Chapter 2
Memorized Nonlinear Discrete Systems

In this chapter, basic concepts of memorized nonlinear discrete systems will be presented. The local theory of stability and bifurcation for memorized nonlinear discrete systems will be discussed. The stability switching and bifurcation on specific eigenvectors of the linearized memorized discrete system at fixed points under specific period will be presented. The higher order singularity and stability for memorized nonlinear discrete systems on the specific eigenvectors will be presented. 1-D memorized discrete dynamical systems will be discussed for an example.

2.1 Definitions

Definition 2.1 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^l$ with $\alpha \in \mathbb{Z}$, consider a vector function with memorized states $\mathbf{f}_\alpha : \Omega_\alpha \times \underbrace{\Omega_{\alpha-1} \times \cdots \times \Omega_{\alpha-s}}_{s} \times \Lambda \to \Omega_{\alpha+1}$ which is $C^r(r \geq 1)$-continuous, and there is a memorized discrete (or difference) equation in a form of

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p}_\alpha), \\ \mathbf{x}_r &= \mathbf{x}_r \text{ for } r = k, k-1, \cdots, k+1-s \end{aligned} \tag{2.1}$$

where $\mathbf{x}_{k-j} \in \Omega_{\alpha-j}, j = -1, 0, \cdots, s$; $k, s \in \mathbb{Z}_+$ and $\mathbf{p}_\alpha \in \Lambda$. The equivalent discrete system of the memorized nonlinear discrete system is defined by

$$\mathbf{y}_{k+1} = \mathbf{F}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha) \tag{2.2}$$

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where the memorized domain and the corresponding vectors are defined as

$$\begin{aligned}
&\boldsymbol{\Omega}_\alpha = \Omega_\alpha \times \Omega_{\alpha-1} \times \cdots \times \Omega_{\alpha-s} \subset \mathscr{R}^{n(s+1)} \\
&\mathbf{y}_k = (\mathbf{x}_k^{\mathrm{T}}, \mathbf{x}_{k-1}^{\mathrm{T}}, \cdots, \mathbf{x}_{k-s}^{\mathrm{T}})^{\mathrm{T}} \equiv (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s}) \in \boldsymbol{\Omega}_\alpha \\
&\mathbf{F}_\alpha = (\mathbf{f}_\alpha^{\mathrm{T}}, \mathbf{x}_k^{\mathrm{T}}, \cdots, \mathbf{x}_{k-s+1}^{\mathrm{T}})^{\mathrm{T}} \equiv (\mathbf{f}_\alpha; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s}) \in \mathscr{R}^{n(s+1)}.
\end{aligned} \tag{2.3}$$

With a memorized initial condition of $\mathbf{y}_0$(or $\mathbf{x}_0, \mathbf{x}_{-1} \cdots, \mathbf{x}_{-s}$), the solution of Eq. (2.2) is given by

$$\begin{aligned}
&\mathbf{y}_k = \underbrace{\mathbf{F}_\alpha(\mathbf{F}_\alpha(\cdots(\mathbf{F}_\alpha}_{k}(\mathbf{y}_0, \mathbf{p}_\alpha)))) \\
&\text{for } \mathbf{y}_k \in \boldsymbol{\Omega}_\alpha = \Omega_\alpha \times \Omega_{\alpha-1} \times \cdots \times \Omega_{\alpha-s}, k \in \mathbb{Z} \text{ and } \mathbf{p}_\alpha \in \Lambda.
\end{aligned} \tag{2.4}$$

(i) The difference equation with the memorized initial condition is called a memorized *discrete dynamical system.*
(ii) The vector function $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p}_\alpha)$ is called a memorized *discrete vector field* on Ω_α.
(iii) The solution $\mathbf{x}_k$ for each $k \in \mathbb{Z}$ is called a *flow* of memorized discrete system.
(iv) The solution $\mathbf{x}_k$ for all $k \in \mathbb{Z}$ on domain $\boldsymbol{\Omega}_\alpha$ is called the trajectory, phase curve or orbit of memorized discrete dynamical system, which is defined as

$$\Gamma = \left\{ \mathbf{x}_k \left| \begin{array}{l} \mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p}_\alpha) \\ \mathbf{x}_r = \mathbf{x}_r \ \text{ for } \ r = k, k-1, \cdots, k+1-s \\ \text{for } k, s \in \mathbb{Z}_+ \text{ and } \mathbf{p}_\alpha \in \Lambda \end{array} \right. \right\} \subseteq \cup_\alpha \boldsymbol{\Omega}_\alpha. \tag{2.5}$$

(v) The memorized discrete dynamical system is called *a uniform memorized discrete system* if

$$\begin{aligned}
&\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p}_\alpha) = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p}) \\
&\text{for } k, s \in \mathbb{Z}_+ \text{ and } \mathbf{x}_{k-j} \in \Omega_{\alpha-j} \equiv \Omega, j = -1, 0, \cdots, s.
\end{aligned} \tag{2.6}$$

Otherwise, the memorized discrete dynamical system is called a *nonuniform memorized discrete system.*

Definition 2.2 For the memorized discrete dynamical system in Eq. (2.1), the relation between two equivalent states $\mathbf{y}_k$ and $\mathbf{y}_{k+1}$ ($k \in \mathbb{Z}$) in the equivalent discrete system in Eq. (2.2) is called a memorized discrete map if

$$P_\alpha : \mathbf{y}_k \xrightarrow{\mathbf{F}_\alpha} \mathbf{y}_{k+1} \text{ and } \mathbf{y}_{k+1} = P_\alpha \mathbf{y}_k \tag{2.7}$$

with the following properties:

$$P_{(k;n)} : \mathbf{y}_k \xrightarrow{\mathbf{F}_{\alpha_1}, \mathbf{F}_{\alpha_2}, \cdots, \mathbf{F}_{\alpha_n}} \mathbf{y}_{k+n} \text{ and } \mathbf{y}_{k+n} = P_{\alpha_n} \circ P_{\alpha_{n-1}} \circ \cdots \circ P_{\alpha_1} \mathbf{y}_k \tag{2.8}$$

where

$$P_{(k;n)} = P_{\alpha_n} \circ P_{\alpha_{n-1}} \circ \cdots \circ P_{\alpha_1}. \tag{2.9}$$

If $P_{\alpha_n} = P_{\alpha_{n-1}} = \cdots = P_{\alpha_1} = P_\alpha$, then

$$P_{(\alpha;n)} \equiv P_\alpha^{(n)} = P_\alpha \circ P_\alpha \circ \cdots \circ P_\alpha \tag{2.10}$$

with

$$P_\alpha^{(n)} = P_\alpha \circ P_\alpha^{(n-1)} \text{ and } P_\alpha^{(0)} = \mathbf{I}. \tag{2.11}$$

The total map with n-different submaps is shown in Fig. 2.1. The map P_{α_k} with the relation function $\mathbf{f}_{\alpha_k}(\alpha_k \in \mathbb{Z})$ is given by Eq. (2.5). The total map $P_{(k,n)}$ is given in Eq. (2.9). The domains $\boldsymbol{\Omega}_{\alpha_k}(\alpha_k \in \mathbb{Z})$ can fully overlap each other or can be completely separated without any intersection.

Definition 2.3 For a vector function $\mathbf{F}_\alpha : \mathscr{R}^{n(s+1)} \to \mathscr{R}^{n(s+1)}$, the operator norm of $\mathbf{F}_\alpha$ is defined by

$$||\mathbf{F}_\alpha|| = \sum\nolimits_{i=1}^{n} \max_{||\mathbf{x}_k|| \leq 1} |F_{\alpha(i)}(\mathbf{y}_k, \mathbf{p}_\alpha)|. \tag{2.12}$$

For an $n(s+1) \times n(s+1)$ matrix $\mathbf{A}_\alpha = (a_{ij})_{n(s+1) \times n(s+1)}$ in $\mathbf{F}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha) = \mathbf{A}_\alpha \mathbf{y}_k$, the corresponding norm is defined by

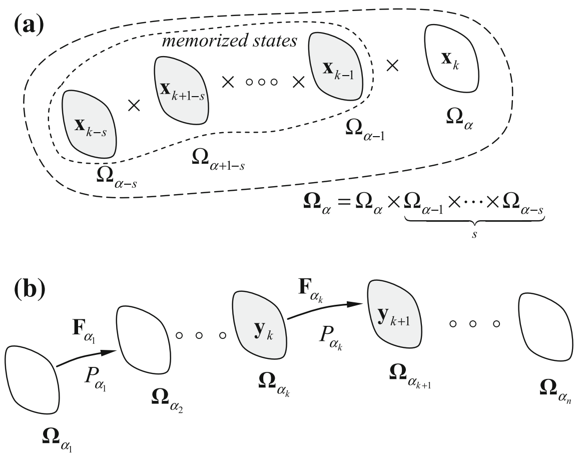

Fig. 2.1 **a** Memorized states and **b** maps and vector functions on each subdomain for a memorized nonlinear discrete dynamical system

$$||\mathbf{A}_\alpha|| = \sum_{i,j=1}^{n} |a_{ij}|. \tag{2.13}$$

Definition 2.4 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$ with $\alpha \in \mathbb{Z}$, consider a vector function $\mathbf{f}_\alpha : \Omega_\alpha \times \Omega_{\alpha-1} \times \cdots \times \Omega_{\alpha-s} \times \Lambda \to \Omega_{\alpha+1}$ where $\mathbf{x}_{k-j} \in \Omega_{\alpha-j}$ $(j = 0, 1, \cdots, s)$ and the resultant memorized vector $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s}) \in \boldsymbol{\Omega}_\alpha$. The memorized vector function $\mathbf{f}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha) \equiv \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots \mathbf{x}_{k-s}, \mathbf{p}_\alpha)$ is called to be differentiable at $\mathbf{y}_k \in \boldsymbol{\Omega}_\alpha$ if

$$\left.\frac{\partial \mathbf{f}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha)}{\partial \mathbf{y}_k}\right|_{(\mathbf{y}_k, \mathbf{p}_\alpha)} = \lim_{\Delta \mathbf{y}_k \to \mathbf{0}} \frac{\mathbf{f}_\alpha(\mathbf{y}_k + \Delta \mathbf{y}_k, \mathbf{p}_\alpha) - \mathbf{f}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha)}{\Delta \mathbf{y}_k} \tag{2.14}$$

where $\mathbf{f}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha) \equiv \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots \mathbf{x}_{k-s}, \mathbf{p}_\alpha)$ with $\mathbf{y}_k = (\mathbf{x}_k^{\mathrm{T}}, \mathbf{x}_{k-1}^{\mathrm{T}}, \cdots, \mathbf{x}_{k-s}^{\mathrm{T}}) \in \boldsymbol{\Omega}_\alpha$. $\partial \mathbf{f}_\alpha / \partial \mathbf{y}_k$ is called the spatial derivative of $\mathbf{f}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha)$ at $\mathbf{y}_k$, and the Jacobian matrix for the memorized discrete dynamical system is defined as

$$D\mathbf{F}_\alpha(\mathbf{y}_k^*, \mathbf{p}) = \frac{\partial \mathbf{F}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha)}{\partial \mathbf{y}_k}\Big|_{\mathbf{y}_k^*} = \left[\frac{\partial F_{\alpha(i)}}{\partial y_{k(j)}}\Big|_{\mathbf{y}_k^*}\right]_{n(s+1)\times n(s+1)} \tag{2.15}$$

with

$$\begin{aligned}
&\mathbf{F}_\alpha = (\mathbf{f}_\alpha^T, \mathbf{x}_k^T, ..., \mathbf{x}_{k+1-s}^T)^T = (\mathbf{F}_\alpha; \mathbf{x}_k; ..., \mathbf{x}_{k+1-s})\\
&D\mathbf{F}_\alpha(\mathbf{y}_k, \mathbf{p}) = \begin{bmatrix} \mathbf{a}_{kk} & \mathbf{a}_{k(k-1)} & \cdots & \mathbf{a}_{k(k-s+1)} & \mathbf{a}_{k(k-s)} \\ \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}_{n(s+1)\times n(s+1)},\\
&\mathbf{a}_{kr} = \left[\frac{\partial \mathbf{f}_\alpha}{\partial \mathbf{x}_r}\Big|_{\mathbf{y}_k}\right]_{n\times n} (r = k, k-1, \cdots, k-s).
\end{aligned} \tag{2.16}$$

Definition 2.5 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^l$ with $\alpha \in \mathbb{Z}$, consider a vector function $\mathbf{f}_\alpha : \Omega_\alpha \times \Omega_{\alpha-1} \times \cdots \times \Omega_{\alpha-s} \times \Lambda \to \Omega_{\alpha+1}$ where $\mathbf{x}_{k-j} \in \Omega_{\alpha-j}$ $(j = 0, 1, \cdots, s)$ and the resultant memorized vector $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s}) \in \boldsymbol{\Omega}_\alpha$. The memorized vector function $\mathbf{f}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha) \equiv \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots \mathbf{x}_{k-s}, \mathbf{p}_\alpha)$ is called to satisfy the Lipschitz condition if

$$||\mathbf{f}_\alpha(\mathbf{z}_k, \mathbf{p}_\alpha) - \mathbf{f}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha)|| \le L||\mathbf{z}_k - \mathbf{y}_k|| \tag{2.17}$$

with $\mathbf{y}_k, \mathbf{z}_k \in \boldsymbol{\Omega}_\alpha$. The constant L is called the Lipschitz constant.

2.2 Fixed Points and Stability

Definition 2.6 For a memorized, discrete system $\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p}_\alpha)$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F}_\alpha = (\mathbf{f}_\alpha; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha)$.

(i) A point $\mathbf{y}_k^* \in \boldsymbol{\Omega}_\alpha$ is called a fixed point or period-1 solution of a memorized discrete nonlinear system $\mathbf{y}_{k+1} = \mathbf{F}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha)$ under a map P_α if for $\mathbf{y}_{k+1} = \mathbf{y}_k = \mathbf{y}_k^*$

$$\mathbf{y}_k^* = \mathbf{F}_\alpha(\mathbf{y}_k^*, \mathbf{p}_\alpha). \tag{2.18}$$

The linearized system of the memorized nonlinear discrete system $\mathbf{y}_{k+1} = \mathbf{F}_\alpha(\mathbf{y}_k, \mathbf{p}_\alpha)$ in Eq. (2.6) at the fixed point $\mathbf{y}_k^*$ is given by

$$\mathbf{z}_{k+1} = DP_\alpha(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k = D\mathbf{F}_\alpha(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k \tag{2.19}$$

where

$$D\mathbf{F}_\alpha(\mathbf{y}_k^*, \mathbf{p}) = \begin{bmatrix} \mathbf{a}_{kk} & \mathbf{a}_{k(k-1)} & \cdots & \mathbf{a}_{k(k-s+1)} & \mathbf{a}_{k(k-s)} \\ \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}_{n(s+1)\times n(s+1)},$$
$$\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^* \text{ and } \mathbf{z}_{k+1} = \mathbf{y}_{k+1} - \mathbf{y}_{k+1}^*,$$
$$\mathbf{a}_{kr} = \frac{\partial \mathbf{f}_\alpha}{\partial \mathbf{x}_r}\Big|_{\mathbf{y}_k^*} (r = k, k-1, \cdots, k-s). \tag{2.20}$$

(ii) A set of points $\mathbf{x}_j^* \in \Omega_{\alpha_j}(\alpha_j \in \mathbb{Z})$ is called the fixed point set or period-1 point set of the total map $P_{(k;n)}$ with n-different submaps in nonlinear discrete system of Eq. (2.7) if

$$\begin{aligned} &\mathbf{y}_{k+j}^* = \mathbf{F}_{\alpha_{j'}}(\mathbf{y}_{k+j-1}^*, \mathbf{p}_{\alpha_j}) \text{ for } j = 1, 2, \cdots, n; \\ &\mathbf{y}_{k+j}^* = \mathbf{y}_k^*. \end{aligned} \tag{2.21}$$

Each linearized equation of the total map $P_{(k;n)}$ gives

$$\begin{aligned} &\mathbf{z}_{k+j} = DP_{\alpha_j}(\mathbf{y}^*_{k+j-1}, \mathbf{p}_{\alpha_j})\mathbf{z}_{k+j-1} = D\mathbf{F}_{\alpha_j}(\mathbf{y}^*_{k+j-1}, \mathbf{p}_{\alpha_j})\mathbf{z}_{k+j-1} \\ &\text{with } \mathbf{z}_{k+j} = \mathbf{y}_{k+j} - \mathbf{y}^*_{k+j} \ \text{ and } \ \mathbf{z}_{k+j-1} = \mathbf{y}_{k+j-1} - \mathbf{y}^*_{k+j-1} \\ &\text{for } j = 1, 2, \ldots, n. \end{aligned} \tag{2.22}$$

The resultant equation for the total map is

$$\mathbf{z}_{k+n} = DP_{(k,n)}(\mathbf{y}^*_k, \mathbf{p})\mathbf{z}_k \tag{2.23}$$

where

$$\begin{aligned} DP_{(k,n)}(\mathbf{y}^*_k, \mathbf{p}) &= \prod\nolimits_{i=n}^{1} DP_{\alpha_i}(\mathbf{y}^*_{k+i-1}, \mathbf{p}) \\ &= DP_{\alpha_n}(\mathbf{y}^*_{k+n-1}, \mathbf{p}_{\alpha_n}) \cdot \ldots \cdot DP_{\alpha_2}(\mathbf{y}^*_{k+1}, \mathbf{p}_{\alpha_2}) \cdot DP_{\alpha_1}(\mathbf{y}^*_k, \mathbf{p}_{\alpha_1}) \\ &= D\mathbf{F}_{\alpha_n}(\mathbf{y}^*_{k+n-1}, \mathbf{p}_{\alpha_n}) \cdot \ldots \cdot D\mathbf{F}_{\alpha_2}(\mathbf{y}^*_{k+1}, \mathbf{p}_{\alpha_2}) \cdot D\mathbf{F}_{\alpha_1}(\mathbf{y}^*_k, \mathbf{p}_{\alpha_1}). \end{aligned} \tag{2.24}$$

The fixed point $\mathbf{y}^*_k$ lies in the intersected set of two memorized domains $\mathbf{\Omega}_k \subset \mathbf{\Omega}_\alpha$ and $\mathbf{\Omega}_{k+1} \subset \mathbf{\Omega}_\alpha$, as shown in Fig. 2.2. In the vicinity of the fixed point $\mathbf{y}^*_k$, the incremental relations in the two memorized domains $\mathbf{\Omega}_k$ and $\mathbf{\Omega}_{k+1}$ are different. In other words, setting $\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}^*_k$ and $\mathbf{z}_{k+1} = \mathbf{y}_{k+1} - \mathbf{y}^*_{k+1}$, the corresponding linearization is generated as in Eq. (2.19). Similarly, The fixed point of the total map with n-different submaps requires the intersection set of two domains $\mathbf{\Omega}_k \subset \mathbf{\Omega}_{\alpha_1}$ and $\mathbf{\Omega}_{k+n} \subset \mathbf{\Omega}_{\alpha_1}$, there are a set of equations to obtain the fixed points from Eq. (2.21). The other values of fixed points lie in different domains, i.e., $\mathbf{y}^*_j \in \mathbf{\Omega}_j \subset \mathbf{\Omega}_{\alpha_r}$ $(j = k+1, k+2, \cdots, k+n-1; \quad r = 2, 3, \cdots, n-1)$, as shown in Fig. 2.3. The corresponding linearized equations of the memorized discrete dynamical systems are given in Eq. (2.21). From Eq. (2.22), the local characteristics of the total map can be discussed as a single map. Thus, the dynamical characteristics for the fixed point of the single map will be discussed comprehensively, and the fixed points for the resultant map are applicable. The results can be extended to any period-m flows with $P^{(m)}$.

