Chapter 2
Assessing the Local Stability Properties of Discrete Three-Dimensional Dynamical Systems: A Geometrical Approach with Triangles and Planes and an Application with Some Cones

Jean-Paul Barinci and Jean-Pierre Drugeon

Abstract The difficulties associated with the appraisal of the determinacy properties of a three-dimensional system are circumvented by the introduction of a new geometrical argument. It brings about a complete typology of the eigenvalues moduli in discrete time three-dimensional dynamical systems and then provides a new apparatus for assessing from a geometrical standpoint the emergence of local bifurcations for parameterised economies. The argument is considered through the extensive characterisation of the stability properties of a benchmark model of inter-temporal economic analysis.

Keywords Discrete time dynamical systems · Geometrical approach · Parameterised economies

JEL Classification: E32 · C62

This research was completed thanks to the supports of the Novo Tempus research grant, ANR-12-BSH1-0007, Program BSH1-2012, and of the Labex MME-DII. The authors would like to thank Jean-Michel Grandmont for the numerous conversations they benefitted from after a seminar they held at Crest. They are also indebted to the anonymous referee for his suggestions. The usual disclaimer nonetheless applies.

J.-P. Barinci
E.P.E.E., Université d'Evry Val d'Essonne, Évry, France

J.-P. Drugeon (✉)
Paris School of Economics and Centre National de la Recherche Scientifique, Paris, France
e-mail: jpdrug@parisschoolofeconomics.eu

K. Nishimura et al. (eds.), *Sunspots and Non-Linear Dynamics*,
Studies in Economic Theory 31, DOI 10.1007/978-3-319-44076-7_2

2.1 Introduction

The difficulties associated with the appraisal of the local determinacy properties of a three-dimensional discrete time dynamical system have long deterred a more widespread use of the associated setups in economic theory. This contribution is intended to introduce graphical methods for assessing the stability and the scope for local bifurcations within such systems. It also provides some illustrations in parameterised economies.

As they reconsider the role of factors substitutability in competitive economies, Grandmont (1998) and Grandmont et al. (1998) have come to introduce a tractable graphical way of assessing local uniqueness or local indeterminacy for dynamical systems of order two. Their approach is based upon a graphical partition of the $(\mathscr{T}\mathscr{D})$-plane defined from the two coefficients $\mathscr{T}$ and $\mathscr{D}$ of the second-order characteristic polynomial $\mathrm{P}(z) = z^2 - \mathscr{T}z + \mathscr{D}$ that is associated with a two-dimensional dynamical system in the neighbourhood of some steady state, these coefficients $\mathscr{T}$ and $\mathscr{D}$ being assumed to depend upon a range of n parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n\}$. Such a partition is then completed by drawing the linear critical loci associated to the occurrence of real and complex eigenvalues with unitary modulus, respectively two straight-lines (AB) and (AC) of slopes $+1$ and -1 and a horizontal segment [BC] over the plane defined from the coefficients $\mathscr{T}$ and $\mathscr{D}$. These critical loci feature boundaries between stability and unstability zones, a *full stability*—all the moduli of the eigenvalues are less than one—zona being noticeably depicted by the interior of the triangle (ABC). A given economy—a set of fundamental preferences and technological parameterisations $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n\}$—was then to be understood as a point over that plane whilst the appraisal of its local dynamics summarised to the localisation of this point. Letting one of its building parameters, say some $\lambda_i \in \{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n\}$, vary gives rise to a family of economies, namely a curve ${}_{\lambda_i}\Delta$, over that plane the localisation of which provided insights about the associated qualitative changes undergone by the dynamical properties of the economy. The crux interest of this construction for economic theory stems from its explicit consideration of meaningful and generic concepts without having to resort to specific parametric formulations. That graphical method was remarkable from its tractability and its potential for significantly easing the appraisal of otherwise complex formal structures.

Two key difficulties however quickly emerge as being associated with the extension of the above approach to three-dimensional dynamical systems and the elaboration of a graphical partition of a three-dimensional $(\mathscr{T}\mathscr{M}\mathscr{D})$-space defined from the three coefficients $\mathscr{T}$, $\mathscr{M}$ and $\mathscr{D}$ of the third-order characteristic polynomial $\mathrm{Q}(z) = -z^3 + \mathscr{T}z^2 - \mathscr{M}z + \mathscr{D}$ in the neighbourhood of some steady state. Firstly, the intricacies of three-dimensional graphs and the geometry of a three-dimensional $(\mathscr{T}\mathscr{M}\mathscr{D})$-space are far more difficult to grasp than the aforementioned two-dimensional pictures in a two-dimensional $(\mathscr{T}\mathscr{D})$-space. Secondly, the elaboration of a graphical partition is anchored on the introduction of critical loci associated with the occurrence of eigenvalues with unitary modulus: whilst linearity keeps on being an attribute of the critical loci associated with real eigenvalues—this results in planes in the $(\mathscr{T}\mathscr{M}\mathscr{D})$-space, one is now faced with the *uprise of a nonlinear critical locus* in order to picture the occurrence of complex eigenvalues with unitary modulus. The first of these issues shall be circumvented by apprehending the original three-dimensional $(\mathscr{T}\mathscr{M}\mathscr{D})$-space through a collection of sections along the $\mathscr{D}$ coordinate and thus of $(\mathscr{T}\mathscr{M})_{\mathscr{D}}$ planes parameterized by $\mathscr{D}$. Fortunately enough, such an approach also entails linear definitions for the three parameterised critical loci: one indeed recovers two straight-lines $(A_{\mathscr{D}}B_{\mathscr{D}})$ and $(A_{\mathscr{D}}C_{\mathscr{D}})$ of slopes $+1$ and -1 and a segment $[B_{\mathscr{D}}C_{\mathscr{D}}]$—its slope is now to vary according to $\mathscr{D}$—over a finite collection of planes $(\mathscr{T}\mathscr{M})_{\mathscr{D}}$ that are also *parameterised* by the coefficient $\mathscr{D}$ and defined for $|\mathscr{D}| < 1$, $\mathscr{D} < -1$ and $\mathscr{D} > 1$, the interior of the parameterised triangle $(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}})$ being accordingly changed from a *full stability* area—all the moduli of the eigenvalues are less than one—to a *full instability* one—all the moduli of the eigenvalues are greater than one.

Assuming further that the coefficients $\mathscr{T}$ and $\mathscr{M}$ depend upon a range of n parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n\}$ while the coefficient $\mathscr{D}$ depends upon *at most* $n-1$ such parameters, say the $n-1$ first ones, $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\}$, the appraisal of the range of configurations admissible for a given economy can anew be completed over a parameterised plane. Letting indeed the parameter λ_n vary gives rise to a family of economies, namely a curve ${}_{\lambda_n}\Delta$, over a parameterised plane—it is *uniquely* defined for a given $\mathscr{D}$—the localisation of which provides insights about the associated qualitative changes undergone by the dynamical properties of the economy when λ_n spans its interval of admissible values.

As an illustration of the appropriateness of this approach, an overlapping generations model is subsequently analysed, the two-period overlapping generations model having become a workhorse for the theory of descriptive fluctuations. The literature on the subject has focused on both the Samuelson's (1958) pure exchange framework, e.g., Grandmont (1985), and the Diamond's (1965) setting with productive capital, e.g., Reichlin (1986). The present contribution more precisely considers a slightly modified Diamond's setting in order to illustrate the easiness of use of the graphical method and the tools it introduces. The first departure from the original framework lies in the consideration of a labor-leisure arbitrage in the first-period of agent's life. The requirement of capital-wealth equality is further relaxed, i.e., it is not any longer assumed that the consumer's wealth is equal to the value of the capital stock. Here private wealth—the sum of private assets–and capital are assumed to be separate entities. This is an extension of the pure-exchange Gale's (1972) model in which the equilibrium value of the private wealth could be nonzero. Taking over the Gale's terminology, economies for which private wealth is smaller, respectively greater, than capital are labelled as *Classical*, respectively *Samuelson*. The local dynamics nearby the steady state of both type of economies are characterized thanks to the aforementioned tools and the parallel use of some infinite cones whose generatrices are defined by a boundary featuring strict concavity and a boundary denoting the border between gross substitutability and complementarity for leisure and first-period consumption.

Beyond the specifics of that setup, the current class of techniques can be applied to quite a large range of parameterised environments that would result in third-order dynamical systems. In models of economics, the applicability of the whole approach revolves around the identification of some fundamental parameter that would not appear into the coefficient $\mathscr{D}$, i.e., the one that corresponds to the product of the eigenvalues. Even though such a qualification may sound as being restrictive, the computation of that coefficient being commonly the most difficult for a given Jacobian Matrix, it typically uncovers a rather simplified analytical form with respect to $\mathscr{T}$ and $\mathscr{M}$ and thus a dependence with respect to a fewer range of coefficients, a property that should prove useful in potential future applications.

The geometrical techniques are introduced in Sect. 2.2. Section 2.3 builds upon an extension of the pure-exchange model of overlapping generations. Some formal details are provided in a final appendix.

2.2 A Geometrical Argument for the Appraisal of the Local Stability Properties of Three-Dimensional Dynamical Systems

2.2.1 A Simple Typology for the Eigenvalues of a Discrete Three-Dimensional Dynamical System

Letting the equilibrium dynamics of an economy be described by a system: $y_{t+1} = \mathrm{G}(y_t)$, $y_t \in \mathbb{R}^3_+$, steady states equilibria are the roots of $\bar{y} - \mathrm{G}(\bar{y}) = 0$. The characterisation of the local dynamics nearby a given steady equilibrium proceeds from the appraisal of an associated linear map $\zeta_{t+1} = \mathscr{J}\zeta_t$, for $\mathscr{J} := \mathrm{DG}(\bar{y})$ the Jacobian matrix of $\mathrm{G}(\cdot)$ evaluated at $\bar{y}$ and $\zeta_t := y_t - \bar{y}$ the deviation from the steady state. The eigenvalues of the matrix $\mathscr{J}$ are the zeroes of the following third order polynomial:

$$\begin{aligned}\mathrm{Q}(z) &= (z_1 - z)(z_2 - z)(z_3 - z) \\ &= -z^3 + \big(z_1 + z_2 + z_3\big)z^2 - \big(z_1z_2 + z_1z_3 + z_2z_3\big)z + z_1z_2z_3 \\ &= -z^3 + \mathscr{T}z^2 - \mathscr{M}z + \mathscr{D},\end{aligned} \tag{2.1}$$

for $\mathscr{T}$, $\mathscr{M}$ and $\mathscr{D}$ that respectively denote the trace, the sum of the principal minors of order two and the determinant of the Jacobian matrix $\mathscr{J} := \mathrm{DG}(\bar{y})$.

