

Chapter 2

Problem Formulation and Tools

2.1 The Governing Equations

The flow of a Newtonian fluid, such as gaseous air, is governed by the Navier–Stokes equations to within sufficient accuracy (Batchelor 1967). These equations are widely accepted as the appropriate vehicle to study laminar and turbulent flows. In a rotating reference frame, a fictitious force, the Coriolis Force, appears to act on the fluid (Chemin et al. 2006). When combined with the conservation laws of energy and mass and an equation of state of the fluid, these equations form a closed coupled system of partial differential equations for three velocity components $(u_i)_{i=1,2,3}$, the pressure p , density ρ and temperature T (see for instance Batchelor 1967)

$$\frac{d\rho u_i}{dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\rho v \left(\tau_{ij} - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{i,j} \right) \right] + 2\rho \epsilon_{ijk} u_j \Omega_k - \rho g \delta_{i,3} \quad (2.1a)$$

$$\frac{d\rho(c_p - R_d)T}{dt} = + \frac{\partial}{\partial x_j} \left(v_H \rho \frac{\partial T}{\partial x_j} \right) + 2v\rho \left[\tau_{ij} \tau_{ij} - \frac{1}{3} \left(\frac{\partial u_i}{\partial x_i} \right)^2 \right] - p \frac{\partial u_i}{\partial x_i} \quad (2.1b)$$

$$\frac{1}{\rho} \frac{d\rho}{dt} = - \frac{\partial u_j}{\partial x_j} \quad (2.1c)$$

$$\frac{p}{\rho} = R_d T. \quad (2.1d)$$

Here, v is the kinematic viscosity of air, and v_H the diffusivity of heat, $(c_p - R_d) = c_v$ is the volumetric heat capacity of air, and

$$\tau_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the rate-of-strain tensor. The coordinate directions $i = 1, 2, 3$ are denoted as Ox, Oy and Oz, and Oz points in the vertical (wall-normal) direction. The velocity components are $u = u_1, v = u_2, w = u_3$.

Hydrostatic balance. In a geophysical context, it is common to use the concept of hydrostatic balance. Using the hydrostatic balance as a background state, density is decomposed into a reference value $\rho_0(z)$ and a fluctuation ρ' such that $\rho(x, y, z, t) = \rho_0 + \rho'(x, y, z, t)$. The density and pressure terms on the right-hand side (RHS) of Eq. (2.1a) for $i = 3$ can be written as

$$-\left(\frac{\partial p}{\partial z} + \rho g\right) = -\left(\frac{\partial p_h}{\partial z} + \rho_0 g + \frac{\partial p'}{\partial z} + \rho' g\right) = -\rho \left(\frac{1}{\rho} \frac{\partial p'}{\partial z} + \frac{\rho'}{\rho} g\right) \quad (2.2)$$

with $\frac{\partial p_h}{\partial z} = -\rho_0 g$,

where p_h is the hydrostatic pressure.

Potential temperature. The potential temperature θ is defined as

$$\theta := T \left(\frac{p_{ref}}{p} \right)^{R_d/c_p} \quad (2.3)$$

with $p_{ref} := 10^5$ Pa. The vertical gradient of θ , $\partial\theta/\partial z$, is a measure of stratification, and in a non-buoyant, i.e. neutrally stratified, atmosphere it is $\partial\theta/\partial z = 0$. θ is a measure of the entropy $s = c_p \ln \theta$, and when the energy Eq. (2.1b) is expressed in terms of θ , the changes of temperature due to density fluctuations $\partial_t(\rho R_d T) = \partial_t(p)$ are absorbed in the potential temperature.