Definition 2.7 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete

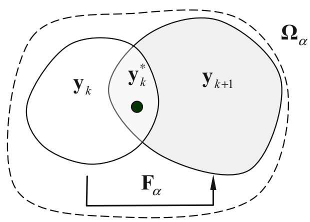

Fig. 2.2 A fixed point between domains $\mathbf{\Omega}_k \subset \mathbf{\Omega}_\alpha$ and $\mathbf{\Omega}_{k+1} \subset \mathbf{\Omega}_\alpha$ for a memorized discrete dynamical system

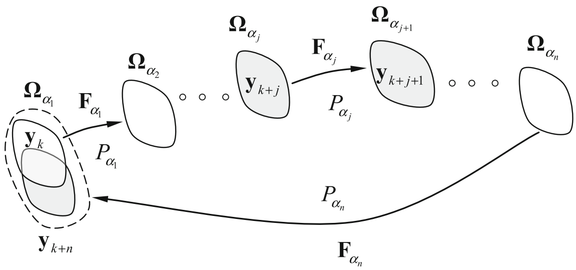

Fig. 2.3 Fixed points with n-maps for a memorized discrete dynamical system with $\mathbf{\Omega}_k \subset \mathbf{\Omega}_{\alpha_1}$ and $\mathbf{\Omega}_{k+n} \subset \mathbf{\Omega}_{\alpha_1}$

system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. The linearized system of the memorized nonlinear discrete system in the neighborhood of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{f}(\mathbf{y}_k^*, \mathbf{p})$ possesses n_1 real eigenvalues $|\lambda_j| < 1$ ($j \in N_1$), n_2 real eigenvalues $|\lambda_j| > 1$ ($j \in N_2$), n_3 real eigenvalues $\lambda_j = 1$ ($j \in N_3$), and n_4 real eigenvalues $\lambda_j = -1$ ($j \in N_4$). $N = \{1, 2, \cdots, n(s+1)\}$ and $N_i = \{i_1, i_2, \cdots, i_{n_i}\} \cup \varnothing$ ($i = 1, 2, 3, 4$) with $i_m \in N$ ($m = 1, 2, \cdots, n_i$), and $\Sigma_{i=1}^4 n_i = n(s+1)$. $N_i \subseteq N \cup \varnothing$, $\cup_{i=1}^4 N_i = N$, $N_i \cap N_p = \varnothing$ ($p \neq i$). $N_i = \varnothing$ if $n_i = 0$. The corresponding eigenvectors for contraction, expansion, invariance, and flip oscillation are $\{\mathbf{v}_j\}$ ($j \in N_i$) ($i = 1, 2, 3, 4$), respectively. The stable, unstable, invariant, and flip subspaces of $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ in Eq. (2.19) are linear subspace spanned by $\{\mathbf{v}_j\}$ ($j \in N_i$) ($i = 1, 2, 3, 4$), respectively, i.e.,

$$\begin{aligned}
\mathscr{E}^{\mathrm{s}} &= span\left\{\mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ |\lambda_j| < 1, j \in N_1 \subseteq N \cup \varnothing \end{array} \right.\right\}; \\
\mathscr{E}^{\mathrm{u}} &= span\left\{\mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ |\lambda_j| > 1, j \in N_2 \subseteq N \cup \varnothing \end{array} \right.\right\}; \\
\mathscr{E}^{\mathrm{i}} &= span\left\{\mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = 1, j \in N_3 \subseteq N \cup \varnothing \end{array} \right.\right\}; \\
\mathscr{E}^{\mathrm{f}} &= span\left\{\mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = -1, j \in N_4 \subseteq N \cup \varnothing \end{array} \right.\right\}.
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
&\mathscr{E}^{\mathrm{s}} = \mathscr{E}^{\mathrm{s}}_{\mathrm{m}} \cup \mathscr{E}^{\mathrm{s}}_{\mathrm{o}} \cup \mathscr{E}^{\mathrm{s}}_{\mathrm{z}} \text{ with} \\
&\mathscr{E}^{\mathrm{s}}_{\mathrm{m}} = span\left\{\mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ 0 < \lambda_j < 1, j \in N_1^{\mathrm{m}} \subseteq N \cup \varnothing \end{array} \right.\right\}; \\
&\mathscr{E}^{\mathrm{s}}_{\mathrm{o}} = span\left\{\mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ -1 < \lambda_j < 0, j \in N_1^{\mathrm{o}} \subseteq N \cup \varnothing \end{array} \right.\right\}, \\
&\mathscr{E}^{\mathrm{s}}_{\mathrm{z}} = span\left\{\mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}; \\ \lambda_j = 0, j \in N_1^{\mathrm{f}} \subseteq N \cup \varnothing \end{array} \right.\right\};
\end{aligned} \tag{2.26}$$

$$\begin{aligned} &\mathscr{E}^{\mathrm{u}} = \mathscr{E}^{\mathrm{u}}_{\mathrm{m}} \cup \mathscr{E}^{\mathrm{u}}_{\mathrm{o}} \text{ with} \\ &\mathscr{E}^{\mathrm{u}}_{\mathrm{m}} = span\left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}^*_k,\mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j > 1, j \in N^{\mathrm{m}}_2 \subseteq N \cup \varnothing \end{array} \right. \right\}; \\ &\mathscr{E}^{\mathrm{u}}_{\mathrm{o}} = span\left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{F}(\mathbf{y}^*_k,\mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j < -1, j \in N^{\mathrm{o}}_2 \subseteq N \cup \varnothing \end{array} \right. \right\}; \end{aligned} \tag{2.27}$$

where subscripts "m" and "o" represent the monotonic and oscillatory evolutions.

Definition 2.8 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. The linearized system of the memorized nonlinear discrete system in the neighborhood of $\mathbf{y}^*_k$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}^*_k$ and $l = k,\ k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p})$ has complex eigenvalues $\alpha_j \pm \mathbf{i}\beta_j$ with eigenvectors $\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j$ ($j \in \{1, 2, \cdots, m\}$) with $2m = n(s+1)$ and the base of vector is

$$\mathbf{B} = \{\mathbf{u}_1, \mathbf{v}_1, \cdots, \mathbf{u}_j, \mathbf{v}_j, \cdots, \mathbf{u}_m, \mathbf{v}_m\}. \tag{2.28}$$

The stable, unstable, center subspaces of $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p})\mathbf{z}_k$ in Eq. (2.19) are linear subspaces spanned by $\{\mathbf{u}_j, \mathbf{v}_j\}(j \in N_i,\quad i = 1, 2, 3)$, respectively. Set $N = \{1, 2, \cdots, m\}$ plus $N_i = \{i_1, i_2, \cdots, i_{n_i}\} \cup \varnothing \subseteq N \cup \varnothing$ with $i_r \in N$ ($r = 1, 2, \cdots, n_i$) and $\Sigma^3_{i=1} n_i = m$. $\cup^3_{i=1} N_i = N$ with $N_i \cap N_p = \varnothing (p \neq i)$. $N_i = \varnothing$ if $n_i = 0$. The stable, unstable, center subspaces of $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p})\mathbf{z}_k$ in Eq. (2.19) are defined by

$$\begin{aligned} &\mathscr{E}^{\mathrm{s}} = span\left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha^2_j + \beta^2_j} < 1, \\ (D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_1 \subseteq \{1, 2, \cdots, m\} \cup \varnothing \end{array} \right. \right\}; \\ &\mathscr{E}^{\mathrm{u}} = span\left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha^2_j + \beta^2_j} > 1, \\ (D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_2 \subseteq \{1, 2, \cdots, m\} \cup \varnothing \end{array} \right. \right\}; \\ &\mathscr{E}^{\mathrm{c}} = span\left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha^2_j + \beta^2_j} = 1, \\ (D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I})(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_3 \subseteq \{1, 2, \cdots, m\} \cup \varnothing \end{array} \right. \right\}. \end{aligned} \tag{2.29}$$

Definition 2.9 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. The linearized system of the memorized nonlinear discrete system in the neighborhood of $\mathbf{y}^*_k$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}^*_k$ and $l = k,\ k+1$) in Eq. (2.19). The fixed point or period-1point is *hyperbolic* if no any eigenvalues of $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ are on the unit circle (i.e., $|\lambda_i| \neq 1$ for $i = 1, 2, \cdots, n(s+1)$).

Theorem 2.1 *For a memorized discrete system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$, *with* $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ *and* $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, *the equivalent discrete system is* $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. *The linearized system of the memorized nonlinear discrete system in the neighborhood of* $\mathbf{y}_k^*$ *is* $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ *and* $l = k,\ k+1$) *in Eq.* (2.19). *The eigenspace of* $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$(*i.e.,* $\mathscr{E} \subseteq \mathscr{R}^{n(s+1)}$) *in the linearized dynamical system is expressed by direct sum of three subspaces*

$$\mathscr{E} = \mathscr{E}^s \oplus \mathscr{E}^u \oplus \mathscr{E}^c. \tag{2.30}$$

where $\mathscr{E}^s, \mathscr{E}^u$ and $\mathscr{E}^c$ *are the stable, unstable and center subspaces, respectively.*

Proof The proof can be referred to Luo (2011). ■

Definition 2.10 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the equilibrium $\mathbf{y}_k^*$ as $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}$, and in the neighborhood $U(\mathbf{y}_k^*)$,

$$\lim_{||\mathbf{z}_k|| \to 0} \frac{||\mathbf{F}(\mathbf{y}_k^* + \mathbf{z}_k, \mathbf{p}) - D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k||}{||\mathbf{z}_k||} = 0. \tag{2.31}$$

and

$$\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k \tag{2.32}$$

(i) A C^r invariant manifold

$$\mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*) = \left\{ \mathbf{y}_k \in U(\mathbf{y}_k^*) \middle| \begin{array}{l} \lim_{j \to +\infty} \mathbf{y}_{k+j} = \mathbf{y}_k^* \text{ and} \\ \mathbf{y}_{k+j} \in U(\mathbf{y}_k^*) \text{ with } j \in \mathbb{Z}_+ \end{array} \right\} \tag{2.33}$$

is called the local stable manifold of $\mathbf{y}_k^*$, and the corresponding global stable manifold is defined as

$$\mathscr{S}(\mathbf{y}_k, \mathbf{y}_k^*) = \cup_{j \in \mathbb{Z}_-} \mathbf{F}(\mathscr{S}_{loc}(\mathbf{y}_{k+j}, \mathbf{y}_{k+j}^*)) = \cup_{j \in \mathbb{Z}_-} \mathbf{F}^{(j)}(\mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)). \tag{2.34}$$

(ii) A C^r invariant manifold $\mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$

$$\mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*) = \left\{ \mathbf{y}_k \in U(\mathbf{y}_k^*) \middle| \begin{array}{l} \lim_{j\to-\infty} \mathbf{y}_{k+j} = \mathbf{y}_k^* \text{ and} \\ \mathbf{y}_{k+j} \in U(\mathbf{y}_k^*) \text{ with } j \in \mathbb{Z}_- \end{array} \right\} \tag{2.35}$$

is called the local unstable manifold of $\mathbf{y}_k^*$, and the corresponding global unstable manifold is defined as

$$\mathscr{U}(\mathbf{y}_k, \mathbf{y}_k^*) = \cup_{j\in\mathbb{Z}_+} \mathbf{F}(\mathscr{U}_{loc}(\mathbf{y}_{k+j}, \mathbf{y}_{k+j}^*)) = \cup_{j\in\mathbb{Z}_+} \mathbf{F}^{(j)}(\mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)). \tag{2.36}$$

(iii) A C^{r-1} invariant manifold $\mathscr{C}_{loc}(\mathbf{y}, \mathbf{y}^*)$ is called the center manifold of $\mathbf{y}^*$ if $\mathscr{C}_{loc}(\mathbf{y}, \mathbf{y}^*)$ possesses the same dimension of $\mathscr{E}^c$ for $\mathbf{y}^* \in \mathscr{C}_{loc}(\mathbf{y}, \mathbf{y}^*)$, and the tangential space of $\mathscr{C}_{loc}(\mathbf{y}, \mathbf{y}^*)$ is identical to $\mathscr{E}^c$.

As in continuous dynamical systems, the stable and unstable manifolds are unique, but the center manifold is not unique. If the nonlinear vector field $\mathbf{f}$ is C^∞-continuous, then a C^r center manifold can be found for any $r < \infty$.

Theorem 2.2 *For a memorized discrete system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$, *with* $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ *and* $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, *the equivalent discrete system is* $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. *Suppose there is a neighborhood of the hyperbolic fixed point* $\mathbf{y}_k^*$ *(i.e.,* $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}$), *and* $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$*-continuous in* $U(\mathbf{y}_k^*)$. *The linearized system of the memorized nonlinear discrete system in the neighborhood* $U(\mathbf{y}_k^*)$ *of* $\mathbf{y}_k^*$ *is* $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ $(\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ *and* $l = k, k+1)$ *in Eq. (2.19). If the homeomorphism between the local invariant subspace* $E(\mathbf{y}_k^*) \subset U(\mathbf{y}_k^*)$ *and the eigenspace E of the linearized system exists with the condition in Eq. (2.31), the local invariant subspace is decomposed by*

$$E(\mathbf{y}_k, \mathbf{y}_k^*) = \mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*) \oplus \mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*). \tag{2.37}$$

(a) *The local stable invariant manifold* $\mathscr{S}_{loc}(\mathbf{y}, \mathbf{y}^*)$ *possesses the following properties:*

(i) *for* $\mathbf{x}_k^* \in \mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$, $\mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$ *possesses the same dimension of* $\mathscr{E}^s$ *and the tangential space of* $\mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$ *is identical to* $\mathscr{E}^s$;

(ii) *for* $\mathbf{y}_k \in \mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$, $\mathbf{y}_{k+j} \in \mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$ *and* $\lim_{j\to\infty} \mathbf{y}_{k+j} = \mathbf{y}_k^*$ *for all* $j \in \mathbb{Z}_+$;

(iii) *for* $\mathbf{y}_k \notin \mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$, $||\mathbf{y}_{k+j} - \mathbf{y}_k^*|| \geq \delta$ *for* $\delta > 0$ *with* $j, j_1 \in \mathbb{Z}_+$ *and* $j \geq j_1 \geq 0$.

(b) *The local unstable invariant manifold* $\mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$ *possesses the following properties:*

(i) *for* $\mathbf{y}_k^* \in \mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$, $\mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$ *possesses the same dimension of* $\mathscr{E}^u$ *and the tangential space of* $\mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$ *is identical to* $\mathscr{E}^u$;

(ii) *for* $\mathbf{y}_k \in \mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$, $\mathbf{y}_{k+j} \in \mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$ *and* $\lim_{j \to -\infty} \mathbf{y}_{k+j} = \mathbf{y}^*$ *for all* $j \in \mathbb{Z}_-$

(iii) *for* $\mathbf{y}_k \notin \mathscr{U}_{loc}(\mathbf{y}, \mathbf{y}^*)$, $||\mathbf{y}_{k+j} - \mathbf{y}_k^*|| \geq \delta$ *for* $\delta > 0$ *with* $j_1, j \in \mathbb{Z}_-$ *and* $j \leq j_1 \ \leq 0$.

Proof The proof is similar to Nitecki (1971). ■

Theorem 2.3 *For a memorized discrete system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$, *with* $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ *and* $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, *the equivalent discrete system is* $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. *Suppose there is a neighborhood of the fixed point* $\mathbf{y}_k^*$ *(i.e.,* $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}$*), and* $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ *is* C^r ($r \geq 1$)*-continuous in* $U(\mathbf{y}_k^*)$. *The linearized system of the memorized nonlinear discrete system in the neighborhood* $U(\mathbf{y}_k^*)$ *of* $\mathbf{y}_k^*$ *is* $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ *and* $l = k, k+1$) *in Eq.* (2.19). *If the homeomorphism between the local invariant subspace* $E(\mathbf{y}_k^*) \subset U(\mathbf{y}_k^*)$ *and the eigenspace* $\mathscr{E}$ *of the linearized system exists with the condition in Eq.* (2.31), *in addition to the local stable and unstable invariant manifolds, there is a* C^{r-1} *center manifold* $\mathscr{C}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$. *If the homeomorphism between the local invariant subspace* $E(\mathbf{y}_k^*) \subset U(\mathbf{y}_k^*)$ *and the eigenspace* $\mathscr{E}$ *of the linearized system exists with the condition in Eq.* (2.28). *The center manifold possesses the same dimension of* $\mathscr{E}^c$ *for* $\mathbf{y}^* \in \mathscr{C}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*)$, *and the tangential space of* $\mathscr{C}_{loc}(\mathbf{y}, \mathbf{y}_k^*)$ *is identical to* $\mathscr{E}^c$. *Thus, the local invariant subspace is decomposed by*

$$E(\mathbf{y}_k, \mathbf{y}_k^*) = \mathscr{S}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*) \oplus \mathscr{U}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*) \oplus \mathscr{C}_{loc}(\mathbf{y}_k, \mathbf{y}_k^*). \tag{2.38}$$

Proof The proof is similar to Guckenhiemer and Holmes (1990). ■

Definition 2.11 *For a memorized discrete system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ *on domain* $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, *with* $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ *and* $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, *the equivalent discrete system is* $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a metric space $(\mathbf{\Omega}_\alpha, \rho)$, then the map P under the vector function $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is called the a contraction map if

$$\rho(\mathbf{y}_{k+1}^{(1)}, \mathbf{y}_{k+1}^{(2)}) = \rho(\mathbf{F}(\mathbf{y}_k^{(1)}, \mathbf{p}), \mathbf{F}(\mathbf{y}_k^{(2)}, \mathbf{p})) \leq \lambda \rho(\mathbf{y}_k^{(1)}, \mathbf{y}_k^{(2)}) \tag{2.39}$$

for $\lambda \in (0, 1)$ and $\mathbf{y}_k^{(1)}, \mathbf{y}_k^{(2)} \in \mathbf{\Omega}_\alpha$ with $\rho(\mathbf{y}_k^{(1)}, \mathbf{y}_k^{(2)}) = ||\mathbf{y}_k^{(1)} - \mathbf{y}_k^{(2)}||$.

Theorem 2.4 *For a memorized discrete system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ *on domain* $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, *with* $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ *and* $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, *the equivalent discrete system is* $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a metric space $(\mathbf{\Omega}_\alpha, \rho)$, *if the map* P *under the vector function* $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ *is the a contraction map, then there is a unique fixed point* $\mathbf{y}_k^*$ *which is globally stable.*

Proof Consider a contraction map as

$$\begin{aligned}\rho(\mathbf{y}_{k+j+1}^{(1)}, \mathbf{y}_{k+j+1}^{(2)}) &= \rho(\mathbf{F}(\mathbf{y}_{k+j}^{(1)}, \mathbf{p}), \mathbf{F}(\mathbf{y}_{k+j}^{(2)}, \mathbf{p})) \leq \lambda\rho(\mathbf{y}_{k+j}^{(1)}, \mathbf{y}_{k+j}^{(2)}) \\ &= \lambda\rho(\mathbf{F}(\mathbf{y}_{k+j-1}^{(1)}, \mathbf{p}), \mathbf{F}(\mathbf{y}_{k+j-1}^{(2)}, \mathbf{p})) \leq \lambda^2\rho(\mathbf{y}_{k+j-1}^{(1)}, \mathbf{y}_{k+j-1}^{(2)}) \\ &\vdots \\ &= \lambda^j\rho(\mathbf{F}(\mathbf{y}_{k+1}^{(1)}, \mathbf{p}), \mathbf{F}(\mathbf{y}_{k+1}^{(2)}, \mathbf{p})) \leq \lambda^{j+1}\rho(\mathbf{y}_{k}^{(1)}, \mathbf{y}_{k}^{(2)}).\end{aligned}$$

As $j \to \infty$ and $0 < \lambda < 1$, thus, we have

$$\lim_{j\to\infty} \rho(\mathbf{y}_{k+j+1}^{(1)}, \mathbf{y}_{k+j+1}^{(2)}) = \lim_{j\to\infty} \lambda^{j+1}\rho(\mathbf{y}_{k}^{(1)}, \mathbf{y}_{k}^{(2)}) = 0.$$

If $\mathbf{y}_{k+j+1}^{(2)} = \mathbf{y}_{k}^{(2)} = \mathbf{y}_k^*$, in domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, we have

$$\lim_{j\to\infty} \rho(\mathbf{y}_{k+j+1}^{(1)}, \mathbf{y}_{k+j+1}^{(2)}) = \lim_{j\to\infty} ||\mathbf{y}_{k+j+1}^{(1)} - \mathbf{y}_k^*|| = 0.$$

Consider two fixed points $\mathbf{y}_{k1}^*$ and $\mathbf{y}_{k2}^*$. The above equation give

$$\begin{aligned}||\mathbf{y}_{k1}^* - \mathbf{y}_{k2}^*|| &= \lim_{j\to\infty} ||\mathbf{y}_{k1}^* - \mathbf{y}_{k+j+1} + \mathbf{y}_{k+j+1} - \mathbf{y}_{k2}^*|| \\ &\leq \lim_{j\to\infty} ||\mathbf{y}_{k1}^* - \mathbf{y}_{k+j+1}|| + \lim_{j\to\infty} ||\mathbf{y}_{k+j+1} - \mathbf{y}_{k2}^*|| = 0.\end{aligned}$$

Therefore, the fixed point is unique and globally stable. This theorem is proved. ■

Definition 2.12 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). Consider a real eigenvalue λ_i of matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ ($i \in N = \{1, 2, \cdots, n(s+1)\}$) and there is a corresponding eigenvector $\mathbf{v}_i$. On the invariant eigenvector $\mathbf{v}_k^{(i)} = \mathbf{v}_i$, consider $\mathbf{z}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{z}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i = \lambda_i c_k^{(i)}\mathbf{v}_i$, thus, $c_{k+1}^{(i)} = \lambda_i c_k^{(i)}$.

(i) $\mathbf{y}_k^{(i)}$ on the direction $\mathbf{v}_i$ is stable if

$$\lim_{k\to\infty} |c_k^{(i)}| = \lim_{k\to\infty} |(\lambda_i)^k| \times |c_0^{(i)}| = 0 \text{ for } |\lambda_i| < 1. \tag{2.40}$$

(ii) $\mathbf{y}_k^{(i)}$ on the direction $\mathbf{v}_i$ is unstable if

$$\lim_{k\to\infty}|c_k^{(i)}| = \lim_{k\to\infty}|(\lambda_i)^k| \times |c_0^{(i)}| = \infty \text{ for } |\lambda_i| > 1. \tag{2.41}$$

(iii) $\mathbf{y}_k^{(i)}$ on the direction $\mathbf{v}_i$ is invariant if

$$\lim_{k\to\infty} c_k^{(i)} = \lim_{k\to\infty}(\lambda_i)^k c_0^{(i)} = c_0^{(i)} \text{ for } \lambda_i = 1. \tag{2.42}$$

(iv) $\mathbf{y}_k^{(i)}$ on the direction $\mathbf{v}_i$ is flipped if

$$\left.\begin{array}{l} \lim\limits_{2k\to\infty} c_k^{(i)} = \lim\limits_{2k\to\infty}(\lambda_i)^{2k} \times c_0^{(i)} = c_0^{(i)} \\ \lim\limits_{2k+1\to\infty} c_k^{(i)} = \lim\limits_{2k+1\to\infty}(\lambda_i)^{2k+1} \times c_0^{(i)} = -c_0^{(i)} \end{array}\right\} \text{for } \lambda_i = -1. \tag{2.43}$$

(v) $\mathbf{y}_k^{(i)}$ on the direction $\mathbf{v}_i$ is degenerate if

$$c_k^{(i)} = (\lambda_i)^k c_0^{(i)} = 0 \text{ for } \lambda_i = 0. \tag{2.44}$$

Definition 2.13 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). Consider a pair of complex eigenvalue $\alpha_i \pm \mathbf{i}\beta_i$ of matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ ($i \in N = \{1, 2, \cdots, n\}$, $\mathbf{i} = \sqrt{-1}$) and there is a corresponding eigenvector $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_k^{(i)}, \mathbf{v}_k^{(i)}) = (\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ with

$$\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{u}_i + d_k^{(i)}\mathbf{v}_i, \mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{u}_i + d_{k+1}^{(i)}\mathbf{v}_i. \tag{2.45}$$

Thus, $\mathbf{c}_k^{(i)} = (c_k^{(i)}, d_k^{(i)})^{\mathrm{T}}$ with

$$\mathbf{c}_{k+1}^{(i)} = \mathbf{E}_i\mathbf{c}_k^{(i)} = r_i\mathbf{R}_i\mathbf{c}_k^{(i)} \tag{2.46}$$

where

$$\begin{gathered} \mathbf{E}_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \text{ and } \mathbf{R}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}, \\ r_i = \sqrt{\alpha_i^2 + \beta_i^2}, \cos\theta_i = \alpha_i/r_i \text{ and } \sin\theta_i = \beta_i/r_i; \end{gathered} \tag{2.47}$$

and

$$\mathbf{E}_i^k = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}^k \text{ and } \mathbf{R}_i^k = \begin{bmatrix} \cos k\theta_i & \sin k\theta_i \\ -\sin k\theta_i & \cos k\theta_i \end{bmatrix}. \tag{2.48}$$

(i) $\mathbf{y}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable if

$$\lim_{k\to\infty} ||\mathbf{c}_k^{(i)}|| = \lim_{k\to\infty} r_i^k ||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = 0 \text{ for } r_i = |\lambda_i| < 1. \tag{2.49}$$

(ii) $\mathbf{y}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable if

$$\lim_{k\to\infty} ||\mathbf{c}_k^{(i)}|| = \lim_{k\to\infty} r_i^k ||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = \infty \text{ for } r_i = |\lambda_i| > 1. \tag{2.50}$$

(iii) $\mathbf{y}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is on the invariant circles if,

$$||\mathbf{c}_k^{(i)}|| = r_i^k ||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = ||\mathbf{c}_0^{(i)}|| \text{ for } r_i = |\lambda_i| = 1. \tag{2.51}$$

(iv) $\mathbf{y}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is degenerate in the direction of $\mathbf{u}_i$ if $\beta_i = 0$.

