

# Chapter 1

## Mathematical Foundation of Quantum System

**Abstract** This book covers several topics in quantum information via group representation. For this purpose, this chapter introduces basic concepts of quantum theory, measurement, state, composite system, many-body system, and entanglement. Since a larger part of this chapter overlaps Chap. 1 of my book, *Group Representations for Quantum Theory* [44], the reader who has already read it can skip this part. However, since Sects. 1.1 and 1.2 contain parts that do not appear in the above book, the reader need to read such a part. Important notations used in this book are introduced in this chapter. Chapter 4 of the other book *Introduction to Quantum Information Science* [43] can be recommended as the detail of this contents.

### 1.1 System, State, and Measurement

In quantum theory that describes microscopic physics, the target is called a **quantum system** or a **system**, and mathematically denoted by a complex Hilbert space  $\mathcal{H}$  with a Hermitian inner product. A complex vector space  $\mathcal{H}$  is called a **Hilbert space** when it equips a Hermitian inner product. Even when its dimension is finite.<sup>1</sup> Since  $\mathcal{H}$  has a Hermitian inner product, there exists a **completely orthonormal system (CONS)**  $\{e_i\}$ . Each normalized base  $e_i$  represents a state in the quantum system that is distinguished from each other. An arbitrary state of the system is given as a normalized vector  $x \in \mathcal{H}$ , which is called a **state vector**. Once we fix a CONS as a standard basis, any vector  $x$  describing a state is written as a superposition (a linear combination)  $\sum_i x_i e_i$ . Quantum theory has two types of notations for an element  $x$  of  $\mathcal{H}$ . One is a ket vector  $|x\rangle$ , and the other is a bra vector  $\langle x|$ . These descriptions are defined so that they have linearity with respect to real coefficients and they satisfy

$$|ax\rangle = a|x\rangle, \quad \langle ax| = \bar{a}\langle x|. \quad (1.1)$$

for a complex number  $a \in \mathbb{C}$ . In particular, for a standard basis  $e_i$ ,  $|e_i\rangle$  and  $\langle e_i|$  are simplified to  $|i\rangle$  and  $\langle i|$ , respectively.

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<sup>1</sup>When its dimension is infinite, we assume that the space satisfies the **completeness** under the given Hermitian inner product.

On the other hand, when the Hermitian inner product  $\langle x|y\rangle$  for  $x, y \in \mathcal{H}$  is defined so that  $\langle ax|by\rangle = \bar{a}b\langle x|y\rangle$  for  $a, b \in \mathbb{C}$ , the inner product  $\langle x|y\rangle$  of  $x$  and  $y$  can be regarded as the product  $\langle x| \cdot |y\rangle$  of the bra vector and the ket vector. In this case, the product  $|y\rangle\langle x|$  with the opposite way can be regarded as a linear map from  $\mathcal{H}$  to  $\mathcal{H}$ . When  $x \in \mathcal{H}$  is a normalized vector,  $|x\rangle\langle x|$  is a one-dimensional projection. We often denote the state corresponding to the normalized vector  $x \in \mathcal{H}$  by the one-dimensional projection  $|x\rangle\langle x|$ . More generally, a state of quantum system  $\mathcal{H}$  is described by a Hermitian matrix  $\rho$  with non-negative eigenvalues and trace 1. Such a Hermitian matrix  $\rho$  is called a **density matrix (density operator)**. Also, a Hermitian matrix is called **positive semi definite** when all of its eigenvalues are non-negative. In particular, when a density matrix  $\rho$  is given a one-dimensional projection, it is called a **pure state**. A density matrix  $\rho$  that is not a pure state is called a **mixed state**. Indeed, a normalized vector  $|x\rangle$  expresses a pure state  $|x\rangle\langle x|$  in the above sense. When we use a normalized vector  $|x\rangle$  to express the pure state, the normalized vector  $|x\rangle$  is called a **vector state**. Further, the density matrix  $\rho_{\text{mix}} := \sum_{i=1}^d \frac{1}{d} |i\rangle\langle i|$  on the  $d$ -dimensional system  $\mathcal{H}$  is called the **completely mixed state**. In particular, when we need to clarify the quantum system  $\mathcal{H}$  of our interest, we denote the completely mixed state by  $\rho_{\text{mix}, \mathcal{H}}$ .