Divergence-free constraint: incompressibility. A total change of pressure can be expressed in terms of potential temperature and density as

$$\frac{dp}{dt} = \frac{\partial p}{\partial \theta} \bigg|_{\rho} \frac{d\theta}{dt} + \frac{\partial p}{\partial \rho} \bigg|_{\theta} \frac{d\rho}{dt}, \quad (2.4)$$

with $\sqrt{(\partial p / \partial \rho)|_{\theta}} =: c$, the speed of sound. Hence, total changes in density ρ are expressed as

$$\frac{d\rho}{dt} = \frac{1}{c^2} \left(\frac{dp}{dt} - \frac{\partial p}{\partial \theta} \bigg|_{\rho} \frac{d\theta}{dt} \right). \quad (2.5)$$

If the speed of sound, c , is large in comparison to any velocity in the flow, changes in density become very small. The neglect of the first term is the *incompressibility assumption*; it implies the limit $c \rightarrow \infty$ and thus infinitely fast propagation of sound waves. Perturbations in the pressure field propagate at infinite speed. This instantaneous communication of pressure perturbations is manifest in a Helmholtz-type equation for the pressure that results if the divergence of Eq. (2.1a) is forced to zero (Sect. 3.1).

The neglect of the second term in Eq. (2.5) is related to density changes in response to internal dissipative heating and density changes in response to molecular conduction of heat (Batchelor (1967), p.170) which are known to be small under typical atmospheric conditions. Together with the neglect of the first term on the RHS of Eq. (2.5), this implies $d\rho/dt = 0$ which together with mass conservation requires that the momentum field has zero divergence, i.e. $\nabla \cdot \mathbf{u} = 0$.

In terms of non-dimensional parameters, all terms in Eq. (2.5) scale with the square of the Mach number

$$Ma = \frac{U_0}{(1/(1 - R_d/c_p)R_d T_0)^{1/2}} = \frac{U_0}{c}. \quad (2.6)$$

For consistency one also needs to neglect the anti-symmetric part of the second term on the RHS of Eq. (2.1b), the dissipative heating itself, which is also of the order of Ma^2 . A consequence of the divergence-free constraint, is that $d \ln p \ll d \ln \theta$, from which $\rho'/\rho \approx -\theta'/\theta$ follows.

Boussinesq approximation. If $\rho'/\rho \ll 1$, one can approximate $\rho = \rho_0$ in the right-hand side of Eq. (2.2). This neglect of density variations is referred to as Boussinesq approximation (Cushman-Roisin and Beckers 2011, p. 83). Introducing the modified pressure $\pi := p/\rho_0$ along with a reference temperature θ_0 that corresponds to the reference density b_0 , Eq. (2.2) becomes

$$-\frac{1}{\rho_0} \left(\frac{\partial p}{\partial z} + \rho g \right) = -\frac{\partial \pi}{\partial z} + \frac{\theta'}{\theta_0} g = -\frac{\partial \pi}{\partial z} + b, \text{ with } b := g \frac{\theta'}{\theta_0} \quad (2.7)$$

where b is the buoyancy and $\theta_0 = \text{const.}$ is the non-buoyant hydrostatic background potential temperature profile.

Equation 2.7 can be plugged into Eq. 2.1 when the latter is divided by ρ . In all other terms, the approximation $\rho = \rho_0$ is applied and density effects are covered by the buoyancy b —the remainder of the gravity term—only.

f-plane approximation, geostrophic balance. On a rotating sphere the local vector of angular rotation depends on the latitude ϕ as $\Omega(\phi) = |\vec{\Omega}| (0, \cos \phi, \sin \phi)$, which makes the Coriolis acceleration

$$-2\vec{\Omega} \times \vec{v} = -2|\vec{\Omega}| \begin{pmatrix} w \cos \phi - v \sin \phi \\ u \sin \phi \\ u \cos \phi \end{pmatrix}. \quad (2.8)$$

At the poles the vector of rotation Ω simplifies to $\Omega = (0, 0, |\Omega|)^T = (0, 0, f/2)^T$ with $f := 2|\Omega| \sin \phi$ the Coriolis parameter, and one may locally consider a horizontal plate rotating around the vertical axis. Hence the name f-plane approximation. Away from the poles this f-plane approximation is a further simplification of Eq. (2.1a), which is justified in particular at high latitudes and for small vertical

velocities. As a consequence of the f-plane approximation, the latitude ϕ drops out as a parameter of the problem.

