

# On the Construction of Radially Symmetric Trivariate Copulas

José De Jesús Arias García, Hans De Meyer and Bernard De Baets

**Abstract** We propose a method to construct a 3-dimensional symmetric function that is radially symmetric, using two symmetric 2-copulas, with one of them being also radially symmetric. We study the properties of the presented construction in some specific cases and provide several examples for different families of copulas.

**Keywords** Copula · Quasi-copula · Radial symmetry · Aggregation function

## 1 Introduction

An  $n$ -dimensional copula (or, for short,  $n$ -copula) is a multivariate distribution function with the property that all its  $n$  univariate marginals are uniform distributions on  $[0, 1]$ . Formally, an  $n$ -copula is a  $[0, 1]^n \rightarrow [0, 1]$  function that satisfies the following conditions:

1.  $C_n(\mathbf{x}) = 0$  if  $\mathbf{x}$  is such that  $x_j = 0$  for some  $j \in \{1, 2, \dots, n\}$ .
2.  $C_n(\mathbf{x}) = x_j$  if  $\mathbf{x}$  is such that  $x_i = 1$  for all  $i \neq j$ .
3.  $C_n$  is  $n$ -increasing, i.e., for any  $n$ -box  $\mathbf{P} = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq [0, 1]^n$  it holds that

$$V_{C_n}(\mathbf{P}) = \sum_{\mathbf{x} \in \text{vertices}(\mathbf{P})} (-1)^{S(\mathbf{x})} C_n(\mathbf{x}) \geq 0,$$

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where  $S(\mathbf{x}) = \#\{j \in \{1, 2, \dots, n\} \mid x_j = a_j\}$ .  $V_{C_n}(P)$  is called the  $C_n$ -volume of  $P$ .

Due to Sklar's theorem, which states that any continuous multivariate distribution function can be written in terms of its  $n$  univariate marginals by means of a unique  $n$ -copula,  $n$ -copulas have become one of the most important tools for the study of certain types of properties of random vectors, such as stochastic dependence (see [4, 13] for more details on  $n$ -copulas). One example of a property that can be directly studied from copulas is the property of radial symmetry. An  $n$ -dimensional random vector  $(X_1, \dots, X_n)$  is said to be radially symmetric about  $(x_1, \dots, x_n)$  if the distribution of the random vector  $(X_1 - x_1, \dots, X_n - x_n)$  is the same as the distribution of the random vector  $(x_1 - X_1, \dots, x_n - X_n)$ . It is easily shown that radial symmetry can be characterized using copulas: a random vector  $(X_1, \dots, X_n)$  is radially symmetric about  $(x_1, \dots, x_n)$  if and only if for any  $j \in \{1, \dots, n\}$   $X_j - x_j$  has the same distribution as  $x_j - X_j$  and the  $n$ -copula  $C_n$  associated to the random vector satisfies the identity  $C_n = \hat{C}_n$ , where  $\hat{C}_n$  denotes the survival  $n$ -copula associated to  $C_n$ , and which can be computed as:

$$\begin{aligned} \hat{C}_n(x_1, \dots, x_n) &= \sum_{j=1}^n x_j - (n-1) + \sum_{i < j}^n C_n(1, \dots, 1 - x_i, \dots, 1 - x_j, \dots, 1) \\ &\quad - \sum_{i < j < k}^n C_n(1, \dots, 1 - x_i, \dots, 1 - x_j, \dots, 1 - x_k, \dots, 1) + \dots \\ &\quad + (-1)^n C_n(1 - x_1, 1 - x_2, \dots, 1 - x_n). \end{aligned} \quad (1)$$

Due to this characterization, we say that an  $n$ -copula  $C_n$  is radially symmetric if it satisfies the identity  $C_n = \hat{C}_n$ . Survival copulas also have a probabilistic interpretation. If the random vector  $(X_1, \dots, X_n)$  has the copula  $C_n$  as its distribution function, then  $\hat{C}_n$  is the distribution function of the random vector  $(1 - X_1, \dots, 1 - X_n)$ . This probabilistic interpretation has led to several studies of the transformations of copulas which are induced by certain types of transformations on random variables (see [6–8]). These transformations have been generalized and studied in the framework of aggregation functions [1–3].

Radially symmetric copulas have a particular importance in stochastic simulation, as they are used as a part of the multivariate version of the antithetic variates method, which is a variance reduction technique used in Monte Carlo methods (see [12]). In the binary case, well-known examples of families of bivariate copulas that are radially symmetric are the Gaussian family, the Frank family and the Farlie-Gumbel-Morgenstern (FGM) family. However, there are few attempts to construct families of  $n$ -copulas with some specific properties for  $n \geq 3$ . In this contribution we propose a construct method for trivariate radially symmetric copulas.