The locus such that the coefficients $\mathscr{T}, \mathscr{M}, \mathscr{D}$ satisfy $\mathrm{Q}(+1) = 0$ is a plane—henceforward referred to as the *saddle-node critical plane*—of the $(\mathscr{T}\mathscr{M}\mathscr{D})$-space whose characteristic equation is given by:

$$-1 + \mathscr{T} - \mathscr{M} + \mathscr{D} = 0. \tag{2.2}$$

Generically, a saddle-node bifurcation[1] will occur when the triple $(\mathscr{T}, \mathscr{M}, \mathscr{D})$ crosses this plane and the uniqueness properties of the steady state will be lost. Similarly, the locus such that the coefficients $\mathscr{T}, \mathscr{M}, \mathscr{D}$ satisfy $\mathrm{Q}(-1) = 0$ is a plane—henceforth mentioned as the *flip critical plane*—of the $(\mathscr{T}\mathscr{M}\mathscr{D})$-space whose characteristic equation is given by:

$$1 + \mathscr{T} + \mathscr{M} + \mathscr{D} = 0. \tag{2.3}$$

A flip bifurcation is bound to occur in its neighbourhood when the triple $(\mathscr{T}, \mathscr{M}, \mathscr{D})$ crosses this plane and two-period cycles will emerge.

Lastly, when a pair of nonreal characteristic roots exhibiting an unitary norm occurs, the remaining eigenvalue, e.g., z_3, summarises to the product of the eigenvalues $\mathscr{D}$.[2] The latter coefficient thus becomes a characteristic root, i.e., $\mathrm{Q}(\mathscr{D}) = 0$. Solving, the characteristic polynomial hence restates as $\mathrm{Q}(z) = (\mathscr{D} - z)\mathrm{P}(z)$, for $\mathrm{P}(z) = z^2 - (\mathscr{T} - \mathscr{D})z + \mathscr{M} - (\mathscr{T} - \mathscr{D})\mathscr{D}$. A standard analysis of $\mathrm{P}(\cdot)$ then indicates that the locus of coefficients $\mathscr{T}, \mathscr{M}$ and $\mathscr{D}$ such that two roots are complex conjugate with unitary modulus is given by:

$$\mathscr{M} - 1 - \big(\mathscr{T} - \mathscr{D}\big)\mathscr{D} = 0, \tag{2.4a}$$

$$\big|\mathscr{T} - \mathscr{D}\big| < 2, \tag{2.4b}$$

Eq. (2.4a) being associated with $\mathrm{P}(\cdot)$ that assumes a pair of roots with a product equal to 1 whereas Eq. (2.4b) follows from the restriction for a negative sign for the discriminant associated to $\mathrm{P}(\cdot)$. This locus defines a hyperbolic paraboloid in the $(\mathscr{T}\mathscr{M}\mathscr{D})$-space. A Poincaré-Hopf bifurcation will occur when the triple $(\mathscr{T}, \mathscr{M}, \mathscr{D})$ crosses the complex interior component of the critical surface (2.4a)–(2.4b) and quasi-periodic equilibria will emerge in its neighbourhood.

For a given $\mathscr{D}$, the depiction of these three critical surfaces is going to be facilitated[3] by the ensued consideration of a collection of sections along the $\mathscr{D}$ coordinate, henceforth denoted as $(\mathscr{T}\mathscr{M})_{\mathscr{D}}$, any of the aforementioned critical loci being then represented through *a straight-line or a segment*.

More explicitly and first introducing the benchmark case $\mathscr{D} = 0$ on Fig. 2.1, the set of coefficients $(\mathscr{T}, \mathscr{M})$ such that $\mathrm{Q}(+1) = 0$ and $\mathrm{Q}(-1) = 0$ respectively correspond to the saddle-node and flip critical lines $\big(\mathrm{A}_0\mathrm{C}_0\big)$ and $\big(\mathrm{A}_0\mathrm{B}_0\big)$—the index 0 refers to the value of the parameter $\mathscr{D}$ under which the whole picture is drawn—whilst the corresponding set for two nonreal eigenvalues with unitary norm is depicted by the horizontal Poincaré-Hopf critical segment $\big[\mathrm{B}_0\mathrm{C}_0\big]$. This gives rise to a construction familiar from the two-dimensional analysis, namely the triangle $\big(\mathrm{A}_0\mathrm{B}_0\mathrm{C}_0\big)$ defined by $|\mathscr{T}| < |1 + \mathscr{M}|$ and $|\mathscr{M}| < 1$.

[1] *Vide* Devaney (1986) or Grandmont (2008) for an extensive typology of local bifurcations.

[2] Letting z_1 and z_2 be two eigenvalues with a unitary norm, it is indeed obtained that $z_1 z_2 z_3 = |z| z_3 = \mathscr{D}$.

[3] Equation (2.4a) depicts a *ruled surface*, i.e., a surface generated by straight-lines in $\mathscr{T}$ and $\mathscr{M}$ for a given value of $\mathscr{D}$.

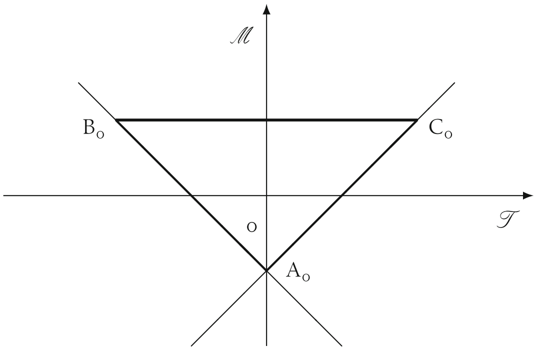

Fig. 2.1 Benchmark case $\mathscr{D} = 0$

The two panels of Fig. 2.2 then assess the status of this construction for various fixed values of $\mathscr{D}$ in the neighbourhood of the benchmark case $\mathscr{D} = 0$, respectively for $\mathscr{D} < 0$ and $\mathscr{D} > 0$. As $\mathscr{D}$ is decreased over $\mathbb{R}_-$ or increased over $\mathbb{R}_+$, the slopes of $\left(A_{\mathscr{D}}C_{\mathscr{D}}\right)$ and $\left(A_{\mathscr{D}}B_{\mathscr{D}}\right)$ are let unmodified. In opposition to this, the segment $\left[B_{\mathscr{D}}C_{\mathscr{D}}\right]$, of slope $\mathscr{D}$, respectively follows a translated clockwise rotation for $\mathscr{D} < 0$ and a translated counter-clockwise rotation for $\mathscr{D} > 0$.

The expressions of the *parameterised* coordinates of $A_{\mathscr{D}}$, $B_{\mathscr{D}}$ and $C_{\mathscr{D}}$ that underlie the definition of the triangle $\left(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}}\right)$ can readily be computed from the solving of (2) and (3), (3) and (4), (2) and (4) and list as:

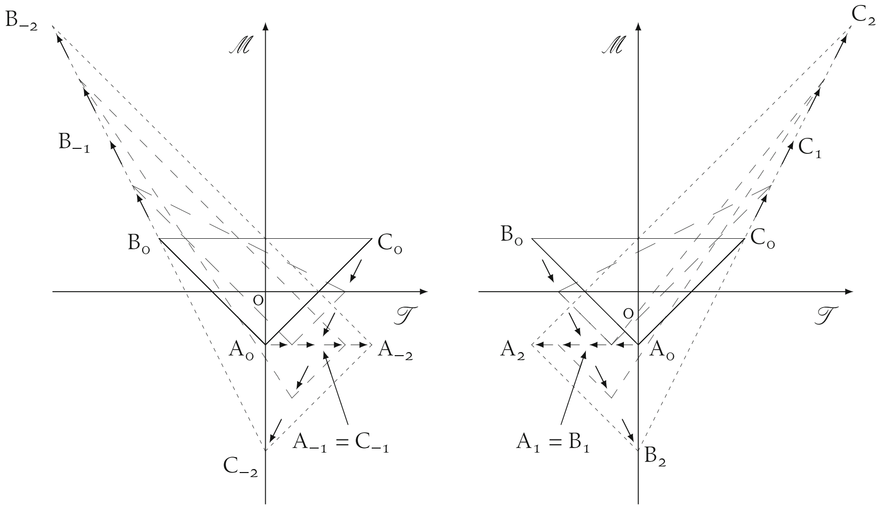

Fig. 2.2 Translated rotations of the benchmark triangle $\left(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}}\right)$

$$\left(\mathscr{T}_{A_\mathscr{D}}, \mathscr{M}_{A_\mathscr{D}}\right) = (-\mathscr{D}, -1), \tag{2.5a}$$
$$\left(\mathscr{T}_{B_\mathscr{D}}, \mathscr{M}_{B_\mathscr{D}}\right) = (-2 + \mathscr{D}, 1 - 2\mathscr{D}), \tag{2.5b}$$
$$\left(\mathscr{T}_{C_\mathscr{D}}, \mathscr{M}_{C_\mathscr{D}}\right) = (2 + \mathscr{D}, 1 + 2\mathscr{D}). \tag{2.5c}$$

Worthwhile noticing is also the non-generic occurrence, for $\mathscr{D} = -1$ and $\mathscr{D} = 1$, of $A_{-1} = B_{-1}$ and $A_1 = B_1$: the Poincaré critical segment becomes respectively part of the flip and the saddle node critical loci. Such occurrences imply that the formal definition of the triangle $\left(A_\mathscr{D} B_\mathscr{D} C_\mathscr{D}\right)$ is modified as $|\mathscr{D}|$ goes through one, namely:

$$\begin{cases} \mathscr{M} < 1 + \left(\mathscr{T} - \mathscr{D}\right)\mathscr{D} \\ |\mathscr{T} + \mathscr{D}| < 1 + \mathscr{M} \end{cases} \quad \text{for} \quad |\mathscr{D}| < 1, \tag{2.6a}$$
$$\begin{cases} \mathscr{M} > 1 + \left(\mathscr{T} - \mathscr{D}\right)\mathscr{D} \\ |1 + \mathscr{M}| > \mathscr{T} + \mathscr{D} \end{cases} \quad \text{for} \quad \mathscr{D} < -1, \tag{2.6b}$$
$$\begin{cases} \mathscr{M} > 1 + \left(\mathscr{T} - \mathscr{D}\right)\mathscr{D} \\ |1 + \mathscr{M}| < \mathscr{T} + \mathscr{D} \end{cases} \quad \text{for} \quad \mathscr{D} > 1. \tag{2.6c}$$

The case $|\mathscr{D}| = 1$ is however non-generic and merely two generic configurations, namely $|\mathscr{D}| < 1$ and $|\mathscr{D}| > 1$, are to be considered, making use, as illustrated by Figs. 2.3 and 2.4, of a finite collection of $(\mathscr{T}\mathscr{M})_\mathscr{D}$ planes. Putting this into perspective and as made clear by Figs. 2.3 and 2.4, there will be *no loss of generality* in considering, for a given sign of $\mathscr{D}$, a finite collection of sections $(\mathscr{T}\mathscr{M})$ of the space $(\mathscr{T}\mathscr{M}\mathscr{D})$ will fully describe the set of admissible geometric configurations.