If a shear-free, homogeneous flow on an f-plane, is exposed to a pressure gradient, the pressure gradient force may only be balanced by the Coriolis force. This balance is the *geostrophic balance*:

$$\mathbf{G}_i = f \epsilon_{i3k} \frac{\partial \Pi}{\partial x_k}, \quad (2.9)$$

where $\mathbf{G} = G_i \hat{e}_i$ is the geostrophic wind vector and $\partial \Pi / \partial x_k$ is the mean pressure gradient along Ox_k . Far away from the wall, this geostrophic balance of the pressure gradient with the mean flow along isobars is a good approximation, and turbulence as well as viscous effects do not play an important role. The physical problem—besides the latitude—also depends on the direction along which the pressure gradient is imposed. Without loss of generality, the coordinate system is aligned such that $\hat{e}_1 = \mathbf{G}/|\mathbf{G}|$ and I refer to the direction of $\mathbf{G}$ as streamwise and to that of $\hat{e}_2 = (\vec{\Omega} \times \mathbf{G})/|\vec{\Omega} \times \mathbf{G}|$ as spanwise direction.

Simplified equations and boundary conditions. Using the simplifications discussed above, Eqs. (2.1a)–(2.1c) become

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial \pi}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + f \epsilon_{ij3} u_j + b \delta_{i,3} \quad (2.10a)$$

$$\frac{\partial b}{\partial t} = -u_j \frac{\partial b}{\partial x_j} + \nu_H \frac{\partial^2 b}{\partial x_j^2} \quad (2.10b)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (2.10c)$$

where the variation of the molecular diffusivities ν and ν_H as a function of the state of the air is ignored. The set of partial differential Equations (2.10a)–(2.10c) is closed, and it describes the time-evolution of the velocity, pressure and buoyancy fields given initial and boundary conditions.

At the lower boundary, a no-slip and no-penetration condition is imposed mimicking a solid wall. At the upper boundary, a free-slip and no-penetration condition is used. This supposes that the upper end of the domain is part of the free stream and as such in geostrophic balance. In the horizontal direction, the domain is doubly-periodic. It needs to be large enough to capture the largest structure relevant for turbulence. Stratification may be imposed via a Dirichlet (fixed-value), Neumann (fixed-gradient) or Robin (mixed) boundary condition. In this work, only Dirichlet boundary conditions are considered.

2.2 Non-dimensionalization and Parameter Space

If a fluid mechanical problem is rescaled, the law of hydrodynamic similarity may be exploited to apply the Buckingham- Π theorem. This allows to draw definite conclusions on classes of flows rather than a single configuration. Therefore it is necessary to identify relevant parameters and group them into non-dimensional combinations.

2.2.1 The Neutrally Stratified Regime

Under neutral stratification, the Ekman-flow dynamics are governed by the quantities $\{G, f, \nu, \nu_H\}$ (we include here the diffusivity of a passive tracer for completeness) once turbulence has fully developed and the flow fields have sufficiently de-correlated from the initial conditions. Following previous studies (Coleman et al. 1992; Spalart et al. 2008, 2009; Marlatt et al. 2010), the Coriolis parameter f is replaced by the laminar Ekman-layer depth $D \equiv \sqrt{2\nu f^{-1}}$ in the dimensional analysis. This yields the Reynolds and Prandtl numbers

$$Re = \frac{GD}{\nu}, \quad Pr = \frac{\nu}{\nu_H}, \quad (2.11)$$

where it is noted that $Re \propto \nu^{-1/2}$. Here, the Prandtl number is fixed as $Pr \equiv 1$. In contrast to the boundary-layer flow over a flat plate, where the shear layer deepens continuously and no steady state is reached, Ekman flow acquires a balance between the production of turbulence due to shear and the suppression of turbulence due to rotation. The neutrally stratified flow has hence a steady-state solution in terms of a statistical description of turbulence which is a function of Re only. This also implies that the definition of an initial condition from a mathematical perspective is irrelevant if the flow is linearly unstable, which is the case for Ekman flow beyond $Re = Re_{\text{crit}} = 115$ (Lilly 1966). For $Re > Re_{\text{crit}}$, the flow transitions to its turbulent statistical equilibrium state.