Definition 2.14 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ $(\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1)$ in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i $(i = 1, 2, \cdots, m$ with $m = n(s+1))$.

(i) The fixed point $\mathbf{y}_k^*$ is called a hyperbolic point if $|\lambda_i| \neq 1$ $(i = 1, 2, \cdots, m)$.
(ii) The fixed point $\mathbf{y}_k^*$ is called a sink if $|\lambda_i| < 1$ $(i = 1, 2, \cdots, m)$.
(iii) The fixed point $\mathbf{y}_k^*$ is called a source if $|\lambda_i| > 1 (i = 1, 2, \cdots, m)$.
(iv) The fixed point $\mathbf{y}_k^*$ is called a center if $|\lambda_i| = 1$ $(i = 1, 2, \cdots, m)$ with distinct eigenvalues.

Definition 2.15 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i ($i = 1, 2, \cdots, m$) with $m = n(s+1)$.

(i) The fixed point $\mathbf{y}_k^*$ is called a stable node if $|\lambda_i| < 1 (i = 1, 2, \cdots, m)$.
(ii) The fixed point $\mathbf{y}_k^*$ is called an unstable node if $|\lambda_i| > 1$ ($i = 1, 2, \cdots, m$).
(iii) The fixed point $\mathbf{y}_k^*$ is called an $(l_1 : l_2)$-saddle if at least one $|\lambda_i| > 1$ ($i \in L_1 \subset \{1, 2, \cdots, m\}$) and the other $|\lambda_j| < 1$ ($j \in L_2 \subset \{1, 2, \cdots n\}$) with $L_1 \cup L_2 = \{1, 2, \cdots, m\}$ and $L_1 \cap L_2 = \varnothing$. $l_1 = \mathrm{span}(L_1)$ and $l_2 = \mathrm{span}(L_2)$.
(iv) The fixed point $\mathbf{y}_k^*$ is called an lth-order degenerate case if $\lambda_i = 0$ for all $i \in L = \{n_1, n_2, \cdots, n_l\} \subseteq \{1, 2, \cdots, m\}$ with $l = \mathrm{span}(L)$.

Definition 2.16 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m complex eigenvalues λ_i ($i = 1, 2, \cdots, m$) with $m = n(s+1)$.

(i) The fixed point $\mathbf{y}_k^*$ is called a spiral sink if $|\lambda_i| < 1$ ($i = 1, 2, \cdots, m$) and $\mathrm{Im}\lambda_j \neq 0$ ($j \in \{1, 2, \cdots, m\}$).
(ii) fixed point $\mathbf{y}_k^*$ is called a spiral source if $|\lambda_i| > 1$ ($i = 1, 2, \cdots, m$) with $\mathrm{Im}\lambda_j \neq 0$ ($j \in \{1, 2, \cdots, m\}$).
(iii) fixed point $\mathbf{y}_k^*$ is called a center if $|\lambda_i| = 1$ with distinct $\mathrm{Im}\lambda_i \neq 0$ ($i = 1, 2, \cdots, m$).

The generalized stability and bifurcation of flows in linearized, nonlinear dynamical systems in Eq. (2.4) will be discussed as follows.

Definition 2.17 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i ($i = 1, 2, \cdots, m$) with $m = n(s+1)$. Set $N = \{1, 2, \cdots, l, l+1, \cdots, (m+l)/2\}$, $N_j = \{j_1, j_2, \cdots, j_{n_j}\} \cup \varnothing$ with $j_p \in N$ ($p = 1, 2, \cdots, n_j; 1, 2, \cdots, n_j; j = 1, 2, \cdots, 7$), $\Sigma_{j=1}^4 n_j = l$ and $2\Sigma_{j=5}^7 n_j = m - l$. $\cup_{j=1}^7 N_j = N$ with $N_j \cap N_r = \varnothing (r \neq j)$. $N_j = \varnothing$ if $n_j = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ ($\alpha = 1, 2$) and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \varnothing$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$ where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant, and n_4-flip real eigenvectors plus n_5-stable,n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_i| = 1 (i \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point $\mathbf{y}_k^*$. With repeated complex eigenvalues of $|\lambda_i| = 1$ $(i \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ flow in the neighborhood of the fixed point $\mathbf{y}_k^*$, $\kappa_\beta \in \{\varnothing, m_\beta\}$ $(\beta = 3, 4)$. $\mathbf{q} = (q_1, q_2, \cdots, q_r)^{\mathrm{T}}$ and $\boldsymbol{\kappa}_7 = (\kappa_{71}, \kappa_{72}, \cdots, \kappa_{7r})^{\mathrm{T}}$ with $\kappa_{7j} \in \{\varnothing, m_{7j}\}$ $(j = 1, 2, \cdots, r)$. $\mathbf{m}_7 = (m_{71}, m_{72}, \cdots, m_{7r})^{\mathrm{T}}$. The meanings of notations in the aforementioned structures are defined as follows:

(i) $[n_1^{\mathrm{m}}, n_1^{\mathrm{o}}]$ represents n_1-sinks with n_1^{m}-monotonic convergence and n_1^{o}-oscillatory convergence among n_1-directions of $\mathbf{v}_i$ if $|\lambda_i| < 1$ $(i \in N_1$ and $1 \le n_1 \le n)$ with distinct or repeated eigenvalues.
(ii) $[n_2^{\mathrm{m}}, n_2^{\mathrm{o}}]$ represents n_2-sources with n_2^{m}-monotonic divergence and n_2^{o}-oscillatory divergence among n_2-directions of $\mathbf{v}_i (i \in N_2)$ if $|\lambda_i| > 1$ $(i \in N_2$ and $1 \le n_2 \le n)$ with distinct or repeated eigenvalues.
(iii) $n_3 = 1$ represents an invariant center on 1-direction of $\mathbf{v}_i$ if $\lambda_i = 1$ $(i \in N_3$ and $n_3 = 1)$.
(iv) $n_4 = 1$ represents a flip center on 1-direction of $\mathbf{v}_i$ if $\lambda_i = -1$ $(i \in N_4$ and $n_4 = 1)$.
(v) n_5 represents n_5-spiral sinks on n_5-pairs of $(\mathbf{u}_i, \mathbf{v}_i)$ if $|\lambda_i| < 1$ and $\mathrm{Im}\lambda_i \neq 0$ $(i \in N_5$ and $1 \le n_5 \le n)$ with distinct or repeated eigenvalues.
(vi) n_6 represents n_6-spiral sources on n_6-directions of $(\mathbf{u}_i, \mathbf{v}_i)$ if $|\lambda_i| > 1$ and $\mathrm{Im}\lambda_i \neq 0$ $(i \in N_6$ and $1 \le n_6 \le n)$ with distinct or repeated eigenvalues.
(vii) n_7 represents n_7-invariant centers on n_7-pairs of $(\mathbf{u}_i, \mathbf{v}_i)$ if $|\lambda_i| = 1$ and $\mathrm{Im}\lambda_i \neq 0$ $(i \in N_7$ and $1 \le n_7 \le n)$ with distinct eigenvalues.
(viii) $\varnothing$ represents an empty set if $n_j = 0$ $(j \in \{1, 2, \cdots, 7\})$.
(ix) $[n_3; \kappa_3]$ represents $(n_3 - \kappa_3)$ invariant centers on $(n_3 - \kappa_3)$ directions of $\mathbf{v}_{i_3}$ $(i_3 \in N_3)$ and κ_3-sources in κ_3-directions of $\mathbf{v}_{j_3} (j_3 \in N_3$ and $j_3 \neq i_3)$ if $\lambda_i = 1$ $(i \in N_3$ and $n_3 \le n)$ with the $(\kappa_3 + 1)$th-order nilpotent matrix $\mathbf{N}_3^{\kappa_3 + 1} = \mathbf{0}$ $(0 < \kappa_3 \le n_3 - 1)$.
(x) $[n_3; \varnothing]$ represents n_3 invariant centers on n_3-directions of $\mathbf{v}_i$ if $\lambda_i = 1$ $(i \in N_3$ and $1 < n_3 \le n)$ with a nilpotent matrix $\mathbf{N}_3 = \mathbf{0}$.
(xi) $[n_4; \kappa_4]$ represents $(n_4 - \kappa_4)$ flip oscillatory centers on $(n_4 - \kappa_4)$ directions of $\mathbf{v}_{i_4}$ $(i_4 \in N_4)$ and κ_4-sources in κ_4-directions of $\mathbf{v}_{j_4}$ $(j_4 \in N_4$ and $j_4 \neq i_4)$ if $\lambda_i = -1$ $(i \in N_4$ and $n_4 \le n)$ with the $(\kappa_4 + 1)$th-order nilpotent matrix $\mathbf{N}_4^{\kappa_4 + 1} = \mathbf{0}$ $(0 < \kappa_4 \le n_4 - 1)$.
(xii) $[n_4; \varnothing]$ represents n_4 flip oscillatory centers on n_4-directions of $\mathbf{v}_i$ if $\lambda_i = -1$ $(i \in N_4$ and $1 < n_4 \le n)$ with a nilpotent matrix $\mathbf{N}_4 = \mathbf{0}$.
(xiii) $[n_7, \mathbf{q}; \boldsymbol{\kappa}_7]$ represents $(n_7 - \Sigma_{j=1}^r \kappa_{7j})$ invariant centers on $(n_7 - \Sigma_{j=1}^r \kappa_{7j})$ pairs of $(\mathbf{u}_{i_7}, \mathbf{v}_{i_7})(i_7 \in N_7)$ and $\Sigma_{j=1}^r \kappa_{7j}$ sources on $\Sigma_{j=1}^r \kappa_{7j}$ pairs of $(\mathbf{u}_{j_7}, \mathbf{v}_{j_7})$ $(j_7 \in N_7$ and $j_7 \neq i_7)$ if $|\lambda_i| = 1$ and $\mathrm{Im}\lambda_i \neq 0$ $(i \in N_7$ and $n_7 \le n)$ for q_j

pairs of repeated eigenvalues with the $(\kappa_{7j}+1)$th-order nilpotent matrix $\mathbf{N}_{7j}^{\kappa_{7j}+1} = \mathbf{0}$ $(0 < \kappa_{7j} \le q_j, j = 1, 2, \cdots, r)$.

(xiv) $[n_7, \mathbf{q}; \varnothing]$ represents n_7-invariant centers on n_7-pairs of $(\mathbf{u}_i, \mathbf{v}_i)$ if $|\lambda_i| = 1$ and $\mathrm{Im}\lambda_i \neq 0$ $(i \in N_7$ and $1 \le n_7 \le n)$ for (q_j+1) pairs of repeated eigenvalues with a nilpotent matrix $\mathbf{N}_{7j} = \mathbf{0}$ $(j = 1, 2, \cdots, r)$.

Definition 2.18 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \ge 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ $(\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1)$ in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i $(i = 1, 2, \cdots, m)$ with $m = n(s+1)$. Set $N = \{1, 2, \cdots, l, l+1, \cdots, (m+l)/2\}$, $N_j = \{j_1, j_2, \cdots, j_{n_j}\} \cup \varnothing$ with $j_p \in N$ $(p = 1, 2, \cdots, n_j;\ j = 1, 2, \cdots, 7)$, $\Sigma_{j=1}^4 n_j = l$ and $2\Sigma_{j=5}^7 n_j = m - l$. $\cup_{j=1}^7 N_j = N$ with $N_j \cap N_r = \varnothing (r \neq j)$. $N_j = \varnothing$ if $n_j = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ $(\alpha = 1, 2)$ and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \varnothing$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$ where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant, and n_4-flip real eigenvectors plus n_5-stable,n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_i| = 1 (i \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point $\mathbf{y}_k^*$. With repeated complex eigenvalues of $|\lambda_i| = 1$ $(i \in N_3 \cup N_4 \ \cup N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ flow in the neighborhood of the fixed point $\mathbf{y}_k^*$, $\kappa_\beta \in \{\varnothing, m_\beta\}$ $(\beta = 3, 4)$. $\mathbf{q} = (q_1, q_2, \cdots, q_r)^{\mathrm{T}}$ and $\boldsymbol{\kappa}_7 = (\kappa_{71}, \kappa_{72}, \cdots, \kappa_{7r})^{\mathrm{T}}$ with $\kappa_{7j} \in \{\varnothing, m_{7j}\}$ $(j = 1, 2, \cdots, r)$. $\mathbf{m}_7 = (m_{71}, m_{72}, \cdots, m_{7r})^{\mathrm{T}}$.

I. *Nondegenerate cases*

(i) The fixed point $\mathbf{y}_k^*$ is an $an([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : \varnothing)$ hyperbolic point.

(ii) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing | n_5 : \varnothing : \varnothing)$-sink .

(iii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | \varnothing : n_6 : \varnothing)$-source.

(iv) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : \varnothing | \varnothing : \varnothing : m/2)$-circular center.

(v) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : \varnothing | \varnothing : \varnothing : [m/2, \mathbf{q}; \varnothing])$-circular center.

(vi) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : \varnothing | \varnothing : \varnothing : [m/2, \mathbf{q}; \mathbf{m}_7])$-point.

(vii) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing | n_5 : \varnothing : n_7)$-point.

(viii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | \varnothing : n_6 : n_7)$-point.

(ix) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$-point.

II. *Simple special cases*

(i) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [m; \varnothing] : \varnothing|\varnothing : \varnothing : \varnothing)$-invariant center (or static center).
(ii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [m; m_3] : \varnothing|\varnothing : \varnothing : \varnothing)$-point.
(iii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : [m; \varnothing]|\varnothing : \varnothing : \varnothing)$-flip center.
(iv) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : [n; m_4]|\varnothing : \varnothing : \varnothing)$-point.
(v) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [n_4; \kappa_4] : |\varnothing : \varnothing : \varnothing)$-point.
(vi) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [1; \varnothing] : [n_4; \kappa_4] : |\varnothing : \varnothing : \varnothing)$-point.
(vii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [1; \varnothing] : |\varnothing : \varnothing : \varnothing)$-point.
(viii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [\varnothing; \varnothing] : |\varnothing : \varnothing : n_7)$-point.
(ix) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [1; \varnothing] : [\varnothing; \varnothing] : |\varnothing : \varnothing : n_7)$-point.
(x) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [\varnothing; \varnothing]|\varnothing : \varnothing : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point.
(xi) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [\varnothing; \varnothing] : [n_4; \kappa_4] : |\varnothing : \varnothing : n_7)$-point.
(xii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [\varnothing; \varnothing] : [n_4; \kappa_4]|\varnothing : \varnothing : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point..
(xiii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [n_4; \kappa_4] : |\varnothing : \varnothing : n_7)$-point.
(xiv) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [n_4; \kappa_4]|\varnothing : \varnothing : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point.

III. *Complex special cases*

(i) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [1; \varnothing] : [\varnothing; \varnothing]|n_5 : n_6 : n_7)$ point.
(ii) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [1; \varnothing] : [\varnothing; \varnothing]|n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point.
(iii) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [\varnothing; \varnothing] : [1; \varnothing]|n_5 : n_6 : n_7)$ point.
(iv) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [\varnothing; \varnothing] : [1; \varnothing]|n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point.
(v) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] : |n_5 : n_6 : n_7)$- point.
(vi) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point.

Definition 2.19 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i $(i = 1, 2, \cdots, m)$ with $m = n(s+1)$. Set $N = \{1, 2, \cdots, m\}$, $N_j = \{j_1, j_2, \cdots, j_{n_j}\} \cup \varnothing$ with $j_p \in N$ $(p = 1, 2, \cdots, n_j,\ j = 1, 2, 3, 4)$ and $\Sigma_{j=1}^4 n_j = m$. $\cup_{j=1}^4 N_j = N$ and $N_j \cap N_l = \varnothing$ $(l \neq j)$. $N_j = \varnothing$ if $n_j = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ $(\alpha = 1, 2)$ and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \varnothing$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$ where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant, and n_4-flip real eigenvectors. An iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|$ flow. $\kappa_j \in \{\varnothing, m_j\}$ $(j = 3, 4)$.

I. *Nondegenerate cases*

(i) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle.
(ii) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing|$-sink.
(iii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$-source.

II. *Simple special cases*

(i) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n; \varnothing] : \varnothing|$-invariant center (or static center).
(ii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n; m] : \varnothing|$-point.
(iii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : [n; \varnothing]|$-flip center.
(iv) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : [n; m]|$-point.
(v) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [n_4; \kappa_4]|$-point.
(vi) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [1; \varnothing] : [n_4; \kappa_4]|$-point.
(vii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [1; \varnothing]|$-point.
(viii) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [\varnothing; \varnothing]|$-point.
(ix) The fixed point $\mathbf{y}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [\varnothing; \varnothing] : [n_4; \kappa_4]|$-point.

III. *Complex special cases*

(i) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [1; \varnothing] : [\varnothing; \varnothing]|$-point.
(ii) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [\varnothing; \varnothing] : [1; \varnothing]|$-point.
(iii) The fixed point $\mathbf{y}_k^*$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|$-point.

Definition 2.20 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses $2m$ eigenvalues $\lambda_i (i = 1, 2, \cdots, 2m)$ with $2m = n(s+1)$. Set $N = \{1, 2, \cdots, n\}$, $N_j = \{j_1, j_2, \cdots, j_{n_j}\} \cup \varnothing$ with $j_p \in N$ ($p = 1, 2, \cdots, n_j$, $j = 5, 6, 7$ and $\Sigma_{j=5}^{7} n_j = n$. $\cup_{j=5}^{7} N_j = N$ and $N_j \cap N_l = \varnothing$ $(l \neq j)$. $N_j = \varnothing$ if $n_j = 0$. The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses n_5-stable, n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_k| = 1 (k \in N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $|n_5 : n_6 : n_7)$ flow. With repeated complex eigenvalues of $|\lambda_k| = 1$ $(k \in N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $|n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ flow. $\mathbf{q} = (q_1,\ q_2, \cdots, q_r)^{\mathrm{T}}$ and $\boldsymbol{\kappa}_7 = (\kappa_{71}, \kappa_{72}, \cdots, \kappa_{7r})^{\mathrm{T}}$ with $\kappa_{7j} \in \{\varnothing, m_{7j}\}$ $(j = 1, 2, \cdots, r)$. $\mathbf{m}_7 = (m_{71}, m_{72}, \cdots, m_{7r})^{\mathrm{T}}$.

I. *Nondegenerate cases*

(i) The fixed point $\mathbf{y}_k^*$ is an $|n_5 : n_6 : \varnothing)$ spiral hyperbolic point.
(ii) The fixed point $\mathbf{y}_k^*$ is an $|n : \varnothing : \varnothing)$ spiral sink.
(iii) The fixed point $\mathbf{y}_k^*$ is an $|\varnothing : m : \varnothing)$ spiral source.
(iv) The fixed point $\mathbf{y}_k^*$ is an $|\varnothing : \varnothing : m)$-circular center.
(v) The fixed point $\mathbf{y}_k^*$ is an $|n_5 : \varnothing : n_7)$-point.
(vi) The fixed point $\mathbf{y}_k^*$ is an $|\varnothing : n_6 : n_7)$-point.
(vii) The fixed point $\mathbf{y}_k^*$ is an $|n_5 : n_6 : n_7)$-point.

II. *Special cases*

(i) The fixed point $\mathbf{y}_k^*$ is an $|\varnothing : \varnothing : [n, \mathbf{q}; \varnothing])$-circular center.
(ii) The fixed point $\mathbf{y}_k^*$ is an $|\varnothing : \varnothing : [n, \mathbf{q}; \mathbf{m}_7])$-point.
(iii) The fixed point $\mathbf{y}_k^*$ is an $|n_5 : \varnothing : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point.
(iv) The fixed point $\mathbf{y}_k^*$ is an $|\varnothing : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point.
(v) The fixed point $\mathbf{y}_k^*$ is an $|n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$-point.

2.3 Bifurcation and Stability Switching

To understand the qualitative changes of dynamical behaviors of memorized discrete systems with parameters in the neighborhood of fixed points, the bifurcation theory for fixed points of memorized nonlinear dynamical system in Eq. (2.2) will be investigated.