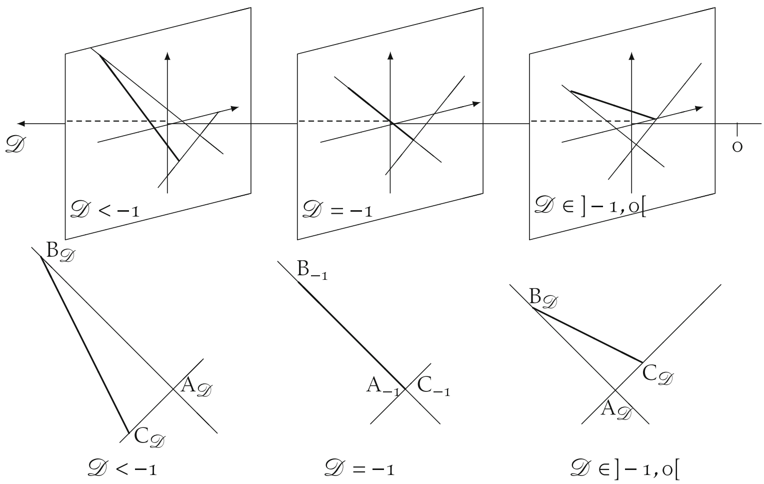

Fig. 2.3 Two generic configurations as $\mathscr{D}$ is decreased over $\mathbb{R}_-$

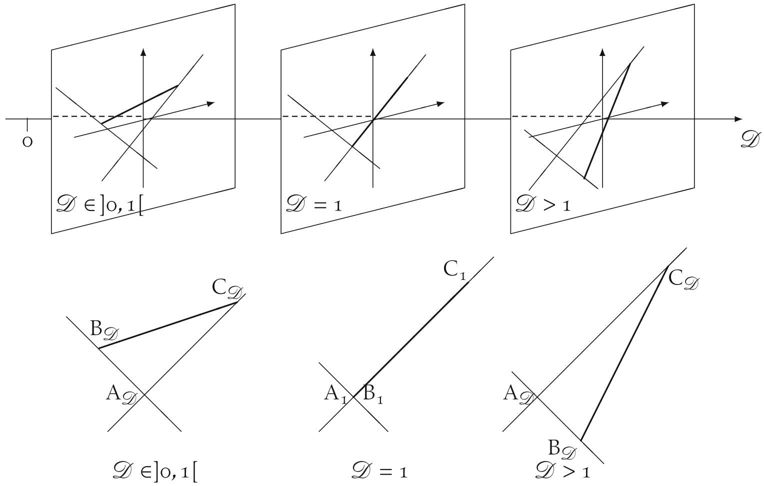

Fig. 2.4 Two generic configurations as $\mathscr{D}$ is increased over $\mathbb{R}_+$

It then remains to characterise any of the generic configurations of Figs. 2.3 and 2.4 in terms of the cardinality of the set of stable roots. It is first noticed that the origin $(\mathscr{T}, \mathscr{M}) = (0, 0)$ belongs to $\left(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}}\right)$ for any $\mathscr{D} \in \mathbb{R} \setminus \{-1, +1\}$: this appears from the translated rotations of Fig. 2.2 but this is also rapidly checked from the analytical definitions (2.6a)–(2.6c) of the triangles for $|\mathscr{D}| < 1$, $\mathscr{D} < -1$ or $\mathscr{D} > 1$. This geometric property translates as the satisfaction of $Q(z) = -z^3 + \mathscr{D} = 0$ by the characteristic polynomial, hence $z^3 = \mathscr{D}$ and the occurrence of a *triple* real eigenvalue at the origin. This will assume an absolute value greater than one for $|\mathscr{D}| > 1$ and an absolute value less than one for $|\mathscr{D}| < 1$.

As long as the system is maintained in the interior of the triangle $\left(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}}\right)$, its stability properties are left unaltered with respect to the ones the origin $(0, 0)$, that eventually establishes the corresponding number of stable eigenvalues between parenthesis for both configurations on Fig. 2.5—equivalently, the dimension of the local stable manifold.

Considering then a perturbation that occasions on Fig. 2.5 the leave from the *unstable* triangle $\left(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}}\right)$ for $|\mathscr{D}| > 1$. A crossing of the Poincaré-Hopf critical segment $\left[B_{\mathscr{D}}C_{\mathscr{D}}\right]$ would then imply that the modulus of the complex eigenvalues enters into the unit circle and an area characterized by two eigenvalues with a norm that is less than one. When such a leave from the *unstable* triangle $\left(A_{\mathscr{D}}B_{\mathscr{D}}C_{\mathscr{D}}\right)$ with zero stable roots rather proceeds through the crossing of the saddle-node critical line $\left(A_{\mathscr{D}}C_{\mathscr{D}}\right)$ or the flip critical line $\left(A_{\mathscr{D}}B_{\mathscr{D}}\right)$, a unique eigenvalue with respect to the unit circle will be modified and the system falls in an area with one stable eigenvalue. Finally, the crossing of the flip critical line after having crossed the saddle node critical line or the reversed sequence will lead the system within an area that exhibits a pair of moduli within the unit circle. A related line of reasoning can straightforwardly

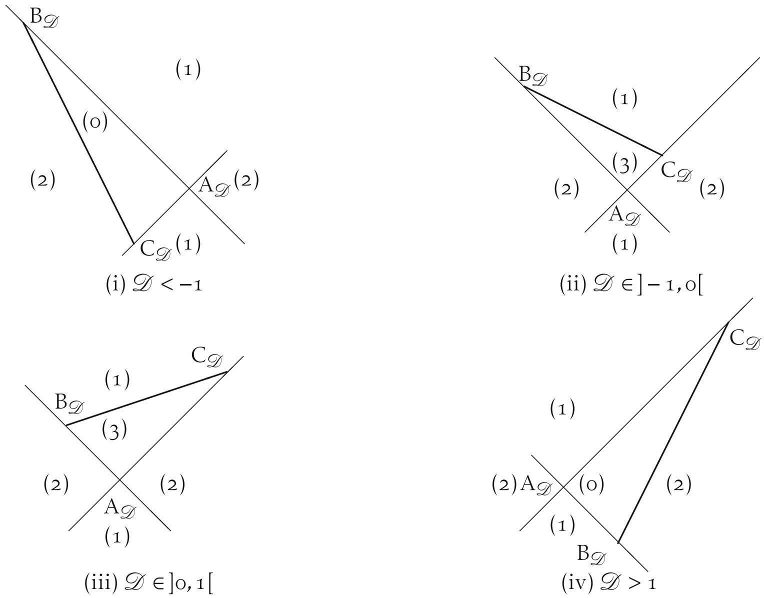

Fig. 2.5 Critical loci and typologies of stable eigenvalues for $|\mathscr{D}| < 1$ and $|\mathscr{D}| > 1$

be completed for the typology of stable eigenvalues associated with $|\mathscr{D}| < 1$ and the *stable* definition of the triangle $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$.

2.2.2 Assessing the Stability Properties of Parameterised Economies

This section shall argue that Fig. 2.5 equips the analysis with a range of tools that are going to facilitate the undertaking of a sensitivity analysis in actual parameterised economies. Consider indeed some characteristic polynomial $\mathrm{Q}(z) = -z^3 + \mathscr{T}z^2 - \mathscr{M}z + \mathscr{D}$ in the neighbourdhood of some steady state and assume that the coefficients $\mathscr{T}$ and $\mathscr{M}$ depend upon a list of four parameters whilst $\mathscr{D}$ merely depends upon one parameter, say the fourth of the list, whence some formulations $\mathscr{T}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $\mathscr{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\mathscr{D}(\lambda_4)$. From Sect. 2.2.1, an appraisal of the local stability properties of the economy $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ will be available from the features of parameterised $(\mathscr{T}\mathscr{M})_{\mathscr{D}}$-planes. For illustration purposes, consider the range of values of the coefficient λ_4 for which the coefficient $\mathscr{D}$ is such that $\mathscr{D}(\lambda_4) > 1$, the typology of the eigenvalues being available from Fig. 2.5(iv). Further let the coefficients $\mathscr{T}$ and $\mathscr{D}$ both assume the same positive linear dependency with respect to parameters λ_1 and λ_2 whose domains are restricted to the positive real line. Letting, e.g., the parameter λ_1 vary, this will result in a parameterised half-line ${}_{\lambda_1}\Delta$—arrowed on

Fig. 2.6—with a slope of $+1$ that is parallel to $(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}})$. It is positioned over the plane $(\mathscr{T}\mathscr{M})_{\mathscr{D}}$ by considering how its *origin*, defined for $\lambda_1 = 0$, would vary with the remaining parameters λ_2 and λ_3, the value of λ_4 being, by definition, given over a plane $(\mathscr{T}\mathscr{M})_{\mathscr{D}}$. Figure 2.6 describes a configuration where the dashed locus ${}_{\lambda_3}\Lambda_{\lambda_1=0}$ follows a counterclockwise rotation whilst the parameter λ_2 is increased, that in turn allows for introducing a dotted locus ${}_{\lambda_2}\Lambda_{\lambda_1=0,\lambda_3=0}$ that is, by assumption, also parallel to $(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}})$.

