Chapter 2
Examining the Role of Prior Experience in the Learning of Algebra

Mercedes McGowen

Abstract Widespread emphasis on developing students' algorithmic competency and symbol manipulation has resulted in students failing to think analytically and critically. If students are not encouraged to think flexibly about arithmetic and algebra in school, then this needs to be addressed by developmental courses and tasks designed to change the procedural orientation and superficial, fragmented knowledge of too many of our undergraduate students. Those who teach mathematics at the postsecondary level often dismiss the increasing number of students enrolled in precollege mathematics courses as "not my problem," not realizing that "just algebra" is the downfall for many college students. Learning "just algebra" is a much more complex task than it appears. In this chapter, prior knowledge will be shown to have become problematic for many students, and we provide evidence of the need to improve the effectiveness of our own teaching and that of our future teachers in ways that help students develop deeper understanding of mathematics and promote mathematical thinking.

Keywords Flexible thinking • Prior knowledge • Problematic met-befores • Remedial mathematics • Developmental algebra • Function machine • The minus sign

2.1 Impact of Current Instructional Practices on Student Learning

"The power of mathematical thinking—pattern recognition, generalization, problem solving, careful analysis, rigorous argument—is important for every citizen" (Barker, Bressoud, Epp, Gantert, Haver, & Pollatsek, 2004, p. 4). Mathematicians and mathematics educators claim that they want their students to think critically, make connections, and see new relationships between mathematical ideas.

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However, in too many classrooms, the ongoing instructional emphasis is predominantly to show students how to use a rule to get the "right" answer. The focus on mastering skills, coupled with the assumption that students understand the related mathematical concepts and terms, has failed many students, leaving them as postgraduates ill prepared for their future careers (Carlson, 1998; DeMarois, 1998; McGowen, 1998; McGowen & Tall, 2010, 2013; Oehrtman, Carlson, & Thompson, 2008; Stigler, Givvin, & Thompson, 2010; Stump, 1999).

There are those who teach mathematics at colleges and universities who dismiss the increasing numbers of students enrolled in precollege mathematics courses as "not my problem." Instructors may rightfully view what follows the initial step(s) in their respective mathematics courses as "just algebra," but in reality, "just algebra" is the downfall for many students in secondary schools and in developmental and college-level mathematics courses.

The increasing growth of undergraduate remedial mathematics courses reveals critical concerns, not only for the mathematics community but for our nation at large. Major growth in 2-year college mathematics enrollments since 1990 has been in the precollege courses (e.g., arithmetic, pre-algebra, elementary algebra, intermediate algebra, and geometry)—courses students have taken previously in elementary and high school and sometimes more than once as undergraduates. The Conference Board of the Mathematical Sciences (CBMS) (2001) survey (Blair, Kirkman, Maxwell, & American Mathematical Society, 2013, p. 136) reports that:

- In 2010, for the first time, enrollment in precollege courses at 2-year colleges totaled more than one million students (1,149,740)—a 19 % increase from 2005 to 2010.
- Arithmetic/basic mathematics, pre-algebra, and geometry course enrollments have grown from 216,000 in 2000 to 378,000 in 2010 at 2-year colleges, an increase of 75 %.
- Beginning and Intermediate Algebra enrollments increased 41 %, from 547,000 in 2000 to 772,000 in 2010.
- During the 5 years from 2005 to 2010, 4-year colleges and universities saw precollege course enrollments increase—though not as dramatically as at 2-year colleges. At 4-year colleges and universities, between 1990 and 2005, precollege course enrollment declined by 30 %.

Developmental education costs are estimated at $1 billion every year (Brothen & Wambach, 2004). Breneman and Haarlow's earlier study (1998) gave a conservative estimate of one to two billion dollars per year spent on remedial education programs at public colleges and universities. Neither of these estimates takes into account the costs incurred in time and money by students enrolled in undergraduate remedial mathematics courses. Based on analysis of data from a nationwide study of community colleges participating in the Achieving the Dream project, Bailey (2009) found that student completion rates in college English and math drop with each additional level of remedial coursework required. Only 10 % of students who placed three levels down from a college-level mathematics course pass a college-level course. Attewell, Lavin, Domina, and Levey (2006) reported that only 28 % of students who take at least one remedial course go on to complete a college credential within 8.5 years. A National Council of State Legislators article by Brenda Bautsch (2013) cites a US Department of Education study which found that only 27 % of students enrolled in remedial mathematics courses earned a bachelor's degree compared with 58 % of students who did not need remedial math. Simple "remediation" focusing on the need for accurate procedural computation does not work: at each stage, more and more students fail.

Increasing numbers of students who intend to become teachers begin their postsecondary academic careers at 2-year colleges, taking the required first 2 years of their mathematics courses at a community college before transferring to a 4-year degree program. Many preservice elementary teachers enroll in one or more developmental mathematics courses prior to taking required math content courses for preservice teachers at these institutions. Their attitudes to mathematics are generally instrumental, focused on formulas and getting correct answers. Educators of these students face a constant challenge—their students' limited understanding of what constitutes mathematics and a mathematical approach to problems.