2.2.2 Uniqueness of the Solution

Given initial and boundary conditions, one might ask the question, Is this equilibrium state deterministic? Despite recent advance (Otelbaev 2014), the general unique solubility of the Navier–Stokes equations still withstands a widely accepted rigid mathematical proof (Gowers 2000; Ladyzhenskaya 2003). So, from a fundamental point of view, it is not clear if there is a single such equilibrium state—multiple such states might exist, and infinitesimal perturbations in the initial condition might determine to which statistical equilibrium the flow transitions. While, in theory,

non-uniqueness of the solution might allow for the existence of multiple relevant equilibria, this is practically not realizable. Not only would additional equilibria be irrelevant for physical realizations of the problem: no more than one equilibrium has been observed both numerically and in laboratory experiment. But also would the existence of two physically realizable equilibria contradict the Navier–Stokes equation’s unique solubility for a finite time (Ladyzhenskaya 2003; Chemin et al. 2006). Hence, in the following it is assumed, there exists only one equilibrium.

If such an equilibrium exists, it is inevitably attained by the flow for $Re > Re_{crit}$. Once the flow is in its turbulent state, the laminar length scale D no longer describes the flow appropriately. Instead, the boundary-layer depth scale $\delta \equiv u_\star/f$ is used, where $u_\star$ is the friction velocity defined below. The following parameters characterize the turbulent flow:

$$u_\star^2 = \nu \frac{\partial \sqrt{\langle u \rangle^2 + \langle v \rangle^2}}{\partial z} \bigg|_{z=0}, \quad \delta = u_\star/f \quad \text{and} \quad Re_\tau = \frac{u_\star \delta}{\nu}. \quad (2.12)$$

In contrast to channel flows, $u_\star$ in the Ekman layer cannot be known a priori but only a posteriori, and $u_\star$ depends weakly on Re (Spalart 1989). Following common practice, the flow is studied in terms of an inner layer (where molecular viscosity plays an important role) and an outer layer (where turbulent viscosity plays an important role). In the inner layer, i.e. also in the surface layer, the wall unit $\nu/u_\star$ and the friction velocity $u_\star$ are chosen for normalization; normalized quantities are denoted by a superscript $+$. In the outer layer, quantities are normalized by $u_\star$ and δ , and correspondingly normalized quantities are denoted by a superscript $-$. Because of its physical meaning, the inertial period $2\pi f^{-1}$ is used as the outer reference time scale instead of f^{-1} .

2.2.3 *Imposing Stratification: Initial and Boundary Conditions*

When a stratified flow is considered, in addition to the boundary conditions for the flow and scalar, an initial condition for the active scalar is needed. For the velocity fields, a turbulent initial condition is employed—depending on the case and question either from an equilibrated neutrally stratified simulation or from a quasi-steady stratified simulation.

Here, the problem is studied for a fixed surface buoyancy only; for a discussion of the impact of flux boundary conditions and more complex set-ups, see Flores and Riley (2011) and van de Wiel et al. (2012). The buoyancy difference between the surface and the far field is B_0 (cf. Coleman et al. 1992); this new parameter combines into the Froude number $Fr = G^2/(B_0 D)$. Fr incorporates the laminar length scale D losing its relevance in turbulent flow. In the turbulent set-up, stratification is expressed more appropriately in terms of the global bulk Richardson number,

$$Ri_B := \frac{B_0 \delta_{\text{neutral}}}{G^2}. \quad (2.13a)$$

Due to the choice of a Dirichlet boundary condition,

$$b(x, y, z = 0, t > 0) = 0 \quad \text{and} \quad b(x, y, z = z_{\text{top}}, t > 0) = B_0, \quad (2.13b)$$

Ri_B is an inviscid external parameter, and used in the following to classify simulations according to their stratification.