Otherwise stated and for the range of values of λ_4 such that $\mathscr{D}(\lambda_4) > 1$, the dotted locus ${}_{\lambda_2}\Lambda_{\lambda_1=0,\lambda_3=0}$ is located above both of the loci $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\right)$ and $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$. As this is clear from Fig. 2.6, for arbitrary small values of λ_3 and whatever the value of λ_2, the parameterised straight-line ${}_{\lambda_1}\Delta$ under consideration will locate in the same area with a unique modulus inside the unit circle, that would, e.g., correspond to a *determinacy property* for a system with a unique predetermined variable. In opposition to this and for larger values of λ_3, the origin of ${}_{\lambda_1}\Delta$ will locate below the locus $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$. If it is also considered for arbitrary small values of λ_2, that origin will be found below $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\right)$ and within an area with, again, one modulus inside the unit circle. The straight-line ${}_{\lambda_1}\Delta$ will assume an intersection with $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\right)$ for larger values of λ_1: if this takes place by the left of $\mathrm{B}_{\mathscr{D}}$, there will exist a range of values of λ_1 for which ${}_{\lambda_1}\Delta$ is located in the interior of the triangle $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$ with no modulus inside the unit circle and an *instability* configuration for a system with a unique predetermined

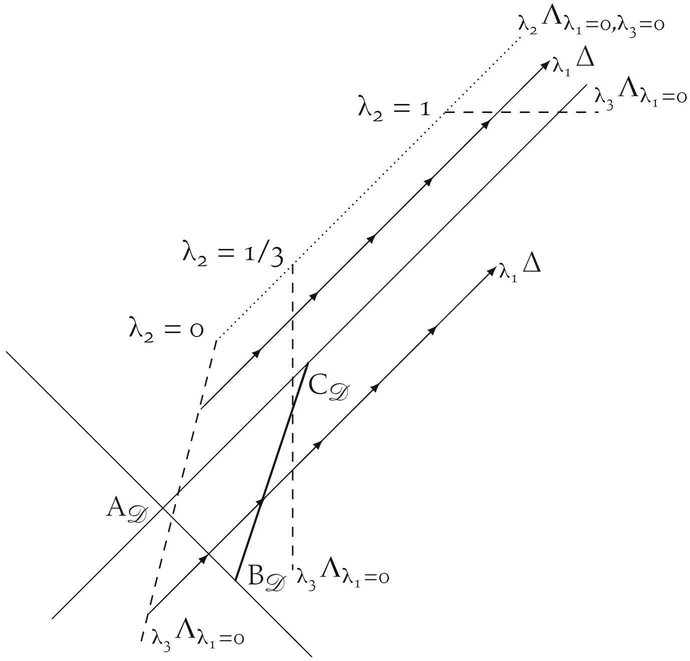

Fig. 2.6 The loci ${}_{\lambda_1}\Delta$, ${}_{\lambda_3}\Lambda_{\lambda_1=0}$ and ${}_{\lambda_2}\Lambda_{\lambda_1=0,\lambda_3=0}$ for $\mathscr{D}(\lambda_4) > 1$

variable. Finally, for still larger values of λ_1 and beyond the segment $\left[\mathrm{B}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right]$, the parameterised line ${}_{\lambda_1}\Delta$ will end up in an area with two moduli inside the unit circle that would correspond to an *indeterminacy* configuration for a system with a unique predetermined variable.

2.3 A Simple Parameterised Economy: The Golden Rule in the Model of Overlapping Generations

This section will consider an economy populated by generations of agents living for two periods. The representative agent works ℓ hours and consumes c when young but then solely consumes c' when old. His preferences are described by a separably additive utility function, i.e., $\gamma_c \mathrm{U}_1(c) + \mathrm{U}_2\left(c'\right) - \gamma_\ell \mathrm{V}(\ell)$, where $\gamma_c > 0$ and $\gamma_\ell > 0$ are scaling parameters. In the sequel it will be assumed that both $\mathrm{U}_1(\cdot)$ and $\mathrm{U}_2(\cdot)$ are increasing and concave whilst $\mathrm{V}(\cdot)$ is increasing and convex. At date $t \geq 1$, the young agent of generation t chooses a consumption vector $\left(c_t^t, c_{t+1}^t\right)$, a supply of labor ℓ_t^t and savings x_{t+1}^t so as to maximize his utility subject to:

$$
\begin{aligned}
c_t^t + x_{t+1}^t &= w_t \ell_t^t, \\
c_{t+1}^t &= \mathscr{R}_{t+1} x_{t+1}^t,
\end{aligned}
$$

and $c_t^t \geq 0$, $c_{t+1}^t \geq 0$, $\ell_t^t \geq 0$, for w_t the wage rate and $\mathscr{R}_{t+1}$ the gross return on savings. The single good is produced by a constant returns neoclassical production function AF(K, L), where K and L are respectively the productive capital and the labor employed, $\mathrm{A} > 0$ being a scaling parameter. In every period, competitive firms maximize profits, given the wage rate and the rental rate—for the sake of simplicity it will be assumed that capital fully depreciates on use within the period. The FOC of the optimisation problem for the young agent, that are necessary and sufficient under the assumed properties of the utility function, list as:

$$
\begin{aligned}
\gamma_\ell \frac{\partial \mathrm{V}}{\partial \ell}\left(\ell_t^t\right) &= w_t \mathscr{R}_{t+1} \frac{\partial \mathrm{U}_2}{\partial c'}\left(c_{t+1}^t\right), \\
\gamma_\ell \frac{\partial \mathrm{V}}{\partial \ell}\left(\ell_t^t\right) &= w_t \gamma_c \frac{\partial \mathrm{U}_1}{\partial c}\left(c_t^t\right), \\
c_t^t + x_{t+1}^t &= w_t \ell_t^t, \\
c_{t+1}^t &= \mathscr{R}_{t+1} x_{t+1}.
\end{aligned}
$$

In the lines of Gale (1973), the agent's sum of the assets will be allowed to differ from the total amount of productive capital. In other words, it will be assumed that capital is not the only channel of inter-temporal exchange. In that perspective, let B_t denote the difference between the savings willingness of the young and the stock of capital, i.e., $\mathrm{B}_t := x_{t+1}^t - \mathrm{K}_{t+1}$. Taking into account the market-clearing conditions for the factor markets, it can be established that, in reduced form, a competitive equilibrium is a sequence $\left(\mathrm{K}_t, \mathrm{L}_t, \mathrm{B}_t\right)$ satisfying:

$$\gamma_\ell \frac{\partial \mathrm{V}}{\partial \mathrm{L}}\left(\mathrm{L}_t\right) = \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}(\mathrm{K}_t, \mathrm{L}_t)\mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{K}}(\mathrm{K}_{t+1}, \mathrm{L}_{t+1})\frac{\partial \mathrm{U}_2}{\partial c'}\left(\mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{K}}(\mathrm{K}_{t+1}, \mathrm{L}_{t+1}) \times \left[\mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}(\mathrm{K}_t, \mathrm{L}_t)\mathrm{L}_t - c_t^t\right]\right), \tag{2.7a}$$

$$\gamma_\ell \frac{\partial \mathrm{V}}{\partial \ell}\left(\mathrm{L}_t\right) = \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}\left(\mathrm{K}_t, \mathrm{L}_t\right)\gamma_c \frac{\partial \mathrm{U}_1}{\partial c}\left(c_t^t\right), \tag{2.7b}$$

$$\mathrm{K}_{t+1} + \mathrm{B}_t + c_t^t = \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}\left(\mathrm{K}_t, \mathrm{L}_t\right)\mathrm{L}_t, \tag{2.7c}$$

$$\mathrm{B}_{t+1} = \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{K}}(\mathrm{K}_{t+1}, \mathrm{L}_{t+1})\mathrm{B}_t, \tag{2.7d}$$

for (2.7b) that defines, for $\partial^2 \mathrm{U}_1/\partial c^2 \neq 0$, c_t^t as a function of $\left(\mathrm{K}_t, \mathrm{L}_t\right)$. Two distinct inter-temporal transfer institutions with a distinct interpretations for the parameter B, can then be considered on top of productive capital: either fiat money in the line of Samuelson (1958), or public debt following the work of Diamond (1965), both giving rise to a reduced form (2.7a)–(2.7d). From the former perspective, assume that *two* assets are available: the productive capital, that is remunerated at rate $\mathscr{R}$, plus outside money with unitary price Q. The maximization problem of the young would then have a solution if and only if the no-arbitrage condition between the two assets holds, i.e., $\mathrm{Q}_{t+1}/\mathrm{Q}_t = \mathscr{R}_{t+1}$. The capital-money portfolio choice being then indeterminate, $x_{t+1}^t \equiv \mathrm{K}_{t+1} + \mathrm{M}_{t+1}\mathrm{Q}_t$. Assuming that the stock of money is in constant supply M, the money market clearing condition writes $\mathrm{M}_t = \mathrm{M}$ for all $t \geq 0$. In this monetary interpretation, $\mathrm{B}_t \equiv \mathrm{MQ}_t$ and the no-arbitage equation (2.7d) is thus determining the equilibrium price of money. From the second perspective, assume that the government has issued at date t a debt G_t to the younger generation. This debt has a one-period maturity and will be repaid with interest at the same rate on return on capital. At date $t+1$, the debt burden is $\mathscr{R}_{t+1}\mathrm{G}_t$. Provided that the policy followed is to maintain a constant zero deficit, the government budget constraint then implies $\mathrm{G}_{t+1} = \mathscr{R}_{t+1}\mathrm{G}_t$. The equilibrium savings willingness of the young must adequate the demand of capital from firms and the debt issued by the government, hence $x_{t+1}^t = \mathrm{K}_{t+1} + \mathrm{G}_t$. For this public debt interpretation, $\mathrm{B}_t \equiv \mathrm{G}_t$ and Eq. (2.7d) hence gives the equilibrium value of the public debt.