To help students gain a more flexible, deeper understanding of arithmetic and algebra requires much more than computational or symbolic fluency. The National Council of Teachers (1989, 1991, 2000), Cohen (1995), National Research Council (2000, 2001), Conference Board of the Mathematical Sciences (2001), American Mathematical Association of Two-Year Colleges (2004), and the Common Core State Standards Initiative (2010) have all published recommendations for teachers, whether elementary, secondary, or postsecondary, directing instructors to:

- Design and implement every instructional activity guided by informed decision-making to actively engage students in the learning of mathematics
- Integrate technology appropriately into teaching to enhance students' understanding of mathematical concepts and skills
- Use results from the ongoing assessment of student learning in mathematics to improve curriculum, materials, and teaching methods

The *Curriculum Foundations Project: Voices of the Partner Disciplines* echoes these recommendations. They advocate that all teachers at every level must be able to "represent concepts in multiple ways, explain why procedures work, or recognize how two ideas are related . . . be able to solve problems and to make connections among mathematical topics. . ." in order to modify instructional strategies and place greater emphasis on learning with understanding and focus on a thorough development of basic mathematical ideas presented in a coherent fashion (Ganter & Barker, 2004, p. xx).

Beliefs about what constitutes mathematics, what skills should be taught, when they should be taught, and to whom vary from individual to individual and community to community. Unfortunately, many mathematics departments have yet to reach a consensus acceptable to all members of the department on these issues. In the absence of mutually agreed definitions and accepted meanings among those who favor a "return to basics" and those who attempt to implement reforms in the teaching and learning of mathematics, the debate continues, with increasingly high costs to students and to our nation. Beaton (1996) cited these conflicting beliefs and practices, describing the current US mathematics curriculum as unfocused, “a splintered vision” which is reflected in our mathematics curricular intentions, textbooks, and teacher practices.

A common theme of programs described in *Models That Work* (Tucker, 1995) is that faculty in effective programs believe in “teaching for the students one has, not the students one wished one had.” Unfortunately, when students lack prerequisite skills, an all-too-common reaction from instructors is “You can’t expect me to reteach the entire prior curriculum. I have to teach the content of my course.” This perspective serves only to block efforts to explore alternative ways of improving mathematics teaching and student learning.

The mathematics needed by first year students enrolled in many college career programs has been described as almost exclusively middle school mathematics—arithmetic, ratio, proportion, expressions, and simple equations (Common Core State Standards Initiative, 2010). Currently, instruction and learning of these topics fall far short of the understanding and competency students will be expected to demonstrate. A National Center on Education and the Economy empirical study (2013) found that introductory courses (a) fail to test complex analytical skills, the ability to synthesize materials, and solve problems not seen before as they demand only memorization of facts and mastery of procedures and (b) are not designed to test students’ ability to think mathematically but instead assess memorization of facts and mastery of procedures, not the higher-order thinking skills of Bloom’s taxonomy. The study also found that many community colleges have low expectations of their students, particularly in developmental courses.

The current focus on modifying technical and career programs that do not require a 4-year or advanced degree cannot result in ignoring the needs of many students who do require a deeper understanding of basic algebra concepts and skills proficiency. These students include not only our prospective elementary, secondary, and postsecondary teachers who are not currently being well-enough prepared but other non-STEM majors who also need solid foundational algebra skills for their future careers.

Evidence of what students actually know and have little or no conceptual understanding of has been reported in many research studies over the years. The word “understanding” is used in the context of mathematics with two different meanings: *relational* understanding, knowing both what to do and why, and *instrumental* understanding, knowing a rule and being able to use it (Skemp, 1987). He referred to instrumental learning as “rules without reason.” It is these alternative meanings of understanding that are at the root of many of the differing perspectives on teaching and learning. Smith (1996) argues that an instrumental approach to teaching mathematics provides a teacher with a robust sense of efficacy.

Many preservice teachers believe this is the only approach to teaching mathematics that will provide them with a sense of competence, proficiency, and know-how. The belief that if a student demonstrates skills proficiency and gets a correct answer, he or she “understands” the concept is not necessarily true.

Autobiographical descriptions of community college developmental algebra students' and preservice teachers' prior instructional experiences reveal the very different attitudes and beliefs about mathematics held by many students compared with that of their instructors and which prove very resistant to change:

> All throughout school, we have been taught that mathematics is simply plugging numbers into a learned equation. The teacher would just show us the equation dealing with what we were studying and we would complete the equation given different numbers because we were shown how to do it.
>
> My previous mathematics experience involved a teacher lecturing. Finding a formula to solve a problem was, in reality, the answer to the problem.
>
> I was taught "how" but not "why". I thought of math as a series of formulas, each of which should be followed in order to find an answer.
>
> When I took Calculus the differentials were what killed me. There were so many equations, and the teacher would go over how to get from one to the other. I tried to go with what I knew, and memorize them. However, that didn't work.

Many prospective teachers have a deeply ingrained procedural orientation to mathematics with its focus on "getting the correct answer" that they have learned to value above all. Changing this orientation is made more difficult when instructors fail to take into account students' prior knowledge and underestimate what their students are capable of. A challenge facing mathematics instructors at undergraduate institutions and community colleges is how to develop a sound conceptual grasp of foundational arithmetic and algebraic topics in these students within the context of their course content.

2.2 The Role of Prior Learning and Its Possible Problematic Consequences

One of the most important findings of cognitive science and brain research is that prior knowledge is the beginning of new knowledge. Ausubel, Novak, and Hanesian (1968) remind us, "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach accordingly." Prior knowledge is a fact, and it is persistent. New experiences that build on prior experience are much better remembered, and what does not fit in prior experience is either not learned or learned temporarily and easily forgotten.