## 2 Radial Symmetry and Associativity

Archimedean  $n$ (-quasi)-copulas are one of the most well-known classes of  $n$ -copulas, which have the additional property that they are symmetric, i.e., for any permutation  $\sigma$  of  $\{1, 2, \dots, n\}$  and for any  $\mathbf{x} \in [0, 1]^n$  it holds that

$$C_n(x_1, x_2, \dots, x_n) = C_n(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}).$$

Archimedean  $n$ (-quasi)-copulas are also associative and can be defined recursively, i.e., for any  $x, y, z \in [0, 1]$  the equality  $C_2(x, C_2(y, z)) = C_2(C_2(x, y), z)$  holds and for any  $n \geq 2$  and  $\mathbf{x} \in [0, 1]^{n+1}$  it holds that

$$\begin{aligned} C_{n+1}(\mathbf{x}) &= C_2(x_1, C_n(x_2, \dots, x_{n+1})) \\ &= C_2(C_n(x_1, \dots, x_n), x_{n+1}). \end{aligned}$$

It is well known that Archimedean  $n$ (-quasi)-copulas can be fully characterized in terms of an additive generator (see [11]). Some further generalizations of Archimedean copulas have been proposed; for example in [10], nested Archimedean  $n$ -copulas are studied, where several bivariate Archimedean copulas are iterated to construct an  $n$ -copula (for example, in the trivariate case  $C_2(x, D_2(y, z))$  would be an example of such a construction, where  $C_2$  and  $D_2$  are bivariate Archimedean copulas).

If a 2-copula is radially symmetric, then for any  $(x, y) \in [0, 1]^2$ , it holds that

$$C_2(x, y) + (1 - C_2(1 - x, 1 - y)) = x + y.$$

If  $C_2$  is an Archimedean copula, the latter equation is a particular case of a functional equation studied by Frank in [5]. More specifically in [5], it is proven that if a continuous function  $F : [0, 1]^2 \rightarrow [0, 1]$  satisfies the following properties

1. for any  $x \in [0, 1]$  it holds that  $F(x, 0) = F(0, x) = 0$ ,
2. for any  $x \in [0, 1]$  it holds that  $F(x, 1) = F(1, x) = x$ ,
3. the functions  $F(x, y)$  and  $G(x, y) = x + y - F(x, y)$  are both associative;

then  $F$  must be a member of the Frank family of copulas or an ordinal sum constructed from members of this family. The Frank family is given by:

$$F^{(\alpha)}(x, y) = -\frac{1}{\alpha} \ln \left( 1 + \frac{(e^{-\alpha x} - 1)(e^{-\alpha y} - 1)}{e^{-\alpha} - 1} \right),$$

where  $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$  (Although the case  $\alpha = \infty$  is not an Archimedean copula).

In [8] this result is complemented by showing that 2-copulas that are both associative and radially symmetric are members of the Frank family of copulas or an ordinal sum of the form  $C_2 = (\langle a_j, b_j, F_{(j)}^{(\alpha_j)} \rangle)_{j \in J}$ , such that for any  $j$ , there exists  $i_j$  with the property that  $\alpha_j = \alpha_{i_j}$ ,  $a_j = 1 - b_{i_j}$  and  $b_j = 1 - a_{i_j}$ .

Note that if a 3-copula is radially symmetric, then its 2-dimensional marginals must also be radially symmetric. This trivially generalizes to higher dimensions. Hence if an associative 3-copula is radially symmetric, then its 2-dimensional marginals must also be solutions of the Frank functional equation. Unfortunately, as shown in [9], for  $n \geq 3$  the only solutions are the product copula  $\Pi_n(x_1, \dots, x_n) = x_1 x_2 \dots x_n$  (which is the copula of independent random variables) and the minimum operator  $M_n(x_1, \dots, x_n) = \min(x_1, \dots, x_n)$  (which is the copula of comonotonic random variables) or ordinal sums constructed using these two  $n$ -copulas. From this it follows that if we want to construct radially symmetric copulas in higher dimensions, we must weaken the condition of associativity (and as a consequence the Archimedean property is lost), as jointly requiring both properties is too restrictive in higher dimensions.

### 3 The Construction

In [8], the authors study several transformations of bivariate copulas. They show that every radially symmetric 2-copula has the following form

$$\frac{C_2(x, y) + \hat{C}_2(x, y)}{2}, \quad (2)$$

where  $C_2$  is a 2-copula. This result is easily generalized to higher dimensions. Keeping this result in mind, we propose the following construction method in three dimensions.