Definition 2.21 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i ($i = 1, 2, \cdots, m$) with $m = n(s+1)$. Set $N = \{1, 2, \cdots, l, l+1, \cdots, (m+l)/2\}, N_j = \{j_1, j_2, \cdots, j_{n_j}\} \cup \varnothing$ with $j_p \in N$ ($p = 1, 2, \cdots, n_j$; $1, 2, \cdots, n_j$;$j = 1, 2, \cdots, 7$), $\Sigma_{j=1}^4 n_j = l$ and $2\Sigma_{j=5}^7 n_j = m - l$. $\cup_{j=1}^7 N_j = N$ with $N_j \cap N_r = \varnothing (r \neq j)$. $N_j = \varnothing$ if $n_j = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ ($\alpha = 1, 2$) and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \varnothing$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$ where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant, and n_4-flip real eigenvectors plus n_5-stable, n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_i| = 1 (i \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point $\mathbf{y}_k^*$. With repeated complex eigenvalues of $|\lambda_i| = 1$ ($i \in N_3 \cup N_4 \cup N_7$), an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ flow in the neighborhood of the fixed point $\mathbf{y}_k^*$, $\kappa_\beta \in \{\varnothing, m_\beta\}$ ($\beta = 3, 4$). $\mathbf{q} = (q_1, q_2, \cdots, q_r)^{\mathrm{T}}$ and $\boldsymbol{\kappa}_7 = (\kappa_{71}, \kappa_{72}, \cdots, \kappa_{7r})^{\mathrm{T}}$ with $\kappa_{7j} \in \{\varnothing, m_{7j}\}$ ($j = 1, 2, \cdots, r$). $\mathbf{m}_7 = (m_{71}, m_{72}, \cdots, m_{7r})^{\mathrm{T}}$.

I. *Simple switching and bifurcation*

(i) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|n_5 : n_6 : \varnothing)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}+1, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|n_5 : n_6 : \varnothing)$ spiral saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}+1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|n_5 : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(ii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1|n_5 : n_6 : \varnothing)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}+1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|n_5 : n_6 : \varnothing)$ spiral saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}+1] : \varnothing : \varnothing|n_5 : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : 1 : \varnothing|n_5 : \varnothing : \varnothing)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a saddle-stable node bifurcation of the $([n_1^{\mathrm{m}}+1, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing|n_5 : \varnothing : \varnothing)$ spiral sink and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [1, \varnothing] : \varnothing : \varnothing|n_5 : \varnothing : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : 1|n_5 : \varnothing : \varnothing)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a stable period-doubling bifurcation of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}+1] : [\varnothing, \varnothing] : \varnothing : \varnothing|n_5 : \varnothing : \varnothing)$ sink and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, 1] : \varnothing : \varnothing|n_5 : \varnothing : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(v) An $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|\varnothing : n_6 : \varnothing)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a saddle-unstable node bifurcation of the $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}+1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|\varnothing : n_6 : \varnothing)$ spiral source and $([1, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|\varnothing : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(vi) An $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1|\varnothing : n_6 : \varnothing)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is an unstable period-doubling bifurcation of the $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}+1] : \varnothing : \varnothing| \varnothing : n_6 : \varnothing)$ spiral source and $([\varnothing, 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|\varnothing : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(vii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 + 1 : n_6 : \varnothing)$ spiral saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 + 1 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(viii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing | n_5 : \varnothing : 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a stable Neimark bifurcation of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing | n_5 + 1 : \varnothing : \varnothing)$ spiral sink and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing | n_5 : 1 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(ix) An $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | \varnothing : n_6 : 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is an unstable Neimark bifurcation of the $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | \varnothing : n_6 + 1 : \varnothing)$ spiral source and $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | 1 : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(x) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + 1, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(xi) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1 | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + 1] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(xii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing | n_5 : n_6 : [n_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + 1, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : [n_7,\ \boldsymbol{q}; \boldsymbol{\kappa}_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : [n_7, \boldsymbol{q}; \boldsymbol{\kappa}_7])$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(xiii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1 | n_5 : n_6 : [n_7, \boldsymbol{q}; \boldsymbol{\kappa}_1])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : [n_7 j; k_1])\ \kappa_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + 1] : \varnothing : \varnothing | n_5 : n_6 : [n_7, \boldsymbol{q}; \boldsymbol{\kappa}_7]))$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(xiv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7 + 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of its $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 + 1 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 + 1 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(xv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : n_7 + 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 + 1 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 + 1 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

II. *Complex switching*

(i) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + n_3, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + n_3, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(ii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + n_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$

state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + n_4] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3 + k_3; \kappa_3] : \varnothing | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + k_3, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + k_3, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4 + k_4; \kappa_4] | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + k_4] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(v) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3 + k_3; \kappa_3] : [n_4 + k_4; \kappa_4] | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + k_3, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + k_3, n_2^{\mathrm{o}} + k_4] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(vi) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3 + k_3; \kappa_3] : \varnothing | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + k_3, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | \; n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + k_3, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(vii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4 + k_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + k_4] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(viii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4 + k_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + k_4] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(ix) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3 + k_3; \kappa_3] : [n_4 + k_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + k_3, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + k_3, n_2^{\mathrm{o}} + k_4] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(x) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7 + k_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 + k_7 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : n_2^{\mathrm{o}} + k_4] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 + k_7 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

Definition 2.22 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ $(\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1)$ in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i $(i = 1, 2, \cdots, m)$ with

$m = n(s+1)$. Set $N = \{1, 2, \cdots, m\}$, $N_j = \{j_1, j_2, \cdots, j_{n_j}\} \cup \varnothing$ with $j_p \in N$ $(p = 1, 2, \cdots, n_j, j = 1, 2, 3, 4)$ and $\Sigma_{j=1}^{4} n_j = m$. $\cup_{j=1}^{4} N_j = N$ and $N_j \cap N_l = \varnothing$ $(l \neq j)$. $N_j = \varnothing$ if $n_j = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ $(\alpha = 1, 2)$ and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \varnothing$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$ where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant, and n_4-flip real eigenvectors. An iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|$ flow. $\kappa_j \in \{\varnothing, m_j\}$ $(j = 3, 4)$.

I. *Simple critical cases*

(i) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + 1, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(ii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + 1] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : 1 : \varnothing|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a *saddle-stable node* bifurcation of the $([n_1^{\mathrm{m}} + 1, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing|$ sink and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [1, \varnothing] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : 1|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a stable period-doubling bifurcation of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + 1] : [\varnothing, \varnothing] : \varnothing : \varnothing|$ sink and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, 1] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(v) An $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a s*addle-unstable node* bifurcation of the $([\varnothing, \varnothing] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ source and $([1, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(vi) An $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is an unstable period-doubling bifurcation of the $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + 1] : \varnothing : \varnothing|$ source and $([\varnothing, 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(vii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + 1, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(viii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + 1] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

II. *Complex switching*

(i) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + n_3, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + n_3, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(ii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4; \kappa_4]|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + n_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + n_4] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iii) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3 + k_3; \kappa_3] : \varnothing|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m} + k_3, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3; \kappa_3] : \varnothing|$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m} + k_3, n_2^{\rm o}] : [n_3; \kappa_3] : \varnothing|$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iv) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : [n_4 + k_4; \kappa_4]|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}, n_1^{\rm o} + k_4] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : [n_4; \kappa_4]|$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o} + k_4] : \varnothing : [n_4; \kappa_4]|$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(v) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3 + k_3; \kappa_3] : [n_4 + k_4; \kappa_4]|$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m} + k_3, n_1^{\rm o} + k_4] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3; \kappa_3] : [n_4; \kappa_4]|$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m} + k_3, n_2^{\rm o} + k_4] : [n_3; \kappa_3] : [n_4; \kappa_4]|$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

Definition 2.23 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses $2m$ eigenvalues $\lambda_i (i = 1, 2, \cdots, 2m)$ with $2m = n(s+1)$. Set $N = \{1, 2, \cdots, n\}$, $N_j = \{j_1, j_2, \cdots, j_{n_j}\} \cup \varnothing$ with $j_p \in N$ $(p = 1, 2, \cdots, n_j, j = 5, 6, 77$ and $\Sigma_{j=5}^7 n_j = n$. $\cup_{j=5}^7 N_j = N$ and $N_j \cap N_l = \varnothing$ $(l \neq j)$. $N_j = \varnothing$ if $n_j = 0$. The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses n_5-stable,n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_k| = 1 (k \in N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $|n_5 : n_6 : n_7)$ flow. With repeated complex eigenvalues of $|\lambda_k| = 1$ $(k \in N_7)$, an iterative response of $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is an $|n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ flow. $\mathbf{q} = (q_1, \ q_2, \cdots, q_r)^{\rm T}$ and $\boldsymbol{\kappa}_7 = (\kappa_{71}, \kappa_{72}, \cdots, \kappa_{7r})^{\rm T}$ with $\kappa_{7j} \in \{\varnothing, m_{7j}\}$ $(j = 1, 2, \cdots, r)$. $\mathbf{m}_7 = (m_{71}, m_{72}, \cdots, m_{7r})^{\rm T}$.

I. *Simple switching and bifurcation*

(i) An $|n_5 : n_6 : 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + 1 : n_6 : \varnothing)$ spiral saddle and $|n_5 : n_6 + 1 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(ii) An $|n_5 : \varnothing : 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a stable Neimark bifurcation of the $|n_5 + 1 : \varnothing : \varnothing)$ spiral sink and $|n_5 : 1 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iii) An $|\varnothing : n_6 : 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is an unstable Neimark bifurcation of the $|\varnothing : n_6 + 1 : \varnothing)$ spiral source and $|1 : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iv) An $|n_5 : n_6 : n_7 + 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + 1 : n_6 : n_7)$ state and $|n_5 : n_6 + 1 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(v) An $|\varnothing : n_6 : n_7 + 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|1 : n_6 : n_7)$ state and $|n_5 : n_6 + 1 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(vi) An $|n_5 : \varnothing : n_7 + 1)$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + 1 : \varnothing : n_7)$ state and $|n_5 : 1 : n_7)$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

II. *Complex switching*

(i) An $|n_5 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + n_7 : n_6 : \varnothing)$ spiral saddle and $|n_5 : n_6 + n_7 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(ii) An $|n_5 : n_6 : [n_7 + k_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + k_7 : n_6 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state and $|n_5 : n_6 + k_7 : [n_7, \mathbf{q}; \boldsymbol{\kappa}_7])$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

(iii) An $|n_5 : n_6 : [n_7 + k_5 - k_6, \mathbf{q}_2; \boldsymbol{\kappa}_7])$ state of the fixed point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + k_5 : n_6 : [n_7, \mathbf{q}_2; \boldsymbol{\kappa}_7])$ state and $|n_5 : n_6 + k_6 : [n_7, \mathbf{q}_3; \boldsymbol{\kappa}_7])$ state for the fixed point $(\mathbf{y}_k^*, \mathbf{p})$.

2.3.1 Stability and Switching

To extend the idea of Definitions 2.11 and 2.12, a new function will be defined to determine the stability and the stability state switching.

Definition 2.24 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i $(i = 1, 2, \cdots, m)$ with $m = n(s+1)$, and there are m linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, m)$. For a perturbation of fixed point $\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$, let $\mathbf{z}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{z}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$. Define

$$s_k^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*) \tag{2.52}$$

where $s_k^{(i)} = c_k^{(i)} ||\mathbf{v}_i||^2$. Define the following functions

$$G_i(\mathbf{y}_k, \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{F}(\mathbf{y}_k, \mathbf{p}) - \mathbf{y}_k^*] \tag{2.53}$$

and

$$\begin{aligned} G^{(1)}_{s_k^{(i)}}(\mathbf{y}_k, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{s_k^{(i)}}\mathbf{F}(\mathbf{y}_k(s_k^{(i)}), \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k(s_k^{(i)}), \mathbf{p})\partial_{c_k^{(i)}}\mathbf{y}_k\partial_{s_k^{(i)}}c_k^{(i)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k(s_k^{(i)}), \mathbf{p})\mathbf{v}_i||\mathbf{v}_i||^{-2} \end{aligned} \tag{2.54}$$

$$G^{(m)}_{s_k^{(i)}}(\mathbf{y}_k, \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot D^{(m)}_{s_k^{(i)}}\mathbf{F}(\mathbf{y}_k(s_k^{(i)}), \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot D_{s_k^{(i)}}(D^{(m-1)}_{s_k^{(i)}}\mathbf{F}(\mathbf{y}_k(s_k^{(i)}), \mathbf{p})) \tag{2.55}$$

where $D_{s_k^{(i)}}(\cdot) = \partial(\cdot)/\partial s_k^{(i)}$ and $D^{(m)}_{s_k^{(i)}}(\cdot) = D_{s_k^{(i)}}(D^{(m-1)}_{s_k^{(i)}}(\cdot))$.

Definition 2.25 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i $(i = 1, 2, \cdots, m)$ with $m = n(s+1)$, and there are m linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, m)$. For a perturbation of fixed point $\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$, let $\mathbf{z}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{z}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is stable if

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \tag{2.56}$$

for $\mathbf{y}_k \in \mathscr{U}(\mathbf{y}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the sink (or stable node) on the direction $\mathbf{v}_i$.

(ii) $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is unstable if

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \tag{2.57}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the source (or unstable node) on the direction $\mathbf{v}_i$.

(iii) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is invariant if

$$\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*) = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*) \tag{2.58}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{y}_k^*$ is called to be degenerate on the direction $\mathbf{v}_i$.

(iv) $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is symmetrically flipped if

$$\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*) = -\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*) \tag{2.59}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \Omega_\alpha$. The equilibrium $\mathbf{y}_k^*$ is called to be degenerate on the direction $\mathbf{v}_i$.

The stability of fixed points for a specific eigenvector is presented in Fig. 2.4. The solid curve is $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. The circular symbol is fixed point. The shaded regions are stable. The horizontal solid line is for a degenerate case. The vertical solid line is for a line with infinite slope. The monotonically stable node (sink) is presented in Fig. 2.4a. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1}$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k$, respectively. From the fixed point, let

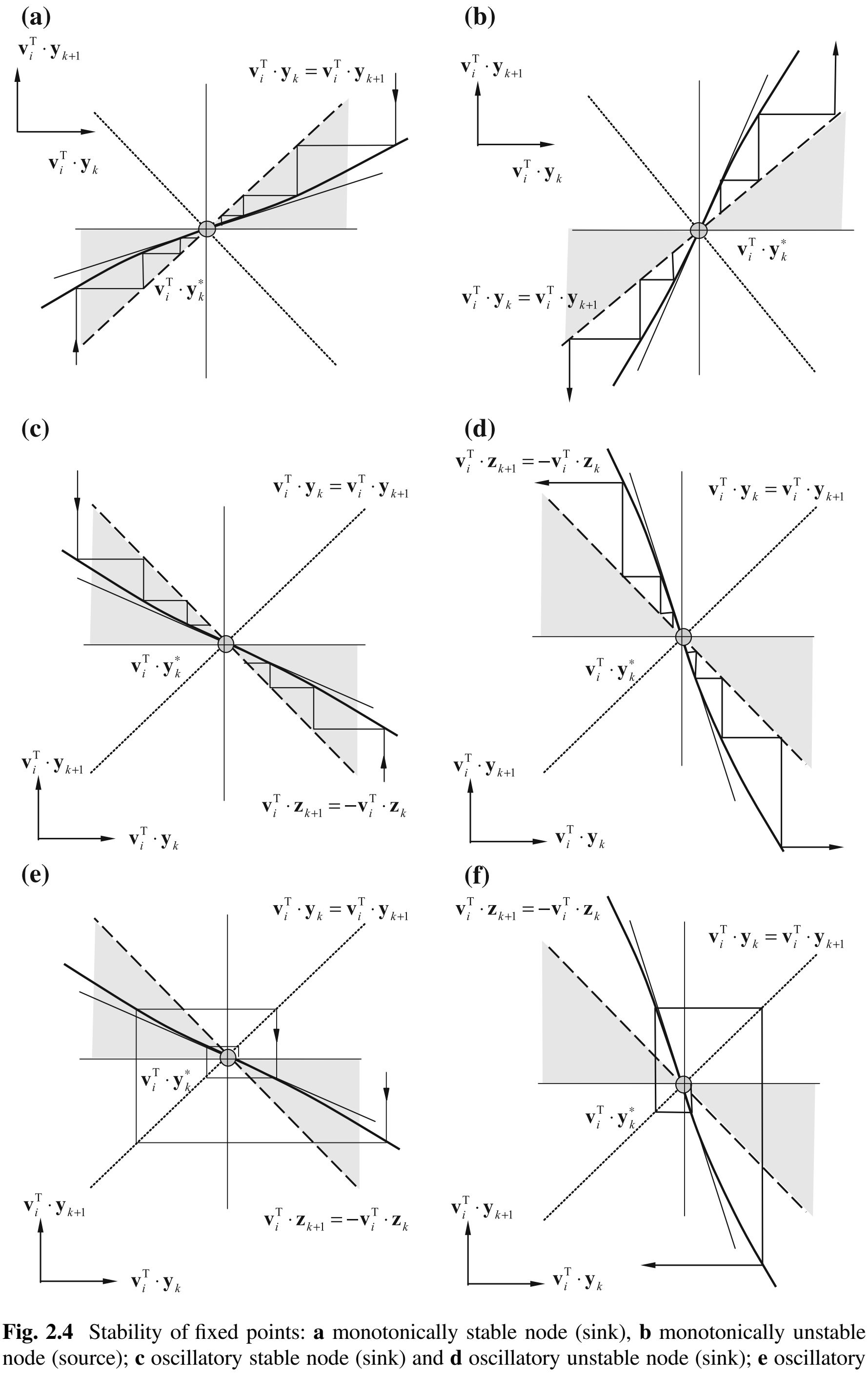

Fig. 2.4 Stability of fixed points: **a** monotonically stable node (sink), **b** monotonically unstable node (source); **c** oscillatory stable node (sink) and **d** oscillatory unstable node (sink); **e** oscillatory stable node (sink) and **f** oscillatory unstable node (sink). *Shaded areas* are stable zones. ($\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$ and $\mathbf{z}_{k+1} = \mathbf{y}_{k+1} - \mathbf{y}_k^*$)

$\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$ and $\mathbf{z}_{k+1} = \mathbf{y}_{k+1} - \mathbf{y}_k^*$. The iterative responses approach the fixed point. However, the monotonically unstable (source) is presented in Fig. 2.4b. The iterative responses go away from the fixed point. Similarly, the oscillatory stable node (sink) after iteration with a flip $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_{k+1}$ is presented in Fig. 2.4c. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1}$, respectively. In a similar fashion, the oscillatory unstable node (source) is presented in Fig. 2.4d. This illustration can be easily observed for the stability of fixed points. In Fig. 2.4e, f, the oscillatory stable and unstable nodes are presented as usual through the two-time iterations.

Theorem 2.5 *For a memorized discrete system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ *on domain* $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, *with* $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ *and* $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, *the equivalent discrete system is* $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. *Suppose there is a neighborhood of the fixed point* $\mathbf{y}_k^*$ *(i.e.,* $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$*), and* $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$*-continuous in* $U(\mathbf{y}_k^*)$. *The linearized system of the memorized nonlinear discrete system in the neighborhood* $U(\mathbf{y}_k^*)$ *of* $\mathbf{y}_k^*$ *is* $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ $(\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ *and* $l = k, k+1)$ *in Eq. (2.19). The matrix* $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ *possesses m eigenvalues* λ_i $(i = 1, 2, \cdots, m)$ *with* $m = n(s+1)$, *and there are m linearly independent vectors* $\mathbf{v}_i$ $(i = 1, 2, \cdots, m)$. *For a perturbation of fixed point* $\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$, *let* $\mathbf{z}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ *and* $\mathbf{z}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is stable if and only if*

$$G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i \in (-1, 1) \tag{2.60}$$

for $\mathbf{y}_k \in \mathscr{U}(\mathbf{y}_k^*) \subset \Omega_\alpha$.

(ii) $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is unstable if and only if*

$$G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i \in (1, \infty) \text{ and } (-\infty, -1) \tag{2.61}$$

for $\mathbf{y}_k \in \mathscr{U}(\mathbf{y}_k^*) \subset \Omega_\alpha$.

(iii) $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is invariant if and only if*

$$G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1 \textit{ and } G_{s_k^{(i)}}^{(m_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \;\; m_i = 2, 3, \cdots \tag{2.62}$$

for $\mathbf{y}_k \in \mathscr{U}(\mathbf{y}_k^*) \subset \Omega_\alpha$.

(iv) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is symmetrically flipped if and only if*

$$G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1 \textit{ and } G_{s_k^{(i)}}^{(m_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \, m_i = 2, 3, \cdots \tag{2.63}$$

for $\mathbf{y}_k \in \mathscr{U}(\mathbf{y}_k^*) \subset \Omega_\alpha$.

Proof Consider the increment on the eigenvector $\mathbf{v}_i$ as

$$\begin{aligned} s_{k+1}^{(i)} &= \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*) = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k^* + G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)} + o(s_k^{(i)}) - \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k^* \\ &= G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)} + o(s_k^{(i)}) \end{aligned}$$

Due to any selection of $s_k^{(i)}$ and $s_{k+1}^{(i)}$ as an infinitesimal, we have

$$\begin{aligned} s_{k+1}^{(i)} &= G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)}, \\ G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{y}_k} \mathbf{F}(\mathbf{y}_k^*, \mathbf{p}) \mathbf{v}_i ||\mathbf{v}_i||^{-2} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \lambda_i \mathbf{v}_i ||\mathbf{v}_i||^{-2} = \lambda_i. \end{aligned}$$

(i) From definition in Eq. (2.56), we have

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \Rightarrow |s_{k+1}^{(i)}| < |s_k^{(i)}|$$

which gives

$$|G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)}| < |s_k^{(i)}|$$

Thus,

$$|G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p})| < 1 \Rightarrow G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i \in (-1, 1)$$

Therefore, $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is stable, vice versa.