The notion of *met-before* (McGowen & Tall, 2010; Nogueira, De Lima & Tall, 2008; Tall, 2004) was introduced to focus on how new learning is affected by the learner's previous experiences and as a way of looking at the effects of prior learning that can support or impede new learning. Prior learning can be supportive in those instances where old ideas can be used to make sense in new contexts and problematic in contexts where the old understanding no longer works. Research studies have examined students' ability to modify their prior knowledge. Discontinuities encountered were reported not only at the developmental arithmetic and

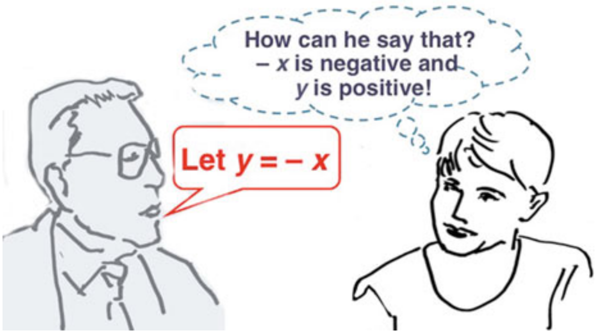


Fig. 2.1 Miscommunication (Thompson, 1996)

algebra levels but also at the undergraduate college level (Davis & McGowen, 2007; McGowen, 1998; McGowen & Tall, 2013).

Prior knowledge of the minus symbol to indicate subtraction, while supportive for whole numbers and positive fractions, becomes problematic for many students when they encounter negative numbers. It becomes even more so in the context of algebra to indicate the additive inverse of an unknown ($-x$). The belief held by many students is that a minus sign in front of it indicates that it has a negative value—a belief that results in increasing difficulties in subsequent mathematics courses (Fig. 2.1).

Ideas encountered at one stage of learning may lead to ways of thinking that are not appropriate later. Mathematics instructors need to take into account the effects of existing knowledge—both positive and negative—that students have *now* as a result of experiences they have met before, *at every level of development*, aware of how earlier mathematical experiences result in students' ideas that can become problematic when context and/or subtle changes in meaning are encountered.

2.2.1 Prior Arithmetic Thinking

The problematic nature of prior arithmetic thinking was revealed when a majority of 128 college freshmen, given the numbers 0, 1, x, y, and $-z$ as marked on the number line below (Fig. 2.2), claimed that $2y$ was larger than y because (a) "$2y$ is larger than y" or (b) "$2y$ is larger because it has a number in front of the variable."

Nearly one-third of the students maintained that $x - y = y - x$, stating that they were "the same problem just switched around" or "because they are both subtracting a variable." One-fifth of the group wrote that $x + y = x - y$ "because adding a positive and a negative is the same as subtracting a positive and a negative number." The other most common response was "because in both equations you are really adding the numbers" (McGowen & Tall, 2010, pp. 175–176).

Fig. 2.2 Quantities on a number line (Bright & Joyner, 2003)

Students' efforts to interpret algebraic and function notation demonstrate how very differently individual students think about notation than do mathematicians. The conceptual requirements for understanding ambiguous expressions, both arithmetic and functional, appear to be far more formidable in their complexity than has generally been recognized by mathematics instructors. Sfard reminds us: "Algebraic symbols do not speak for themselves. What one actually sees in them depends on the requirements of the problem to which they are applied. Not less important, it depends on what one is *able* to perceive and *prepared* to notice" (Sfard, 1991, p. 17).

That students have difficulties with fractions, negative numbers, decimals, and the transition to algebra is well documented. How the changes in meaning of the successive number systems N, F, Z, Q, and R impact individual students at the undergraduate level—particularly where previous experiences involve some aspects that are supportive and generalize while others are problematic and impede new learning—has been less widely researched and reported. The multiple meanings of the minus symbol and the need to interpret symbolism flexibly were identified as major sources of difficulties for undergraduates, particularly when the symbolism changes meaning in new contexts (Davis & McGowen, 2007; McGowen, 1998; McGowen & Tall, 2013).

The minus symbol becomes problematic for many students when they encounter negative numbers, the precedence of division and multiplication over addition, and the precedence of powers over taking the additive inverse (McGowen, 1998). They are faced with new conventions when combining two operations: (a) the minus sign and the power operation, and (b)the order of operations of two *unary* processes, squaring negative three and taking the additive inverse of the square of three. Their prior knowledge consists of the order of *binary* operations and a mnemonic such as Please Excuse My Dear Aunt Sally (PEMDAS) to indicate the order of priority of operations (parentheses, exponent, multiply, divide, add, and subtract). Students who focus on the qualitatively different features of -3^2 and $(-3)^2$ are able to make sense of the notation and correctly manipulate these expressions. These students connect new knowledge with their prior knowledge in a way which results in a reconceptualization of the two processes of squaring a negative number and finding the opposite of a number squared. A student's typical explanation is:

> When I see the sign $(-)$ it is a change for me to know that it means "the opposite of," I always though it meant a negative number or, $-(-x)$ *a positive* x. The reflection assignment enhanced my understanding of the opposite of a square by looking at it as two functions, and then order of operations would have exponents first, then the opposite of the value . . . Exponentiation takes precedence over oppositing in the absence of grouping symbols.

Data on students' difficulties evaluating -3^2 and $(-3)^2$ have been collected during various research studies. Pre- and post-surveys administered over several years[1] to 516 community college and university students enrolled in a developmental algebra course showed that, at the completion of the course, 81 % (418/516) of the students correctly evaluated $(-3)^2$, but only 49 % (251/516) of the students could correctly evaluate -3^2. The latter may simply relate to the way that the symbols are read from left to right "minus" "3" "squared." The minus and the three are taken together as "minus three," and the student has been told that the square of a negative number is positive.