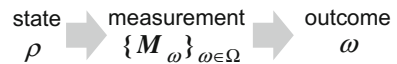
A measurement is given as a decomposition  $\{M_\omega\}_{\omega \in \Omega}$  of the unit matrix  $I$  by positive semi-definite matrices on  $\mathcal{H}$  (i.e.,  $\sum_{\omega \in \Omega} M_\omega = I$ ) [72]. Here,  $\Omega$  is the set of possible outcomes, and is called the probability space. When the state of the system is given as the density matrix  $\rho$  as Fig. 1.1, the probability to obtain the outcome  $\omega \in \Omega$  is  $\text{Tr } \rho M_\omega$ . Since  $\rho$  and  $M_\omega$  are positive semi definite, the value  $\text{Tr } \rho M_\omega$  is always non-negative. Further, the above conditions guarantees that  $\sum_{\omega \in \Omega} \text{Tr } \rho M_\omega = \text{Tr } \rho \sum_{\omega \in \Omega} M_\omega = \text{Tr } \rho I = 1$ , we find that  $\text{Tr } \rho M_\omega$  satisfies the axioms of the probability. This probability distribution is written as  $P_M^\rho$ . Such a decomposition  $\{M_\omega\}_{\omega \in \Omega}$  of the unit matrix  $I$  by positive semi definite matrices is called a **Positive operator-valued measure (POVM)**. Especially, when all of  $M_\omega$  are projections, it is called a **Projection valued measure (PVM)**.

Next, let us discuss the case when the set of outcomes is a general topological space  $\Omega$ . In this case, we cannot assign the matrix  $M_\omega$  corresponding to the outcome  $\omega$  in the same way. When a measure  $\nu(d\omega)$  on  $\Omega$ , the integral

$$\int_{\Omega} M_\omega \nu(d\omega) = I \quad (1.2)$$

can be regarded as the decomposition of the unit matrix  $I$ . So,  $\{M_\omega\}$  form a POVM. However, a POVM on a general topological space  $\Omega$  does not necessarily have the above form. Hence, we need to treat a function of a subset of  $\Omega$  for describing an arbitrary POVM. Since  $\Omega$  is a topological space, we consider the Borel sets  $\mathcal{B}(\Omega)$

Fig. 1.1 Measuring process



that are generated by open sets of  $\Omega$ . Then, a map  $M$  from  $\mathcal{B}(\Omega)$  to  $\mathcal{B}_+(\mathcal{H})$  is called a POVM when it satisfies the following conditions. Here, we denote the set of positive semi definite matrices by  $\mathcal{B}_+(\mathcal{H})$ . When  $\mathcal{H}$  is infinite-dimensional,  $\mathcal{B}_+(\mathcal{H})$  is the set of positive semi definite bounded operators.

- (M1) When the sets  $B_j \in \mathcal{B}(\Omega)$  satisfy  $B_j \cap B_l = \emptyset$  for distinct elements  $j, l$ , we have  $\sum_j M(B_j) = M(\cup_j B_j)$ .
- (M2)  $M(\emptyset) = 0$ .
- (M3)  $M(\Omega) = I$ .

In particular,  $M$  is a PVM when  $M(B)$  is a projection for any  $B \in \mathcal{B}(\Omega)$ . Given a POVM  $M$ , we can extend the Hilbert space so that  $M$  can be written as a restriction of a PVM  $E$  on the enlarged system [99].

Here, we introduce notations for probability. We denote the probability that the occurring event belongs to the set  $S$  under the distribution  $Q$  by  $Q(S)$ . When  $S$  is given as  $\{\omega \mid \text{condition for } \omega\}$ , this notation is simplified to  $Q\{\omega \mid \text{condition for } \omega\}$ . When we focus on the random variable  $X(\omega)$ , we denote its expectation under the distribution by  $E_Q[X(\omega)]$ .