2.3.1 The Golden Rule Steady State: Existence and Normalisation

Under the previous assumptions, the system (2.7a)–(2.7d) defines an implicit three-dimensional dynamical system $\left(\mathrm{K}_{t+1}, \mathrm{L}_{t+1}, \mathrm{B}_{t+1}\right)' = \Upsilon\left(\mathrm{K}_t, \mathrm{L}_t, \mathrm{B}_t\right)$. A steady state is a fixed point of the map $\Upsilon(\cdot)$, i.e., a triple $\left(\mathrm{K}^\star, \mathrm{L}^\star, \mathrm{B}^\star\right)$ such that $\left(\mathrm{K}^\star, \mathrm{L}^\star, \mathrm{B}^\star\right)' = \Upsilon\left(\mathrm{K}^\star, \mathrm{L}^\star, \mathrm{B}^\star\right)$. The economy under study possesses two steady states, namely the wealth-capital or balanced steady state in which $\mathrm{B}^\star = 0$ and the *Golden Rule*—its understanding is detailed below—in which $\mathrm{B}^\star \neq 0$. In the sequel, the focus will be exclusively upon the latter. Formally, the *Golden Rule* is a triple $(\mathrm{K}^\star, \mathrm{L}^\star, \mathrm{B}^\star) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R} \setminus \{0\}$ such that:

$$\gamma_\ell \frac{\partial \mathrm{V}}{\partial \ell}\left(\mathrm{L}^\star\right) = \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}\left(\mathrm{K}^\star, \mathrm{L}^\star\right)\frac{\partial \mathrm{U}_2}{\partial c'}\left[\mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}\left(\mathrm{K}^\star, \mathrm{L}^\star\right)\mathrm{L}^\star - c^\star\right], \tag{2.8a}$$

$$\gamma_\ell \frac{\partial \mathrm{V}}{\partial \ell}\left(\mathrm{L}^\star\right) = \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}\left(\mathrm{K}^\star, \mathrm{L}^\star\right)\gamma_c \frac{\partial \mathrm{U}_1}{\partial c}\left(c^\star\right), \tag{2.8b}$$

$$\mathrm{K}^\star + \mathrm{B}^\star + c^\star = \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}\left(\mathrm{K}^\star, \mathrm{L}^\star\right)\mathrm{L}^\star, \tag{2.8c}$$

$$1 = \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{K}}\left(\mathrm{K}^\star, \mathrm{L}^\star\right). \tag{2.8d}$$

Noticing that (2.8a), (2.8b) and (2.8d) are the FOC of a constrained stationary second-best program:

$$\max_{\{c, c', \mathrm{L}, \mathrm{K}\}} \gamma_c \mathrm{U}_1(c) + \mathrm{U}_2(c') - \gamma_\ell \mathrm{V}(\mathrm{L}) \quad \text{subject to} \quad c + c' \leq \mathrm{AF}(\mathrm{K}, \mathrm{L}) - \mathrm{K},$$

the above defined competitive equilibrium steady state is then efficient, i.e., it coincides with the Golden Rule. The quantity of money or the government debt required to sustain the Golden Rule is given by (2.8c). Following the terminology coined by Gale (1973), an economy in which this quantity is negative, respectively positive, is termed *Classical*, respectively *Samuelson*.[4] Currently, the fact that $\mathrm{B} < 0$ means that the sum of the young agents' assets, namely their savings, is smaller that the total amount of capital; sustaining the Golden Rule stock of capital hence requires transfers towards the young agents. Oppositely, for $\mathrm{B} > 0$, the sustainment of the Golden Rule stock of capital requires transfers from the young.

In order to simplify the analysis, and making use of the scaling parameters, conditions for the existence of a *normalised* Golden Rule will be explicitly detailed. Let then

$$\alpha_c\left(\mathrm{K}^\star, \mathrm{L}^\star, c^\star\right) := c^\star \Big/ \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{L}}\left(\mathrm{K}^\star, \mathrm{L}^\star\right)\mathrm{L}^\star,$$

$$s\left(\mathrm{K}^\star, \mathrm{L}^\star\right) := \mathrm{A}\frac{\partial \mathrm{F}}{\partial \mathrm{K}}\left(\mathrm{K}^\star, \mathrm{L}^\star\right)\mathrm{K}^\star \Big/ \mathrm{F}\left(\mathrm{K}^\star, \mathrm{L}^\star\right),$$

respectively denote the share of first-period consumption in wage income and the share of capital in output, both being evaluated at the steady state and henceforward compactly referred to as α_c and s.

[4]See, e.g., the enlightening discussion in Weil (2008).

Now, fix arbitrarily a vector $(\mathrm{K}^\star, \mathrm{L}^\star, c^\star) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*$. Solving the Eqs. (2.8a), (2.8b) and (2.8d) in $\big(\gamma_c, \gamma_\ell, \mathrm{A}\big)$, it is obtained that:

$$\gamma_c^\star = \frac{\partial \mathrm{U}_2}{\partial c'}\left[\frac{(1-s)\mathrm{K}^\star}{s} - c^\star\right] \Big/ \frac{\partial \mathrm{U}_1}{\partial c}\big(c^\star\big), \tag{2.9a}$$

$$\gamma_\ell^\star = (1-s)\mathrm{K}^\star \frac{\partial \mathrm{U}_2}{\partial c'}\left[\frac{(1-s)\mathrm{K}^\star}{s} - c^\star\right] \Big/ \mathrm{L}^\star \frac{\partial V}{\partial L}\big(\mathrm{L}^\star\big)s, \tag{2.9b}$$

$$\mathrm{A}^\star = 1 \Big/ \frac{\partial \mathrm{F}}{\partial \mathrm{K}}\big(\mathrm{K}^\star, \mathrm{L}^\star\big). \tag{2.9c}$$

The steady state value $\mathrm{B}^\star$ then follows from (2.8c):

$$\mathrm{B}^\star = \left[\frac{1-s}{s}\big(1-\alpha_c\big) - 1\right]\mathrm{K}^\star.$$

Aside from $\mathrm{B}^\star$, whose sign is unrestricted, and under the earlier assumptions on preferences, $\gamma_c^\star$, $\gamma_\ell^\star$ and $\mathrm{A}^\star$ are unambiguously positive. It follows that a unique restriction is to be imposed on the arbitrary choice of $\big(\mathrm{K}^\star, \mathrm{L}^\star, c^\star\big) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*$, in order to ensure the existence of the normalized steady state, namely: $c'^\star = (1-s)\mathrm{K}^\star/s - c^\star > 0$. Note however that the latter is equivalent to the holding of the restriction $\alpha_c < 1$ that will be hereafter assumed to prevail. To sum up, choose arbitrarily $(\mathrm{K}^\star, \mathrm{L}^\star, c^\star) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*$ such that $\alpha_c \in [0, 1[$. Let $\big(\gamma_c, \gamma_\ell, \mathrm{A}\big) = \big(\gamma_c^\star, \gamma_\ell^\star, \mathrm{A}^\star\big)$. By construction, $\big(\mathrm{K}^\star, \mathrm{L}^\star, \mathrm{B}^\star\big) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}^*$ is a Golden Rule steady state. In the sequel, the local dynamics will be characterised in the neighbourhood of the normalised steady state.

2.3.2 Some Parameterised Curves

The coefficients of the characteristic polynomial list, letting $\mathrm{V}(\ell) = \ell$, along:

$$\mathscr{T} = 1 + \frac{1-s}{s} + \frac{s}{1-s} + (1-\mathscr{D})\frac{s}{1-s}\frac{1}{1-\alpha_c}\left[\frac{\big(1+\eta_c\big)\alpha_c}{\eta_c} - \frac{\varsigma}{s}\right], \tag{2.10a}$$

$$\mathscr{M} = \mathscr{T} - (1-\mathscr{D})\left(\frac{1}{s} - \frac{1}{1-\alpha_c}\right), \tag{2.10b}$$

$$\mathscr{D} = \frac{1}{1+\eta_{c'}}. \tag{2.10c}$$

for

$$\varsigma := \frac{\partial \mathrm{F}}{\partial \mathrm{K}} \cdot \frac{\partial \mathrm{F}}{\partial \mathrm{L}} \Big/ \mathrm{F} \cdot \frac{\partial^2 \mathrm{F}}{\partial \mathrm{K} \partial \mathrm{L}}, \quad 1 - s := \frac{\partial \mathrm{F}}{\partial \mathrm{L}} \cdot \mathrm{L} \Big/ \mathrm{F}, \; s := \frac{\partial \mathrm{F}}{\partial \mathrm{K}} \cdot \mathrm{K} \Big/ \mathrm{F},$$
$$\eta_c := c_t^t \frac{\partial^2 \mathrm{U}_1}{\partial \left(c_t^t\right)^2} \Big/ \frac{\partial \mathrm{U}_1}{\partial c_t^t}, \quad \eta_{c'} := c_{t+1}^t \frac{\partial^2 \mathrm{U}_2}{\partial \left(c_{t+1}^t\right)^2} \Big/ \frac{\partial \mathrm{U}_2}{\partial c_{t+1}^t},$$

that are evaluated at $\left(\mathrm{K}^\star, \mathrm{L}^\star, c^\star\right)$ and where ς denotes the elasticity of substitution between the productive factors. While concavity assumptions ensure that $\eta_c < 0$ and $\eta_{c'} < 0$, gross substitutability properties would correspond to $1 + \eta_c > 0$ and $1 + \eta_{c'} > 0$. It is also worth emphasising the gross substitutability on second-period consumption $1 + \eta_{c'} > 0$ translates as $\mathscr{D} > 1$ whereas its violation would result in $\mathscr{D} < 0$. Let further, and for convenience, $1/\eta_1 := \eta_c$ $1/\eta_2 := \eta_{c'}$. In a more concise form, the coefficients of the characteristic polynomial may be understood as a triple of functions of structural parameters, namely

$$\left\{\mathscr{T}\left(\eta_1, \eta_2, \varsigma, \alpha_c, s\right), \mathscr{M}\left(\eta_1, \eta_2, \varsigma, \alpha_c, s\right), \mathscr{D}\left(\eta_2\right)\right\},$$

These functions can be seen as *parametric equations* for curves in the $(\mathscr{T}\mathscr{M}\mathscr{D})$-space. It is then noticed that $\mathscr{D}$ does neither depend upon the parameters describing the technology, namely s and ς, nor on the ones that relate to first-period consumption, namely α_c and η_1. The current approach being based upon diagrams over the planes $(\mathscr{T}\mathscr{M})_{\mathscr{D}}$, the following parameterised curve is *generated* by the variations of η_1 for fixed $\left(\cdot, \eta_2, \varsigma, \alpha_c, s\right)$ and hence leaves unaffected the coefficient $\mathscr{D}$:

$${}_{\eta_1}\Delta := \left\{\left(\mathscr{T}\left(\eta_1, \eta_2, \varsigma, \alpha_c, s\right), \mathscr{M}\left(\eta_1, \eta_2, \varsigma, \alpha_c, s, \right), \mathscr{D}\left(\eta_2\right)\right) : \eta_1 \in] - \infty, 0[\right\}. \tag{2.11}$$

From (2.10b), it is observed that ${}_{\eta_1}\Delta$ is a half-line starting from ${}_0\Delta$ that is parallel to $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$ and whose direction vector is available from:

$$\mathscr{T}' = \mathscr{M}' = (1 - \mathscr{D}) \frac{s}{1 - s} \frac{\alpha_c}{1 - \alpha_c} \gtrless 0 \quad \text{for} \quad 1 - \mathscr{D} \gtrless 0. \tag{2.12}$$

It is further noticed that the position of this straight-line with respect to the locus $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$ is ruled by the sign of:

$$(1 - \mathscr{D}) \frac{1}{1 - \alpha_c} \left[\frac{1 - s}{s}\left(1 - \alpha_c\right) - 1\right]. \tag{2.13}$$

One is then to undertake a sensitivity analysis on ${}_0\Delta$ and as the share of first period consumption α_c spans its interval $[0, 1[$, hence the locus:

$$
\begin{aligned}
&{}_{\alpha_c}\Lambda_{\eta_1=0} := \Big\{\Big(\mathscr{T}\big(0,\eta_2,\varsigma,\alpha_c,s\big),\mathscr{M}\big(0,\eta_2,\varsigma,\alpha_c,s\big),\mathscr{D}\big(\eta_2\big)\Big) : \alpha_c \in [0,1[\Big\},\\
&\text{for} \quad \mathscr{T}\big(0,\eta_2,\varsigma,\alpha_c,s\big) = 1 + \frac{1-s}{s} + \frac{s}{1-s} + (1-\mathscr{D})\frac{s}{1-s}\frac{1}{1-\alpha_c}\Big(\alpha_c - \frac{\varsigma}{s}\Big),\\
&\qquad \mathscr{M}\big(0,\eta_2,\varsigma,\alpha_c,s\big) = \mathscr{T}\big(0,\varsigma,\alpha_c,s,\eta_2\big) - (1-\mathscr{D})\Big(\frac{1}{s} - \frac{1}{1-\alpha_c}\Big).
\end{aligned}
$$

It is further established in Appendix A that this locus does in turn correspond to a half-line that starts from $\big(\mathscr{T}\big(0,\eta_2,\varsigma,0,s\big),\mathscr{M}\big(0,\eta_2,\varsigma,0,s\big)\big)$ and assumes a slope of $(1-\varsigma)/(s-\varsigma)$.

