Chapter 2
Conformal Kinematics

2.1 Projective Null Cone

In Chap. 1 we motivated the study of Scale Invariant (SI) fixed points. We also argued that the SI is generically enhanced to Conformal Invariance (CI). We saw that in $D \geqslant 3$ the group of Conformal Transformations (CT) is finite dimensional and is generated by the Poincaré transformations plus dilatations plus Special Conformal Transformations (SCT).

Then we introduced the concept of primary operators, which transform under CT as

$$\phi(x) \to \tilde{\phi}(x') = \frac{1}{b(x)^\Delta}\phi(x) , \tag{2.1}$$

if ϕ is scalar, or

$$\phi(x) \to \tilde{\phi}(x') = \frac{1}{b(x)^\Delta} R[M^\mu_{\ \nu}(x)]\phi(x) , \tag{2.2}$$

if ϕ has intrinsic spin, i.e. belongs to an irreducible representation R of $SO(D)$. Here the local scale factor $b(x)$ and the local rotation matrix $M^\mu{}_\nu$ appear in the Jacobian of the conformal transformation, see (1.55). The correlation functions of $\tilde{\phi}$ are the same as those of ϕ, as $\tilde{\phi}$ can be thought of as an image of ϕ under a non-uniform RG transformation leaving the Hamiltonian invariant. Below we will sometimes omit the tilde from $\tilde{\phi}$.

Operationally, the above transformation property simply means that the n-point correlation functions of ϕ must satisfy

$$\langle \phi(x')\phi(y')\ldots\rangle = \frac{1}{b(x)^\Delta}\frac{1}{b(y)^\Delta}\cdots\langle\phi(x)\phi(y)\ldots\rangle . \tag{2.3}$$

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This condition is clearly an important constraint on the correlation functions of the theory, and in this chapter we will study its consequences. There are several ways to work them out, some more pedestrian than others. We will present a method which gives the most information in the least possible time.

First of all we have to understand better the conformal algebra. Last time we wrote the formulas for the vector fields that correspond to the generators of the group

$$\begin{aligned} &P_\mu = i\partial_\mu \;\rightarrow\; \text{translations},\\ &M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \;\rightarrow\; \text{rotations},\\ &D = ix^\mu\partial_\mu \;\rightarrow\; \text{dilatations},\\ &K_\mu = i(2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu) \;\rightarrow\; \text{SCTs}\,. \end{aligned} \tag{2.4}$$

From these expressions for the generators we can compute the commutation relations. Part of it is the Poincaré algebra

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = -i(\delta_{\mu\rho}M_{\nu\sigma} \pm \text{permutations})\\ &[M_{\mu\nu}, P_\rho] = i(\delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu)\,, \end{aligned} \tag{2.5}$$

The interesting new relations are

$$\begin{aligned} &[D, P_\mu] = -iP_\mu\\ &[D, K_\mu] = iK_\mu\\ &[P_\mu, K_\nu] = 2i(\delta_{\mu\nu}D - M_{\mu\nu})\,. \end{aligned} \tag{2.6}$$

It turns out that this "conformal algebra" is in fact isomorphic to $SO(D+1, 1)$, the algebra of Lorentz transformations in $\mathbb{R}^{D+1,1}$ Minkowski space. Let us demonstrate this fact.

Consider in the latter space the coordinates

$$X^1, \ldots X^D, X^{D+1},\; X^{D+2}\,, \tag{2.7}$$

where X^{D+2} is the timelike direction. We will also use the lightcone coordinates

$$X^+ = X^{D+2} + X^{D+1},\; X^- = X^{D+2} - X^{D+1}\,. \tag{2.8}$$

In terms of the above, the mostly plus metric η_{MN} in $\mathbb{R}^{D+1,1}$ is

$$ds^2 = \sum_{i=1}^{D}(dX^i)^2 - dX^+dX^-\,. \tag{2.9}$$

The conformal algebra generators will be identified with the $SO(D+1,1)$ generators as follows

$$
\begin{aligned}
J_{\mu\nu} &= M_{\mu\nu} \ , \\
J_{\mu+} &= P_\mu \ , \\
J_{\mu-} &= K_\mu \ , \\
J_{+-} &= D \ ,
\end{aligned}
\tag{2.10}
$$

with $\mu, \nu = 1, \ldots, D$. It is understood that $J_{\mu\nu}$ is antisymmetric under $\mu \leftrightarrow \nu$. Then one can check that the conformal algebra commutation relations coincide with the $SO(D+1,1)$ ones:

$$[J_{MN}, J_{RS}] = -i(\eta_{MR} J_{NS} \pm \text{permutations}) \, . \tag{2.11}$$

One example is

$$[J_{\mu+}, J_{\nu-}] \propto \delta_{\mu\nu} J_{+-} + \delta_{+-} J_{\mu\nu} \, . \tag{2.12}$$

Exercise: Check and fix all the constants in the above identification.