For a Hermitian matrix  $A$ , we denote the eigenvalues by  $\{a_i\}$ , and the projection to the eigenspace with the eigenvalue  $a_i$  by  $E_i$ . Then, we have a PVM  $E = \{E_i\}_i$ . Further, we have the relation  $A = \sum_i a_i E_i$ , which is called the spectral decomposition of the Hermitian matrix  $A$ . When the state is given as a density operator  $\rho$ , the expectation and the variance are given by  $\text{Tr } A\rho$  and  $\Delta_\rho^2 A := \text{Tr}(A - (\text{Tr } A\rho)I)^2 \rho$ , respectively. That is, we have  $E_{P_E}[a_i] = \text{Tr } A\rho$  and  $E_{P_E}[(a_i - \text{Tr } A\rho)^2] = \Delta_\rho^2 A$ .

When  $\mathcal{H}$  is infinite-dimensional, using a PVM, we can define the spectral decomposition of an operator on  $\mathcal{H}$ , which can be regarded as the infinite-dimensional extension of diagonalization. In this case, a self-adjoint operator  $A$  on  $\mathcal{H}$  does not necessarily have an eigenvector. However, for a self-adjoint operator  $A$  on  $\mathcal{H}$ , there uniquely exists a PVM  $E$  with the probability space  $\mathbb{R}$  such that

$$A = \int_{\mathbb{R}} x E(dx). \quad (1.3)$$

Here, this fact holds even when the self-adjoint operator  $A$  is not bounded [109]. When  $A$  is a unitary, the same fact holds by replacing  $\mathbb{R}$  by the unit circle  $U = \{z \in \mathbb{C} \mid |z| = 1\}$ . In this way, we describe a measurement of a physical quantity given as a self-adjoint operator  $A$  on a infinite dimensional space  $\mathcal{H}$ . When the state is given as a density operator  $\rho$ , the expectation and the variance are similarly given by  $\text{Tr } A\rho$  and  $\Delta_\rho^2 A$ , respectively.

In fact, a POVM not only gives a probability distribution for the measurement outcome, but also gives a convex decomposition of a density matrix  $\rho$ . That is, since  $\sum_\omega M_\omega = I$ , we have

$$(\text{Tr } M_\omega \rho) \sum_\omega \frac{\sqrt{\rho} M_\omega \sqrt{\rho}}{\text{Tr } M_\omega \rho} = \sum_\omega \sqrt{\rho} M_\omega \sqrt{\rho} = \sqrt{\rho} I \sqrt{\rho} = \rho. \quad (1.4)$$

For a convex decomposition  $\rho = \sum_{\omega} p_{\omega} \rho_{\omega}$  of a density matrix  $\rho$ , we can define the POVM  $M_{\omega} := \rho^{-\frac{1}{2}} p_{\omega} \rho_{\omega} \rho^{-\frac{1}{2}}$ . Then, we can reconstruct the convex decomposition  $\rho = \sum_{\omega} p_{\omega} \rho_{\omega}$  by using (1.4).

**Exercise 1.1** Consider the case when  $M_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} \end{pmatrix}$ ,  $M_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} \end{pmatrix}$ ,  $M_3 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$ , and  $\rho = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ . Calculate the distribution  $P_M^{\rho}$ .

**Exercise 1.2** Give the spectral decomposition of  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

**Exercise 1.3** Calculate the average when the measurement is the spectral decomposition of  $A$  given in Exercise 1.2 and the state is  $\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

**Exercise 1.4** Calculate the variance in Exercise 1.3.

**Exercise 1.5** Focus on the probabilistic decomposition of density matrix  $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ . Give a POVM based on the decomposition.