(ii) From definition in Eq. (2.57),

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \Rightarrow |s_{k+1}^{(i)}| > |s_k^{(i)}|$$

which requires

$$|G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)}| > |s_k^{(i)}|$$

Thus,

$$|G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p})| > 1 \Rightarrow G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i \in (-\infty, -1) \text{ and } (1, \infty).$$

Therefore, $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is unstable, vice versa.

(iii) Because

$$\begin{aligned} s_{k+1}^{(i)} &= \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*) \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k^* + G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)} + \sum\nolimits_{m_i=2}^{\infty} \frac{1}{m_i!} G_{s_k^{(i)}}^{(m_i)}(\mathbf{y}_k^*, \mathbf{p}) (s_k^{(i)})^{m_i} - \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k^* \\ &= G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)} + \sum\nolimits_{m_i=2}^{\infty} \frac{1}{m_i!} G_{s_k^{(i)}}^{(m_i)}(\mathbf{y}_k^*, \mathbf{p}) (s_k^{(i)})^{m_i} \end{aligned}$$

From definition in Eq. (2.58)

$$\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*) = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*) \;\Rightarrow\; s_{k+1}^{(i)} = s_k^{(i)}$$

Due to any selection of $s_k^{(i)}$ and $s_{k+1}^{(i)}$ as an infinitesimal, we have

$$G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1 \text{ and } G_{s_k^{(i)}}^{(m_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \;\text{ for } m_i = 2, 3, \cdots.$$

Therefore, $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is invariant, vice versa.

(iv) From definition in Eq. (2.59)

$$\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*) = -\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*) \Rightarrow s_{k+1}^{(i)} = -s_k^{(i)}$$

Due to any selection of $s_k^{(i)}$ and $s_{k+1}^{(i)}$ as an infinitesimal, we have

$$G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1 \text{ and } G_{s_k^{(i)}}^{(m_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } m_i = 2, 3, \cdots.$$

Therefore, $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is flipped, vice versa. The theorem is proved. ■

Definition 2.26 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i $(i = 1, 2, \cdots, m)$ with $m = n(s+1)$, and there are m linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, m)$. For a perturbation of fixed point $\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$, let $\mathbf{z}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{z}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is monotonically stable of the $(2m_i+1)$th-order if

$$\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \,\text{for}\, r_i = 2, 3, \cdots, 2m_i, \\
&G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p}) \neq 0, \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|
\end{aligned} \tag{2.64}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the monotonic sink (or stable node) of the $(2m_i+1)$th-order on the direction $\mathbf{v}_i$.

(ii) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i+1)$th-order if

$$\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \,\text{for}\, r_i = 2, 3, \cdots, 2m_i; \\
&G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p}) \neq 0; \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|
\end{aligned} \tag{2.65}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the monotonic source (or unstable node) of the $(2m_i+1)$th-order on the direction $\mathbf{v}_i$.

(iii) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, lower saddle if

$$\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \,\text{for}\, r_i = 2, 3, \cdots, 2m_i - 1; \\
&G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) \neq 0, \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \,\text{for}\, s_k^{(i)} > 0 \\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \,\text{for}\, s_k^{(i)} < 0
\end{aligned} \tag{2.66}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the monotonic, lower saddle of the $(2m_i)$th-order on the direction $\mathbf{v}_i$.

(iv) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, upper saddle if

$$\left.\begin{array}{l}
G^{(1)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1, \\
G^{(r_i)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1; \\
G^{(2m_i)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) \neq 0, \\
|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \text{ for } s_k^{(i)} > 0 \\
|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \text{ for } s_k^{(i)} < 0
\end{array}\right\} \tag{2.67}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the monotonic, upper saddle of the $(2m_i)$th-order on the direction $\mathbf{v}_i$.

(v) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory stable of the $(2m_i + 1)$th-order if

$$\left.\begin{array}{l}
G^{(1)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1, \\
G^{(r_i)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i; \\
G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) \neq 0; \\
|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|
\end{array}\right\} \tag{2.68}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the oscillatory sink (or stable node) of the $(2m_i + 1)$th-order on the direction $\mathbf{v}_i$.

(vi) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i + 1)$th-order if

$$\left.\begin{array}{l}
G^{(1)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1; \\
G^{(r_i)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i; \\
G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) \neq 0, \\
|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|
\end{array}\right\} \tag{2.69}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the oscillatory source (or unstable node) of the $(2m_i + 1)$th-order on the direction $\mathbf{v}_i$.

(vii) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i)$th-order, lower saddle if

$$
\begin{aligned}
& G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1, \\
& G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1; \\
& G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) \neq 0, \\
& |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \text{ for } s_k^{(i)} > 0, \\
& |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \text{ for } s_k^{(i)} < 0
\end{aligned}
\tag{2.70}
$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the oscillatory lower saddle of the $(2m_i)$th-order on the direction $\mathbf{v}_i$.

(viii) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i)$th-order, upper saddle if

$$
\begin{aligned}
& G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1, \\
& G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1; \\
& G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) \neq 0, \\
& |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \text{ for } s_k^{(i)} > 0, \\
& |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)| \text{ for } s_k^{(i)} < 0
\end{aligned}
\tag{2.71}
$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$. The fixed point $\mathbf{y}_k^*$ is called the oscillatory, upper saddle of the $(2m_i)$th-order on the direction $\mathbf{v}_i$.

The monotonic stability of fixed points with higher order singularity for a specific eigenvector is presented in Fig. 2.5. The solid curve is $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. The circular symbol is fixed pointed. The shaded regions are stable. The horizontal solid line is also for the degenerate case. The vertical solid line is for a line with infinite slope. The monotonically stable node (sink) of the $(2m_i + 1)$th order is sketched in Fig. 2.5a. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1}$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k$, respectively. The nonlinear curve lies in the stable zone, and the iterative responses approach the fixed point. However, the monotonically unstable node source of the $(2m_i + 1)$th order is presented in Fig. 2.5b. The nonlinear curve lies in the unstable zone, and the iterative responses go away from the fixed point. The monotonically lower saddle of the $(2m_i)$th order is presented in Fig. 2.5c. The nonlinear curve is tangential to the line of $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1}$ with the $(2m_i)$th order, and the one branch is in the stable zone and another branch is in the unstable zone. Similarly, the monotonically upper saddle of

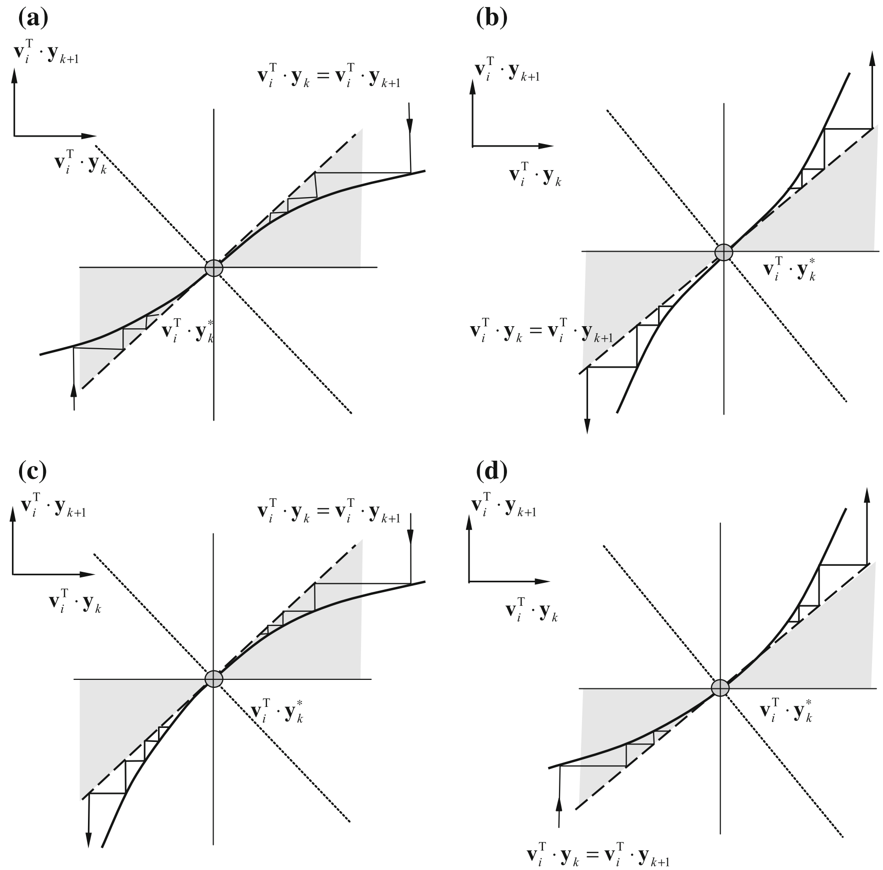

Fig. 2.5 Monotonic stability of fixed points with higher order singularity: **a** monotonically stable node (sink) of $(2m_i + 1)$th-order, **b** monotonically unstable node (source) of $(2m_i + 1)$th-order, **c** monotonically lower saddle of $(2m_i)$th-order and **d** monotonically upper saddle of $(2m_i)$th-order. *Shaded areas* are stable zones. ($\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$ and $\mathbf{z}_{k+1} = \mathbf{y}_{k+1} - \mathbf{y}_k^*$)

the $(2m_i)$th order is presented in Fig. 2.5d. The oscillatory stability of fixed points with higher order singularity for a specific eigenvector after iteration with a flip $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_{k+1}$ is presented in Fig. 2.6. The oscillatory stable node (sink) of the $(2m_i + 1)$th order is sketched in Fig. 2.6a. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1}$, respectively. The nonlinear curve lies in the stable zone, and the iterative responses approach the fixed point. However, the oscillatory unstable node (source) of the $(2m_i + 1)$th order is presented in Fig. 2.6b. The nonlinear curve lies in the unstable zone, and the iterative responses go away from the fixed point. The oscillatory lower saddle of the $(2m_i)$th order is

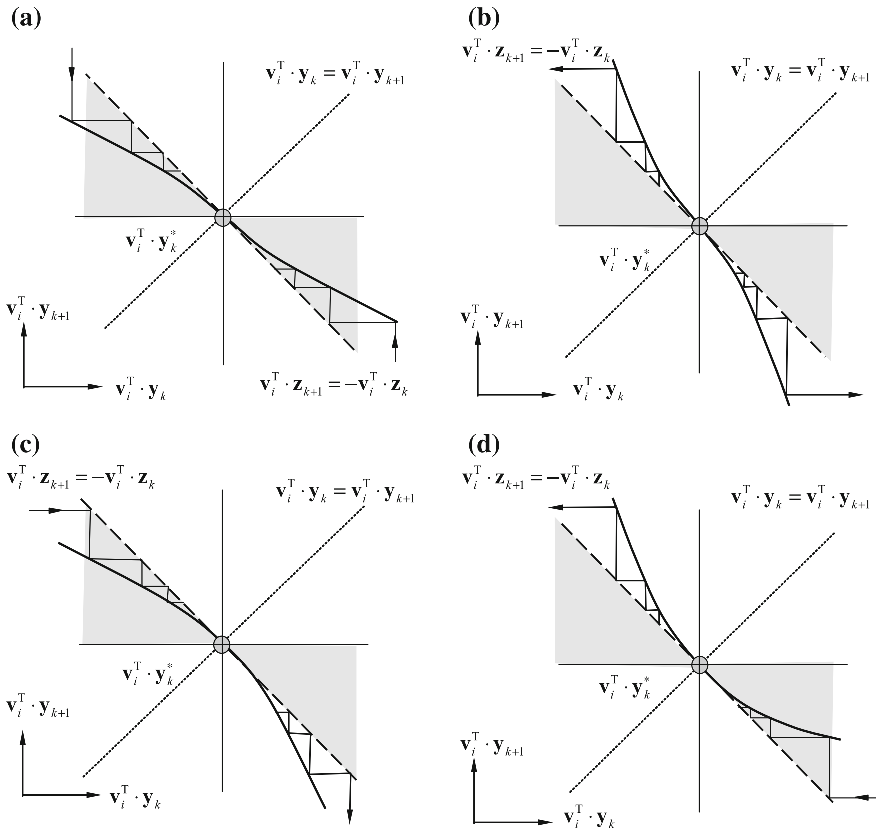

Fig. 2.6 Oscillatory stability of fixed points with higher order singularity after iteration with a flip $\mathbf{v}_i^T \cdot \mathbf{z}_k = -\mathbf{v}_i^T \cdot \mathbf{z}_{k+1}$: **a** oscillatory stable node (sink) of $(2m_i+1)$th-order, **b** oscillatory unstable node (source) of $(2m_i+1)$th-order, **c** oscillatory lower saddle of $(2m_i)$th-order and **d** oscillatory upper saddle of $(2m_i)$th-order. Shaded areas are stable zones. ($\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$ and $\mathbf{z}_{k+1} = \mathbf{y}_{k+1} - \mathbf{y}_k^*$)

presented in Fig. 2.6c. The nonlinear curve is tangential to and below the line of $\mathbf{v}_i^T \cdot \mathbf{z}_{k+1} = -\mathbf{v}_i^T \cdot \mathbf{z}_k$ with the $(2m_i)$th order, and the one branch is in the stable zone and another branch is in the unstable zone. Finally, the oscillatory upper saddle of the $(2m_i)$th order is presented in Fig. 2.6d. For clear illustrations, oscillatory stability of fixed points with higher order singularity for the two-time iterations are presented in Fig. 2.7.

Theorem 2.6 *For a memorized discrete system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ *on domain* $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, *with* $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ *and* $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$,

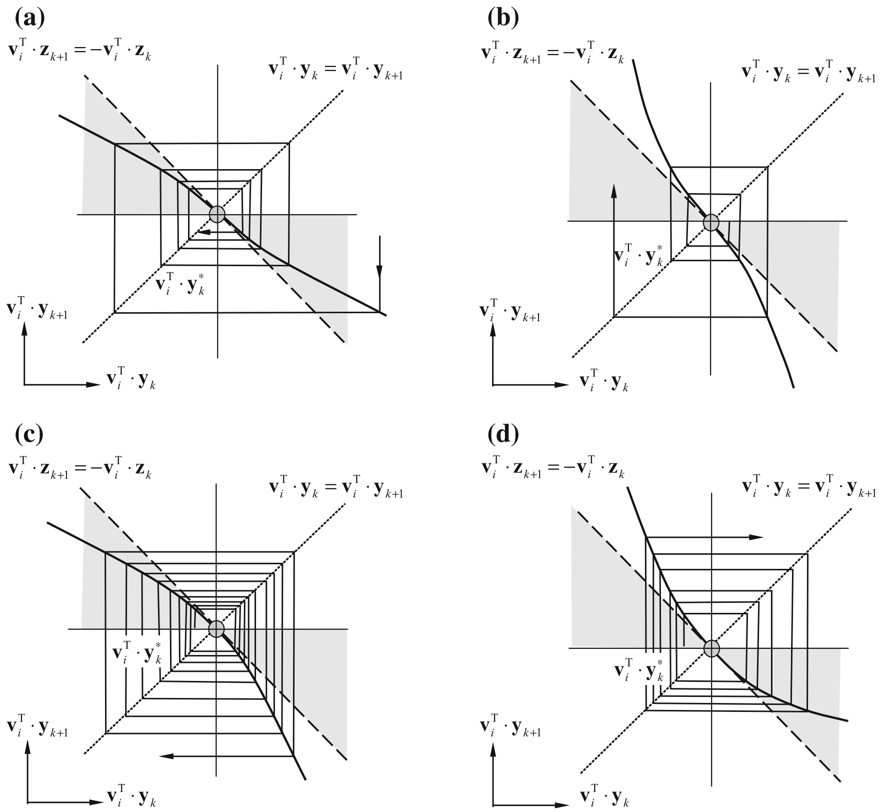

Fig. 2.7 Oscillatory stability of fixed points with higher order singularity for the two-time iterations: **a** oscillatory stable node (sink) of $(2m_i + 1)$th-order, **b** oscillatory unstable node (source) of $(2m_i + 1)$th-order, **c** oscillatory lower saddle of $(2m_i)$th-order and **d** oscillatory upper saddle of $(2m_i)$th-order. *Shaded areas* are stable zones. ($\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$ and $\mathbf{z}_{k+1} = \mathbf{y}_{k+1} - \mathbf{y}_k^*$)

the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. *Suppose there is a neighborhood of the fixed point* $\mathbf{y}_k^*$ (*i.e.*, $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), *and* $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ *is* C^r ($r \geq 1$)*-continuous in* $U(\mathbf{y}_k^*)$. *The linearized system of the memorized nonlinear discrete system in the neighborhood* $U(\mathbf{y}_k^*)$ *of* $\mathbf{y}_k^*$ *is* $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ *and* $l = k, k+1$) *in Eq.* (2.19). *The matrix* $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ *possesses m-eigenvalues* λ_i ($i = 1, 2, \cdots, m$) *with* $m = n(s+1)$, *and there are m linearly independent vectors* $\mathbf{v}_i (i = 1, 2, \cdots, m)$. *For a perturbation of fixed point* $\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$, *let* $\mathbf{z}_k^{(i)} = c_k^{(i)} \mathbf{v}_i$ *and* $\mathbf{z}_{k+1}^{(i)} = c_{k+1}^{(i)} \mathbf{v}_i$.

(i) $\mathbf{y}_{k+j} (j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is monotonically stable of the* $(2m_i + 1)$th-*order if and only if*

$$\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i, \\
&G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p}) < 0
\end{aligned} \tag{2.72}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$.

(ii) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is monotonically unstable of the* $(2m_i + 1)$th-*order if and only if*

$$\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i, \\
&G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p}) > 0
\end{aligned} \tag{2.73}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$.

(iii) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is monotonically unstable of the* $(2m_i)$th-*order, lower saddle if and only if*

$$\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1, \\
&G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) < 0 \text{ stable for } s_k^{(i)} > 0; \\
&G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) < 0 \text{ unstable for } s_k^{(i)} < 0
\end{aligned} \tag{2.74}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iv) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is monotonically unstable of the* $(2m_i)$th-*order if and only if*

$$\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = 1, \\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1, \\
&G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) > 0 \text{ unstable for } s_k^{(i)} > 0; \\
&G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) > 0 \text{ stable for } s_k^{(i)} < 0
\end{aligned} \tag{2.75}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$.

(v) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is oscillatory stable of the* $(2m_i+1)$th-*order if and only if*

$$\begin{aligned} &G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1, \\ &G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i, \\ &G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p}) > 0 \end{aligned} \tag{2.76}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$.

(vi) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i+1)$th-*order if and only if*

$$\begin{aligned} &G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1, \\ &G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i, \\ &G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p}) < 0 \end{aligned} \tag{2.77}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$.

(vii) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i)$th-*order, upper saddle if and only if*

$$\begin{aligned} &G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1, \\ &G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1, \\ &G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) > 0 \text{ stable for } s_k^{(i)} > 0; \\ &G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) > 0 \text{ unstable for } s_k^{(i)} < 0 \end{aligned} \tag{2.78}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$.

(viii) $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{y}_k^*$ *on the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i)$th-*order lower saddle if and only if*

$$\begin{aligned} &G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = \lambda_i = -1, \\ &G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p}) = 0 \text{for } r_i = 2, 3, \cdots, 2m_i - 1, \\ &G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) < 0 \text{ stable for } s_k^{(i)} < 0; \\ &G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) < 0 \text{ unstable for } s_k^{(i)} > 0 \end{aligned} \tag{2.79}$$

for $\mathbf{y}_k \in U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$.

Proof Because

$$\begin{aligned} s_{k+1}^{(i)} &= \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*) \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k^* + G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)} + \sum\nolimits_{r_i=2}^{2m_i+1} \frac{1}{r_i!} G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{r_i} \\ &\quad - \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k^* + o((s_k^{(i)})^{2m_i+1}) \\ &= G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)} + \sum\nolimits_{r_i=2}^{2m_i} \frac{1}{r_i!} G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{r_i} \\ &\quad + \frac{1}{(2m_i+1)!} G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i+1} + o((s_k^{(i)})^{2m_i+1}) \end{aligned}$$

and

$$s_k^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)$$

(i) From the first two equations of Eq. (2.72), for the infinitesimal $s_k^{(i)}$, one obtains

$$s_{k+1}^{(i)} = [G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) + \frac{1}{(2m_i+1)!} G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}] s_k^{(i)}$$

Since

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|,$$

we have

$$\begin{aligned} |s_{k+1}^{(i)}| &= \left| [G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) + \frac{1}{(2m_i+1)!} G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}] s_k^{(i)} \right| \\ &= \left| G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) + \frac{1}{(2m_i+1)!} G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i} \right| |s_k^{(i)}| \\ &< |s_k^{(i)}|. \end{aligned}$$

For $G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = 1$, we have

$$|1 + \frac{1}{(2m_i+1)!} G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}| < 1.$$

Since the infinitesimal $s_k^{(i)}$ is arbitrarily selected, the foregoing equation gives

$$G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p}) < 0.$$

Therefore, $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is monotonically stable of the $(2m_i+1)$th-order, vice versa.

(ii) Similarly, since

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|,$$

we have

$$\left| 1 + \frac{1}{(2m_i+1)!} G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i} \right| > 1.$$

For the arbitrarily infinitesimal $s_k^{(i)}$, the foregoing equation requires

$$G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{y}_k^*, \mathbf{p}) > 0.$$

Therefore, $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i+1)$th-order, vice versa.