This result means that the D-dimensional conformal group, while acting in a somewhat non-trivial way on $\mathbb{R}^D$, acts much more naturally (linearly) on the $\mathbb{R}^{D+1,1}$ space. In the vector representation we can write

$$X^M \to \Lambda^M_{\ N} X^N \ , \tag{2.13}$$

with $\Lambda^M_{\ N}$ an $SO(D+1,1)$ matrix. If we could somehow get an action on $\mathbb{R}^D$ out of this simple action, then the implications of CI would be easier to understand. To do that, we have to embed the D-dimensional space into $\mathbb{R}^{D+1,1}$ dimensional space, i.e. to get rid of two extra coordinates.

First of all let's restrict the attention to the null cone in $\mathbb{R}^{D+1,1}$:

$$X^2 = 0 \tag{2.14}$$

This gets rid of one coordinate. Since this constraint is preserved by the action of the group, we don't lose simplicity.

To get down to D dimensions, we take a generic section of the light-cone:

$$X^+ = f(X^\mu), \tag{2.15}$$

The section is parametrized by X^μ which we identify with the $\mathbb{R}^D$ coordinates x^μ.

The group $SO(D+1,1)$ acts on the section as follows (see Fig. 2.1). A point x^μ on the section defines a lightray. Applying a Lorentz transformation, this lightray is mapped to a new one which passes through another point x'^μ. Thus

$$x^\mu \to \ \text{light ray} \xrightarrow{\Lambda^M_{\ N} \in SO(D+1,1)} \text{light ray}' \to x'^\mu \, . \tag{2.16}$$

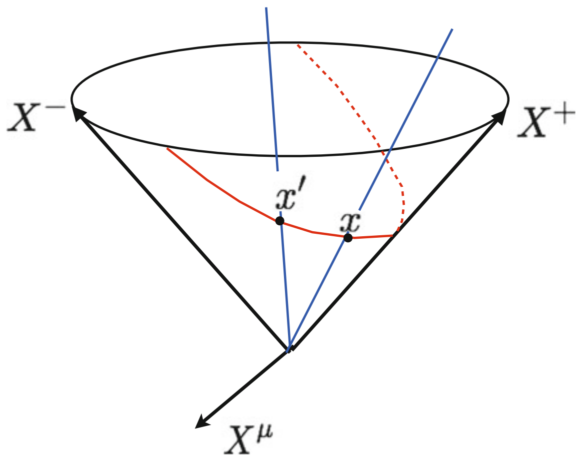

Fig. 2.1 *Red* section, *blue* light ray and light ray$'$

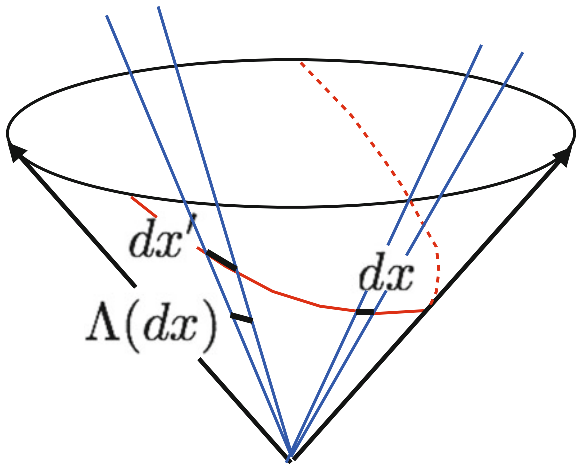

Fig. 2.2 How the infinitesimal interval transforms under the defined $SO(D+1,1)$ action

This defines an action of $SO(D+1,1)$ on $\mathbb{R}^D$, and we will now check that this action corresponds to a CT. Consider the metric ds^2 on the section induced from the Minkowski metric in $\mathbb{R}^{D+1,1}$. We have:

$$ds^2 = dx^2 - dX^+ dX^-|_{X^+=f(x),X^-=x^2/X^+} = g_{\mu\nu}(x)dx^\mu dx^\nu \,, \tag{2.17}$$

where $g_{\mu\nu}(x)$ is a metric we could compute explicitly in terms of $f(x)$ but we won't need it.