## 1.2 Composite System

Consider two quantum systems  $\mathcal{H}_A$  and  $\mathcal{H}_B$  that are distinguishable from each other. For example, when the quantum system  $\mathcal{H}_A$  expresses the internal system of a particle and the quantum system  $\mathcal{H}_B$  expresses the position of the particle, we need the quantum system that describes the freedom of the whole particle. Such a quantum system is called the composite system of the systems  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . The composite system is given by the tensor product space  $\mathcal{H}_A \otimes \mathcal{H}_B$ . The tensor product space  $\mathcal{H}_A \otimes \mathcal{H}_B$  is defined as the space spanned by the CONS  $\{|v_i, u_j\rangle\}_{1 \leq i \leq k, 1 \leq j \leq l}$  when  $\{|v_i\rangle\}_{i=1}^k$  and  $\{|u_j\rangle\}_{j=1}^l$  are CONS of  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively.

Next, we consider the case when the states of the systems  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are prepared to be the density matrices  $\rho$  and  $\sigma$ , respectively. In this case, when  $\rho = \sum_{i,i'=1}^k a_{i,i'} |v_i\rangle \langle v_{i'}|$  and  $\sigma = \sum_{j,j'=1}^l b_{j,j'} |u_j\rangle \langle u_{j'}|$ , the state of the composite system is the **tensor product state**  $\rho \otimes \sigma := \sum_{i,i'=1}^k \sum_{j,j'=1}^l a_{i,i'} b_{j,j'} |v_i, u_j\rangle \langle v_{i'}, u_{j'}|$ . A state of the composite system  $\mathcal{H}_A \otimes \mathcal{H}_B$  cannot be restricted to a tensor product state  $\rho \otimes \sigma$  or its convex combination  $\sum_j p_j \rho_j \otimes \sigma_j$ , which is called a **separable state**. (Here,  $p_j$  expresses the probability distribution and  $\rho_j$  and  $\sigma_j$  express density matrices on  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively.) For example, when a state is given by the pure state corresponding to the vector  $\sum_j \sqrt{p_j} |v_j, u_j\rangle$ , it cannot be written as a convex combination of tensor product state, then, is not separable. Such a state is called

**entangled**, and has been studied very actively. Especially, such a property of a state is called **entanglement**.

When a state of the composite system  $\mathcal{H}_A \otimes \mathcal{H}_B$  is given by

$$\rho = \sum_{i,i'=1}^k \sum_{j,j'=1}^l c_{i,j,i',j'} |v_i, u_j\rangle \langle v_{i'}, u_{j'}|,$$

there exists a density matrix  $\text{Tr}_{\mathcal{H}_B} \rho$  on the system  $\mathcal{H}_A$  such that the relation

$$\text{Tr}(\text{Tr}_{\mathcal{H}_B} \rho) X = \text{Tr} \rho (X \otimes I_{\mathcal{H}_B})$$

holds for a matrix  $X$  on the system  $\mathcal{H}_A$ . In this case, when we focus on only the system  $\mathcal{H}_A$ , it is suitable to consider that the state of the system  $\mathcal{H}_A$  is the density matrix  $\text{Tr}_{\mathcal{H}_B} \rho$ . The state  $\text{Tr}_{\mathcal{H}_B} \rho$  is called the **reduced density matrix** of  $\rho$ , and is calculated by

$$\text{Tr}_{\mathcal{H}_B} \rho = \sum_{i,i'=1}^k \sum_{j=1}^l c_{i,j,i',j} |v_i\rangle \langle v_{i'}|.$$

The operation of taking the reduced density matrix is called the **partial trace**. When the density matrix  $\rho$  is diagonal with respect to the basis  $\{|v_i, u_j\rangle\}_{i,j}$ , the reduced density matrix is the same as the marginal distribution of the probability distribution composed of the diagonal elements. When there is a possibility of confusion, we simplify  $\text{Tr}_{\mathcal{H}_B} \rho$  to  $\text{Tr}_B \rho$ .