The Obukhov length L_O (Obukhov 1971) is an alternative measure to characterize stratification. In particular, the ratio of the Obukhov length L_O with the wall unit $\nu/u_\star$,

$$L_O^+ = L_O \frac{u_\star}{\nu} := \left(-\frac{\theta_0}{g} \frac{u_\star^3}{\theta w|_{z=z_0}} \right)^+ = \left(\frac{u_\star^3}{u_\star b_\star} \right)^+ \quad \text{with} \quad u_\star b_\star = \nu \left. \frac{\partial \langle b \rangle}{\partial z} \right|_{z=0} \quad (2.13c)$$

determines the character of turbulence (Flores and Riley 2011): L_O^+ measures the biggest possible scale separation in a stratified flow (Flores and Riley 2011), and is therefore an appropriate Reynolds number in a stratified environment. Interestingly, the critical value of $L_O^+ \simeq 100$ estimated by Flores and Riley is very similar to the critical Reynolds number for linear instability of Ekman flow ($Re_{\text{crit}} = 115$). Recent work (Deusebio 2015) indeed suggests this threshold is almost universal.

This is consistent with the research on minimal flow units suggesting that the near-wall cycle of turbulence can only be sustained for scales larger than 100 wall units (Jiménez and Pinelli 1999).

Over a smooth wall, this parameter can also be interpreted in terms of the gradient Richardson number

$$Ri_G := \frac{\partial_z B|_{z=0}}{(\partial_z U)^2|_{z=0}} = \nu \frac{b_\star}{u_\star^3} = (L_O^+)^{-1}, \quad (2.13d)$$

where $Pr = 1$, as defined above, is used. Therefore, L_O^+ contains information about the stability character in the near-wall region. Large L_O^+ implies a small Ri_G and hence turbulence can develop in the lower part of the SBL. In Ekman flow over a surface at fixed temperature, both L_O and Ri_G are not external parameters to the problem, but they are time-dependent measures describing the evolution of the system. For that reason Ri_B is used as control parameter.

Besides the strength of stratification, the profile of stratification imposed as the initial condition has a significant impact during the initial phase: the initial profile determines the duration of the initial transient. The focus here is on the cases where almost the entire stratification concentrates initially within the viscous sub-layer of

the flow, mimicking a sudden cooling of the surface. Following Coleman et al. (1992), the initial condition for the buoyancy is

$$b(x, y, z, t_0) = 1 - \operatorname{erf} \left[\frac{z/D}{2a} \right] = 1 - \operatorname{erf} \left[\frac{z}{2a} \left(\frac{\delta_{neutral}}{\sqrt{2} Re_\tau} \right)^{-1} \right], \quad (2.14)$$

where b is the buoyancy normalized with the surface boundary value B_0 , Re_τ is defined in Eq. (2.12) and $a = 0.15$ is the non-dimensional thickness. Also, $\delta_{neutral}$ refers to the value of the neutrally stratified case used for the initialization of the velocity fields.

This particular choice of the initial condition through the concentration of the entire buoyancy gradient into the surface layer bears the dilemma that stratification may be very strong close to the surface even if Ri_B , which is expected to control the long-time evolution of the system, is sub-critical. In that case, the initial transient can contain periods of time wherein the turbulence is shut off through a very efficient cut of production in the buffer layer; this initial phase is most appropriately characterized by Ri_G . Ri_G is indicative of the respective stratification in the production region (surface layer) and more appropriate to study the initial transient when a stratification is imposed in an initially very thin layer at the surface. Given the large spread between the two parameters Ri_B and Ri_G ($Ri_G(t=0)/Ri_B = G^2/u_\star^2$), the range of stratifications over which this might occur is potentially large. Thereafter molecular mixing slowly diminishes the buoyancy gradient until the flow may become unstable again. Although the focus of this work is the long-time evolution of the system, the ratios of these transition time scales to the integral time scale of turbulence, $2\pi f^{-1}$ (or $f^{-1} = \delta/u_\star$, see above), are relevant in the context of the atmosphere. For instance, they determine whether a fully-developed boundary layer has time to reach its quasi-steady state over the course of a night (or other externally set time scales). In Sect. 7.1, I show that the key findings are, at least in the range considered here, independent of the choice of initial condition.