The action of $SO(D+1,1)$ on a point x can be split into two steps, (1) $X \to \Lambda.X$ and (2) then rescale to get back into the section. We want to understand how this action changes the infinitesimal interval length (see Fig. 2.2). The first step is an isometry and does not change ds^2. The second step changes the metric by an

x-dependent scale factor. Indeed, assuming that we have to rescale by λ to get back into the section, where λ in general depends on X, we have:

$$(d(\lambda(X)X))^2 = (\lambda dX + X(\nabla\lambda.dX))^2 = \lambda^2 dX^2\,, \tag{2.18}$$

the other terms vanishing by $X^2 = 0$, $X.dX = 0$.

We conclude that the metric transformation is of the form:

$$ds'^2 = c(x)ds^2, \quad c(x) = \lambda(X)^2\,. \tag{2.19}$$

This will exactly agree with the definition of the conformal transformations as long as ds^2 is flat. From the definition of ds^2 it's easy to guess which $f(X)$ achieves this: it is $f(X) = const$, so that $dX^+ = 0$. For simplicity and without loss of generality we take $const = 1$. Thus our *Euclidean* section is parametrized as:

$$X^M = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{2.20}$$

We note in passing that by taking this section and rescaling it in the radial direction by an x-dependent factor we can reproduce any metric which is a Weyl transformation of the flat space metric (for example the metric on the sphere, de Sitter or Anti de Sitter spaces).

We would now like to extend the above action to fields. We thus consider fields $\phi(X)$ defined on the cone. The most natural action of the Lorentz group on such scalar fields is

$$X \to X'\,,\ \phi(X) \to \tilde{\phi}(X') = \phi(X)\,. \tag{2.21}$$

The field on the Euclidean section will be assumed to coincide with the D-dimensional field:

$$\phi(X)|_{\text{section}} = \phi(x)\,. \tag{2.22}$$

Finally, we will assume that ϕ depends homogeneously on X:

$$\phi(\lambda X) = \lambda^{-\Delta}\phi(X)\,, \tag{2.23}$$

Let us show that these conditions imply the correct transformation rule for the fields on $\mathbb{R}^D$:

$$\phi(x') = b(x)^{-\Delta}\phi(x)\,. \tag{2.24}$$

Indeed, $b(x)$ in this equation is the local expansion factor, and according to Eq. (2.19) it must be identified with $\lambda(X)$, the scale factor in the second phase of $SO(D+1, 1)$ action. Since $\phi(X)$ scales homogeneously with λ, we get exactly what we need.

Using this "projective light cone" formalism, any conformally invariant quantity (e.g. correlation function) in $\mathbb{R}^D$ can be lifted to an $SO(D+1,1)$–invariant quantity in $\mathbb{R}^{D+1,1}$. Basically, this formalism makes CI kinematics as simple as the Lorentz-invariant kinematics.

2.2 Simple Applications

2.2.1 Primary Scalar 2pt Function

The expression of the 2pt function on the light-cone is

$$\langle\phi(X)\phi(Y)\rangle = \frac{c}{(X\cdot Y)^{\Delta}}\ , \tag{2.25}$$

with c a constant and Δ the field's scaling dimension. The above is the most general Lorentz invariant expression consistent with scaling of both $\phi(X)$ and $\phi(Y)$ with degree Δ. Note that $X^2 = Y^2 = 0$ cannot appear. To write the 2pt function in the physical space, we project X and Y on the section, i.e.

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \text{ and } Y = (Y^+, Y^-, Y^\mu) = (1, y^2, y^\mu)\ . \tag{2.26}$$

We get

$$\begin{aligned} X\cdot Y &= X^\mu Y_\mu - \frac{1}{2}(X^+Y^- + X^-Y^+) \\ &= x^\mu y_\mu - \frac{1}{2}(x^2+y^2) \\ &= -\frac{1}{2}(x-y)^2\ . \end{aligned} \tag{2.27}$$

The 2pt function (2.25) is therefore projected to

$$\langle\phi(x)\phi(y)\rangle \propto \frac{1}{(x-y)^{2\Delta}} \tag{2.28}$$

That this expression is consistent with the SI of the field $\phi(x)$ is obvious. It's less obvious that it's conformally invariant but our construction guarantees it. If we wanted to check this in a pedestrian way, without using the projective light cone, we would have to show that

$$|x'-y'|^2 = b(x)b(y)|x-y|^2\ . \tag{2.29}$$

for any CT, which does not look obvious at first sight. The standard way to derive this is to show first that this holds for the inversion transformation