Further, if there is a possibility of confusion, for a matrix  $X$  on  $\mathcal{H}_A$  and a matrix  $Y$  on  $\mathcal{H}_B$ , we simplify the matrices  $X \otimes I_{\mathcal{H}_B}$  and  $I_{\mathcal{H}_A} \otimes Y$  on  $\mathcal{H}_A \otimes \mathcal{H}_B$  to  $X$  and  $Y$ , respectively. So, the matrix  $X \otimes Y$  is simplified to  $XY$ . Now, we explain our abbreviations that are applied to the case when standard bases of the systems  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . In this case, we denote the ket vectors of standard basis of both systems by  $|j\rangle_A$  and  $|j\rangle_B$  so that the system of the ket vector can be distinguished. Hence, the standard basis of the composite system  $\mathcal{H}_A \otimes \mathcal{H}_B$  is given by  $|k\rangle_A \otimes |j\rangle_B$ , which is simplified to  $|k, j\rangle_{A,B}$ .

In the following discussion, for a matrix  $X = (x_{k,j})$ , we denote the matrix composed of the **complex conjugate**  $\overline{x_{k,j}}$  of each entries with respect to the standard basis by  $\overline{X}$ , and denote the **transposed matrix** with respect to the standard basis by  $X^T$ . Then,  $X^\dagger$  expresses the **transposed complex conjugate matrix** of  $X$ . The matrices  $\overline{X}$  and  $X^T$  depend on the choice of the standard basis, however, the matrix  $X^\dagger$  depends only on the definition of the Hermitian matrix. Then, we denote the vector  $\sum_{k,j} x_{k,j} |k, j\rangle_{A,B}$  on the composite system  $\mathcal{H}_A \otimes \mathcal{H}_B$  by  $|X\rangle_{A,B}$ . So, we have

$$Y \otimes Z |X\rangle_{A,B} = |YXZ^T\rangle_{A,B}. \quad (1.5)$$

The inner product of two vectors  $|X\rangle\rangle_{A,B}$  and  $|Y\rangle\rangle_{A,B}$  is calculated to be  ${}_{A,B}\langle\langle X|Y\rangle\rangle_{A,B} = \text{Tr } X^\dagger Y$ . Hence, a vector  $|X\rangle\rangle_{A,B}$  is normalized (the norm is 1) if and only if  $\text{Tr } X^\dagger X = 1$ . Thus, we have the following formula with respect to the partial trace:

$$\text{Tr}_B |X\rangle\rangle_{A,B} {}_{A,B}\langle\langle X| = XX^\dagger, \quad \text{Tr}_A |X\rangle\rangle_{A,B} {}_{A,B}\langle\langle X| = X^T \bar{X}. \quad (1.6)$$

Next, let us diagonalize the matrices  $XX^\dagger$  and  $X^\dagger X$  by using the isometries  $U$  and  $V$  in the following way:

$$U^\dagger X V V^\dagger X^\dagger U = U^\dagger X X^\dagger U = D, \quad V^\dagger X^\dagger U U^\dagger X V = V^\dagger X^\dagger X V = D,$$

where  $D$  is the diagonal matrix whose diagonal elements are in the decreasing order and are non-negative. The rank of  $D$  is equal or less than the dimensions of  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . In this composite system, the rank of the matrix  $D$  is called the **Schmidt rank** of the state vector  $|X\rangle\rangle$ . So, the Schmidt rank is equal or less than the dimensions of  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . The diagonal elements of  $\sqrt{D}$  is called the **Schmidt coefficient** of the state vector  $|X\rangle\rangle$ . Since the matrix  $U^\dagger X V$  and its transposed complex conjugate matrix  $V^\dagger X^\dagger U$  are commutative with each other, the squares of the absolute values of the diagonal elements of  $U^\dagger X V$  equal the eigenvalues of  $\sqrt{D}$ . Then, considering a diagonal matrix  $D'$  whose diagonal elements have the absolute value 1, we obtain  $X = U D' \sqrt{D} V^\dagger$ . Rewriting the isometry  $U D'$  to  $U$ , we obtain the **Schmidt decomposition** of  $X$  as

$$X = U \sqrt{D} V^\dagger. \quad (1.7)$$

Let us apply this fact to an arbitrary state vector  $|a\rangle$  on the composite system  $\mathcal{H}_A \otimes \mathcal{H}_B$ . There exist a CONS  $\{|v_i\rangle\}$ ,  $\{|u_j\rangle\}$  and Schmidt coefficients  $d_1, \dots, d_k$  such that

