k^{\mathrm{T}} &= \mathbf{v}_k^{\mathrm{T}} \cdot \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} \neq \mathbf{0},
\end{aligned} \tag{2.110}$$

$$G_k^{(2,0)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) \times [\mathbf{b}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.111}$$

such a bifurcation at point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ is called the *saddle-node* bifurcation on the eigenvector of $\mathbf{v}_k$.

(ii) If

$$\begin{aligned}
&\mathbf{b}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0, \\
&(\mathbf{a}_k^{(1,0,1)} + \mathbf{a}_k^{(0,1,1)}) \cdot (\mathbf{p} - \mathbf{p}_0)(s_k^* - s_{k0}^*) + \frac{1}{2!}G_k^{(2,0)}(s_k^* - s_{k0}^*)^2 = 0 \\
&\text{or} \\
&\mathbf{G}_k^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)(s_k^* - s_{k0}^*) + \frac{1}{2!}G_k^{(2,0)}(s_k^* - s_{k0}^*)^2 = 0
\end{aligned} \tag{2.112}$$

where

$$\mathbf{G}_k^{(1,1)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) = \mathbf{v}_k^{\mathrm{T}} \cdot [\partial_{s_k}^{(1)}(\cdot) + \partial_{s_k^\tau}^{(1)}(\cdot)]\partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}, \mathbf{x}^\tau, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} \neq \mathbf{0}, \tag{2.113}$$

$$G_k^{(2,0)} \times [\mathbf{G}_k^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)] \neq 0, \tag{2.114}$$

such a bifurcation at point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ is called the *transcritical* bifurcation on the eigenvector of $\mathbf{v}_k$.

(iii) If

$$\begin{aligned} &\mathbf{b}_{\alpha}^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0, \\ &G_k^{(2,0)} = 0,\ \mathbf{G}_k^{(2,1)} = 0,\ \mathbf{G}_k^{(1,2)} = 0, \\ &\mathbf{G}_k^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)(s_k^* - s_{k0}^*) + G_k^{(3,0)} (s_k^* - s_{k0}^*)^3 = 0 \end{aligned} \tag{2.115}$$

where

$$\begin{aligned} &G_k^{(3,0)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) = \mathbf{v}_k^{\mathrm{T}} \cdot [\partial_{s_k}^{(1)}(\cdot) + \partial_{s_k^{\tau}}^{(1)}(\cdot)]^3 \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} \neq 0, \\ &\mathbf{G}_k^{(2,1)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) = \mathbf{v}_k^{\mathrm{T}} \cdot [\partial_{s_k}^{(1)}(\cdot) + \partial_{s_k^{\tau}}^{(1)}(\cdot)]^2 \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} = \mathbf{0}, \\ &\mathbf{G}_k^{(1,2)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) = \mathbf{v}_k^{\mathrm{T}} \cdot [\partial_{s_k}^{(1)}(\cdot) + \partial_{s_k^{\tau}}^{(1)}(\cdot)] \partial_{\mathbf{p}}^{(2)} \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} = \mathbf{0}, \\ &\mathbf{G}_k^{(0,3)}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0) = \mathbf{v}_k^{\mathrm{T}} \cdot \partial_{\mathbf{p}}^{(3)} \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})\Big|_{(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)} = \mathbf{0}, \end{aligned} \tag{2.116}$$

$$G_k^{(3,0)} \times [\mathbf{a}_k^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.117}$$

such a bifurcation at point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ is called the *pitchfork* bifurcation on the eigenvector of $\mathbf{v}_k$.

The bifurcation points possess the higher-order singularity of a flow in a dynamical system. For the saddle-node bifurcation, the $(2m)$th-order singularity of the flow at the bifurcation point exists as a saddle of the $(2m)$th-order. For the transcritical bifurcation, the $(2m)$th-order singularity of the flow at the bifurcation point exists as a saddle of the $(2m)$th-order. However, for the stable pitchfork bifurcation, the $(2m+1)$th-order singularity of the flow at the bifurcation point exists as a sink of the $(2m+1)$th-order. For the unstable pitchfork bifurcation, the $(2m+1)$th-order singularity of the flow at the bifurcation point exists as a source of the $(2m+1)$th-order.