$$x'^{\mu} \to \frac{x^{\mu}}{x^2} , \tag{2.30}$$

which is a CT. Indeed the Jacobian is given by

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{1}{x^2}\left(\delta^{\mu\nu} - \frac{2x^{\mu}x^{\nu}}{x^2}\right) \equiv b_{\text{inv}}(x) I^{\mu\nu}(x) , \tag{2.31}$$

where $b_{inv}(x) = 1/x^2$ and $I^{\mu\nu}(x)$ an orthogonal matrix. This can be easily seen if we go to a particular frame where x lies on the x_1 direction. Then the matrix is diagonal

$$I^{\mu\nu}(x) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} , \tag{2.32}$$

and is clearly an $O(D)$ matrix. But it is not in $SO(D)$. This means that inversion is not in the connected part of the conformal group, i.e. it cannot be obtained by exponentiating a Lie algebra element.

If we apply inversion twice we get back to the connected component. In fact we can reproduce SCT this way:

$$\text{SCT}_{\mu} = \text{inversion} \circ \text{translation}_{\mu} \circ \text{inversion} . \tag{2.33}$$

Going back to Eq. (2.29), it is not difficult to verify for the inversion:

$$|x' - y'|^2 = \left|\frac{x^{\mu}}{x^2} - \frac{y^{\mu}}{y^2}\right|^2 = \frac{|x-y|^2}{x^2 y^2} = b_{\text{inv}}(x) b_{\text{inv}}(y) |x-y|^2 . \tag{2.34}$$

By (2.33), it then holds for SCTs, and by extension for all other CT's. This way of checking the invariance may look a bit ad hoc, and it would become more and more awkward as we go to fields with spin and higher order correlation functions. As we will see, the projective lightcone formalism extends rather easily to such more general situations.

But first two more comments about the 2pt functions of scalar primaries. If the fields have different scaling dimensions, $\Delta_1 \neq \Delta_2$, the 2pt function vanishes

$$\langle \phi_1(x) \phi_2(y) \rangle = 0 . \tag{2.35}$$

This is clear from the lightcone as we cannot construct the analogue of (2.25) if $\Delta_1 \neq \Delta_2$.

In a theory with several fields ϕ_i with same scaling dimension Δ, the 2pt function is

$$\langle \phi_i(x)\phi_j(y)\rangle = \frac{M_{ij}}{(x-y)^{2\Delta}} \ , \tag{2.36}$$

As we will see in Chap. 3, the matrix M_{ij} will be positive definite in a unitary theory. This implies that there exists a field basis such that M_{ij} becomes unit-diagonal:

$$\langle \phi_i(x)\phi_j(y)\rangle = \frac{\delta_{ij}}{(x-y)^{2\Delta}} \ . \tag{2.37}$$

We will always assume that such a basis is chosen.

2.2.2 Primary Scalar 3pt Function

The 3pt function of three primary scalar fields with scaling dimensions Δ_1, Δ_2, Δ_3 (which could be equal or different) must have the following form on the cone

$$\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\rangle = \frac{const.}{(X_1X_2)^{\alpha_{123}}(X_1X_3)^{\alpha_{132}}(X_2X_3)^{\alpha_{231}}} \ . \tag{2.38}$$

As in the 2pt function case, the above is the most general Lorentz-invariant expression. To make it consistent with scaling, we should impose the constraints

$$\begin{aligned} \alpha_{123} + \alpha_{132} &= \Delta_1 \ , \\ \alpha_{123} + \alpha_{231} &= \Delta_2 \ , \\ \alpha_{132} + \alpha_{231} &= \Delta_3 \ . \end{aligned} \tag{2.39}$$

This linear system of three equations for three unknowns admits a unique solution:

$$\alpha_{ijk} = \frac{\Delta_i + \Delta_j - \Delta_k}{2} \ . \tag{2.40}$$

Thus the 3pt function is uniquely determined up to a constant. Projecting to the Euclidean section, we find

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{2\alpha_{123}}|x_{13}|^{2\alpha_{132}}|x_{23}|^{2\alpha_{231}}} \ , \tag{2.41}$$

where λ_{123} a free parameter (a CFT analogue of a "coupling constant"), and

$$x_{ij} = x_i - x_j \ . \tag{2.42}$$

This remarkable formula derived by Polyakov in 1970 gave birth to CFT. To understand its significance, we should compare it with infinitely many functional forms which would be allowed if we imposed only SI:

$$\sum \frac{const.}{|x_{12}|^a |x_{13}|^b |x_{23}|^c}\ , \ a+b+c = \Delta_1 + \Delta_2 + \Delta_3\ , \tag{2.43}$$

whereas there is only one term consistent with CI.