2.2.4 Parameter Space of the Non-dimensionalized Problem

The problem as formulated hereunto depends only on two parameters, a Reynolds and a Richardson number. The Reynolds number measures the separation of scales between the large scales (in this case, these are the scales forcing the system) and the smallest scales contributing to the spectrum of turbulent motion. Some relevant aspects of the flow are commonly expected to scale self-similarly with respect to the Reynolds number—at least in the limit of the (high) Reynolds numbers typical for the PBL. The Richardson number represents stratification, and it is well known that turbulent flow reacts sensitively to changes in the stratification from theoretical considerations, observations and numerical studies (Richardson 1920; Fernando 1991; Peltier and Caulfield 2003; Ivey et al. 2008).

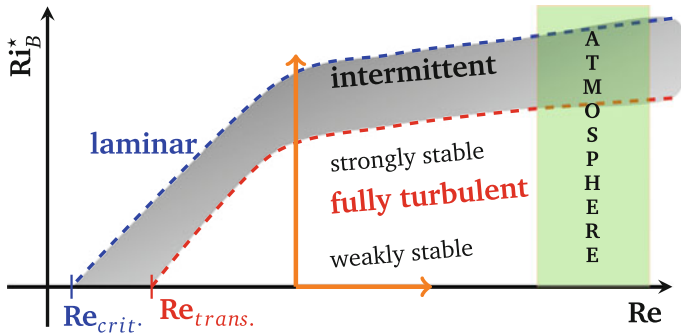


Fig. 2.1 Schematic of the problem's parameter space as introduced in the text

Previous work suggests to partition the problem's parameter space as sketched in Fig. 2.1. Under neutral stratification the only parameter of the problem is the Reynolds number, and once in the fully turbulent regime, low-order statistics are well-known to be to a certain degree independent of the Reynolds number Re (Moin and Mahesh 1998). If stratification is increased at fixed Re , the regimes of weak, intermediate and strong stratification are covered before the flow laminarizes. While there is no doubt as to the existence of those regimes, the thresholds and dynamics of transitions and their nature are under debate (Tritton and Davies 1981; Manneville 2008); in particular no conclusive statements have been made on the limit and dynamics of stratification for the re-laminarization of a turbulent flow and the associated phenomenon of global intermittency, where laminar patches exist in an otherwise turbulent flow.

2.3 Analysis Tools

The direct numerical simulation of a turbulent flow consists of the solution of Eqs. 2.1a–2.1d, and it may be interpreted as a merely technical problem or numerical exercise (Chaps. 3–5). Once these solutions are available, they may be studied to gain physical insight (Chaps. 6–8). Commonly, this is not achieved via the analysis of a single datum (even though this constitutes a reasonable option for some kind of analyses) but rather by means of analysis methods that have solid theoretical foundations and are widely applied. Besides the very common approach of a spectral decomposition (Finnegan and Kaimal 1994) and the analysis of probability density functions, conditional sampling and virtual tower measurements are used in this work and described in the following sections.

2.3.1 *Conditional Sampling*

Free turbulent flows are ubiquitous in engineering, and they are even more so in geophysical applications of turbulence research. The possible coexistence of non-turbulent and turbulent fluid in a statistically homogeneous manifold of a turbulent flow domain was already recognized by Corrsin (1943, cf. Chap. 1 of the present work). Phenomena at the turbulent–non-turbulent interface—namely external intermittency and entrainment—play a pivotal role in the development of boundary layers with a free stream. The successful (experimental) distinction between turbulent and non-turbulent fluid based on the instantaneous vorticity magnitude allows to separate the flow domain into a turbulent and non-turbulent part (Corrsin and Kistler 1955). If turbulent processes or quantities—such as the turbulent diffusion and the associated turbulent diffusivity—are of interest, one may argue that a consideration of statistics based on the turbulent or non-turbulent subsets alone is useful. A sample is then based on a subset of the domain that suffices a particular condition—hence the term conditional sampling for this procedure that is widely used in the experimental study of turbulent flows (Antonia 1981). The capability to sample a flow conditionally allows a whole new set of analysis methods embraced under the term of conditional statistics; these statistics may be averages or higher moments, but they may also be more complex such as probability density functions or wavelets.