In an expression with more terms to put together, the problem may become more complicated. When given $f(x) = x^2 - 3x + 5$, find $f(-3)$; a College Algebra student explained his work as follows:

> "*I used up the negative sign. I have to do parentheses first*" and wrote: $f(-3) = -3^2 - 3(-3) + 5 = 9 + 9 + 5 = 23$ followed by: $9 + 5$. "*Now I have to do this* (indicates the -3^2), "*but I can't remember if it's negative nine or just nine. I never know which to use*." He wrote down -9 and stopped. "*There's no sign in front of this* (pointing at $9 + 5$), *so I need to multiply*," writing $-9(14) = 136$.

2.2.2 Interpreting the Minus Sign in Linear Factors

Many students are confused as to whether the minus sign in $(x - c)$ represents subtraction or is the sign attached to c. An instructor who participated in a recent formative assessment pilot study reported that many of her College Algebra students had difficulty using the linear factors of a polynomial correctly (McGowen & Tall, 2013). Working from a graph using the zeros of a function to determine a quadratic function's linear factors, several students viewed c in $(x - \mathrm{c})$ as a negative value, a belief which results in many sign errors when writing the factors and/or zeros of a function (McGowen, 1998):

> The value of c is negative because of the minus sign in front of c. c will subtract from any number that comes before the "$-$" symbol.
>
> I used up the negative sign.

Students used the subtraction operator of a linear factor as the sign of c, and many thought the x- and y-intercepts were the coefficients in the equation of the function. Given the graph of a line, only one in five students was able to determine whether the slope and y-intercept should be positive or negative. Inflexible in their thinking, they were only able to answer questions from one direction—unable to reverse the process.

Prior experience involving the minus sign also proves problematic for undergraduates in other contexts. Students are faced with making sense of function notation as well as interpreting the minus symbol in expressions such as $f(-x)$ and $-f(x)$. Many students believe that $f(-x)$ represents "f of negative x" or "a

[1] Portions of the collected data have been previously reported (McGowen, 1998; McGowen & Tall, 2013), but the accumulated data of 516 students have not been reported previously.

negative input value" and that $-f(x)$ represents "negative f of x" or "a negative output value." Some students interpret $-f(x)$ as "the entire function is negative" and $f(-x)$ as "only the x is negative." There are also students who interpret function notation as indicating multiplication: $-f(x)$ means $-f$ times x and $f(-x)$ as f times $-x$.

2.2.3 Understanding Basic Algebraic Terms and Concepts

A lack of understanding of basic algebraic terms contributes to students making other errors. College Algebra students' written comments indicate that many of them have only a vague understanding of the meanings of foundational concepts, such as slopes, coefficients, and intercepts. Given the graph of a line, only one in five was able to determine whether the slope and y-intercept should be positive or negative. Some interpreted *slope* as an ordered pair and plotted it as an intercept. Other students equated the x- and y-coefficients in the equation with the intercepts on the graph. Several students wrote that an intercept is a number value. Many of them wrote the slope as an ordered pair and did not view slope as a ratio. Still others used the value of the slope as the x-intercept value. They were only able to answer questions from one direction—unable to reverse the process—indicating the inflexibility of their thinking (Davis & McGowen, 2007).

In their capstone course, a class of senior mathematics majors intending to be secondary math teachers were asked to describe what they knew about slope, covariation, rate of change, tangent, and derivative and which, if any of these ideas, are related. Only one student identified *covariation* as the fundamental link among the five ideas. Many made no attempt to provide any meaning for covariation as it was not a recognizable term in the curriculum. Rate of change was connected to derivative only because derivative is an instant rate of change and only a few students explicitly mentioned what changed.

2.3 Identifying and Addressing Problematic Prior Met-Befores

Black and William (1998) examined approximately 250 studies and found that gains in student learning resulted from a variety of methods all of which had a common feature: formative assessment (assessment that uses the data acquired to adapt instruction to better meet student need). They found that when teachers understand what students know and how they think and then use that knowledge to make more effective instructional decisions, significant increases in student learning occur. For instructors at all levels unaware of the knowledge and understanding of basic mathematical concepts and terms students lack when they enter our courses, Krutetskii's advice is appropriate: "Don't make a hasty conclusion about the incapacity of children in mathematics on the basis of the fact that they are not successful in this subject. First, *clarify the reason for their lack of success*" (Krutetskii, 1969, p. 122).

Clarifying the reasons for a student's previous lack of success—identifying *what precisely is lacking* in an individual student's development—is a challenge facing mathematics instructors. If students' prior knowledge (met-befores) impedes new learning and has resulted in misconceptions, instructors need to adapt instructional strategies that overcome and transform students' problematic met-befores. Some classroom assessment strategies that have proven effective in identifying what students understand and useful in addressing problematic met-befores are:

- Explicitly discussing prior understandings and how it changes in a different context
- Asking basic questions that instructors assume students know the answers
- Comparing students' written responses to two or more questions dealing the same concept or with related concepts, revealing of their ability to think flexibly
- Pre- and posttesting that offers a measure of individual student growth over time
- Using a function machine representation and the graphing calculator to make sense of notation and a deeper understanding of binary and unary arithmetic operations
- Identifying one's own met-befores and examining how they impact one's teaching and beliefs about curriculum and students

As Thompson (1994) reminds us:

> An instructor who fails to understand how students are thinking about a situation will probably speak past their difficulties. Students need a different kind of remediation, a remediation that orients them to construct the situation in a mathematically more appropriate way.