The 3pt function $\langle \sigma(x)\sigma(y)\epsilon(z)\rangle$ for two spins and energy in the 2-dimensional Ising model at the critical point can be extracted from the exact Onsager's lattice solution. Polyakov noticed that it agrees with his formula and conjectured the CI of the critical 2d Ising model, which as we now know is indeed true.

2.2.3 Four Point Function

Moving to the 4pt function, consider four identical fields for simplicity. Requiring consistency under Lorentz transformations and scaling, we get on the cone

$$\langle \phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4)\rangle = \frac{1}{(X_1\cdot X_2)^{\Delta}(X_3\cdot X_4)^{\Delta}} f(u,v)\ , \tag{2.44}$$

Here u and v are *conformally invariant cross-ratios* which on the light-cone are given by Lorentz-invariant expressions

$$u = \frac{(X_1\cdot X_2)(X_3\cdot X_4)}{(X_1\cdot X_3)(X_2\cdot X_4)}\ , \quad \text{and} \quad v = u|_{2\leftrightarrow 4} = \frac{(X_1\cdot X_4)(X_2\cdot X_3)}{(X_1\cdot X_3)(X_2\cdot X_4)}\ . \tag{2.45}$$

Notice that they have scaling zero in every variable. Since the prefactor in the RHS of (2.44) takes care of the scaling, any function $f(u,v)$ of u and v can appear.

Projecting to the Euclidean section, u and v become

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad \text{and} \quad v = u|_{2\leftrightarrow 4} = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\ , \tag{2.46}$$

and the full 4pt function:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} f(u,v)\ . \tag{2.47}$$

That the 4pt function must be given by a simple expression times a function of the conformally invariant cross-ratios is an enormous reduction of the functional freedom, although not as large as for the 3pt functions where the functional form was completely fixed.

We will later see that $f(u, v)$ is not an independent quantity but is related in a non-trivial way to the 3pt function. This will require a set of arguments going beyond just conformal kinematics.

For the moment let us notice a functional constraint on $f(u, v)$ which comes from the crossing symmetry of the 4pt function. The prefactor of (2.44) or (2.47) groups the external points as (12)(34), but this is an arbitrary choice. If we interchange $2 \leftrightarrow 4$, we get

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{14}^{2\Delta}x_{23}^{2\Delta}} f(\tilde{u}, \tilde{v}) , \tag{2.48}$$

where now $f(\tilde{u}, \tilde{v})$ depends on the conformally invariant crossratios calculated with the interchanged indices:

$$\tilde{u} = v , \quad \tilde{v} = u . \tag{2.49}$$

Notice that the same function f appears in (2.47) and (2.48), since the 4pt function is totally symmetric under permutations. Moreover, (2.47) and (2.48) must agree:

$$\frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}} f(u, v) = \frac{1}{x_{14}^{2\Delta}x_{23}^{2\Delta}} f(v, u) \tag{2.50}$$

Multiplying by $x_{14}^{2\Delta}x_{23}^{2\Delta}$. we find that $f(u, v)$ must satisfy:

$$\left(\frac{v}{u}\right)^{\Delta} f(u, v) = f(v, u) . \tag{2.51}$$

This constraint will play an important role in Chap. 4.

2.3 Fields with Spin

2.3.1 Extending the Null Cone Formalism

So far we only talked about scalar primaries. Let us now consider primaries with spin.

We will consider symmetric traceless primary fields living on the D dimensional space.[1] We will put such a field in correspondence with a field which lives on the light-cone and is also symmetric and traceless:

$$\phi_{\mu\nu\lambda\ldots}(x) \leftrightarrow \phi_{MNL\ldots}(X) . \tag{2.52}$$

[1]Primaries in other representations of $SO(D)$, like antisymmetric tensors of fermions, can also be considered.

We notice that the fields on the light-cone have more components that the D dimensional ones (roughly two extra components per index). For this correspondence to be useful, we have to eliminate these extra components. Let's first of all impose transversality of the null cone fields

$$X^M \phi_{MNL\ldots}(X) = 0 \,. \tag{2.53}$$

This condition eliminates one extra component per index. We will see below how the remaining ones are dealt with.