(iii) The Taylor expansion of $s_{k+1}^{(i)}$ keeps up to the $(2m_i)$th term of $s_k^{(i)}$

$$\begin{aligned} s_{k+1}^{(i)} &= \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*) \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k^* + G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)} + \sum\nolimits_{r_i=2}^{2m_i} \frac{1}{r_i!} G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{r_i} \\ &\quad - \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k^* + o((s_k^{(i)})^{2m_i}) \\ &= G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) s_k^{(i)} + \sum\nolimits_{r_i=2}^{2m_i-1} \frac{1}{r_i!} G_{s_k^{(i)}}^{(r_i)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{r_i} \\ &\quad + \frac{1}{(2m_i)!} G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i} + o((s_k^{(i)})^{2m_i}). \end{aligned}$$

From the first two equations of Eq. (2.74), for the infinitesimal $s_k^{(i)}$, one obtains

$$s_{k+1}^{(i)} = s_k^{(i)} + \frac{1}{(2m_i)!} G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}.$$

Thus

$$|s_{k+1}^{(i)}| = \left|[1 + \frac{1}{(2m_i)!} G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i - 1}]s_k^{(i)}\right|$$
$$= \left|1 + \frac{1}{(2m_i)!} G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i - 1}\right| |s_k^{(i)}|.$$

For $G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) < 0$, if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| < |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|, \text{and}$$

if $s_k^{(i)} < 0$, we have

$$|s_{k+1}^{(i)}| > |s_k^{(i)}| \;\Rightarrow\; |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|.$$

Thus, $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, lower saddle, vice versa.

(iv) Similar to (iii), for $G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) > 0$, if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| > |s_k^{(i)}| \;\Rightarrow\; |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|,$$

if $s_k^{(i)} < 0$, we have

$$|s_{k+1}^{(i)}| < |s_k^{(i)}| \;\Rightarrow\; |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|,$$

Thus, $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, upper saddle, vice versa.

(v) Similar to case (i), consider

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|,$$

For $G_{s_k^{(i)}}^{(1)}(\mathbf{y}_k^*, \mathbf{p}) = -1$, we have

$$|-1 + \frac{1}{(2m_i + 1)!} G_{s_k^{(i)}}^{(2m_i + 1)}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}| < 1.$$

Since the infinitesimal $s_k^{(i)}$ is arbitrarily selected, the foregoing equation gives

$$G_{s_k^{(i)}}^{(2m_i + 1)}(\mathbf{y}_k^*, \mathbf{p}) > 0.$$

Therefore, $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory stable of the $(2m_i + 1)$th-order, vice versa.

(vi) Similar to case (ii), consider

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|,$$

For $G^{(1)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) = -1$, we have

$$|-1 + \frac{1}{(2m_i+1)!} G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}| > 1.$$

Since the infinitesimal $s_k^{(i)}$ is arbitrarily selected, the foregoing equation gives

$$G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) < 0.$$

Therefore, $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i + 1)$th-order, vice versa.

(vii) Similar to (iii), from the first two equations of Eq. (2.78), for the infinitesimal $s_k^{(i)}$, one obtains

$$s_{k+1}^{(i)} = -s_k^{(i)} + \frac{1}{(2m_i)!} G^{(2m_i)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}$$

Thus

$$\begin{aligned} |s_{k+1}^{(i)}| &= \left| [-1 + \frac{1}{(2m_i)!} G^{(2m_i)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i - 1}] s_k^{(i)} \right| \\ &= \left| -1 + \frac{1}{(2m_i)!} G^{(2m_i)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i - 1} \right| |s_k^{(i)}|. \end{aligned}$$

For $G^{(2m_i)}_{s_k^{(i)}}(\mathbf{y}_k^*, \mathbf{p}) > 0$, if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| < |s_k^{(i)}| \;\Rightarrow\; |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|, \text{and}$$

if $s_k^{(i)} < 0$, we have

$$|s_{k+1}^{(i)}| > |s_k^{(i)}| \;\Rightarrow\; |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|.$$

Thus, $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i)$th-order, upper saddle, vice versa.

(viii) Similar to (vii), for $G_{s_k^{(i)}}^{(2m_i)}(\mathbf{y}_k^*, \mathbf{p}) < 0$, if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| > |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|,$$

if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| < |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_{k+1} - \mathbf{y}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*)|.$$

Thus, $\mathbf{y}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{y}_k^*$ on the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, lower saddle, vice versa. This theorem is proved. ■

Definition 2.27 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$, and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). Consider, a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ from m eigenvalues $\lambda_j (j = 1, 2, \cdots, m)$ with $m = n(s+1)$ of matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{z}_k^{(i)} = \mathbf{z}_{k+}^{(i)} + \mathbf{z}_{k-}^{(i)}$ with

$$\begin{aligned} \mathbf{r}_k^{(i)} &= c_k^{(i)} \mathbf{u}_i + d_k^{(i)} \mathbf{v}_i, \\ \mathbf{r}_{k+1}^{(i)} &= c_{k+1}^{(i)} \mathbf{u}_i + d_{k+1}^{(i)} \mathbf{v}_i \end{aligned} \tag{2.80}$$

and

$$\begin{aligned} c_k^{(i)} &= \frac{1}{\Delta}[\Delta_2(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{z}_k) - \Delta_{12}(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k)], \\ d_k^{(i)} &= \frac{1}{\Delta}[\Delta_1(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k) - \Delta_{12}(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{z}_k)]; \\ \Delta_1 &= ||\mathbf{u}_i||^2, \Delta_2 = ||\mathbf{v}_i||^2, \Delta_{12} = \mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{v}_i; \\ \Delta &= \Delta_1 \Delta_2 - \Delta_{12}^2. \end{aligned} \tag{2.81}$$

Consider a polar coordinate of (r_k, θ_k) defined by

$$\begin{aligned} & c_k^{(i)} = r_k^{(i)} \cos \theta_k^{(i)}, \text{ and } d_k^{(i)} = r_k^{(i)} \sin \theta_k^{(i)}; \\ & r_k^{(i)} = \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2}, \text{ and } \theta_k^{(i)} = \arctan\,(d_k^{(i)}/c_k^{(i)}). \end{aligned} \tag{2.82}$$

Thus

$$\begin{aligned} c_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}(\mathbf{y}_k,\mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{y}_k,\mathbf{p})] \\ d_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}(\mathbf{y}_k,\mathbf{p}) - \Delta_{12} G_{c_k^{(i)}}(\mathbf{y}_k,\mathbf{p})] \end{aligned} \tag{2.83}$$

where

$$\begin{aligned} G_{c_k^{(i)}}(\mathbf{y}_k,\mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}} \cdot [\mathbf{F}(\mathbf{y}_k,\mathbf{p}) - \mathbf{y}_k^*] = \sum\nolimits_{m_i=1}^{\infty} \frac{1}{m_i!} G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)})(r_k^{(i)})^{m_i}, \\ G_{d_k^{(i)}}(\mathbf{y}_k,\mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{F}(\mathbf{y}_k,\mathbf{p}) - \mathbf{y}_k^*] = \sum\nolimits_{m_i=1}^{\infty} \frac{1}{m_i!} G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)})(r_k^{(i)})^{m_i}; \end{aligned} \tag{2.84}$$

$$\begin{aligned} G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \mathbf{u}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(m_i)} \mathbf{F}(\mathbf{y}_k,\mathbf{p})[\mathbf{u}_i \cos\theta_k^{(i)} + \mathbf{v}_i \sin\theta_k^{(i)}]^{m_i}\Big|_{(\mathbf{y}_k^*,\mathbf{p})}, \\ G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(m_i)} \mathbf{F}(\mathbf{y}_k,\mathbf{p})[\mathbf{u}_i \cos\theta_k^{(i)} + \mathbf{v}_i \sin\theta_k^{(i)}]^{m_i}\Big|_{(\mathbf{y}_k^*,\mathbf{p})}. \end{aligned} \tag{2.85}$$

Thus

$$\begin{aligned} r_{k+1}^{(i)} &= \sqrt{(c_{k+1}^{(i)})^2 + (d_{k+1}^{(i)})^2} = \sqrt{\sum\nolimits_{m=2}^{\infty} (r_k^{(i)})^{m_i} G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)})} \\ &= \sqrt{G_{r_{k+1}^{(i)}}^{(2)}}\, r_k^{(i)} \sqrt{1 + (G_{r_{k+1}^{(i)}}^{(2)})^{-1} \sum\nolimits_{m_i=3}^{\infty} (r_k^{(i)})^{m_i-2} G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)})} \\ \theta_{k+1}^{(i)} &= \arctan(d_{k+1}^{(i)}/c_{k+1}^{(i)}) \end{aligned} \tag{2.86}$$

where

$$G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) = \frac{1}{m_i!} \sum\nolimits_{q=1}^{m_i-1} C_{m_i}^q \left[G_{c_{k+1}^{(i)}}^{(m_i-q)}(\theta_k^{(i)}) G_{c_{k+1}^{(i)}}^{(q)}(\theta_k^{(i)}) + G_{d_{k+1}^{(i)}}^{(m_i-q)}(\theta_k^{(i)}) G_{d_{k+1}^{(i)}}^{(q)}(\theta_k^{(i)}) \right] \tag{2.87}$$

and

$$\begin{aligned} G_{c_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) - \Delta_{12} G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)})], \\ G_{d_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) - \Delta_{12} G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)})], \\ C_{m_i}^q &= \frac{m_i!}{q!(m_i-q)!}. \end{aligned} \tag{2.88}$$

From the foregoing definition, consider the first-order terms of G-function

$$\begin{aligned} G^{(1)}_{c^{(i)}_k}(\mathbf{y}_k,\mathbf{p}) &= G^{(1)}_{c^{(i)}_k 1}(\mathbf{y}_k,\mathbf{p}) + G^{(1)}_{c^{(i)}_k 2}(\mathbf{y}_k,\mathbf{p}),\\ G^{(1)}_{d^{(i)}_k}(\mathbf{y}_k,\mathbf{p}) &= G^{(1)}_{d^{(i)}_k 1}(\mathbf{y}_k,\mathbf{p}) + G^{(1)}_{d^{(i)}_k 2}(\mathbf{y}_k,\mathbf{p}) \end{aligned} \tag{2.89}$$

where

$$\begin{aligned} G^{(1)}_{c^{(i)}_k 1}(\mathbf{y}_k,\mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}}\cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\partial_{c^{(i)}_k}\mathbf{y}_k = \mathbf{u}_i^{\mathrm{T}}\cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\mathbf{u}_i\\ &= \mathbf{u}_i^{\mathrm{T}}\cdot(-\beta_i\mathbf{v}_i+\alpha_i\mathbf{u}_i) = \alpha_i\Delta_1-\beta_i\Delta_{12},\\ G^{(1)}_{c^{(i)}_k 2}(\mathbf{y}_k,\mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}}\cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\partial_{d^{(i)}_k}\mathbf{y}_k = \mathbf{u}_i^{\mathrm{T}}\cdot D_{\mathbf{x}_k}D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\mathbf{v}_i\\ &= \mathbf{u}_i^{\mathrm{T}}\cdot(\beta_i\mathbf{u}_i+\alpha_i\mathbf{v}_i) = \alpha_i\Delta_{12}+\beta_i\Delta_1; \end{aligned} \tag{2.90}$$

and

$$\begin{aligned} G^{(1)}_{d^{(i)}_k 1}(\mathbf{y}_k,\mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}}\cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\partial_{c^{(i)}_k}\mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}}\cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\mathbf{u}_i\\ &= \mathbf{v}_i^{\mathrm{T}}\cdot(-\beta_i\mathbf{v}_i+\alpha_i\mathbf{u}_i) = -\beta_i\Delta_2+\alpha_i\Delta_{12},\\ G^{(1)}_{d^{(i)}_k 2}(\mathbf{y},\mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}}\cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\partial_{d^{(i)}_k}\mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}}\cdot D_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\mathbf{v}_i\\ &= \mathbf{v}_i^{\mathrm{T}}\cdot(\beta_i\mathbf{u}_i+\alpha_i\mathbf{v}_i) = \alpha_i\Delta_2+\beta_i\Delta_{12}. \end{aligned} \tag{2.91}$$

Substitution of Eqs. (2.89)–(2.91) into Eq. (2.85) gives

$$\begin{aligned} G^{(1)}_{c^{(i)}_k}(\theta^{(i)}_k) &= G^{(1)}_{c^{(i)}_k 1}(\mathbf{y}_k,\mathbf{p})\cos\theta^{(i)}_k + G^{(1)}_{c^{(i)}_k 2}(\mathbf{y}_k,\mathbf{p})\sin\theta^{(i)}_k\\ &= (\alpha_i\Delta_1-\beta_i\Delta_{12})\cos\theta^{(i)}_k + (\alpha_i\Delta_{12}+\beta_i\Delta_1)\sin\theta^{(i)}_k,\\ G^{(1)}_{d^{(i)}_k}(\theta^{(i)}_k) &= G^{(1)}_{d^{(i)}_k 1}(\mathbf{y}_k,\mathbf{p})\cos\theta^{(i)}_k + G^{(1)}_{d^{(i)}_k 2}(\mathbf{y}_k,\mathbf{p})\sin\theta^{(i)}_k\\ &= (-\beta_i\Delta_2+\alpha_i\Delta_{12})\cos\theta^{(i)}_k + (\alpha_i\Delta_2+\beta_i\Delta_{12})\sin\theta^{(i)}_k. \end{aligned} \tag{2.92}$$

From Eq. (2.85), we have

$$\begin{aligned} G^{(1)}_{c^{(i)}_{k+1}}(\theta^{(i)}_k) &= \frac{1}{\Delta}[\Delta_2 G^{(1)}_{c^{(i)}_k}(\theta^{(i)}_k) - \Delta_{12}G^{(1)}_{d^{(i)}_k}(\theta^{(i)}_k)]\\ &= \alpha_i\cos\theta^{(i)}_k + \beta_i\sin\theta^{(i)}_k,\\ G^{(1)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k) &= \frac{1}{\Delta}[\Delta_1 G^{(1)}_{d^{(i)}_k}(\theta^{(i)}_k) - \Delta_{12}G^{(1)}_{c^{(i)}_k}(\theta^{(i)}_k)]\\ &= \alpha_i\sin\theta^{(i)}_k - \beta_i\cos\theta^{(i)}_k. \end{aligned} \tag{2.93}$$

Thus

$$\begin{aligned} G^{(2)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k) &= [G^{(1)}_{c^{(i)}_{k+1}}(\theta^{(i)}_k)G^{(1)}_{c^{(i)}_{k+1}}(\theta^{(i)}_k) + G^{(1)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k)G^{(1)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k)] \\ &= \alpha_i^2 + \beta_i^2. \end{aligned} \tag{2.94}$$

Furthermore, Eq. (2.86) gives

$$r^{(i)}_{k+1} = \rho_i r^{(i)}_k + o(r^{(i)}_k) \text{ and } \theta^{(i)}_{k+1} = \theta^{(i)}_k - \vartheta_i + o(r^{(i)}_k). \tag{2.95}$$

where

$$\vartheta_i = \arctan(\beta_i/\alpha_i) \text{ and } \rho_i = \sqrt{\alpha_i^2 + \beta_i^2}. \tag{2.96}$$

As $r^{(i)}_k \ll 1$ and $r_k \to 0$, we have

$$r^{(i)}_{k+1} = \rho_i r^{(i)}_k \text{ and } \theta^{(i)}_{k+1} = \vartheta_i - \theta^{(i)}_k. \tag{2.97}$$

With an initial condition of $r^{(i)}_k = r^0_k$ and $\theta^{(i)}_k = \theta^{(i)}_k$, the corresponding solution of Eq. (2.97) is

$$r^{(i)}_{k+j} = (\rho_i)^j r^0_k \text{ and } \theta^{(i)}_{k+j} = j\vartheta_i - \theta^{(i)}_k. \tag{2.98}$$

From Eqs. (2.83), (2.84) and (2.93), we have

$$\begin{aligned} c^{(i)}_{k+1} &= \alpha_i r^{(i)}_k \cos\theta^{(i)}_k + \beta_i r^{(i)}_k \sin\theta^{(i)}_k = \alpha_i c^{(i)}_k + \beta_i d^{(i)}_k, \\ d^{(i)}_{k+1} &= \alpha_i r^{(i)}_k \sin\theta^{(i)}_k - \beta_i r^{(i)}_k \cos\theta^{(i)}_k = -\beta_i c^{(i)}_k + \alpha_i d^{(i)}_k. \end{aligned} \tag{2.99}$$

That is,

$$\begin{Bmatrix} c^{(i)}_{k+1} \\ d^{(i)}_{k+1} \end{Bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \begin{Bmatrix} c^{(i)}_k \\ d^{(i)}_k \end{Bmatrix} = \rho_i \begin{bmatrix} \cos\vartheta_i & \sin\vartheta_i \\ -\sin\vartheta_i & \cos\vartheta_i \end{bmatrix} \begin{Bmatrix} c^{(i)}_k \\ d^{(i)}_k \end{Bmatrix}. \tag{2.100}$$

From the foregoing equation, we have

$$\begin{Bmatrix} c^{(i)}_{k+j} \\ d^{(i)}_{k+j} \end{Bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}^j \begin{Bmatrix} c^{(i)}_k \\ d^{(i)}_k \end{Bmatrix} = (\rho_i)^j \begin{bmatrix} \cos j\vartheta_i & \sin j\vartheta_i \\ -\sin j\vartheta_i & \cos j\vartheta_i \end{bmatrix} \begin{Bmatrix} c^{(i)}_k \\ d^{(i)}_k \end{Bmatrix}. \tag{2.101}$$

Definition 2.28 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ from m eigenvalues $\lambda_j (j = 1, 2, \cdots, m)$ with $m = n(s+1)$ of matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{z}_k^{(i)} = \mathbf{z}_{k+}^{(i)} + \mathbf{z}_{k-}^{(i)}$ with Eqs. (2.80) and (2.82). For any arbitrarily small $\varepsilon > 0$, the stability of the equilibrium $\mathbf{y}_k^*$ on the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$ can be determined.

(i) $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$is spirally stable if

$$r_{k+1}^{(i)} - r_k^{(i)} < 0. \tag{2.102}$$

(ii) $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$is spirally unstable if

$$r_{k+1}^{(i)} - r_k^{(i)} > 0. \tag{2.103}$$

(iii) $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable with the m_ith-order singularity if for $\theta_k^{(i)} \in [0, 2\pi)$

$$\begin{aligned} &\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ &G_{r_{k+1}^{(i)}}^{(s_k^{(i)})}(\theta_k) = 0 \text{ for } s_k^{(i)} = 1, 2, \cdots, m_i - 1 \\ &r_{k+1}^{(i)} - r_k^{(i)} < 0. \end{aligned} \tag{2.104}$$

(iv) $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable with the m_ith-order singularity if for $\theta_k^{(i)} \in [0, 2\pi)$

$$\begin{aligned} &\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ &G_{r_{k+1}^{(i)}}^{(s_k^{(i)})}(\theta_k) = 0 \text{ for } s_k^{(i)} = 1, 2, \cdots, m_i - 1 \\ &r_{k+1}^{(i)} - r_k^{(i)} > 0. \end{aligned} \tag{2.105}$$

(v) $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$is circular if for $\theta_k^{(i)} \in [0, 2\pi)$

$$r_{k+1}^{(i)} - r_k^{(i)} = 0. \tag{2.106}$$

(vi) $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is degenerate in the direction of $\mathbf{u}_i$ if

$$\beta_i = 0 \text{ and } \theta_{k+1}^{(i)} - \theta_k^{(i)} = 0. \tag{2.107}$$

Theorem 2.7 *For a memorized discrete system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ *on domain* $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, *with* $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ *and* $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, *the equivalent discrete system is* $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. *Suppose there is a neighborhood of the fixed point* $\mathbf{y}_k^*$ *(i.e.,* $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$*), and* $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$*-continuous in* $U(\mathbf{y}_k^*)$. *The linearized system of the memorized nonlinear discrete system in the neighborhood* $U(\mathbf{y}_k^*)$ *of* $\mathbf{y}_k^*$ *is* $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ $(\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ *and* $l = k, k+1)$ *in Eq.* (2.19). *Consider a pair of complex eigenvalues* $\alpha_i \pm \mathbf{i}\beta_i$ *from* m *eigenvalues* $\lambda_j (j = 1, 2, \cdots, m)$ *with* $m = n(s+1)$ *of matrix* $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ *with a pair of eigenvectors* $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. *On the invariant plane of* $(\mathbf{u}_i, \mathbf{v}_i)$, *consider* $\mathbf{r}_k^{(i)} = \mathbf{z}_k^{(i)} = \mathbf{z}_{k+}^{(i)} + \mathbf{z}_{k-}^{(i)}$ *with Eqs.* (2.80) *and* (2.82). *For any arbitrarily small* $\varepsilon > 0$, *the stability of the equilibrium* $\mathbf{y}_k^*$ *on the invariant plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *can be determined.*

(i) $\mathbf{y}_k^{(i)}$ *at the fixed point* $\mathbf{y}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally stable if and only if*

$$\rho_i < 1. \tag{2.108}$$

(ii) $\mathbf{y}_k^{(i)}$ *at the fixed point* $\mathbf{y}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally unstable if and only if*

$$\rho_i > 1. \tag{2.109}$$

(iii) $\mathbf{y}_k^{(i)}$ *at the fixed point* $\mathbf{y}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally stable with the* m_ith*-order singularity if and only if for* $\theta_k^{(i)} \in [0, 2\pi)$

$$\begin{aligned} &\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ &G_{r_k^{(i)}}^{(s_k^{(i)})}(\theta_k^{(i)}) = 0 \text{ for } s_k = 1, 2, \cdots, m_i - 1 \\ &G_{r_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) < 0. \end{aligned} \tag{2.110}$$

(iv) $\mathbf{y}_k^{(i)}$ *at the fixed point* $\mathbf{y}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally unstable with the* m_ith*-order singularity if and only if for* $\theta_k^{(i)} \in [0, 2\pi)$

$$\begin{aligned} &\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ &G_{r_k^{(i)}}^{(s_k^{(i)})}(\theta_k^{(i)}) = 0 \text{ for } s_k^{(i)} = 0, 1, 2, \cdots, m_i - 1 \\ &G_{r_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) > 0. \end{aligned} \tag{2.111}$$

(v) $\mathbf{y}_k^{(i)}$ *at the fixed point* $\mathbf{y}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is circular if and only if for* $\theta_k^{(i)} \in [0, 2\pi)$

$$\begin{aligned} \rho_i &= \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ G_{r_k^{(i)}}^{(s_k^{(i)})}(\theta_k^{(i)}) &= 0 \text{ for } s_k^{(i)} = 0, 1, 2, \cdots. \end{aligned} \tag{2.112}$$