**Definition 1** Let  $C_2, D_2$  be two symmetric 2-copulas, such that  $C_2$  is also radially symmetric. We define the symmetric function  $S_{C_2, D_2} : [0, 1]^3 \rightarrow \mathbb{R}$  associated to the pair  $C_2, D_2$  as

$$\begin{aligned} S_{C_2, D_2}(x, y, z) = & \frac{1}{2}[1 - x - y - z + C_2(x, y) + C_2(x, z) + C_2(y, z)] \\ & + \frac{1}{2}[H_{C_2, D_2}(x, y, z) - H_{C_2, D_2}(1 - x, 1 - y, 1 - z)], \end{aligned} \quad (3)$$

where

$$\begin{aligned} H_{C_2, D_2}(x, y, z) = & D_2(x, C_2(y, z)) + D_2(y, C_2(x, z)) + D_2(z, C_2(x, y)) \\ & - \frac{2}{3}[D_2(x, D_2(y, z)) + D_2(y, D_2(x, z)) + D_2(z, D_2(x, y))]. \end{aligned}$$

Note that the function  $H_{C_2, D_2}$  satisfies the boundary conditions of a 3-copula, and from this is easy to prove that  $S_{C_2, D_2}$  also satisfies the boundary conditions of a copula, and that the bivariate marginals of  $S_{C_2, D_2}$  are all equal to  $C_2$ .

**Proposition 1** *Let  $C_2, D_2$  be two symmetric 2-copulas, such that  $C_2$  is also radially symmetric. Let  $S_{C_2, D_2}$  be defined as in Eq. (3). If  $S_{C_2, D_2}$  is a 3-copula, then it is radially symmetric.*

*Proof* It can be shown after some tedious computations that we can rewrite  $S_{C_2, D_2}$  as

$$\begin{aligned} S_{C_2, D_2}(x, y, z) = & x + y + z - 2 + S_{C_2, D_2}(1 - x, 1 - y, 1) \\ & + S_{C_2, D_2}(1 - x, 1, 1 - z) + S_{C_2, D_2}(1, 1 - y, 1 - z) \\ & - S_{C_2, D_2}(1 - x, 1 - y, 1 - z), \end{aligned}$$

i.e., from the definition of survival copula given by Eq. (1), it follows that if  $S_{C_2, D_2}$  is a 3-copula, then  $S_{C_2, D_2}$  is a radially symmetric 3-copula because it coincides with its associated survival copula.  $\square$

However,  $S_{C_2, D_2}$  is not necessarily a 3-copula, as it may even not be an increasing function. For example, if  $C_2 = D_2 = F^{(-2)}$ , we can easily see that  $S_{F^{(-2)}, F^{(-2)}}(\frac{1}{2}, \frac{1}{10}, \frac{1}{10}) < 0 = S_{F^{(-2)}, F^{(-2)}}(0, \frac{1}{10}, \frac{1}{10})$ . We now provide some examples where the construction effectively yields a 3-copula.

*Example 1* Consider the Frank family of 2-copulas. From the results in [11], we know that for  $\alpha \geq -\ln(2)$ , the 3-dimensional version of the Frank 2-copula, given by  $F_3^{(\alpha)}(x, y, z) = F^{(\alpha)}(x, F^{(\alpha)}(y, z))$ , is a 3-copula. From this, it follows immediately that if  $\alpha \geq -\ln(2)$  then  $S_{F^{(\alpha)}, F^{(\alpha)}}$  is a 3-copula. However, with some computational help, it can be shown that  $S_{F^{(\alpha)}, F^{(\alpha)}}$  is also a 3-copula for  $\alpha \geq -\ln(3)$ .

*Example 2* The FGM family of bivariate copulas is given by

$$F^{(\theta)}(x, y) = xy + \theta xy(1 - x)(1 - y), \quad \theta \in [-1, 1].$$

In this case, some computations show that  $S_{F^{(\theta)}, F^{(\theta)}}$  is a 3-copula if and only if  $\theta \in [-1/2(3 - \sqrt{5}), 1/2(\sqrt{21} - 3)]$ .