Then we define $\phi_{\mu\nu\lambda\ldots}(x)$ to be related to $\phi_{MNL\ldots}(X)$ by projection on the Euclidean section

$$\phi_{\mu\nu\lambda\ldots}(x) = \phi_{MNL\ldots}(X) \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} \ldots \,, \tag{2.54}$$

where $X^M = (1, x^2, x^\mu)$ is the parametrization of the section, so

$$\frac{\partial X^M}{\partial x^\nu} = (0, 2x_\nu, \delta^\mu_\nu) \,. \tag{2.55}$$

Notice that this rule preserves the tracelessness condition: if we start from a traceless $(D+2)$-dimensional tensor, we will end up with a traceless D-dimensional tensor. Indeed, to compute the trace of $\phi_{\mu\nu\ldots}$ we have to evaluate the contraction

$$\delta^{\mu\nu} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} \tag{2.56}$$

which can be shown to be equal to

$$\eta^{MN} + X^M K^N + X^N K^M \,, \tag{2.57}$$

with $K_M = (0, 2, 0)$ an auxiliary vector. Contracted with $\phi_{MN\ldots}$, it will vanish by tracelessness and transversality.

Notice also that anything proportional to X^M projects to zero, since

$$X^2 = 0 \Rightarrow X^M \frac{\partial X^M}{\partial x^\mu} = 0 \,. \tag{2.58}$$

This means that, for the purposes of this correspondence, $\phi_{MNL\ldots}$ is defined up to adding an arbitrary tensor proportional to X^M. This "gauge invariance" further reduces the number of degrees of freedom, exactly to what we need.

Let's discuss the transformation properties. Under an $SO(D+1, 1)$ transformation, the field on the null cone transforms in the standard Lorentz-invariant way:

$$\tilde{\phi}_{MNL\ldots}(X') = \Lambda_M^{\ M'} \Lambda_N^{\ N'} \ldots \phi_{M'N'L'\ldots}(X) \,. \tag{2.59}$$

Just like for primary scalars, we will impose that the null cone fields are homogeneous in X:

$$\phi_{\dots}(\lambda X) = \lambda^{-\Delta}\phi_{\dots}(X)\ , \tag{2.60}$$

We claim that the resulting transformations for the fields on the section is what we need:

$$\tilde{\phi}_{\mu\dots}(x') = \frac{1}{b(x)^{\Delta}} M^{\mu'}{}_{\mu}(x)\cdots\phi_{\mu'\dots}(x)\ . \tag{2.61}$$

The line element transforms as

$$dx' = b(x)M(x).dx \tag{2.62}$$

To show that (2.61) is true, it's enough to show that (consider spin 1 case for simplicity)

$$\tilde{\phi}(x').dx' = \frac{1}{b(x)^{\Delta-1}}\phi(x).dx \tag{2.63}$$

Now, the projection rule implies that

$$\phi(x).dx = \phi(X).dX \tag{2.64}$$

When $X \to \Lambda.X$ the scalar product $\phi(X).dX$ is preserved:

$$\phi(Y).dY = \phi(X).dX, \qquad Y = \Lambda.X \tag{2.65}$$

To get from Y back into the section we have to rescale: $X' = bY$. When we do it $\phi(Y)$ simply rescales. dY rescales plus gets a contribution proportional to Y if b is not a constant. This extra contribution vanishes when contracted with $\phi(Y)$ because of transversality. The end result is exactly (2.63).

2.3.2 Two Point Function

To see the consequences, consider the 2pt function of a vector field. On the cone we have:

$$\langle\phi_M(X)\phi_N(Y)\rangle = \frac{\eta_{MN} + \alpha\frac{Y_M X_N}{XY}}{(XY)^{\Delta}}\ , \tag{2.66}$$

where we once again considered the most general Lorentz invariant form consistent with scaling. Notice that we don't write terms proportional to X_M or Y_N, since they anyway project to zero.

We have to impose transversality which fixes the value of the constant α

$$X^M(\quad) = Y^N(\quad) = 0 \Rightarrow \alpha = -1\ . \tag{2.67}$$

Projecting the 2pt function in the physical space, we find:

$$\begin{aligned}
\eta_{MN} &\to \delta_{\mu\nu} \\
Y_M &\to -x_\mu + y_\mu \\
X_N &\to x_\nu - y_\nu \\
X \cdot Y &\to -\frac{1}{2}(x-y)^2 \,,
\end{aligned} \tag{2.68}$$

therefore

$$\frac{\eta_{MN} - \frac{Y_M X_N}{XY}}{(XY)^\Delta} \to \frac{I_{\mu\nu}(x-y)}{(x-y)^{2\Delta}} \,, \tag{2.69}$$

with

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \,. \tag{2.70}$$

CI fixed the relative coefficient -2 between the two terms here. If we had SI only, the relative coefficient would be free.