Definition 2.31 Consider an n-dimensional, autonomous, nonlinear, time-delay system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ in Eq. (2.4) with an equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ and $\mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in a neighborhood of $\mathbf{x}^* = \mathbf{x}^{\tau*}$. The corresponding solution is $\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x}_0, t - t_0, \mathbf{p})$. Suppose $U(\mathbf{x}^*) \subset \Omega$ is a neighborhood of equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$, and there are n linearly independent vectors $\mathbf{v}_k$ $(k = 1, 2, \cdots, n)$. For a linearized time-delay system in Eq. (2.19), consider a pair of complex eigenvalues $\alpha_k \pm \mathbf{i}\beta_k$ $(k \in N = \{1, 2, \cdots, n\}, \mathbf{i} = \sqrt{-1})$ of matrix $D\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_k \pm \mathbf{i}\mathbf{v}_k$. On the invariant plane of $(\mathbf{u}_k, \mathbf{v}_k)$, consider $\mathbf{r}_k = \mathbf{y}_+^{(k)} + \mathbf{y}_-^{(k)}$ with

$$\begin{aligned} \mathbf{r}_k &= c_k \mathbf{u}_k + d_k \mathbf{v}_k = r_k \mathbf{e}_{r_k}, \\ \dot{\mathbf{r}}_k &= \dot{c}_k \mathbf{u}_k + \dot{d}_k \mathbf{v}_k = \dot{r}_k \mathbf{e}_{r_k} + r_k \dot{\mathbf{e}}_{r_k} \end{aligned} \tag{2.118}$$

and

$$
\begin{aligned}
&c_k = \frac{1}{\Delta}[\Delta_2(\mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{y}) - \Delta_{12}(\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y})] \text{ and } d_k = \frac{1}{\Delta}[\Delta_1(\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{y}) - \Delta_{12}(\mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{y})] \\
&\Delta_1 = ||\mathbf{u}_k||^2,\ \Delta_2 = ||\mathbf{v}_k||^2, \Delta_{12} = \mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{v}_k \text{ and } \Delta = \Delta_1\Delta_2 - \Delta_{12}^2
\end{aligned}
\tag{2.119}
$$

Consider a polar coordinate of (r_k, θ_k) defined by

$$
\begin{aligned}
c_k &= r_k \cos\theta_k, \text{and } d_k = r_k \sin\theta_k; \\
r_k &= \sqrt{c_k^2 + d_k^2}, \text{and } \theta_k = \arctan d_k/c_k; \\
\mathbf{e}_{r_k} &= \cos\theta_k \mathbf{u}_k + \sin\theta_k \mathbf{v}_k \text{ and} \\
\mathbf{e}_{\theta_k} &= -\cos\theta_k \mathbf{u}_k^{\perp}\Delta_3 + \sin\theta_k \mathbf{v}_k^{\perp}\Delta_4, \\
\Delta_3 &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{u}_k^{\perp} \text{ and } \Delta_4 = \mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{v}_k^{\perp}.
\end{aligned}
\tag{2.120}
$$

Thus,

$$
\begin{aligned}
\dot{c}_k &= \frac{1}{\Delta}[\Delta_2 G_{c_k}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) - \Delta_{12} G_{d_k}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})], \\
\dot{d}_k &= \frac{1}{\Delta}[\Delta_1 G_{d_k}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) - \Delta_{12} G_{d_k}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p})]
\end{aligned}
\tag{2.121}
$$