2.3.1 Asking Basic Questions

Asking questions about basic mathematical concepts and terms of which one assumes students have good understanding often reveals problematic knowledge which interferes with new learning. A lack of understanding of basic mathematical terms like "solve" and "evaluate" is often not recognized. Many students believe that they "solve an equation" whenever "x" is part of an expression. As part of a formative assessment pilot project at a local community college, undergraduates were asked to complete the following:

A. Finding the output when the input is known is the process of:

(a) simplifying (b) evaluating (c) factoring (d) solving
B. Finding the input when the output is known is the process of:

(a) simplifying (b) evaluating (c) factoring (d) solving

Only nine of 75 (13 %) Introductory Algebra students and 15 of 114 (12 %) College Algebra students selected (b) *evaluating* as the correct response to question A. On question B, less than 19 % of Introductory Algebra students and only 31 % of College Algebra students chose option (d) *solving*.

Developmental algebra students at a community college participating in a recent online survey were asked, "What does it mean to solve an equation?" The response "to get a variable by itself" was given by 35 % of the participating students. A second question asked "Is $x = y$ an equation: Why or why not?" The most common student responses were as follows: "No, because you need a number"; "No, because there are no known numbers on both sides"; and "No, because x and y are variables."

It is not only developmental algebra students enrolled in precollege courses that lack understanding of basic mathematical concepts and processes. A Ph.D. student completing his doctorate in mathematics and working as an online tutor for a textbook company when asked to explain the difference between solving an equation and evaluating an expression replied:

> If a book asks you to evaluate $x^2 - 2x + 1$, what they are asking for is a simplified version of this polynomial, which would be $(x - 1)^2$.
>
> Solving an equation or expression is actually plugging in a particular value to come up with a solution.
>
> For example:
>
> $$F(x) = x^2 - 2x + 1 \text{ Solve for } f(4).$$
>
> $$F(4) = 4^2 - 2(4) + 1 = 16 - 8 + 1 = 9$$
>
> Is this helping you feel a little bit better about the difference between the two? (McGowen, 2006, p. 25)

2.3.2 Comparing Responses to Two or More Questions

The inability of students to correctly answer *two or more* questions on related content suggests that students do not see the questions as being intimately connected. One indication that what students have learned and remembered is fragmented and unconnected is that they are unable to apply what they know when *confronted with a different context*. Noticeable differences in students' responses to related questions dealing with slopes of linear equations were reported in a study by Davis and McGowen (2007).

The questions and responses of 92 community college students enrolled in an Introductory Algebra course are shown below (Table 2.1). Note that only 20 of the 92 students were able to answer three of the five questions correctly, suggesting these students lack robust understanding of the mathematics they are learning (p. 24).

Table 2.1 Related responses to questions on linear equations, slopes, and intercepts

Question	$n = 92$	Correct (%)
7. Given slope -3 and y-intercept 5, select linear equation	65	71
4. Determine the x-intercept of the equation $2x - 7y = 12$	38	41
5. What is the vertical intercept of $y = mx + b$?	37	40
3. What is the slope of $Ax + By = C$?	24	26
9. Given the view window and graph, what is the equation?	13	14
Three of five questions answered correctly	20	22
Four of five questions answered correctly	6	7
All five questions answered correctly	1	1

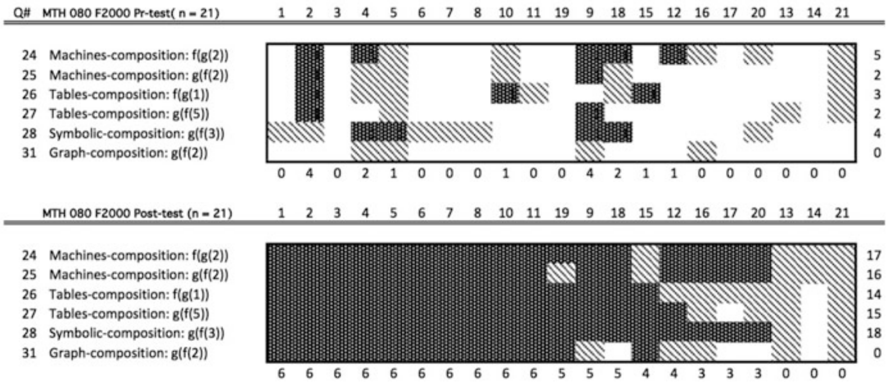


Fig. 2.3 Pre- and posttest responses on various representations of function composition

2.3.3 *Comparing Pre- and Posttest Responses*

A visual analysis comparing individual students' pre- and posttest responses to related questions is informative for students as well as for their instructor. Shown in Fig. 2.3 is an example of an analysis of students' responses to questions on various representations of function composition. An analysis of all pretest responses is shared with each student, identified by the column number which corresponds with a number on the individual student's returned test. Each student receives an analysis comparing all pretest responses to all posttest responses near the end of the course. Each column represents an individual student's responses, and each row represents the responses to a given question. A black cell indicates a correct response, a striped cell indicates an incorrect response, and a blank cell indicates no attempt to answer the question.

2.4 Function as an Organizing Lens: A Function Machine Representation and Technology

A NSF-funded developmental algebra curriculum (DeMarois, McGowen & Whitkanack, 1996a) has been shown to deepen developmental algebra students' understanding of mathematics, make sense of mathematical notation that increase skills proficiency, and provide them with opportunities to examine and reconstruct problematic prior learning (Davis & McGowen, 2002; DeMarois, 1998; DeMarois & McGowen, 1996b; McGowen, 1998; McGowen, DeMarois, & Tall, 2000; Tall, McGowen, & DeMarois, 2000). The unifying concept of function and difference equations facilitated students' ability to see connections and link fundamental ideas. Constant finite differences and ratios were used to develop sequences as functions, determine parameters, and develop models of linear, exponential, and polynomial functions.