Proof Consider

$$\begin{aligned} c_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}(\mathbf{y}_k, \mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{y}_k, \mathbf{p})], \\ d_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}(\mathbf{y}_k, \mathbf{p}) - \Delta_{12} G_{c_k^{(i)}}(\mathbf{y}_k, \mathbf{p})]. \end{aligned}$$

For $\mathbf{y}_{k+1} = \mathbf{y}_k = \mathbf{y}_k^*$, $r_k = 0$. The first order approximation of $c_{k+1}^{(i)}$ and $d_{k+1}^{(i)}$ in the Taylor series expansion gives

$$\begin{aligned} c_{k+1}^{(i)} &= G_{c_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) r_k^{(i)} + o(r_k^{(i)}), \\ d_{k+1}^{(i)} &= G_{d_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) r_k^{(i)} + o(r_k^{(i)}) \end{aligned}$$

where $r_k^{(i)} = \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2}$ and $\theta_k^{(i)} = \arctan(d_k^{(i)}/c_k^{(i)})$

$$\begin{aligned} G_{c_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}^{(1)}(\theta_k^{(i)}) - \Delta_{12} G_{d_k^{(i)}}^{(1)}(\theta_k^{(i)})] \\ G_{d_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}^{(1)}(\theta_k^{(i)}) - \Delta_{12} G_{c_k^{(i)}}^{(1)}(\theta_k^{(i)})] \end{aligned}$$

and

$$\begin{aligned} G_{c_k^{(i)}}^{(1)}(\theta_k^{(i)}) &= (\alpha_i \Delta_1 - \beta_i \Delta_{12}) \cos\theta_k^{(i)} + (\alpha_i \Delta_{12} + \beta_i \Delta_1) \sin\theta_k^{(i)}, \\ G_{d_k^{(i)}}^{(1)}(\theta_k^{(i)}) &= (-\beta_i \Delta_2 + \alpha_i \Delta_{12}) \cos\theta_k^{(i)} + (\alpha_i \Delta_2 + \beta_i \Delta_{12}) \sin\theta_k^{(i)}. \end{aligned}$$

Therefore,

$$\begin{aligned} G_{c_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) &= \alpha_i \cos\theta_k^{(i)} + \beta_i \sin\theta_k^{(i)}, \\ G_{d_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) &= \alpha_i \sin\theta_k^{(i)} - \beta_i \cos\theta_k^{(i)}. \end{aligned}$$

Further

$$\begin{aligned} c_{k+1}^{(i)} &= \alpha_i r_k^{(i)} \cos\theta_k^{(i)} + \beta_i r_k^{(i)} \sin\theta_k^{(i)} = \alpha_i c_k^{(i)} + \beta_i d_k^{(i)}, \\ d_{k+1}^{(i)} &= \alpha_i r_k^{(i)} \sin\theta_k^{(i)} - \beta_i r_k^{(i)} \cos\theta_k^{(i)} = -\beta_i c_k^{(i)} + \alpha_i d_k^{(i)}. \end{aligned}$$

That is,

$$\begin{Bmatrix} c_{k+1}^{(i)} \\ d_{k+1}^{(i)} \end{Bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \begin{Bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{Bmatrix} = \rho_i \begin{bmatrix} \cos\vartheta_i & \sin\vartheta_i \\ -\sin\vartheta_i & \cos\vartheta_i \end{bmatrix} \begin{Bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{Bmatrix}.$$

From the foregoing equation,

$$r_{k+1}^{(i)} = \rho_i r_k^{(i)} + o(r_k^{(i)}) \text{ and } \theta_{k+1}^{(i)} = \theta_k^{(i)} - \vartheta_i + o(r_k^{(i)}).$$

where

$$\vartheta_i = \arctan(\beta_i/\alpha_i) \text{ and } \rho_i = \sqrt{\alpha_i^2 + \beta_i^2}$$

As $r_k^{(i)} \ll 1$ and $r_k \to 0$, we have

$$r_{k+1}^{(i)} = \rho_i r_k^{(i)} \text{ and } \theta_{k+1}^{(i)} = \vartheta_i - \theta_k^{(i)}.$$

(i) For fixed point stability, if $\rho_i < 1$, then

$$r_{k+1}^{(i)} < r_k^{(i)}$$

which implies $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable, vice versa.

(ii) If $\rho_i > 1$, then

$$r_{k+1}^{(i)} > r_k^{(i)}$$

which implies $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable, vice versa.

(iii) If for $\theta_k^{(i)} \in [0, 2\pi]$ the following conditions exist

$$\begin{aligned} &\rho_i = \sqrt{G^{(2)}_{r_{k+1}^{(i)}}} = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ &G^{(s_k^{(i)})}_{r_{k+1}^{(i)}}(\theta_k^{(i)}) = 0 \text{ for } s_k^{(i)} = 1, 2, \cdots, m_i - 1 \\ &G^{(m_i)}_{r_{k+1}^{(i)}}(\theta_k^{(i)}) \neq 0, \text{ and } |G^{(s_k^{(i)})}_{r_{k+1}^{(i)}}(\theta_k^{(i)})| < \infty \text{ for } s_k^{(i)} = m_i + 1, m + 2 \cdots. \end{aligned}$$

then the higher terms can be ignored, i.e.,

$$r_{k+1}^{(i)} = r_k^{(i)} \sqrt{1 + \sum\nolimits_{m_i=3}^{\infty} (r_k^{(i)})^{m_i-2} G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)})}$$

If $G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)})$ is independent of $\theta_k^{(i)}$ (i.e., $G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) = \text{const}$), it can be used to determine the equilibrium stability. If $G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) < 0$, then

$$r_{k+1}^{(i)} < r_k^{(i)}.$$

In other words, $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable, vice versa.

(iv) If $G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) > 0$, then

$$r_{k+1}^{(i)} > r_k^{(i)}$$

That is, $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable with the $(m_k - 1)$th-order singularity, vice versa.

(v) If for $\theta_k^{(i)} \in [0, 2\pi]$ the following conditions exist

$$G_{r_{k+1}^{(i)}}^{(s_k^{(i)})}(\theta_k^{(i)}) = 0 \text{for } s_k^{(i)} = 1, 2, \cdots,$$

then

$$r_{k+1}^{(i)} = r_k^{(i)}.$$

Therefore $\mathbf{y}_k^{(i)}$ at the fixed point $\mathbf{y}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is circular vice versa. This theorem is proved. ■

2.3.2 Bifurcations

Definition 2.29 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\mathbf{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \mathbf{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in

Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i $(i = 1, 2, \cdots, m)$ with $m = n(s+1)$, and there are m linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, m)$. For a perturbation of fixed point $\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$, let $\mathbf{z}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{z}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

$$s_k^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{y}_k - \mathbf{y}_k^*) \tag{2.113}$$

where $s_k^{(i)} = c_k^{(i)}||\mathbf{v}_i||^2$.

$$s_{k+1}^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{F}(\mathbf{y}_k, \mathbf{p}) - \mathbf{y}_k^*]. \tag{2.114}$$

In the vicinity of point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$, $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ can be expended for $(0<\theta<1)$ as

$$\begin{aligned}
\mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{F}(\mathbf{y}_k, \mathbf{p}) - \mathbf{y}_{k(0)}^*] = {} & a_i(s_k^{(i)} - s_{k(0)}^{(i)*}) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\
& + \sum_{q=2}^{m}\sum_{r=0}^{q}\frac{1}{q!}C_q^r \mathbf{a}_i^{(q-r,r)}(s_k^{(i)*} - s_{k(0)}^{(i)*})^{q-r}(\mathbf{p} - \mathbf{p}_0)^r \\
& + \frac{1}{(m+1)!}[(s_k^{(i)} - s_{k(0)}^{(i)*})\partial_{s_k^{(i)}} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
& \times (\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{F}(\mathbf{y}_{k(0)}^* + \boldsymbol{\theta}_1^T \cdot \Delta\mathbf{y}_k, \mathbf{p}_0 + \boldsymbol{\theta}_2^T \cdot \Delta\mathbf{p}))
\end{aligned} \tag{2.115}$$

where

$$\begin{aligned}
& a_i = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}\mathbf{F}(\mathbf{y}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}, \mathbf{b}_i^{\mathrm{T}} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{p}}\mathbf{F}(\mathbf{y}_k, \mathbf{p})\big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}, \\
& \mathbf{a}_i^{(r,s)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(r)}\partial_{\mathbf{p}}^{(s)}\mathbf{F}(\mathbf{y}_k, \mathbf{p})\Bigg|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}, C_q^r = \frac{q!}{r!(q-r)!}.
\end{aligned} \tag{2.116}$$

If $a_i = 1$ and $\mathbf{p} = \mathbf{p}_0$, the stability of fixed point $\mathbf{y}_k^*$ on an eigenvector $\mathbf{v}_i$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold on the direction of $\mathbf{v}_i$ is determined by

$$\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + \sum_{q=2}^{m}\sum_{r=0}^{q}\frac{1}{q!}C_q^r \mathbf{a}_i^{(r,q-r)}(\mathbf{p} - \mathbf{p}_0)^{q-r}(s_k^{(i)} - s_{k(0)}^{(i)*})^r = 0. \tag{2.117}$$

In the neighborhood of $(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)$, when other components of fixed point $\mathbf{y}_k^*$ on the eigenvector of $\mathbf{v}_j$ for all $j \neq i$, $(i, j \in N)$ do not change their stability states, Eq. (2.114) possesses l-branch solutions of equilibrium $s_k^{(i)*}$ $(0<l\leq m)$ with l_1-stable and l_2-unstable solutions $(l_1, l_2 \in \{0, 1, 2, \cdots, l\})$. Such l-branch solutions are called the bifurcation solutions of fixed point $\mathbf{y}_k^*$ on the eigenvector of $\mathbf{v}_i$ in the neighborhood of $(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)$. Such a bifurcation at point $(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)$ is called the

hyperbolic bifurcation of mth-order on the eigenvector of $\mathbf{v}_i$. Consider two special cases herein.

(i) If

$$\mathbf{a}_i^{(1,1)} = \mathbf{0}\,\text{and}\,\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + \frac{1}{2!} a_i^{(2,0)} (s_k^{(i)*} - s_{k(0)}^{(i)*})^2 = 0 \tag{2.118}$$

where

$$\begin{aligned} a_i^{(2,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(2)} \partial_{\mathbf{p}}^{(0)} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(2)} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(2)} \mathbf{F}(\mathbf{y}_k, \mathbf{p})(\mathbf{v}_i \mathbf{v}_i) \big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = G_{s_k^{(i)}}^{(2)}(\mathbf{y}_{k(0)}^*, \mathbf{p}_0) \neq 0, \\ \mathbf{b}_i^{\mathrm{T}} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{p}} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \neq \mathbf{0}, \end{aligned} \tag{2.119}$$

$$a_i^{(2,0)} \times [\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.120}$$

such a bifurcation at point $(\mathbf{x}_0^*, \mathbf{p}_0)$ is called the *saddle-node* bifurcation on the eigenvector of $\mathbf{v}_i$.

(ii) If

$$\begin{aligned} &\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0\,\text{and} \\ &\mathbf{a}_i^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)(s_k^{(i)*} - s_{k(0)}^{(i)*}) + \frac{1}{2!} a_i^{(2,0)} (s_k^{(i)*} - s_{k(0)}^{(i)*})^2 = 0 \end{aligned} \tag{2.121}$$

where

$$\begin{aligned} a_i^{(2,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(2)} \partial_{\mathbf{p}}^{(0)} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(2)} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(2)} \mathbf{F}(\mathbf{y}_k, \mathbf{p})(\mathbf{v}_i \mathbf{v}_i) \big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = G_{s_k^{(i)}}^{(2)}(\mathbf{y}_{k(0)}^*, \mathbf{p}_0) \neq 0, \\ \mathbf{a}_i^{(1,1)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(1)} \partial_{\mathbf{p}}^{(1)} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}} \partial_{\mathbf{p}} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k} \partial_{\mathbf{p}} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \mathbf{v}_i \big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \neq \mathbf{0}, \end{aligned} \tag{2.122}$$

such a bifurcation at point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$ is called the *transcritical* bifurcation on the eigenvector of $\mathbf{v}_i$.

Definition 2.30 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). The matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ possesses m eigenvalues λ_i ($i = 1, 2, \cdots, m$) with $m = n(s+1)$, and there are m linearly independent vectors $\mathbf{v}_i$ ($i = 1, 2, \cdots, m$). For a perturbation of fixed point $\mathbf{z}_k = \mathbf{y}_k - \mathbf{y}_k^*$, let $\mathbf{z}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{z}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$. Equations (2.113), (2.114) and (2.116) hold. In the vicinity of point $(\mathbf{y}_{k0}^*, \mathbf{p}_0)$, $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{F}(\mathbf{y}_k, \mathbf{p})$ can be expended for $\boldsymbol{\theta}_1 = (\theta_{11}, \theta_{12}, \cdots, \theta_{1m})^{\mathrm{T}}$ $(0 < \theta_{1j} < 1, j = 1, 2, \cdots, m)$ and $\boldsymbol{\theta}_2 = (\theta_{21}, \theta_{22}, \cdots, \theta_{2m})^{\mathrm{T}}$ $(0 < \theta_{2j} < 1, j = 1, 2, \cdots, l)$ as

$$
\begin{aligned}
\mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{F}(\mathbf{y}_k, \mathbf{p}) - \mathbf{y}_{k+1(0)}^*] = {} & a_i(s_k^{(i)} - s_{k(0)}^{(i)*}) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\
& + \sum_{q=2}^{m} \sum_{r=0}^{q} \frac{1}{q!} C_q^r \mathbf{a}_i^{(q-r,r)} (s_k^{(i)} - s_{k(0)}^{(i)*})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r \\
& + \frac{1}{(m+1)!} [(s_k^{(i)} - s_{k(0)}^{(i)*})\partial_{s_k^{(i)}} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
& \times (\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{F}(\mathbf{y}_{k(0)}^* + \boldsymbol{\theta}_1^{\mathrm{T}} \cdot \Delta\mathbf{y}_k, \mathbf{p}_0 + \boldsymbol{\theta}_2^{\mathrm{T}} \cdot \Delta\mathbf{p}))
\end{aligned}
\tag{2.123}
$$

and

$$
\begin{aligned}
\mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{F}(\mathbf{y}_{k+1}, \mathbf{p}) - \mathbf{y}_{k(0)}^*] = {} & a_i(s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*}) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\
& + \sum_{q=2}^{m} \sum_{r=0}^{q} \frac{1}{q!} C_q^r \mathbf{a}_i^{(q-r,r)} (s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*})^{q-r} (\mathbf{p} - \mathbf{p}_0)^r \\
& + \frac{1}{(m+1)!} [(s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*})\partial_{s_{k+1}^{(i)}} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
& \times (\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{F}(\mathbf{y}_{k+1(0)}^* + \boldsymbol{\theta}_1^{T} \cdot \Delta\mathbf{y}_{k+1}, \mathbf{p}_0 + \boldsymbol{\theta}_2^{T} \cdot \Delta\mathbf{p})).
\end{aligned}
\tag{2.124}
$$

If $a_i = -1$ and $\mathbf{p} = \mathbf{p}_0$, the stability of current equilibrium $\mathbf{y}_k^*$ on an eigenvector $\mathbf{v}_i$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_i$ is determined by

$$\begin{aligned}
&\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p}-\mathbf{p}_0) + a_i(s_k^{(i)*} - s_{k(0)}^{(i)*}) \\
&\quad + \sum_{q=2}^{m}\sum_{r=0}^{q} \frac{1}{q!} C_q^r \mathbf{a}_i^{(q-r,r)} (s_k^{(i)} - s_{k(0)}^{(i)*})^{q-r} (\mathbf{p}-\mathbf{p}_0)^r = (s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}); \\
&\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p}-\mathbf{p}_0) + a_i(s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}) \\
&\quad + \sum_{q=2}^{m}\sum_{r=0}^{q} \frac{1}{q!} C_q^r \mathbf{a}_i^{(q-r,r)} (s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*})^{q-r} (\mathbf{p}-\mathbf{p}_0)^r = (s_k^{(i)*} - s_{k(0)}^{(i)*}).
\end{aligned} \tag{2.125}$$

In the neighborhood of $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$, when other components of fixed point $\mathbf{x}_{k(0)}^*$ on the eigenvector of $\mathbf{v}_j$ for all $j \neq i$, $(j, i \in N)$ do not change their stability states, Eq. (2.125) possesses l-branch solutions of equilibrium $s_k^{(i)*} (0 < l \leq m)$ with l_1-stable and l_2-unstable solutions ($l_1, l_2 \in \{0, 1, 2, \cdots, l\}$). Such l-branch solutions are called the bifurcation solutions of fixed point $\mathbf{x}_k^*$ on the eigenvector of $\mathbf{v}_i$ in the neighborhood of $(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)$. Such a bifurcation at point $(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)$ is called the *hyperbolic bifurcation* of mth-order with doubling iterations on the eigenvector of $\mathbf{v}_i$. Consider a special case. If

$$\begin{aligned}
&\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p}-\mathbf{p}_0) = 0, a_i = -1, a_i^{(2,0)} = 0, \mathbf{a}_i^{(2,1)} = \mathbf{0}, \mathbf{a}_i^{(1,2)} = \mathbf{0}, \\
&[\mathbf{a}^{(1,1)} \cdot (\mathbf{p}-\mathbf{p}_0) + a_i](s_k^{(i)*} - s_{k(0)}^{(i)*}) + \frac{1}{3!} a_i^{(3,0)} (s_k^* - s_{k(0)}^*)^3 \\
&\qquad = (s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}), \\
&[\mathbf{a}^{(1,1)} \cdot (\mathbf{p}-\mathbf{p}_0) + a_i](s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}) + \frac{1}{3!} a_i^{(3,0)} (s_{k+1}^* - s_{k+1(0)}^*)^3 \\
&\qquad (s_k^{(i)*} - s_{k(0)}^{(i)*})
\end{aligned} \tag{2.126}$$

where

$$\begin{aligned}
a_i^{(3,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(3)} \partial_{\mathbf{p}}^{(0)} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(3)} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)} \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(3)} \mathbf{F}(\mathbf{y}_k, \mathbf{p})(\mathbf{v}_i\mathbf{v}_i\mathbf{v}_i)\big|_{(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)} = G_i^{(3)}(\mathbf{y}_{k(0)}^*, \mathbf{p}_0) \neq 0, \\
\mathbf{a}_i^{(1,1)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(1)} \partial_{\mathbf{p}}^{(1)} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}} \partial_{\mathbf{p}} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \Big|_{(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)} \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k} \partial_{\mathbf{p}} \mathbf{F}(\mathbf{y}_k, \mathbf{p}) \mathbf{v}_i \big|_{(\mathbf{y}_{k(0)}^*, \mathbf{p}_0)} \neq \mathbf{0},
\end{aligned} \tag{2.127}$$

$$a_i^{(3,0)} \times [\mathbf{a}_i^{(1,1)} \cdot (\mathbf{p}-\mathbf{p}_0)] < 0, \tag{2.128}$$

such a bifurcation at point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$ is called the *pitchfork* bifurcation (or period-doubling bifurcation) on the eigenvector of $\mathbf{v}_i$.