$$|a\rangle = \sum_{j=1}^k d_j |v_j\rangle \otimes |u_j\rangle \quad (1.8)$$

when the dimension  $k$  of  $\mathcal{H}_A$  is not greater than the dimension  $l$  of  $\mathcal{H}_B$ . In some literatures of quantum information, (1.8) is often called the Schmidt decomposition rather than (1.7). The bases  $\{|v_i\rangle\}$  and  $\{|u_j\rangle\}$  given in (1.8) are called the Schmidt bases.

Given a mixed state  $\rho$  on the system  $\mathcal{H}_A$ , a vector state  $|X\rangle\rangle_{A,B}$  on the composite system  $\mathcal{H}_A \otimes \mathcal{H}_B$  is called a **purification** of  $\rho$  with the **reference system**  $\mathcal{H}_B$  when

$$\rho = \text{Tr}_B |X\rangle\rangle_{A,B} {}_{A,B}\langle\langle X|. \quad (1.9)$$

Hence, a state vector  $|X\rangle\rangle_{A,B}$  is a purification of  $\rho$  if and only if  $XX^\dagger = \rho$ . This condition is equivalent to the following condition: There exists a partial isometry  $V$  such that the support of  $V$  is the range of  $X^\dagger X$  and  $X$  is written as

$$X = \sqrt{\rho} V. \quad (1.10)$$

This fact shows that the purification of  $\rho$  is uniquely determined up to the freedom with respect to application of the partial isometry on the reference system.

Especially, when the dimensions of both systems  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are  $d$  and all of the diagonal elements of  $D$  satisfying (1.7) are  $\frac{1}{d}$ , the state vector  $|X\rangle_{A,B}$  is called a **maximally entangled state**. This condition is equivalent to the condition of  $X$  being a unitary matrix.

**Exercise 1.6** Calculate the Schmidt rank and the Schmidt coefficients of  $|X\rangle$  when

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

**Exercise 1.7** Show (1.5).

**Exercise 1.8** Show  $\text{Tr}_B |X\rangle_{A,B} \langle Y|_{A,B} = XY^\dagger$ .

**Exercise 1.9** Give a purification of  $\rho = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$ .

**Exercise 1.10** Let  $\rho$  be a full-rank density matrix on  $\mathcal{H}_A$ , and  $|X\rangle$  be its purification with the reference  $\mathcal{H}_B$ , whose dimension equal that of  $\mathcal{H}_A$ . Show that for any probabilistic decomposition  $\rho = \sum_i p_i \rho_i$ , there exists a POVM  $M = \{M_i\}_i$  on  $\mathcal{H}_B$  such that  $p_i \rho_i = \text{Tr}_B M_i |X\rangle \langle X|$ .

**Exercise 1.11** Let  $\rho$  be a density matrix on  $\mathcal{H}_A$ . Show that, for any probabilistic decomposition  $\rho = \sum_i p_i |x_i\rangle \langle x_i|$  with pure states, there exist another system  $\mathcal{H}_B$ , a purification  $|X\rangle$  of  $\rho$  on  $\mathcal{H}_B$ , and a PVM  $E = \{E_i\}_i$  on  $\mathcal{H}_B$  such that  $p_i |x_i\rangle \langle x_i| = \text{Tr}_B E_i |X\rangle \langle X|$ .

**Exercise 1.12** Calculate the distribution of the outcome when  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are  $d$ -dimensional systems, the state is  $\rho_A \otimes \rho_{\text{mix},B}$ , and the measurement is written as  $\{\frac{1}{d} |U_i\rangle \langle U_i|\}_{i=1}^{d^2}$  by using  $d^2$  unitaries  $U_i$ .