[It would not be so easy to check that the found 2pt function transforms correctly under CT without using the cone. As usual, it would be sufficient to check that it transforms correctly under inversion. This in turn would be equivalent to the identify

$$I(x)I(x-y)I(y) = I(x'-y') \,, \tag{2.71}$$

with $x' = x/x^2$. One can check this by an explicit computation, expanding throughout, but done this way it looks rather accidental.]

The 2pt function for higher spin primaries can be computed similarly. Interestingly, apart from $I_{\mu\nu}$, no new conformally covariant tensors appear. All the 2pt functions are made of $I_{\mu\nu}$'s connecting different points, and $\delta_{\mu\nu}$'s if the indices μ, ν are associated with the same point. For example, the 2pt function for a symmetric traceless field will be

$$\langle \phi_{\mu\nu}(x)\phi_{\lambda\sigma}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}[I_{\mu\lambda}(x-y)I_{\nu\sigma}(x-y) + (\mu \leftrightarrow \nu) + \beta\delta_{\mu\nu}\delta_{\lambda\sigma}] \,, \tag{2.72}$$

where the term with the delta functions (transforming correctly under CT since both indices get multiplied by the same M^μ_ν) was inserted in order to satisfy the tracelessness condition, which fixes the value of α:

$$\beta = -\frac{2}{D} \,. \tag{2.73}$$

To summarize, the 2pt functions are completely fixed for higher spin primaries just like for the scalar.

2.3.3 Remark About Inversion

In "pedestrian" CFT calculations, not based on the projective null cone formalism, one often checks invariance under the inversion rather than under SCT. However, as we mentioned, inversion is not in the connected part of the conformal group. One may wonder if assuming invariance under the inversion is in fact an extra assumption.

On the cone, the inversion corresponds to the transformation

$$X^{D+1} \to -X^{D+1}, \text{ i.e. } X^+ \leftrightarrow X^- \quad (X^\pm = X^{D+2} \pm X^{D+1})\,. \tag{2.74}$$

Indeed, this maps the point $X^M = (1, x^2, x^\mu)$ on the Euclidean section to

$$(1, x^2, x^\mu) \to (x^2, 1, x^\mu) \xrightarrow{\text{rescale}} (1, 1/x^2, x^\mu/x^2)\,, \tag{2.75}$$

which contains inversion in the last component.

Notice that the transformation (2.74) belongs to $O(D+1, 1)$ but not to $SO(D+1, 1)$, i.e. it is not in the connected component.

Another transformation in the same class is a simple spatial reflection (parity transformation)

$$X^1 \to -X^1\,. \tag{2.76}$$

The two discrete symmetries, parity and inversion, are conjugate by $SO(D+1, 1)$. This implies that a CFT invariant under parity will be invariant under inversion and vice versa.

There are CFTs which break parity (and inversion). Correlators in those theories, lifted to the null cone, will involve the $(D+2)$-dimensional ϵ-tensor, or Γ_{D+3} (the analog of the γ_5 matrix) for fermions if $D+2$ is even. Since we only considered scalars and symmetric tensors, these structures did not occur in our calculations.

2.3.4 Remark on Conservation

We have seen that for spin-1 and spin-2 primary fields, the form of the 2pt correlation functions is fixed by CI in terms of just one parameter: the scaling dimension of the field. Canonical dimensions

$$\Delta = D - 1\,, \quad \text{for} \quad l = 1\,, \quad \Delta = D\,, \quad \text{for} \quad l = 2\,. \tag{2.77}$$

would correspond to the conserved currents and the stress tensor. We expect their 2pt functions to be conserved objects. This should happen automatically since there is nothing to be adjusted. And indeed one can check that this is true. E.g. for the currents.

$$\partial^\mu \frac{I_{\mu\nu}(x)}{x^{2\Delta}} = 0\,, \quad \text{for} \quad \Delta = D - 1\,, \tag{2.78}$$

Notice that the null cone formalism is simply a way to compute constraints imposed by CI. For example, current and stress tensor conservation may be more convenient to check in the physical space rather than on the null cone. There is no reason to insist in doing everything on the null cone. The two points of view—null cone and physical space—can be used interchangeably depending what one wants to compute.