It is finally also appropriate to first clarify how the half-line points ${}_{\alpha_c}\Lambda_{\eta_1=0}$ moves while the parameter ς spans its interval $[0,+\infty]$. This ${}_{\varsigma}\Lambda_{\eta_1=0,\alpha_c=0}$ locus is formally defined as follows:

$$
\begin{aligned}
&{}_{\varsigma}\Lambda_{\eta_1=0,\alpha_c=0} := \Big\{\Big(\mathscr{T}\big(0,\eta_2,\varsigma,0,s\big),\mathscr{M}\big(0,\eta_2,\varsigma,0,s\big),\mathscr{D}\big(\eta_2\big)\Big) : \varsigma \in]0,+\infty[\Big\},\\
&\text{for} \quad \mathscr{T}\big(0,\eta_2,\varsigma,0,s\big) = 1 + \frac{1-s}{s} + \frac{s}{1-s} + (1-\mathscr{D})\frac{s}{1-s}\Big(-\frac{\varsigma}{s}\Big),\\
&\qquad \mathscr{M}\big(0,\eta_2,\varsigma,0,s\big) = \mathscr{T}\big(0,\eta_2,\varsigma,0,s\big) - (1-\mathscr{D})\frac{1}{s}.
\end{aligned}
$$

This, once again, results in the obtention of a half-line starting from $\big(\mathscr{T}\big(0,\eta_2,0,0,s\big),\mathscr{M}\big(0,\eta_2,0,0,s\big)\big)$ whose properties are detailed in Appendix B.

The diagrams on Figs. 2.7 and 2.8 picture the three half-lines ${}_{\eta_1}\Delta$, ${}_{\alpha_c}\Lambda_{\eta_1=0}$ and ${}_{\varsigma}\Lambda_{\eta_1=0,\alpha_c=0}$. They clarify their dependency with respect to the admissible values of $\mathscr{D}$. They allow at a glance to picture the dependency of ${}_{\eta_1}\Delta$ with respect to ς and α_c. Letting ς be fixed sums up to select a point of a point of ${}_{\varsigma}\Lambda_{\eta_1=0,\alpha_c=0}$, that will in its turn gives rise to a specific half-line ${}_{\alpha_c}\Lambda_{\eta_1=0}$. Subsequently selecting a value of α_c, i.e., a point of ${}_{\alpha_c}\Lambda_{\eta_1=0}$, eventually defines the starting point of a *particular* set of economies, i.e., the ${}_{\eta_1}\Delta$ associated to these given values of α_c and ς. Diagrams such as Figs. 2.7 and 2.8 hence allow for contemplating the whole set of such economies for the range of admissible values of α_c and ς: as a simple illustration and for $\mathscr{D} > 1$, a larger substitutability between the factors would, e.g., uniformly translate into a north-east move for the economies.

It is finally of interest to introduce a last, economically relevant, locus that separates, in the course of the line ${}_{\eta_1}\Delta$, the areas where gross substitutability prevails between first-period consumption and leisure from the ones where it is gross complementarity that prevails. As for the origin of the parameterized line ${}_{\alpha_c}\Delta_{\eta_1=0}$ and from Appendix C, the border between these two areas emerges as being described by the curve $\big(\mathscr{T}\big(-1,\eta_2,\varsigma,\alpha_c,s\big),\mathscr{M}\big(-1,\eta_2,\varsigma,\alpha_c,s\big)\big)$ as α_c spans its interval $[0,1[$, namely:

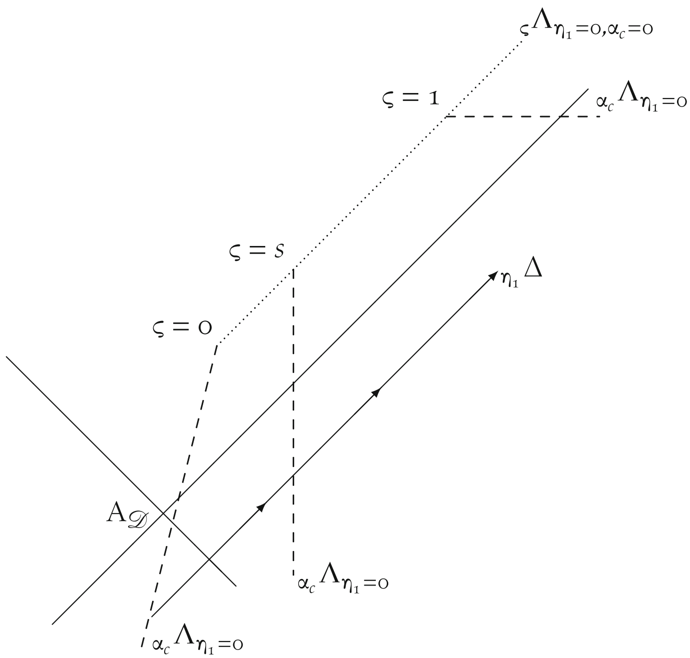

Fig. 2.7 The loci ${}_{\eta_1}\Delta$, ${}_{\alpha_c}\Lambda_{\eta_1=0}$ and ${}_{\varsigma}\Lambda_{\eta_1=0,\alpha_c=0}$ for $\mathscr{D} > 1$

$$
{}_{\alpha_c}\Lambda_{\eta_1=-1} := \Big\{ \Big(\mathscr{T}\big(-1, \eta_2, \varsigma, \alpha_c, s\big), \mathscr{M}\big(-1, \eta_2, \varsigma, \alpha_c, s\big), \mathscr{D}\big(\eta_2\big) : \alpha_c \in \; [0, 1[\Big\},
$$

$$
\text{for} \quad \mathscr{T}\big(-1, \eta_2, \varsigma, \alpha_c, s\big) = 1 + \frac{1-s}{s} + \frac{s}{1-s} - (1-\mathscr{D})\frac{1-s}{s}\frac{1}{1-\alpha_c}\frac{\varsigma}{s},
$$

$$
\mathscr{M}\big(-1, \eta_2, \varsigma, \alpha_c, s\big) = \mathscr{T}\big(-1, \eta_2, \varsigma, \alpha_c, s\big) - (1-\mathscr{D})\left(\frac{1}{s} - \frac{1}{1-\alpha_c}\right).
$$

As this is formally established in Appendix C and pictured on Figs. 2.9 and 2.10, the loci ${}_{\alpha_c}\Lambda_{\eta_1=0}$ and ${}_{\alpha_c}\Lambda_{\eta_1=-1}$ share a common origin for $\alpha_c = 0$, that further defines an infinite cone with an apex at that point and two generatrices that correspond to the two loci. The interior of a cone corresponds to an area where gross complementarity prevails between first-period consumption and leisure. In opposition to this, the gross substitutability property prevails beyond the locus ${}_{\alpha_c}\Lambda_{\eta_1=-1}$. Consider then an economy starting from given point on ${}_{\alpha_c}\Lambda_{\eta_1=0}$: while, by construction, it is first characterised by gross complementarity, as soon as it crosses ${}_{\alpha_c}\Lambda_{\eta_1=-1}$, it falls into an area where gross substitutability is recovered.

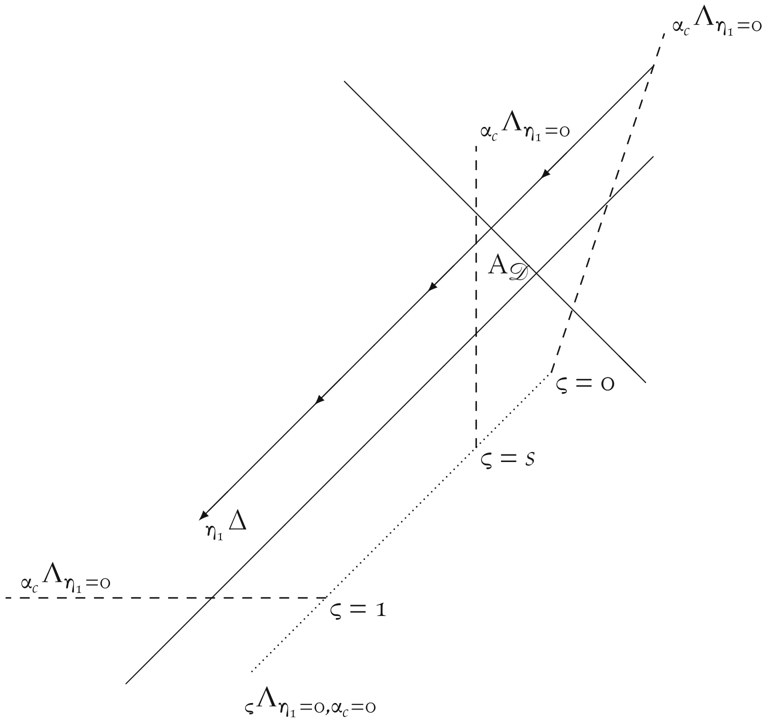

Fig. 2.8 The loci $_{\eta_1}\Delta$, $_{\alpha_c}\Lambda_{\eta_1=0}$ and $_{\varsigma}\Lambda_{\eta_1=0,\alpha_c=0}$ for $\mathscr{D} < 0$