where

$$
\begin{aligned}
G_{c_k}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) \\
&= \mathbf{a}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + a_{k11}(c_k - c_{k0}^*) + a_{k12}(d_k - d_{k0}^*) \\
&\quad + \sum_{q=2}^{m}\sum_{r=0}^{q} \frac{1}{q!} C_q^r \mathbf{G}_{c_k}^{(q-r,r)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}_0)(\mathbf{p} - \mathbf{p}_0)^r r_k^{q-r} \\
&\quad + \frac{1}{(m+1)!}[(c_k - c_{k0}^*)\partial_{c_k} + (d_k - d_{k0}^*)\partial_{d_k} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
&\quad \times (\mathbf{u}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_0^* + \theta\Delta\mathbf{x}, \mathbf{x}_0^{\tau*} + \theta\Delta\mathbf{x}^{\tau}, \mathbf{p}_0 + \theta\Delta\mathbf{p})), \\
G_{d_k}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}, \mathbf{x}^{\tau}, \mathbf{p}) \\
&= \mathbf{b}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + a_{k21}(c_k - c_{k0}^*) + a_{k22}(d_k - d_{k0}^*) \\
&\quad + \sum_{q=2}^{m}\sum_{r=0}^{q} \frac{1}{q!} C_q^r \mathbf{G}_{c_k}^{(q-r,r)}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p}_0)(\mathbf{p} - \mathbf{p}_0)^r r_k^{q-r} \\
&\quad + \frac{1}{(m+1)!}[(c_k - c_{k0}^*)\partial_{c_k} + (d_k - d_{k0}^*)\partial_{d_k} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
&\quad \times (\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_0^* + \theta\Delta\mathbf{x}, \mathbf{x}_0^{\tau*} + \theta\Delta\mathbf{x}^{\tau}, \mathbf{p}_0 + \theta\Delta\mathbf{p}));
\end{aligned}
\tag{2.122}
$$

and

$$\begin{aligned}
\mathbf{G}_{c_k}^{(s,r)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}) &= \mathbf{u}_k^{\mathrm{T}}\cdot\{[\partial_{\mathbf{x}}()+\frac{c_k^{\tau}-c_{k0}^{\tau*}}{c_k-c_{k0}^{*}}\partial_{\mathbf{x}^{\tau}}()]\mathbf{u}_k\cos\theta_k \\
&\quad +[\partial_{\mathbf{x}}()+\frac{d_k^{\tau}-d_k^{\tau*}}{d_k-d_{k0}^{*}}\partial_{\mathbf{x}^{\tau}}()]\mathbf{v}_k\sin\theta_k]\}^s\partial_{\mathbf{p}}^{(r)}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\Big|_{(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p})}, \\
\mathbf{G}_{d_k}^{(s,r)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}) &= \mathbf{v}_k^{\mathrm{T}}\cdot\{[\partial_{\mathbf{x}}()+\frac{c_k^{\tau}-c_{k0}^{\tau*}}{c_k-c_{k0}^{*}}\partial_{\mathbf{x}^{\tau}}()]\mathbf{u}_k\cos\theta_k \\
&\quad +[\partial_{\mathbf{x}}()+\frac{d_k^{\tau}-d_{k0}^{\tau*}}{d_k-d_{k0}^{*}}\partial_{\mathbf{x}^{\tau}}()]\mathbf{v}_k\sin\theta_k]^s\partial_{\mathbf{p}}^{(r)}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\Big|_{(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p})};
\end{aligned} \tag{2.123}$$

$$\begin{aligned}
\mathbf{a}_k^{\mathrm{T}} &= \mathbf{u}_k^{\mathrm{T}}\cdot\partial_{\mathbf{p}}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p}),\ \mathbf{b}_k^{\mathrm{T}}=\mathbf{v}_k^{\mathrm{T}}\cdot\partial_{\mathbf{p}}\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p}); \\
a_{k11} &= \mathbf{u}_k^{\mathrm{T}}\cdot[\partial_{\mathbf{x}}()+\frac{c_k^{\tau}-c_{k0}^{\tau*}}{c_k-c_{k0}^{*}}\partial_{\mathbf{x}^{\tau}}()]\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\mathbf{u}_k, \\
a_{k12} &= \mathbf{u}_k^{\mathrm{T}}\cdot[\partial_{\mathbf{x}}()+\frac{d_k^{\tau}-d_k^{\tau*}}{d_k-d_{k0}^{*}}\partial_{\mathbf{x}^{\tau}}()]\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\mathbf{v}_k; \\
a_{k21} &= \mathbf{v}_k^{\mathrm{T}}\cdot[\partial_{\mathbf{x}}()+\frac{c_k^{\tau}-c_{k0}^{\tau*}}{c_k-c_{k0}^{*}}\partial_{\mathbf{x}^{\tau}}()]\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\mathbf{u}_k, \\
a_{k22} &= \mathbf{v}_k^{\mathrm{T}}\cdot[\partial_{\mathbf{x}}()+\frac{d_k^{\tau}-d_k^{\tau*}}{d_k-d_{k0}^{*}}\partial_{\mathbf{x}^{\tau}}()]\mathbf{f}(\mathbf{x},\mathbf{x}^{\tau},\mathbf{p})\mathbf{v}_k.
\end{aligned} \tag{2.124}$$