A significant finding from this research on Introductory and Intermediate Algebra students is that initially there was a spectrum of interpretations from students who saw only a process, such as $2 + 4$ meaning "two is added to four," to those who could view notation flexibly, seeing the expression $2 + 4$ not only as a process but as the concept, "a sum," and $2x + 5$ not only as the process of addition but also as the concept "expression." Students who successfully completed the course were found to be prepared for continued study of increasingly sophisticated mathematical ideas in both STEM and non-STEM courses.

Using the function concept and the graphing calculator results in a coherent sequence different from the traditional ordering of algebraic topics as shown below in Fig. 2.4. Developmental algebra students were able to make connections and generalize algebraic linear, exponential, and quadratic models from data. They gained a deeper understanding of parameters and increased flexibility of thinking.

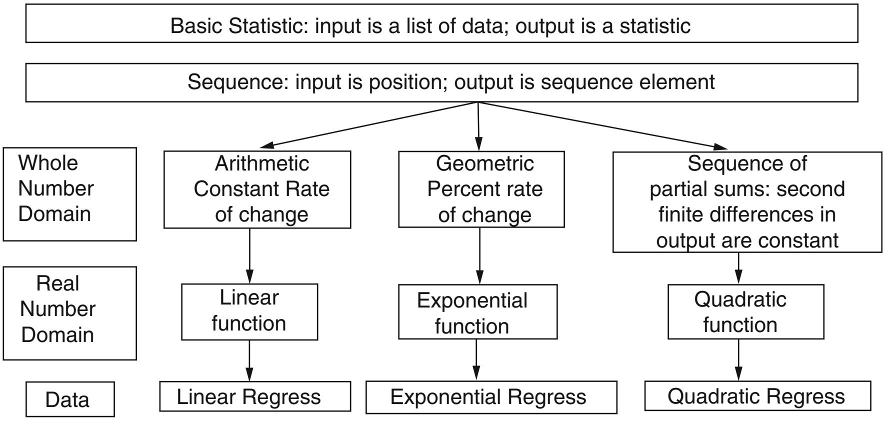


Fig. 2.4 Intermediate algebra course sequence

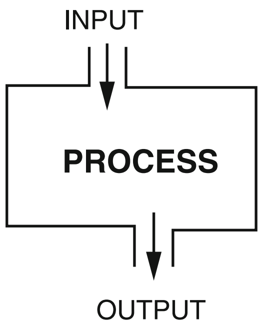


Fig. 2.5 Function machine representation

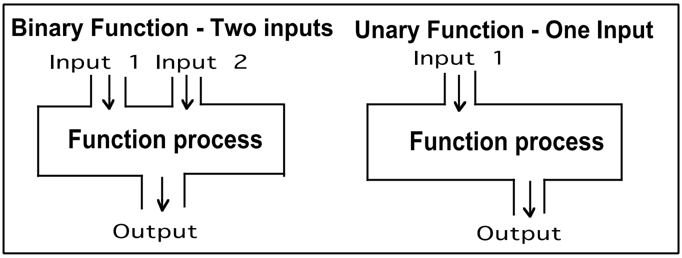


Fig. 2.6 Function machine representations: binary and unary processes

Functions viewed as input-output machines were studied in mathematics education as far back as 1965 by Peter Braunfeld. The function machine representation shown in Fig. 2.5, introduced as a visual representation for the concept of function seen as an input/output process, was found to be an accessible starting point for many developmental students (Davis & McGowen, 2002; DeMarois, 1998; McGowen, 1998; McGowen, 2006; McGowen, DeMarois & Tall, 2000). The function machine representation becomes a meaningful unit of core knowledge leading to more meaningful understanding of function, domain, range, and notation and was found to be a supportive met-before (McGowen & Tall, 2010; Tall, McGowen & DeMarois, 2000).

2.4.1 Understanding Binary and Unary Arithmetic Processes

The function machine representation in Fig. 2.6 is an effective visual means of distinguishing between binary and unary arithmetic operations (DeMarois, McGowen, & Whitkanack, 1996a, p. 23).

Students also find the function machine representation, together with the graphing calculator, helpful when reexamining their prior understanding of the minus sign and how meaning changes in different contexts. In their text, DeMarois,

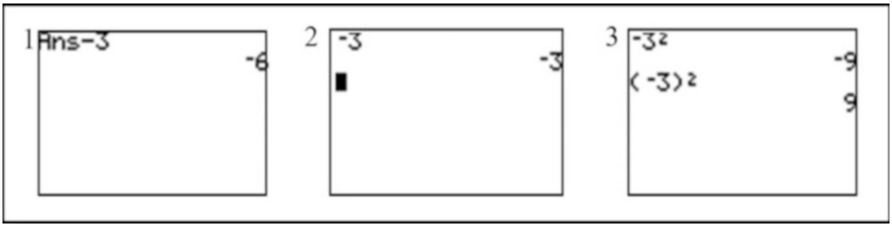


Fig. 2.7 TI-83 view screen of binary and unary operations

McGowen, and Whitkanack (1996a) discuss three distinct meanings of the minus sign: a binary arithmetic function requiring two inputs (subtraction), an object (a negative number), and a unary function requiring a single input (the "opposite of x" denoted by $-x$). Two different calculator keys input the minus sign: one indicating a binary operation, subtraction, and the other signifying a negative number or a unary operation, the additive inverse. Together with the function machine representation, students investigate three tasks: (1) subtract three; (2) type in the opposite of three; and (3) compare the results of entering -3^2 without using brackets and with brackets, typing $(3)^2$ (Fig. 2.7).