2.3.3 A Local Stability Analysis

Taking advantage of the previous constructions, a global qualitative picture becomes available from the mere localisation of the apex of the cone when it is defined for $\varsigma = 0$. Its position with respect to the critical lines $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$ and $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\right)$ in turn derives from:

$$-1 + \mathscr{T}\big(0, \eta_2, 0, 0, s\big) - \mathscr{M}\big(0, \eta_2, 0, 0, s\big) + \mathscr{D} = (1 - \mathscr{D})\left(\frac{1-s}{s} - 1\right), \tag{2.14a}$$

$$1 + \mathscr{T}\big(0, \eta_2, 0, 0, s\big) + \mathscr{M}\big(0, \eta_2, 0, 0, s\big) + \mathscr{D} = (1 + \mathscr{D})\frac{1}{s} + 2\frac{1}{1-s}. \tag{2.14b}$$

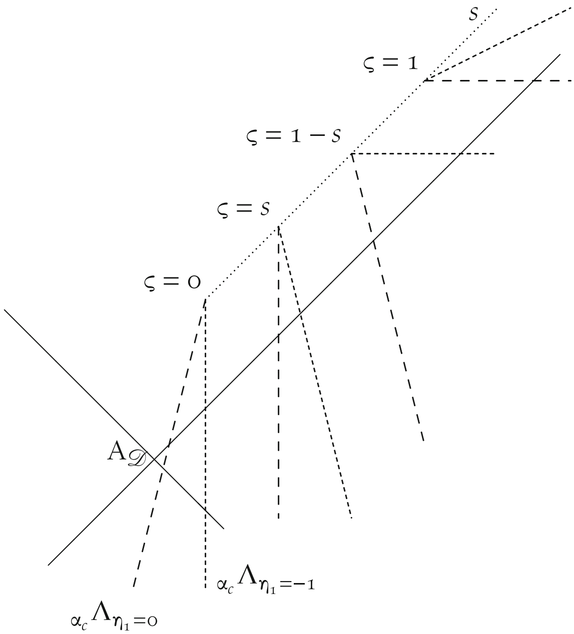

Fig. 2.9 The cones for $\mathscr{D} > 1$

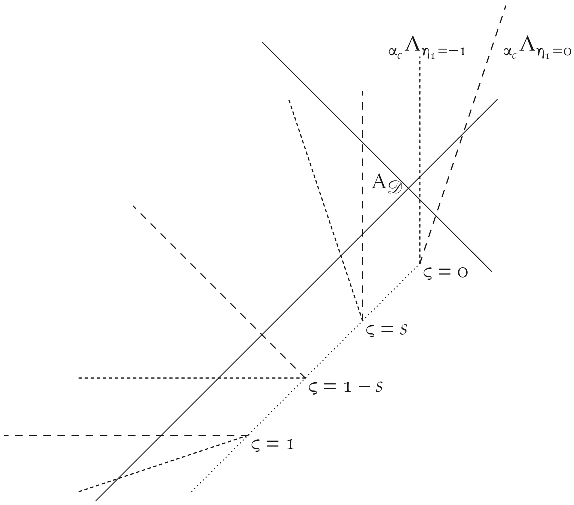

Fig. 2.10 The cones for $\mathscr{D} < 0$

2.3.3.1 Gross Substitutability Between Second-Period Consumption and Leisure ($\mathscr{D} > 1$)

Under a gross substitutability assumption, $\mathscr{D} > 1$, whence, from (2.14a), $\mathrm{Q}(+1) < 0$, and, from (2.14b) and for $s < 1/2$, $\mathrm{Q}(-1) > 0$. Otherwise stated, the apex of the cone for $\varsigma = 0$ is located above both $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$ and $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\right)$ on Fig. 2.11. From

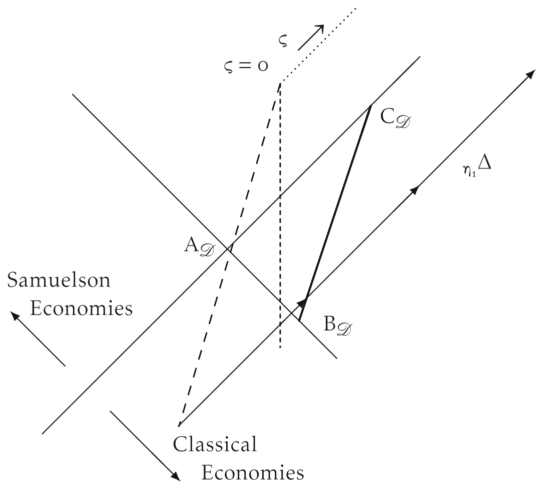

Fig. 2.11 The ${}_{\eta_1}\Delta$ line for $\mathscr{D} > 1$

Appendices A and C, the direction vectors of the generatrices of the cone are of negative sign: the generatrices ${}_{\alpha_c}\Lambda_{\eta_1=0}$ and ${}_{\alpha_c}\Lambda_{\eta_1=-1}$ are then to cross $\big(A_{\mathscr{D}}C_{\mathscr{D}}\big)$. It however remains to check on which side of $A_{\mathscr{D}}$ this is to occur.

For that purpose, observe from (2.13) that, if $\alpha_c \in [0, 1[$ is such that $1/(1-\alpha_c) = (1-s)/s$ and the pair $\big(\mathscr{T}\big(0, \eta_2, 0, \alpha_c, s\big), \mathscr{M}\big(0, \eta_2, 0, \alpha_c, s\big)\big)$ is on $\big(A_{\mathscr{D}}C_{\mathscr{D}}\big)$ that currently corresponds to the borderline between classical and Samuelson economies, then

$$\mathscr{T}\big(0, \eta_2, 0, \alpha_c, s\big) = 1 + \frac{1-s}{s} + \frac{s}{1-s} + (1-\mathscr{D})\left[1 - \frac{s}{1-s}\right].$$

Recalling that, from (2.5a), $\mathscr{T}_{A_{\mathscr{D}}} = -\mathscr{D}$, it is derived that:

$$\mathscr{T}\big(0, \eta_2, 0, \alpha_c, s\big) - \mathscr{T}_{A_{\mathscr{D}}} = 1 + \frac{1-s}{s} + 1 + \mathscr{D}\frac{s}{1-s} > 0,$$

that eventually implies that the generatrices ${}_{\alpha_c}\Lambda_{\eta_1=0}$ and ${}_{\alpha_c}\Lambda_{\eta_1=-1}$ intersect $\big(A_{\mathscr{D}}C_{\mathscr{D}}\big)$ on the right hand side of $A_{\mathscr{D}}$.

In the same vein, while it is known that the generatrices are to intersect $\big(A_{\mathscr{D}}B_{\mathscr{D}}\big)$, it remains to check on which side of $B_{\mathscr{D}}$ this is to occur. To get some insight about it, consider now

$$\mathscr{T}\big(-1, \eta_2, 0, 0, s\big) = 1 + \frac{1-s}{s} + \frac{s}{1-s}$$

and compare this with $\mathscr{T}_{B_{\mathscr{D}}} = -2 + \mathscr{D}$ that derives from Eq. (2.5b). It is obtained that:

$$\mathscr{T}\big(-1, \eta_2, 0, 0, s\big) - \mathscr{T}_{B_{\mathscr{D}}} = 1 + \frac{1-s}{s} + \frac{s}{1-s} + 2 - \mathscr{D}.$$

Figure 2.11 depicts a configuration with sufficiently large values of $\mathscr{D}$ for which $\mathscr{T}\left(-1, \eta_2, 0, 0, s\right) < \mathscr{T}_{\mathrm{B}_{\mathscr{D}}}$. In such a case, the curve ${}_{\eta_1}\Delta$ will be associated with a succession of flip and Poincaré-Hopf bifurcations that will both take place below the line $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$, i.e., from (2.13), for classical economies. Interestingly, both bifurcation phenomena take place in an area with a gross substitutability property between first-period consumption and leisure. It is however to be recalled that these conclusions hold for $\varsigma = 0$, that is for fixed coefficients Leontief-type technologies. Letting ς undergo positive values and from Fig. 2.9, the apexes of the cones are to move north-east along the dotted line while the generatrices are to follow a counter clockwise translation. The scope for bifurcations is then first to shrink and then to disappear with an increased substitutability between the productive factors.

Focusing then on the uniqueness issue and taking advantage of the cardinality of the stable eigenvalues available from Fig. 2.4, Samuelson economies will unambiguously be associated with a unique modulus inside the unit circle and a local uniqueness property. Scenarios for Classical economies are in their turn conditional to both α_c and ς, their key-feature being that locally indeterminate steady states are admissible under a gross substitutability property between leisure and first-period consumption.

To put these results into the perspective of the endogenous fluctuations literature, an inter-temporal consumption arbitrage on top of an inter-temporal consumption-leisure arbitrage has been proved to generate a new *degree* of instability for Classical economies. In spite of a gross substitutability assumption on preferences, flip cycles are indeed allowed for sufficiently large values of the share of first-period consumption. With that regard, it is worth recalling that for a formulation without first-period consumption, savings must be a decreasing function of the interest rate in order for flip cycles to exist—*vide* the extensive discussion in Benhabib and Laroque (1988, Proposition III.1, p.154). The retainment of a gross substitutability assumption on preferences would have then uniformly ruled out any area for flip cycles. As for the Poincaré-Hopf bifurcation, though it is uniformly precluded for Samuelson economies, it reveals as a robust phenomenon in Classical economies.

2.3.3.2 Gross Complementarity Between Second-Period Consumption and Leisure ($\mathscr{D} < 0$)

Under a gross complementarity assumption, $\mathscr{D} < 0$: from (2.14a), $\mathrm{Q}(+1) > 0$. From (2.14b), the sign of $\mathrm{Q}(-1)$ remains ambiguous. Henceforward focusing on the most interesting case with $\mathscr{D} > -1$—the interior of the triangle $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$ is now associated with a strong indeterminacy configuration three moduli inside the unit circle, $\mathrm{Q}(-1) > 0$ and the apex of the cone is located below $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$ but above $\left(\mathrm{A}_{\mathscr{D}}\mathrm{B}_{\mathscr{D}}\right)$ on Fig. 2.12. From Appendices A and C, the direction vectors of the generatrices are of positive sign and they will cross $\left(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\right)$ from below. In addition to this, as $\mathscr{T}\left(-1, \eta_2, 0, 0, s\right) > \mathscr{T}_{\mathrm{C}_{\mathscr{D}}}$, these crossings will occur on the right hand side of $\mathrm{C}_{\mathscr{D}}$. This configuration with strong indeterminacies is represented on Fig. 2.12.