Thus,

$$\begin{aligned}
\dot r_k &= \dot c_k\cos\theta_k+\dot d_k\sin\theta_k \\
&= \sum_{q=1}^{m}\sum_{r=0}^{q}\frac{1}{q!}C_q^r\mathbf{G}_{r_k}^{(q-r,r)}(\theta_k,\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0)^{q-r}r_k^r, \\
\dot\theta_k &= r_k^{-1}(\dot d_k\cos\theta_k-\dot c_k\sin\theta_k) \\
&= \sum_{q=1}^{m}\sum_{r=0}^{q}\frac{1}{q!}C_q^r\mathbf{G}_{\theta_k}^{(q-r,r)}(\theta_k,\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0)^{q-r}r_k^r
\end{aligned} \tag{2.125}$$

where

$$\begin{aligned}
\mathbf{G}_{r_k}^{(m-r,r)}(\theta_k,\mathbf{p}_0) &= \frac{1}{\Delta}[(\Delta_2\cos\theta_k-\Delta_{12}\sin\theta_k)\mathbf{G}_{c_k}^{(m-r,r)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}_0) \\
&\quad +(\Delta_2\sin\theta_k-\Delta_{12}\cos\theta_k)\mathbf{G}_{d_k}^{(m-r,r)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}_0)], \\
\mathbf{G}_{\theta_k}^{(m-r,r)}(\theta_k,\mathbf{p}_0) &= -\frac{1}{\Delta}[(\Delta_2\sin\theta_k+\Delta_{12}\cos\theta_k)\mathbf{G}_{c_k}^{(m-r,r)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}_0) \\
&\quad -(\Delta_1\cos\theta_k-\Delta_{12}\sin\theta_k)\mathbf{G}_{d_k}^{(m-r,r)}(\mathbf{x}^*,\mathbf{x}^{\tau*},\mathbf{p}_0)].
\end{aligned} \tag{2.126}$$

Suppose

$$\mathbf{a}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0 \text{ and } \mathbf{b}_k^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0, \tag{2.127}$$

then

$$\begin{aligned} \dot{r}_k &= (\alpha_k + \mathbf{G}_{r_k}^{(1,1)}(\theta_k, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0))r_k + \frac{1}{3!} G_{r_k}^{(3,0)}(\theta_k, \mathbf{p}_0) r_k^3 + o(r_k^3), \\ \dot{\theta}_k &= \beta_k + \mathbf{G}_{\theta_k}^{(1,1)}(\theta_k, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0) + \frac{1}{3!} \mathrm{G}_{\theta_k}^{(3,0)}(\theta_k, \mathbf{p}_0) r_k^2 + o(r_k^2) \end{aligned} \tag{2.128}$$

where

$$\begin{aligned} \mathbf{G}_{r_k}^{(1,1)}(\theta_k, \mathbf{p}_0) &= \mathbf{G}_{r_k}^{(1,1)}(\mathbf{p}_0) \text{ and } G_{r_k}^{(3,0)}(\theta_k, \mathbf{p}_0) = G_{r_k}^{(3,0)}(\mathbf{p}_0), \\ \mathbf{G}_{\theta_k}^{(1,1)}(\theta_k, \mathbf{p}_0) &= \mathbf{G}_{\theta_k}^{(1,1)}(\mathbf{p}_0) \text{ and } G_{\theta_k}^{(3,0)}(\theta_k, \mathbf{p}_0) = G_{\theta_k}^{(3,0)}(\mathbf{p}_0). \end{aligned} \tag{2.129}$$