Pre- and posttests which ask students to evaluate $(5)^2$ and -5^2 were administered to two cohorts of students: 121 community college Intermediate Algebra students using the function machine approach and 140 university students using a traditional textbook. The responses of both cohorts were reported by McGowen and Tall (2013). Overall, both sets of students perform poorly in solving both questions correctly on the pretest. Both groups improved from pretest to posttest, with the community college students who had experienced the function machine strategy and the graphic calculator, improving more.

2.4.2 Clarifying Understanding of Terms

Students' understandings of "evaluating an expression" and "solving an equation" are clarified by examining them as functional processes, shown in Fig. 2.8.

In a preservice elementary teacher content course, an initial focus on building connections between different representations of a problem of binary choice led to students describing connections between building towers, grid walks, and binomial expansions. Followed by investigations of sequences as functions similar to those in the developmental algebra course, the prospective elementary teachers demonstrated deeper understanding of content, improved ability to generalize various arithmetic and geometric sequences, and improved skill competency, as well as changed attitudes and beliefs about mathematics (Davis & McGowen, 2002).

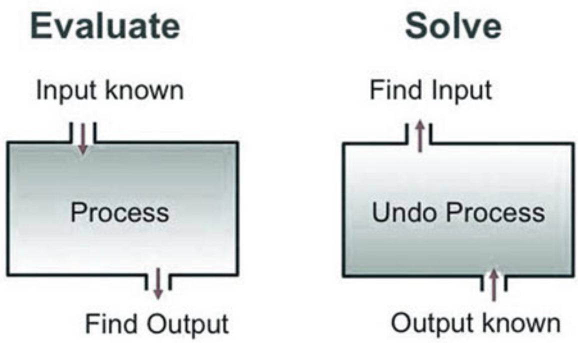


Fig. 2.8 Comparing the processes of evaluating vs. solving

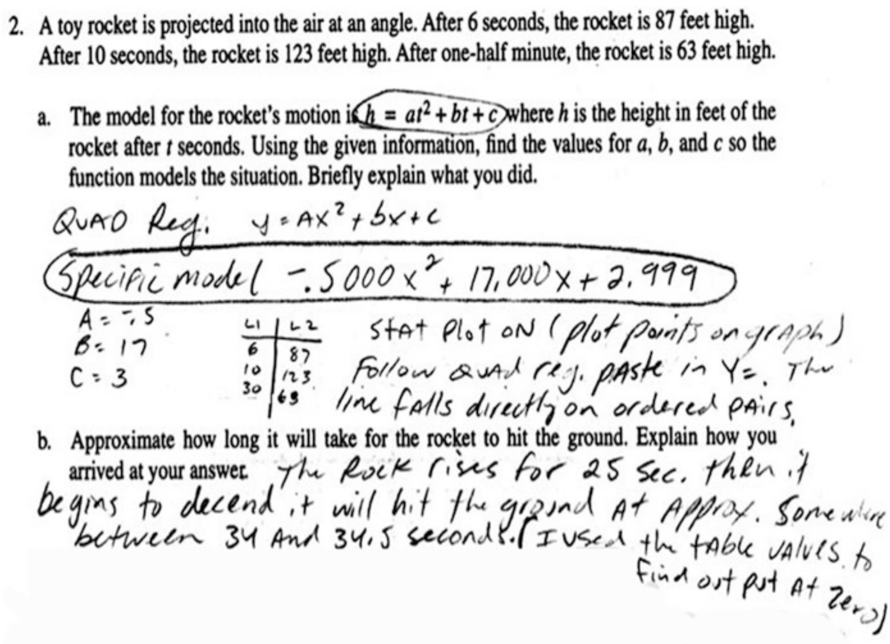
2.5 An Open Question: Which Computational Skills Are Essential with Today's Technology?

The effective utilization of graphing calculators in the teaching and learning of mathematics has not yet been incorporated into the classroom by many instructors of developmental mathematics. Which pencil and paper skills students in a given course must be able to demonstrate remains an open and divisive question in mathematics departments, given the technology available today. Instructors, depending upon their beliefs and assumptions, have different orientations about the purpose and use of graphing calculators. Some instructors consider graphing calculators only useful as a means to check homework and do not permit their use on exams, fearing students' computational skills will deteriorate. Some use them to teach traditional mathematics topics sequenced in the traditional order with a more efficient, dynamic, or appealing presentation. Still others realize that using the graphing calculator effectively transforms the curriculum, necessarily altering the character of knowledge as well as the sequence and content of the curriculum, thus raising questions as to which computational skills are essential.

Heated exchanges still occur among instructors who hold differing beliefs about what students should know. At a university faculty workshop, participants were provided with the work of a developmental algebra student on the following problem and asked how they would evaluate the student's work:

> A toy rocket is projected into the air at an angle. After 6 seconds, the rocket is 87 feet high. After 10 seconds, the rocket is 123 feet high. After one-half minute, the rocket is 63 feet high.
>
> (a) The model for the rocket's motion is $h(\mathrm{t}) = at^2 + bt + c$ where h is the height in feet of the rocket after t seconds. Using the given information, find the values for a, b, and c, so the function models the situation. Briefly explain what you did.
> (b) Approximate how long it will take for the rocket to hit the ground. Explain how you arrived at your answer.