### 1.3 Many-Body System

Let us consider the case when  $n$  particles are given and these particles are characterized by the quantum systems  $\mathcal{H}_i$  ( $i = 1, \dots, n$ ). When these particles are distinguishable from each other, the composite system corresponding to  $n$  particles is given by  $((\mathcal{H}_1 \otimes \mathcal{H}_2) \cdots) \otimes \mathcal{H}_n$ . Since the tensor product does not depend on the order of the tensor product, the above space is the same as  $(\mathcal{H}_1 \cdots (\mathcal{H}_{n-1} \otimes \mathcal{H}_n))$ . Hence, we simplify it to  $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ . Especially, when each system  $\mathcal{H}_i$  is isometric to  $\mathcal{H}$ , the tensor product space is simplified to  $\mathcal{H}^{\otimes n}$ . When the state of each system  $\mathcal{H}_i$  is independently prepared to be the density matrix  $\rho_i$ , the state of the composite system is given as the density matrix  $((\rho_1 \otimes \rho_2) \cdots) \otimes \rho_n$ . Since the

tensor product of the matrices does not depend on the order of the tensor product, this density matrix is denoted by  $\rho_1 \otimes \rho_2 \cdots \otimes \rho_n$ . Especially, when  $\rho_i = \rho$ , i.e., the  $n$  particles are independently prepared in the same  $\rho$ , the density matrix of the total system is denoted by  $\rho^{\otimes n}$ , and is called the  $n$ -fold **tensor product state** of the density matrix  $\rho$ . The above notation will be applied to the case when  $\rho$  and  $\rho_i$  are not necessarily density matrices, i.e., are general matrices.

Given a tensor product system  $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$  and matrix  $A$  on the tensor product system, when we take the partial trace with respect to the specific system  $\mathcal{H}_i$ , the reduced density matrix of  $A$  is denoted by  $\text{Tr}_{\mathcal{H}_i} A$ . Conversely, when we take the partial trace with respect to all of other systems except for the specific system  $\mathcal{H}_i$ , the reduced density matrix of  $A$  is denoted by  $\text{Tr}_{\check{\mathcal{H}}_i} A$ . When  $A$  is a density matrix  $\rho$ , the reduced density matrix  $\text{Tr}_{\check{\mathcal{H}}_i} \rho$  is simplified to  $\rho_{\mathcal{H}_i}$  or  $\rho_i$ .

On the other hand, when a matrix  $A_i$  on the system  $\mathcal{H}_i$ , the matrix  $I^{\otimes i-1} \otimes A_i \otimes I^{\otimes n-i}$  on the tensor product system  $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$  is simplified to  $A_i$ . Then, when  $A_i = A$ , we simplify  $\sum_{i=1}^n A_i$  to  $A^{(n)}$ .

In fact, this kind of description can be applied to more general case. That is, when we have  $n$  systems, we can define the composite system composed of these  $n$  systems. when  $n \geq 3$ , such a system is called a **many-body system**, a **multipartite system**, or a **multi-party system**. The term “many-body system” is mostly used when each system is given as one particle. Other two terms are used for a more general case. The term “multi-party system” is more used in the viewpoint of information science. The term “multipartite system” is used more often in quantum information. So, this book mainly uses the term “multipartite system”. To identify the number  $n$ , we use the term “ $n$ -partite system” to express the multipartite system. In contrast, when  $n = 2$ , to distinguish the case with  $n \geq 3$ , this system is called, a **two-body system**, a **bipartite system**, or a **two-party system**.

## 1.4 Guide for Related Literatures

Here, we introduce literatures that treat quantum information in the viewpoint of the representation theory. Although there are many books for quantum information [12, 42, 43, 45, 97, 101, 129], books with this viewpoint are limited. The first book with this viewpoint is Holevo’s book [72], whose Chaps. 3 and 4 discuss the optimization problems with respect to quantum measurement by using the representation theory on a finite-dimensional system. After Holevo’s book, several papers employed the representation theory for analysis of quantum information. As the next book of this viewpoint, we can list Christandl [22], which deals with quantum information based on the representation theory on a finite-dimensional system. Chapters 5 and 6 of [72] deals with quantum information in the Bosonic system. Latest progress of this direction is reviewed in Wang [125].



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