*Example 3* For any 2-copula  $C_2$ ,  $S_{C_2, \Pi_2}$  is a 3-copula if and only if for any  $x_1, x_2, y_1, y_2, z_1, z_2 \in [0, 1]$  such that  $x_1 \leq x_2, y_1 \leq y_2, z_1 \leq z_2$  it holds that

$$\begin{aligned} & (x_2 - x_1)V_{C_2}([y_1, y_2] \times [z_1, z_2]) + (y_2 - y_1)V_{C_2}([x_1, x_2] \times [z_1, z_2]) \\ & + (z_2 - z_1)V_{C_2}([x_1, x_2] \times [y_1, y_2]) \\ & \geq 2(x_2 - x_1)(y_2 - y_1)(z_2 - z_1). \end{aligned}$$

If  $C_2$  is absolutely continuous, then this last condition is equivalent to:

$$\frac{\partial^2 C_2}{\partial x \partial y}(x, y) + \frac{\partial^2 C_2}{\partial x \partial z}(x, z) + \frac{\partial^2 C_2}{\partial y \partial z}(y, z) \geq 2. \quad (4)$$

for almost every  $x, y, z \in [0, 1]$ . An example of a family of 2-copulas that satisfies Eq. (4) is the FGM copulas  $F^{(\theta)}$  for  $\theta \in [-1/3, 1/3]$ .

We note that Example 3 can be generalized to higher dimensions. Given a symmetric  $n$ -copula  $C_n$  that satisfies  $C_n = \hat{C}_n$ , we define the  $(n + 1)$ -dimensional function  $S_{C_n}$  as

$$\begin{aligned} S_{C_n}(x_1, \dots, x_{n+1}) = & \frac{1}{2} \left[ \sum_{j=1}^{n+1} x_j - n + \sum_{i < j}^{n+1} C_n(1, \dots, 1 - x_i, \dots, 1 - x_j, \dots, 1) \right. \\ & - \sum_{i < j < k}^{n+1} C_n(1, \dots, 1 - x_i, \dots, 1 - x_j, \dots, 1 - x_k, \dots, 1) + \dots \\ & + (-1)^n \sum_{j=1}^{n+1} C_n(1 - x_1, \dots, 1 - x_{j-1}, 1 - x_{j+1}, \dots, 1 - x_{n+1}) \left. \right] \\ & + \frac{1}{2} \left[ H(x_1, \dots, x_{n+1}) + (-1)^{n+1} H(1 - x_1, \dots, 1 - x_{n+1}) \right], \end{aligned}$$

where

$$\begin{aligned} H(x_1, \dots, x_{n+1}) = & \sum_{j=1}^{n+1} x_j C_n(\mathbf{x}_{\{j\}}) \\ & - \sum_{i < j}^{n+1} x_i x_j C_{n-1}(\mathbf{x}_{\{i, j\}}) \\ & \dots \\ & + (-1)^n \sum_{i < j}^{n+1} \left( \prod_{k \neq i, j} x_k \right) C_2(x_i, x_j) \\ & + n(-1)^{n+1} x_1 x_2 \dots x_{n+1}, \end{aligned}$$

and  $\mathbf{x}_A$  denotes the vector whose components take the values of the elements  $x_1, \dots, x_{n+1}$ , except of those elements  $x_j$  for which  $j$  is in the set  $A$  of indices. It can be proven that the function  $S_{C_n}$  is such that if  $S_{C_n}$  is an  $(n + 1)$ -copula, then it is an  $(n + 1)$ -dimensional radially symmetric copula, with  $n$ -dimensional marginals given by  $C_n$ . The characterization in the absolutely continuous case is also simple, since after doing some combinatorial analysis, it is easy to prove that if  $C_n$  is absolutely continuous, then  $S_{C_n}$  is an  $(n + 1)$ -copula if and only if for any  $x_1, \dots, x_{n+1} \in [0, 1]$ , it holds that

$$\sum_{j=1}^{n+1} \frac{\partial^n C_n}{\partial x_1 \dots \partial x_{j-1} \partial x_{j+1} \dots \partial x_{n+1}}(\mathbf{x}_{\{j\}})$$

$$\begin{aligned}
& - \sum_{i < j}^{n+1} \frac{\partial^{n-1} C_{n-1}}{\partial x_1 \dots \partial x_{i-1} \partial x_{i+1}, \dots, \partial x_{j-1} \partial x_{j+1}, \dots \partial x_{n+1}}(\mathbf{x}_{\{i,j\}}) \\
& \dots \\
& + (-1)^n \sum_{i < j}^{n+1} \frac{\partial^2 C_2}{\partial x_i \partial x_j}(x_i, x_j) + n(-1)^{n+1} \geq 0.
\end{aligned}$$

## 4 Conclusions and Future Work

We proposed a way of constructing a symmetric and radially symmetric trivariate copula with given bivariate marginals, and provided some examples of this construction. However, it remains an open problem to determine for which pairs of 2-copulas  $C_2$  and  $D_2$ , the function  $S_{C_2, D_2}$  is a 3-copula. A first step in this study would be to characterize the set of ternary aggregation functions for which it holds that their ‘survival transformation’ is also an aggregation function (see [1–3]). A final task is to analyse whether the presented construction can be properly generalized to any dimension  $n > 3$ .

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