The student set up a linear system in three variables by creating an input/output list on his calculator and used quadratic regression to find the parameter values. He



2. A toy rocket is projected into the air at an angle. After 6 seconds, the rocket is 87 feet high. After 10 seconds, the rocket is 123 feet high. After one-half minute, the rocket is 63 feet high.

a. The model for the rocket's motion is $h = at^2 + bt + c$ where h is the height in feet of the rocket after t seconds. Using the given information, find the values for a, b, and c so the function models the situation. Briefly explain what you did.

QUAD Reg. $y = Ax^2 + bx + c$

Specific model $-.5000x^2 + 17.000x + 2.999$

A = −.5
B = 17
C = 3

L1	L2
6	87
10	123
30	63

Stat Plot ON (plot points on graph) Follow QUAD reg. paste in Y=. The line falls directly on ordered pairs.

b. Approximate how long it will take for the rocket to hit the ground. Explain how you arrived at your answer. The Rock rises for 25 sec. then it begins to decend, it will hit the ground At Approx. Somewhere between 34 And 34.5 seconds. (I used the table values to find out put at zero)

Fig. 2.9 Intermediate algebra student's work

then wrote the equation that models the situation, describing how far the rocket would rise before it began its descent and when it would hit the ground. His work is shown below (Fig. 2.9).

Faculty participants discussed how they would grade this work. Some instructors maintained that they would give the students no credit because he hadn't solved the linear system algebraically. Other instructors argued that the student should receive full credit as the response demonstrated very good understanding of the problem and his responses were correct. When the workshop facilitator asked participants: "Given the technology available today when will students be asked to solve a 3 × 3 linear system using pencil and paper outside of the classroom?" No one provided an answer to the question.

2.6 Conclusion

Teachers at all levels are faced with the result of the accumulated detritus of students' fragmented prior knowledge as a result of their earlier mathematical experiences. Though teachers cannot be expected to deal with all the problems that arise from previous learning, *at each stage* they need to be aware of the problematic met-befores of their individual students and counsel them accordingly. The failure to understand how students are thinking results in speaking past their difficulties and increasing disaffection on the part of students. Effective instruction is dependent on *how students use what they have learned before* and how what is taught *will affect what the students learn later*.

The problematic met-befores discussed in this chapter, including prior learning of arithmetic and the ambiguous minus symbol when faced with changes in meaning in algebraic and function notation, illustrate some of the complex aspects of learning algebra. A lack of conceptual understanding of the meaning of “solving an equation” compared with “evaluating an expression” and the failure to distinguish slope from *x*- and *y*-intercepts or coefficients contribute to the lack of success. These problematic aspects of learning algebra generally go undiagnosed and unaddressed by instructors and curriculum developers at all levels.

Explicit rethinking and reflecting on the longer-term effects of learning are essential if we are to improve student learning and success at every level—from the elementary grades through college and university. A focus on the specific changes in meaning as mathematics becomes more sophisticated and contexts change the meaning of what has been learned previously is a critical component in these efforts. Incorporating the use of formative assessment is essential in order to clarify students’ difficulties at each stage of their development and adapt instruction to better meet their needs.

Following the Curriculum Foundations report, several alternative innovative courses designed to better meet the needs of students in technical and non-STEM career programs have been developed and adopted in many undergraduate and community college programs. A similar initiative is required to address the critical mathematical needs of our many future teachers and non-STEM students who will be required to have a much more solid mathematical foundation in arithmetic and algebra than they graduate with currently. As Einstein is reported to have said:

> Insanity is doing the same thing over and over again and expecting different results.
>
> We can’t solve problems by using the same kind of thinking we used when we created them.

Currently lacking for students who need a deeper understanding of arithmetic and algebra are developmental courses and tasks designed to change the procedural orientation and superficial, fragmented knowledge of too many of our students. The curriculum, particularly for non-STEM undergraduate students along with those who intend to become teachers of mathematics, whether at the elementary, high school, or college level, should include a course that requires students to examine the long-term cognitive development of mathematical thinking and understanding of foundational arithmetic and algebraic concepts. Such a course would provide experiences in which students identify and analyze prior experiences that support new learning and identify situations in which prior learning can become problematic or be supportive.

Changing students’ beliefs about the nature of mathematics and what it means to learn mathematics remains as much of a challenge today as it was more than 500 years ago when Robert Record (1543), in *The Grounde of Artes*, described the danger of rote learning and the resulting conflicting perspectives of teachers and students:

> Master: . . .I wil propounde here ii examples to you whiche if you often doo practice, you shall be rype and perfect to subtract any other summe lightly. . .
>
> Scholar: Sir, I thanke you, but I thynke I might the better doo it, if you did showe me the workinge of it.
>
> Master: Yes, but you muste prove yourselfe to doo som thnges that you were never taught, or els you shall not be able to doo any more than you were taught, and were rather to learne by rote (as they cal it) than by reason.

Mathematicians and mathematics educators must be willing to adjust their beliefs and assumptions about students' learning and incorporate what is known about the long-term cognitive development of mathematical thinking into their instructional practices. As Stephen Crane (1972) wrote: "The wayfarer, perceiving the pathway to truth, was struck with astonishment. It was thickly grown with weeds. . . Later he saw that each weed was a singular knife. 'Well,' he mumbled at last, 'Doubtless there are other roads'."

Complaining about what students can't do is no longer an option. Too many students are failing. Our challenge is to provide tasks and opportunities for students to engage in explicit examination of their prior knowledge when confronted with new situations and contexts. Recognizing that learning "just algebra" is a much more complex task than it appears. Incorporating what is known about the long-term cognitive development of mathematical thinking into instruction can result in changing how and what is taught so that more students who need to learn algebra can do so meaningfully and effectively.

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