Definition 2.31 For a memorized discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k-1}, \cdots, \mathbf{x}_{k-s}, \mathbf{p})$ on domain $\boldsymbol{\Omega}_\alpha \in \mathscr{R}^{n(s+1)}$, with $\mathbf{y}_k = (\mathbf{x}_k; \mathbf{x}_{k-1}; \cdots; \mathbf{x}_{k-s})$ and $\mathbf{F} = (\mathbf{f}; \mathbf{x}_k; \cdots; \mathbf{x}_{k+1-s})$, the equivalent discrete system is $\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p})$. Suppose there is a neighborhood of the fixed point $\mathbf{y}_k^*$ (i.e., $U(\mathbf{y}_k^*) \subset \boldsymbol{\Omega}_\alpha$), and $\mathbf{F}(\mathbf{y}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U(\mathbf{y}_k^*)$. The linearized system of the memorized nonlinear discrete system in the neighborhood $U(\mathbf{y}_k^*)$ of $\mathbf{y}_k^*$ is $\mathbf{z}_{k+1} = D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})\mathbf{z}_k$ ($\mathbf{z}_l = \mathbf{y}_l - \mathbf{y}_k^*$ and $l = k, k+1$) in Eq. (2.19). Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ from m eigenvalues $\lambda_j (j = 1, 2, \cdots, m)$ with $m = n(s+1)$ of matrix $D\mathbf{F}(\mathbf{y}_k^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{z}_k^{(i)} = \mathbf{z}_{k+}^{(i)} + \mathbf{z}_{k-}^{(i)}$ with

$$\begin{aligned} \mathbf{r}_k^{(i)} &= c_k^{(i)}\mathbf{u}_i + d_k^{(i)}\mathbf{v}_i \\ \mathbf{r}_{k+1}^{(i)} &= c_{k+1}^{(i)}\mathbf{u}_i + d_{k+1}^{(i)}\mathbf{v}_i. \end{aligned} \tag{2.129}$$

and

$$\begin{aligned} c_k^{(i)} &= \frac{1}{\Delta}[\Delta_2(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{z}_k) - \Delta_{12}(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k)], \\ d_k^{(i)} &= \frac{1}{\Delta}[\Delta_1(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{z}_k) - \Delta_{12}(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{z}_k)]; \\ \Delta_1 &= ||\mathbf{u}_i||^2, \Delta_2 = ||\mathbf{v}_i||^2, \Delta_{12} = \mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{v}_i; \\ \Delta &= \Delta_1\Delta_2 - \Delta_{12}^2. \end{aligned} \tag{2.130}$$

Consider a polar coordinate of (r_k, θ_k) defined by

$$\begin{aligned} c_k^{(i)} &= r_k^{(i)} \cos \theta_k^{(i)}, \\ d_k^{(i)} &= r_k^{(i)} \sin \theta_k^{(i)}; \\ r_k^{(i)} &= \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2}, \\ \theta_k^{(i)} &= \arctan\, (d_k^{(i)}/c_k^{(i)}). \end{aligned} \tag{2.131}$$

Thus

$$\begin{aligned} c_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}(\mathbf{y}_k, \mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{y}_k, \mathbf{p})] \\ d_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}(\mathbf{y}_k, \mathbf{p}) - \Delta_{12} G_{c_k^{(i)}}(\mathbf{y}_k, \mathbf{p})] \end{aligned} \tag{2.132}$$

where

$$\begin{aligned}
G_{c_k^{(i)}}(\mathbf{y}_k,\mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}}\cdot[\mathbf{F}(\mathbf{y}_k,\mathbf{p})-\mathbf{y}_{k(0)}^*] \\
&= \mathbf{a}_i^{\mathrm{T}}\cdot(\mathbf{p}-\mathbf{p}_0)+a_{i11}(c_k^{(i)}-c_{k(0)}^{(i)*})+a_{i12}(d_k^{(i)}-d_{k(0)}^{(i)*}) \\
&\quad+\sum_{q=2}^{m_i}\sum_{r_i=0}^{q}\frac{1}{q!}C_q^{r_i}\mathbf{G}_{c_k^{(i)}}^{(q-r_i,r_i)}(\mathbf{y}_{k(0)}^*,\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0)^{r_i}(r_k^{(i)})^{q-r_i} \\
&\quad+\frac{1}{(m_i+1)!}[(c_k^{(i)}-c_{k(0)}^{(i)*})\partial_{c_k^{(i)}}+(d_k^{(i)}-d_{k(0)}^{(i)*})\partial_{d_k^{(i)}}+(\mathbf{p}-\mathbf{p}_0)\partial_{\mathbf{p}}]^{m_i+1} \\
&\quad\times(\mathbf{u}_i^{\mathrm{T}}\cdot\mathbf{F}(\mathbf{y}_{k(0)}^*+\boldsymbol{\theta}_1^{\mathrm{T}}\cdot\Delta\mathbf{y}_k,\mathbf{p}_0+\boldsymbol{\theta}_2^{\mathrm{T}}\cdot\Delta\mathbf{p})),
\end{aligned} \tag{2.133}$$

$$\begin{aligned}
G_{d_k^{(i)}}(\mathbf{y}_k,\mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}}\cdot[\mathbf{F}(\mathbf{y}_k,\mathbf{p})-\mathbf{y}_{k(0)}^*] \\
&= \mathbf{b}_i^{\mathrm{T}}\cdot(\mathbf{p}-\mathbf{p}_0)+a_{i21}(c_k^{(i)}-c_{k(0)}^{(i)*})+a_{i22}(d_k^{(i)}-d_{k(0)}^{(i)*}) \\
&\quad+\sum_{q=2}^{m_i}\sum_{r_i=0}^{q}\frac{1}{q!}C_q^{r_i}\mathbf{G}_{d_k^{(i)}}^{(q-r_i,r_i)}(\mathbf{y}_{k(0)}^*,\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0)^{r_i}(r_k^{(i)})^{q-r_i} \\
&\quad+\frac{1}{(m_i+1)!}[(c_k^{(i)}-c_{k(0)}^{(i)*})\partial_{c_k^{(i)}}+(d_k^{(i)}-d_{k(0)}^{(i)*})\partial_{d_k^{(i)}}+(\mathbf{p}-\mathbf{p}_0)\partial_{\mathbf{p}}]^{m_i+1} \\
&\quad\times(\mathbf{v}_i^{\mathrm{T}}\cdot\mathbf{F}(\mathbf{y}_{k(0)}^*+\boldsymbol{\theta}_1^{\mathrm{T}}\cdot\Delta\mathbf{y},\mathbf{p}_0+\boldsymbol{\theta}_2^{\mathrm{T}}\cdot\Delta\mathbf{p}));
\end{aligned} \tag{2.134}$$

and

$$\begin{aligned}
\mathbf{G}_{c_k^{(i)}}^{(s,r)}(\mathbf{y}_{k(0)}^*,\mathbf{p}_0) &= \mathbf{u}_i^{\mathrm{T}}\cdot[\partial_{\mathbf{y}_k}()\mathbf{u}_i\cos\theta_k^{(i)}+\partial_{\mathbf{y}_k}()\mathbf{v}_i\sin\theta_k^{(i)}]^s\partial_{\mathbf{p}}^{(r)}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\Big|_{(\mathbf{y}_{k(0)}^*,\mathbf{p}_0)}, \\
\mathbf{G}_{d_k^{(i)}}^{(s,r)}(\mathbf{y}_{k(0)}^*,\mathbf{p}_0) &= \mathbf{v}_i^{\mathrm{T}}\cdot[\partial_{\mathbf{y}_k}()\mathbf{u}_i\cos\theta_k^{(i)}+\partial_{\mathbf{y}_k}()\mathbf{v}_i\sin\theta_k^{(i)}]^s\partial_{\mathbf{p}}^{(r)}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\Big|_{(\mathbf{y}_{k(0)}^*,\mathbf{p}_0)};
\end{aligned} \tag{2.135}$$

$$\begin{aligned}
\mathbf{a}_i^{\mathrm{T}} &= \mathbf{u}_i^{\mathrm{T}}\cdot\partial_{\mathbf{p}}\mathbf{F}(\mathbf{y}_k,\mathbf{p}),\mathbf{b}_i^{\mathrm{T}}=\mathbf{v}_i^{\mathrm{T}}\cdot\partial_{\mathbf{p}}\mathbf{F}(\mathbf{y}_k,\mathbf{p}); \\
a_{i11} &= \mathbf{u}_i^{\mathrm{T}}\cdot\partial_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\mathbf{u}_i,a_{i12}=\mathbf{u}_i^{\mathrm{T}}\cdot\partial_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\mathbf{u}_i; \\
a_{i21} &= \mathbf{v}_i^{\mathrm{T}}\cdot\partial_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\mathbf{u}_i,a_{i22}=\mathbf{v}_i^{\mathrm{T}}\cdot\partial_{\mathbf{y}_k}\mathbf{F}(\mathbf{y}_k,\mathbf{p})\mathbf{v}_i.
\end{aligned} \tag{2.136}$$

Suppose

$$\mathbf{a}_i=\mathbf{0},\mathbf{b}_i=\mathbf{0},\mathbf{G}_{c_k^{(i)}}^{(0,r)}(\mathbf{x}_k^*,\mathbf{p}_0)=\mathbf{0}\text{ and }\mathbf{G}_{d_k^{(i)}}^{(0,r)}(\mathbf{x}_k^*,\mathbf{p}_0)=\mathbf{0} \tag{2.137}$$

then

$$\begin{aligned}
r_{k+1}^{(i)} &= \sqrt{(c_{k+1}^{(i)})^2 + (d_{k+1}^{(i)})^2} \\
&= \sqrt{\sum\nolimits_{m=2}^{\infty} (r_k^{(i)})^m G_{r_{k+1}^{(i)}}^{(m)}} \\
&= \sqrt{G_{r_{k+1}^{(i)}}^{(2,0)}}\, r_k^{(i)} \sqrt{1 + \lambda^{(i)} + \sum\nolimits_{m=3}^{\infty} \lambda_m^{(i)} (r_k^{(i)})^{m-2}} \\
\theta_{k+1}^{(i)} &= \arctan(d_{k+1}^{(i)}/c_{k+1}^{(i)})
\end{aligned} \tag{2.138}$$

where

$$\begin{aligned}
G_{r_{k+1}^{(i)}}^{(2)} &= G_{r_{k+1}^{(i)}}^{(2,0)} + G_{r_{k+1}^{(i)}}^{(1,1)} \text{ and } \lambda^{(i)} = G_{r_{k+1}^{(i)}}^{(1,1)}/G_{r_{k+1}^{(i)}}^{(2,0)} \text{ with} \\
G_{r_{k+1}^{(i)}}^{(2,0)} &= [G_{c_{k+1}^{(i)}}^{(1,0)}(\theta_k^{(i)}, \mathbf{p}_0)]^2 + [G_{d_{k+1}^{(i)}}^{(1,0)}(\theta_k^{(i)}, \mathbf{p}_0)]^2,
\end{aligned}$$

$$\begin{aligned}
G_{r_{k+1}^{(i)}}^{(1,1)} = \sum\nolimits_{m_i=2}^{M} \sum\nolimits_{m_j=2}^{M} \frac{1}{(m_i+m_j-2)!} C_{m_i+m_j-2}^{m_i-1} \\
([\mathbf{G}_{c_{k+1}^{(i)}}^{(1,m_i-1)}(\theta_k^{(i)}, \mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_i-1}][\mathbf{G}_{c_{k+1}^{(i)}}^{(1,m_j-1)}(\theta_k^{(i)}, \mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_j-1}] \\
+ [\mathbf{G}_{d_{k+1}^{(i)}}^{(1,m_i-1)}(\theta_k^{(i)}, \mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_i-1}][\mathbf{G}_{d_{k+1}^{(i)}}^{(1,m_j-1)}(\theta_k^{(i)}, \mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_j-1}])
\end{aligned} \tag{2.139}$$

and

$$\begin{aligned}
\lambda_m^{(i)} &= G_{r_{k+1}^{(i)}}^{(m)}/G_{r_{k+1}^{(i)}}^{(2,0)} \text{ with} \\
G_{r_{k+1}^{(i)}}^{(m)} &= \frac{1}{m!}\sum\nolimits_{s=1}^{m-1} C_m^s \sum\nolimits_{m_i=1}^{M} \sum\nolimits_{m_j=1}^{M} \frac{1}{(m_i+m_j-m)!} C_{m_i+m_j-m}^{m_i-s} \\
&\quad [\mathbf{G}_{c_{k+1}^{(i)}}^{(s,m_i-s)}(\theta_k^{(i)}, \mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_i-s}\mathbf{G}_{c_{k+1}^{(i)}}^{(m-s,m_j-m+s)}(\theta_k^{(i)}, \mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_j-m+s} \\
&\quad + \mathbf{G}_{d_{k+1}^{(i)}}^{(s,m_i-s)}(\theta_k^{(i)}, \mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_i-s}\mathbf{G}_{d_{k+1}^{(i)}}^{(m-s,m_j-m+s)}(\theta_k^{(i)}, \mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_j-m+s}].
\end{aligned} \tag{2.140}$$

$$\begin{aligned}
\mathbf{G}_{c_{k+1}^{(i)}}^{(r,m-r)}(\theta_k, \mathbf{p}_0) &= \frac{1}{\Delta}[\Delta_2 \mathbf{G}_{c_k^{(i)}}^{(r,m-r)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0) - \Delta_{12}\mathbf{G}_{d_k^{(i)}}^{(r,m-r)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)], \\
\mathbf{G}_{d_{k+1}^{(i)}}^{(r,m-r)}(\theta_k, \mathbf{p}_0) &= \frac{1}{\Delta}[\Delta_1 \mathbf{G}_{d_k^{(i)}}^{(r,m-r)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0) - \Delta_{12}\mathbf{G}_{c_k^{(i)}}^{(r,m-r)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)].
\end{aligned} \tag{2.141}$$

If $G^{(2,0)}_{r^{(i)}_{k+1}} = 1$ and $\mathbf{p} = \mathbf{p}_0$, the stability of current fixed point $\mathbf{x}^*_k$ on an eigenvector plane of $(\mathbf{u}_i, \mathbf{v}_i)$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_i$ is determined by

$$\lambda^{(i)} + \sum\nolimits_{m=3}^{\infty} \lambda^{(i)}_m (r^{(i)}_k)^{m-2} = 0. \tag{2.142}$$

Such a bifurcation at the fixed point $(\mathbf{x}^*_{k(0)}, \mathbf{p}_0)$ is called the generalized Neimark bifurcation on the eigenvector plane of $(\mathbf{u}_i, \mathbf{v}_i)$.

For a special case, if

$$\lambda^{(i)} + \lambda^{(i)}_4 (r^{(i)}_k)^2 = 0, \text{for} \lambda^{(i)} \times \lambda^{(i)}_4 < 0 \text{ and } \lambda^{(i)}_3 = 0 \tag{2.143}$$

such a bifurcation at point $(\mathbf{x}^*_{k(0)}, \mathbf{p}_0)$ is called the Neimark bifurcation on the eigenvector plane of $(\mathbf{u}_i, \mathbf{v}_i)$.

2.3.3 1-D Memorized Nonlinear Discrete Systems

Consider a 1-D memorized nonlinear discrete system

$$\begin{aligned} x_{k+1} &= f(x_k, x_{k-1}, \cdots, x_{k-s}, \mathbf{p}), \\ x_j &= x_j (j = k, k-1, \cdots, k-s+1). \end{aligned} \tag{2.144}$$

The memorized vectors are introduced as

$$\begin{aligned} \mathbf{y}_{k+1} &= (x_{k+1}, x_k, \cdots, x_{k-s+1})^{\mathrm{T}} \equiv (x_{k+1}; x_k; \cdots; x_{k-s+1}) \\ \mathbf{y}_k &= (x_k, x_{k-1}, \cdots, x_{k-s})^{\mathrm{T}} \equiv (x_k; x_{k-1}; \cdots; x_{k-s}) \\ \mathbf{F} &= (f, x_k, \cdots, x_{k-s+1})^{\mathrm{T}} \equiv (f; x_k; \cdots; x_{k-s+1}) \end{aligned} \tag{2.145}$$

Thus the equivalent discrete systems of Eq. (2.144) is

$$\mathbf{y}_{k+1} = \mathbf{F}(\mathbf{y}_k, \mathbf{p}). \tag{2.146}$$

The fixed point $\mathbf{y}^*_{k+1} = \mathbf{y}^*_k$ is determined by $\mathbf{y}^*_k = \mathbf{F}(\mathbf{y}^*_k, \mathbf{p})$, i.e.,

$$\begin{aligned} x^*_{k+1} &= x^*_k = x^*_{k-1} = \cdots = x^*_{k-s}, \\ x^*_{k+1} &= f(x^*_k, x^*_{k-1}, \cdots, x^*_{k-s}, \mathbf{p}). \end{aligned} \tag{2.147}$$

The linearized equation at the fixed point is

$$\Delta \mathbf{y}_{k+1} = D\mathbf{F}(\mathbf{y}^*_k, \mathbf{p}) \Delta \mathbf{y}_k \tag{2.148}$$

where

$$DF(\mathbf{y}_k^*, \mathbf{p}) = \begin{bmatrix} a_{kk} & a_{k(k-1)} & \cdots & a_{k(k-s+1)} & a_{k(k-s)} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$
$$a_{kj} = \frac{\partial f}{\partial x_j}(j = k, k-1, \cdots, k-s). \tag{2.149}$$

The eigenvalue analysis of Eq. (2.148) is completed by

$$|DF(\mathbf{y}_k^*, \mathbf{p}) - \lambda \mathbf{I}| = 0. \tag{2.150}$$

For $s = 1$, we have

$$\begin{aligned} x_{k+1} &= f(x_k, x_{k-1}, \mathbf{p}), \\ x_k &= x_k. \end{aligned} \tag{2.151}$$

Based on the notations of

$$\begin{aligned} &\mathbf{y}_{k+1} = (x_{k+1}, x_k)^{\mathrm{T}} \equiv (x_{k+1}; x_k), \mathbf{y}_k = (x_k, x_{k-1})^{\mathrm{T}} \equiv (x_k; x_{k-1}) \\ &\mathbf{F} = (f, x_k)^{\mathrm{T}} \equiv (f; x_k), \end{aligned} \tag{2.152}$$

the equivalent discrete system is obtained in Eq. (2.146). From the equivalent discrete system, the corresponding fixed points of $\mathbf{y}_{k+1}^* = \mathbf{y}_k^*$ are achieved. The linearized discrete system in the vicinity of $\mathbf{y}_{k+1}^* = \mathbf{y}_k^*$ is

$$\Delta \mathbf{y}_{k+1} = DF(\mathbf{y}_k^*, \mathbf{p}) \Delta \mathbf{y}_k \tag{2.153}$$

where

$$\begin{gathered} DF(\mathbf{y}_k^*, \mathbf{p}) = \begin{bmatrix} a_{kk} & a_{k(k-1)} \\ 1 & 0 \end{bmatrix} \\ a_{kk} = \frac{\partial f}{\partial x_k} \text{ and } a_{k(k-1)} = \frac{\partial f}{\partial x_{k-1}} \end{gathered} \tag{2.154}$$

From the eigenvalue analysis

$$|DF(\mathbf{y}_k^*, \mathbf{p}) - \lambda \mathbf{I}| = \begin{vmatrix} a_{kk} - \lambda & a_{k(k-1)} \\ 1 & -\lambda \end{vmatrix} = 0, \tag{2.155}$$

we have

$$\lambda^2 - a_{kk}\lambda - a_{k(k-1)} = 0, \tag{2.156}$$

and

$$\lambda_{1,2} = \frac{a_{kk} \pm \sqrt{a_{kk}^2 + 4a_{k(k-1)}}}{2}. \tag{2.157}$$

The stability and bifurcation conditions for 1-D memorized systems with 1-step memorization are given as follows.

(i) period-doubling (flip or pitchfork) bifurcation

$$a_{kk} - a_{k(k-1)} + 1 = 0, \frac{\partial f(x_k, x_{k-1})}{\partial x_k}|_{(x_k^*, x_{k-1}^*)} - \frac{\partial f(x_k, x_{k-1})}{\partial x_{k-1}}|_{(x_k^*, x_{k-1}^*)} + 1 = 0 \tag{2.158}$$

(ii) saddle-node bifurcation

$$\begin{aligned} &1 - a_{k(k-1)} = a_{kk}, \\ &1 - \frac{\partial f(x_k, x_{k-1})}{\partial x_{k-1}}|_{(x_k^*, x_{k-1}^*)} = \frac{\partial f(x_k, x_{k-1})}{\partial x_k}|_{(x_k^*, x_{k-1}^*)} \end{aligned} \tag{2.159}$$

(iii) Neimark bifurcation

$$\begin{aligned} &a_{k(k-1)} = -1, \\ &\frac{\partial f(x_k, x_{k-1})}{\partial x_{k-1}}|_{(x_k^*, x_{k-1}^*)} = -1. \end{aligned} \tag{2.160}$$

The bifurcation and stability conditions 1-D memorized systems with 1-step memorization are summarized in Fig. 2.8 with $\det(DP^{(n)}) = \det(DP^{(n)}\ (\mathbf{x}_{k(0)}^*, \mathbf{p}_0))$ and $\mathrm{tr}(DP^{(n)}) = \mathrm{tr}(DP^{(n)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0))$. The thick dashed lines are bifurcation lines. The stability of fixed point is given by the eigenvalues in complex plane. The stability of fixed point for higher dimensional systems can be identified by using a naming of stability for linear dynamical systems in Chap. 1. The saddle-node bifurcation possesses stable saddle-node bifurcation (critical) and unstable saddle-node bifurcation (degenerate).

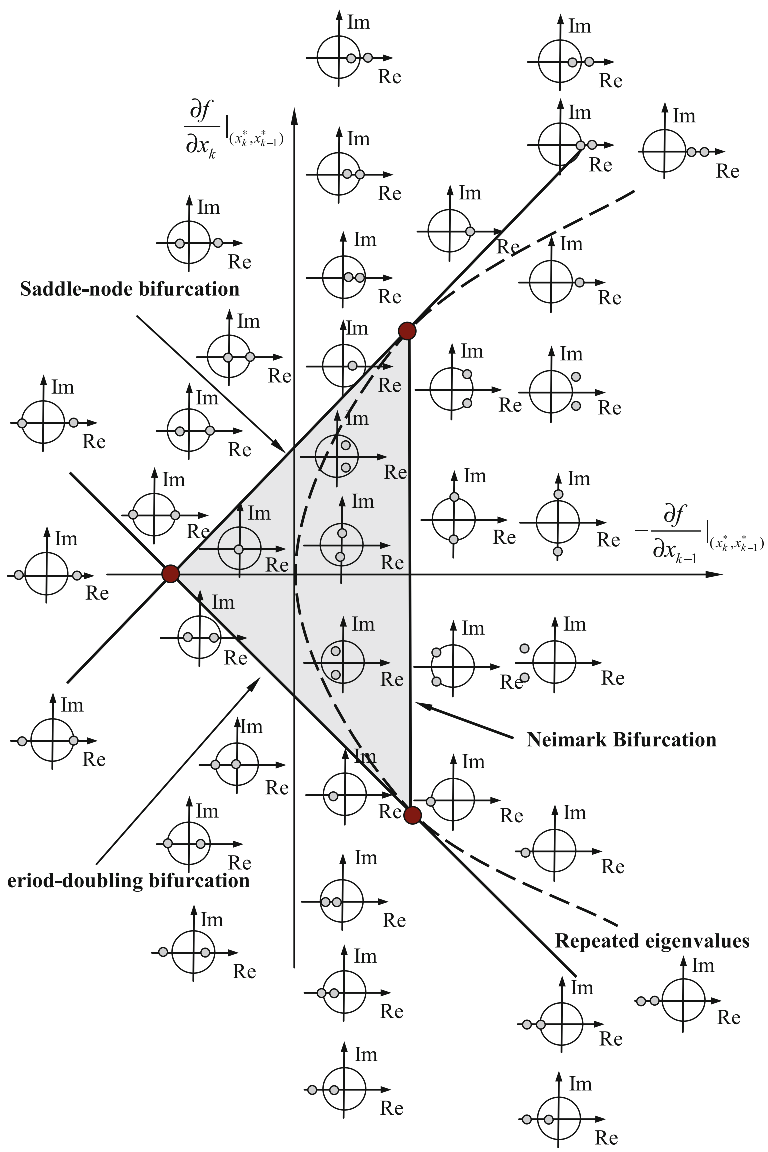

Fig. 2.8 Stability and bifurcation diagrams through the complex plane of eigenvalues for 1D-discrete dynamical systems with 1-step memorization

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