2.3.5 *Scalar-scalar-(spin l)*

The last correlator that we will study in this chapter is the 3pt function of two scalars and one spin l operator. Start with spin one. On the null cone we will have

$$\langle \phi_1(X)\phi_2(Y)\phi_{3M}(Z)\rangle = \text{scalar factor} \times (\text{tensor structure})_M\,, \tag{2.79}$$

The scalar factor will be the same as for the scalars

$$\frac{const.}{(XY)^{\alpha_{123}}(YZ)^{\alpha_{231}}(XZ)^{\alpha_{132}}}\,, \tag{2.80}$$

where the powers are fixed by the dimensions of the fields in order to get the correct scaling. The tensor structure must then have scaling 0 in all variables, and will also have to be transverse: $Z^M(\ldots)_M = 0$. Moreover we don't need to include a term proportional to Z_M since it will project to zero. It's then easy to see that the tensor part must be equal to

$$\frac{(YZ)X_M - (XZ)Y_M}{(XZ)^{1/2}(XY)^{1/2}(YZ)^{1/2}}\,, \tag{2.81}$$

where the relative coefficient was fixed from the transversality constraint. We now have to project the tensor part into the physical space, i.e. multiply by $\partial Z^M/\partial z^\mu$. We find that

$$X_M \to (x-z)_\mu \quad \text{and} \quad Y_M \to (y-z)_\mu\,, \tag{2.82}$$

therefore the expression projects into

$$\frac{(x-z)_\mu|y-z|^2 - (y-z)_\mu|x-z|^2}{|x-z||y-z||x-y|}\,, \tag{2.83}$$

or in a nicer form

$$\frac{|y-z||x-z|}{|x-y|}\left(\frac{(x-z)_\mu}{|x-z|^2} - \frac{(y-z)_\mu}{|y-z|^2}\right) \equiv R_\mu(x,y|z)\,. \tag{2.84}$$

The quantity R_μ transforms correctly under CT. It turns out that this is the only indexed object for three points with this property ($I_{\mu\nu}$ is not useful here since it has two indices at different points). For spin l fields the above formula is generalized to

$$\langle \phi_1(x)\phi_2(y)\phi_{3\mu\nu\lambda\ldots}(z)\rangle \propto \text{scalar part} \times (R_\mu R_\nu R_\lambda \cdots - \text{traces})\,. \tag{2.85}$$

We see that the 3pt function is again completely fixed up to an arbitrary constant.

An important special case is 3pt functions of two scalars with the current J_μ and the stress tensor $T_{\mu\nu}$:

$$\langle \phi_1(x)\phi_2(y)J_\mu(z)\rangle\,, \quad \langle \phi_1(x)\phi_2(y)T_{\mu\nu}(z)\rangle\,. \tag{2.86}$$

The above construction gives conformally invariant candidate expressions for these 3pt functions. What if we impose the conservation condition? For 2pt functions conservation was automatic, but here it is not so. In fact, the candidate 3pt functions are conserved if and only if the scalars have equal dimensions, $\Delta_1 = \Delta_2$. Intuitively this happens because the 3pt function must satisfy the Ward identities, which relate it to the 2pt function. As we have seen, the 2pt function is non-zero if and only if $\Delta_1 = \Delta_2$. The conclusion is that the coupling of the stress tensor and conserved currents to two scalar primaries of unequal dimensions must vanish.

In this chapter we learned to dominate the kinematics of the conformal group. Even though it gets us a long way, by itself it is not enough to solve a theory. In the next chapter we will introduce dynamics in the guise of Operator Product Expansion (OPE).

2.4 Appendix: An Elementary Property of CTs

It's good to know the following simple property of conformal transformations: they map circles to circles (considering straight lines as circles with infinite radius).

On the lightcone this is completely obvious. The Euclidean circle is represented on the lightcone by a set of points X satisfying $X.X_0 = c$ where X_0 is a fixed vector and c is a constant. A Lorentz transformation maps this to $X.X_0' = c$, which is another circle.

This property of CT is well known and useful in classical geometry. Here is an amusing geometrical problem which becomes easy if one uses it but would be tricky to solve otherwise: Given a circle γ and two points A, B outside of it, construct a circle tangent to γ that passes through A and B.

Idea of solution: Apply an inversion with respect to any point on γ. The circle is now mapped to a straight line. In these coordinates, the problem is reduced to solving a quadratic equation. Then map back.

2.5 Literature

The projective null cone idea is due to Dirac, and was used by Mack and Salam, Ferrara et al., Siegel and others. More recently it was used by Cornalba, Costa and Penedones [1] in its essentially modern form. It was then discussed by Weinberg [2] (who apparently rediscovered it independently of all the previous literature). This formalism is also developed further in our work [3].

For the pedestrian approach to conformal correlators (as opposed to the null cone) see e.g. Osborn and Petkou [4].

The classic 1970 paper by Polyakov is [5]. It gave birth to CFTs, together with Mack and Salam's [6].

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