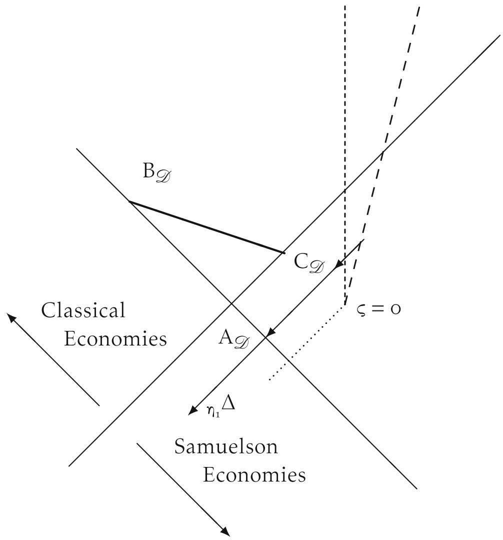

Fig. 2.12 The ${}_{\eta_1}\Delta$ approach for $\mathscr{D} \in]-1, 0[$

To sum up, the allowance for gross complementaries between leisure and second-period consumption had dramatic implications since Samuelson economies, even though they kept on being associated with an gross substitutability between leisure and first-period consumption, were no longer immune to local indeterminacies and the occurrence of a flip bifurcation. Classical economies were then still more prone to multiplicities with configurations with *two* degrees of indeterminacy, the latters being further potentially available under a gross complementarity between leisure and second-period consumption. Extrapolating the south west move of the cone along Fig. 2.10 and as ς undergoes positive values, the generatrices are to follow a counter-clockwise translation. This strong indeterminacy phenomenon and the scope for bifurcations are to disappear.

Appendix A: The Origin of ${}_{\eta_1}\Delta$

The starting point of ${}_{\eta_1}\Delta$ is defined from the following set parameterised by α_c:

$$\Big\{\Big(\mathscr{T}\big(0,\eta_2,\varsigma,\alpha_c,s\big),\mathscr{M}\big(0,\eta_2,\varsigma,\alpha_c,s\big),\mathscr{D}\big(\eta_2\big)\Big) : \alpha_c \in [0,1[\Big\},$$

$$\text{for}\quad \mathscr{T}\big(0,\eta_2,\varsigma,\alpha_c,s\big) = 1 + \frac{1-s}{s} + \frac{s}{1-s} + (1-\mathscr{D})\frac{s}{1-s}\frac{1}{1-\alpha_c}\left(\alpha_c - \frac{\varsigma}{s}\right),$$

$$\mathscr{M}\big(0,\eta_2,\varsigma,\alpha_c,s\big) = \mathscr{T}\big(0,\eta_2,\varsigma,\alpha_c,s\big) - (1-\mathscr{D})\left(\frac{1}{s} - \frac{1}{1-\alpha_c}\right).$$

Letting $\mathscr{T}|_{\eta_1=0} := \mathscr{T}\left(0, \eta_2, \varsigma, \alpha_c, s\right)$ and $\mathscr{M}|_{\eta_1=0} := \mathscr{M}\left(0, \eta_2, \varsigma, \alpha_c, s\right)$ and from their above expressions, their dependency with respect to α_c are available as:

$$\left(\mathscr{T}|_{\eta_1=0}\right)'_{\alpha_c} = (1-\mathscr{D})\frac{s}{1-s}\frac{1}{\left(1-\alpha_c\right)^2}\left(1-\frac{\varsigma}{s}\right) \gtrless 0,$$
$$\left(\mathscr{M}|_{\eta_1=0}\right)'_{\alpha_c} = (1-\mathscr{D})\frac{1}{\left(1-\alpha_c\right)^2}\frac{1}{1-s}\left(1-\varsigma\right) \gtrless 0,$$

whence a slope available as

$$_{\alpha_c}\Lambda'_{\eta_1=0} = \frac{\left(\mathscr{M}|_{\eta_1=0}\right)'_{\alpha_c}}{\left(\mathscr{T}|_{\eta_1=0}\right)'_{\alpha_c}} = \frac{1-\varsigma}{s-\varsigma}.$$

This again indicates that $_{\alpha_c}\Lambda_{\eta_1=0}$ depicts a straight-line, its origin being derived by letting $\alpha_c = 0$:

$$\mathscr{T}|_{\eta_1=0,\alpha_c=0} = 1 + \frac{1-s}{s} + \frac{s}{(1-s)} - (1-\mathscr{D})\frac{\varsigma}{1-s},$$
$$\mathscr{M}|_{\eta_1=0,\alpha_c=0} = \mathscr{T}|_{\eta_1=0,\alpha_c=0} - (1-\mathscr{D})\frac{1-s}{s}.$$

Appendix B: A ς-Sensitivity Analysis

The origin is in its turn fully described by letting ς span its interval $[0, +\infty]$ through $_{\varsigma}\Lambda_{\eta_0=0,\alpha_c=0}$. As it is readily checked that both $\mathscr{T}|_{\eta_1=0,\alpha_c=0}$ and $\mathscr{M}|_{\eta_1=0,\alpha_c=0}$ increase as functions of ς for $\mathscr{D} > 1$ but decrease as functions of ς for $\mathscr{D} < 0$:

$$\left(\mathscr{T}|_{\eta_1=0,\alpha_c=0}\right)'_{\varsigma} = -(1-\mathscr{D})\frac{1}{1-s},$$
$$\left(\mathscr{M}|_{\eta_1=0,\alpha_c=0}\right)'_{\varsigma} = -(1-\mathscr{D})\frac{1}{1-s},$$

whence a slope available as

$$\begin{aligned} _{\varsigma}\Lambda'_{\eta_1=0,\alpha_c=0} &= \frac{\left(\mathscr{M}|_{\eta_1=0,\alpha_c=0}\right)'_{\varsigma}}{\left(\mathscr{T}|_{\eta_1=0,\alpha_c=0}\right)'_{\varsigma}} \\ &= 1, \end{aligned}$$

this latter curve being parallel to $\big(\mathrm{A}_{\mathscr{D}}\mathrm{C}_{\mathscr{D}}\big)$, its origin ${}_0\Lambda_{\eta_0=0,\alpha_c=0}$ being given by:

$$\mathscr{T}|^{\varsigma=0}_{\eta_1=0,\alpha_c=0} = 1 + \frac{1-s}{s} + \frac{s}{1-s},$$
$$\mathscr{M}|^{\varsigma=0}_{\eta_1=0,\alpha_c=0} = \mathscr{T}|^{\varsigma=0}_{\eta_1=0,\alpha_c=0} - (1-\mathscr{D})\frac{1-s}{s}.$$

Appendix C: The Border Between Gross Substitutability and Gross Complementarity

The gross substitutability and gross complementarity zones are separated by the following set:

$$\Big\{\Big(\mathscr{T}\big(-1,\eta_2,\varsigma,\alpha_c,s\big),\mathscr{M}\big(-1,\eta_2,\varsigma,\alpha_c,s\big),\mathscr{D}\big(\eta_2\big)\Big):\alpha_c\in\ [0,1[\Big\},$$

for $$\mathscr{T}\big(-1,\eta_2,\varsigma,\alpha_c,s\big) = 1 + \frac{1-s}{s} + \frac{s}{1-s} - (1-\mathscr{D})\frac{1-s}{s}\frac{1}{1-\alpha_c}\frac{\varsigma}{s},$$
$$\mathscr{M}\big(-1,\eta_2,\varsigma,\alpha_c,s\big) = \mathscr{T}\big(-1,\eta_2,\varsigma,\alpha_c,s\big) - (1-\mathscr{D})\bigg(\frac{1}{s}-\frac{1}{1-\alpha_c}\bigg).$$

Letting $\mathscr{T}|_{\eta_1=-1} := \mathscr{T}\big(-1,\eta_2,\varsigma,\alpha_c,s\big)$ and $\mathscr{M}|_{\eta_1=-1} := \mathscr{M}\big(-1,\eta_2,\varsigma,\alpha_c,s\big)$ and from their above expressions, their dependency with respect to α_c is in turn available from:

$$\big(\mathscr{T}|_{\eta_1=-1}\big)'_{\alpha_c} = -(1-\mathscr{D})\frac{\varsigma}{1-s}\frac{1}{\big(1-\alpha_c\big)^2} \lesseqgtr 0 \quad \text{for} \quad \mathscr{D} \lesseqgtr 1,$$
$$\big(\mathscr{M}|_{\eta_1=-1}\big)'_{\alpha_c} = (1-\mathscr{D})\frac{1}{\big(1-\alpha_c\big)^2}\bigg(1-\frac{\varsigma}{1-s}\bigg) \gtreqless 0 \quad \text{for} \quad \mathscr{D} \lesseqgtr 1,$$

that in turn indicates a slope of

$$\frac{\big(\mathscr{M}|_{\eta_1=-1}\big)'_{\alpha_c}}{\big(\mathscr{T}|_{\eta_1=-1}\big)'_{\alpha_c}} = 1 - \frac{1-s}{\varsigma}$$

for the locus ${}_{\alpha_c}\Lambda_{\eta_1=-1}$, the latter being a straight-line with a nil slope for $\varsigma = 1-s$, that provides a first critical threshold value for ς. The coordinates of the origin of the latter derive by letting $\alpha_c = 0$:

$$\mathscr{T}|_{\eta_1=-1,\alpha_c=0} = 1 + \frac{1-s}{s} + \frac{s}{(1-s)} - (1-\mathscr{D})\frac{\varsigma}{1-s},$$
$$\mathscr{M}|_{\eta_1=-1,\alpha_c=0} = \mathscr{T}|_{\eta_1=-1,\alpha_c=0} - (1-\mathscr{D})\frac{1-s}{s}.$$

Interestingly, the origins of ${}_{\alpha_c}\Lambda_{\eta_1=-1}$ and of ${}_{\alpha_c}\Lambda_{\eta_1=0}$ coincide along ${}_0\Lambda_{\eta_1=-1} \equiv {}_0\Lambda_{\eta_1=0}$, this latter locus corresponding to the apex of the cone defined by the generatrices ${}_{\alpha_c}\Lambda_{\eta_1=-1}$ and ${}_{\alpha_c}\Lambda_{\eta_1=0}$ for a given ς.

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http://www.springer.com/978-3-319-44074-3

Sunspots and Non-Linear Dynamics
Essays in Honor of Jean-Michel Grandmont
Nishimura, K.; Venditti, A.; Yannelis, N.C. (Eds.)
2017, VI, 409 p. 62 illus., 28 illus. in color., Hardcover
ISBN: 978-3-319-44074-3