With respect to atmospheric boundary layers, conditional sampling so far has almost exclusively been used in the context of field measurements in the surface layer under convective conditions (Singh Khalsa 1980; Singh Khalsa and Greenhut 1985; Katul et al. 1994). A weakly intermittent nocturnal surface inversion layer and a Bora flow have been investigated using conditional statistics with respect to the eigen-structure of micro-fronts by Mahrt and Frank (1988).

2.3.2 *Temporally Resolved Probes*

While the details of an individual realization of a turbulent flow are often not of particular interest when investigating the turbulent dynamics, also the vast number of degrees of freedom in a DNS prohibits to analyze all the data that potentially become available throughout a simulation. For technical reasons, namely the restart of a simulation, fully-resolved three-dimensional fields are saved at fixed iteration points, but otherwise only a predefined set of statistics is extracted and saved. Hence, for an a posteriori analysis there are data at very high spatial but rather coarse temporal resolution. While this is generally sufficient for the study of the bulk turbulence in a statistical sense, i.e. based on ensemble averages, investigation of some processes at high temporal frequency demands higher temporal resolution. Also, observational data in the atmosphere are mostly obtained from fixed-in-space probes (towers) at a high temporal resolution, and generally very coarse spatial resolution—if at all

there is more than one tower available for the analysis of a flow within the spatial separation relevant to turbulent processes.

During this work, I implemented a tower-sampling at high temporal resolution at selected locations. In the largest set-up (at $Re = 1000$ with a horizontal box size $L_{xy}/\delta \simeq 20$), the flow is sampled at the full temporal resolution, i.e. at every iteration at 32×32 locations in each horizontal plane. This corresponds to a spacing of $(0.64\delta \times 0.64\delta)$ or $(1.01\delta_{95} \times 1.01\delta_{95})$ (cf. Chap. 6). This spatial separation allows to consider each tower as an approximately independent realizations of the flow.

Let $\mathbf{x}_i(t, z)$ the instantaneous realization of a flow realization at time t , height z and location $\mathbf{r}_i$. Let further $(t_n)_{n \in \mathbb{N}}$ the discrete series of times through which the flow is integrated, and $(\mathbf{r}_i)_{i \in \{1 \dots 1024\} \subset \mathbb{N}}$. Further, let

$$\overline{x_i(z)}^T := \frac{1}{n(T)} \sum_{n=1}^{n(T)} \mathbf{x}_i(t_n, z) \quad (2.15)$$

with $n(T)$ such that $t_{n(T)-1} < T \leq t_{n(T)}$,

that is, $\overline{x_i(z)}^T$ is the average of the tower at $\mathbf{r}_i$ over a period of time of length T .

Ergodicity of the stationary, four-dimensional turbulent field implies

$$\langle \mathbf{x}_i(t, z) \rangle = \lim_{T \rightarrow \infty} \overline{x_i(z)}^T. \quad (2.16)$$

Moreover, in the limit of an infinite number of towers, it is $\langle \mathbf{x}(z) \rangle = \langle \mathbf{x}_i(t_n, z) \rangle$.

2.4 Summary

In this chapter, the physical problem is formulated using the governing equations. The Navier–Stokes equations in the Boussinesq limit are identified as the appropriate set of equations to study the flow. These equations are introduced along some common assumptions and simplifications, and their simplified version is subsequently non-dimensionalized. The non-dimensional parameter space of the problem is introduced and qualitatively characterized, and the main analysis methods are presented.

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