If $\alpha_k = 0$ and $\mathbf{p} = \mathbf{p}_0$, the stability of current equilibrium $\mathbf{x}^* = \mathbf{x}^{\tau*}$ on an eigenvector plane of $(\mathbf{u}_k, \mathbf{v}_k)$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_k$ is determined by

$$\begin{aligned} (\alpha_{k0} + \mathbf{G}_{r_k}^{(1,1)}(\theta_k, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0))r_k + \frac{1}{3!} G_{r_k}^{(3,0)}(\theta_k, \mathbf{p}_0) r_k^3 &= 0, \\ \beta_{k0} + \mathbf{G}_{\theta_k}^{(1,1)}(\theta_k, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0) + \frac{1}{3!} \mathrm{G}_{\theta_k}^{(3,0)}(\theta_k, \mathbf{p}_0) r_k^2 &= 0 \end{aligned} \tag{2.130}$$

where

$$\begin{aligned} &\mathbf{G}_{r_k}^{(1,1)}(\theta_k, \mathbf{p}_0) = \partial_{\mathbf{p}} \alpha_k |_{(\mathbf{x}_0^*, \mathbf{p}_0)} \neq \mathbf{0}, \\ &[\mathbf{G}_{r_k}^{(1,1)}(\theta_k, \mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)] \times G_{r_k}^{(3,0)}(\theta_k, \mathbf{p}_0) < 0 \end{aligned} \tag{2.131}$$

Such a bifurcation at point $(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0)$ is called the Hopf bifurcation on the eigenvector plane of $(\mathbf{u}_k, \mathbf{v}_k)$.

For the repeated eigenvalues of $D\mathbf{f}(\mathbf{x}^*, \mathbf{x}^{\tau*}, \mathbf{p})$, the bifurcation of equilibrium can be similarly discussed in the foregoing two Theorems 2.9 and 2.10. Herein, such a procedure will not be repeated.

As in Luo (2012), the Hopf bifurcation points possess the higher-order singularity of the flow in dynamical system in the corresponding radial direction. For the stable Hopf bifurcation, the mth-order singularity of the flow at the bifurcation point exists as a sink of the mth-order in the radial direction. For the unstable Hopf bifurcation, the mth-order singularity of the flow at the bifurcation point exists as a source of the mth-order in the radial direction.

The stability and bifurcation of equilibriums are summarized in Fig. 2.1 with $\det(D\mathbf{f}) = \det(D\mathbf{f}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0))$ and $\mathrm{tr}(D\mathbf{f}) = \mathrm{tr}(D\mathbf{f}(\mathbf{x}_0^*, \mathbf{x}_0^{\tau*}, \mathbf{p}_0))$ for 2D nonlinear time-delay system. The thick dashed lines are bifurcation lines. The stability of

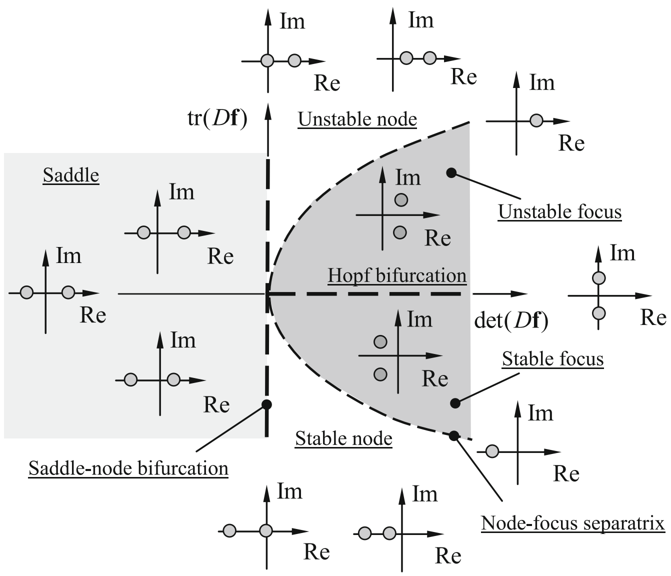

Fig. 2.1 Stability and bifurcation diagrams through the complex plane of eigenvalues for 2D nonlinear time-delay systems

equilibriums is given by the eigenvalues in complex plane. The stability of equilibriums for higher dimensional systems can be identified by using a naming of stability for linear dynamical systems in Luo (2012). The saddle-node bifurcation possesses stable saddle-node bifurcation (critical) and unstable saddle-node bifurcation (degenerate).

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