

Chapter 2

Theoretical Foundation of Finite Frequency Control

Finite frequency control strategy has been proven to be an important method for modern control system. Combined with the particular frequency characteristics of the plant, many control specifications in the full frequency domain can be simplified into finite frequency ones. Commonly used tools in the frequency division are the weighting function and general Kalman-Yakubovich-Popov (GKYP) Lemma. In this chapter, some background information and useful lemmas in the field of finite frequency control have been investigated in detail.

2.1 The Laplace Transform

The Laplace transform is used to solve linear constant coefficient differential equations, which acts as a function of a positive real variable t (often time) to a function of a complex variable s (frequency). On the other hand, the inverse Laplace transform is used to obtain a solution in terms of the original variables, which takes a function of a complex variable s to a positive real variable t . This techniques can be applied to both single differential equation and simultaneous differential equations. The Laplace transform can be used to produce TFMs to describe the elements of an engineering system. As the system elements, blocks are connected together to form the closed-loop diagram and to represent the characteristics of systems. Through decomposing a system in this way, it is much easier to visualize that how the various parts of the system interact. As a result, a TF model can be used to describe a time-domain model, which is particularly important in the control system design.

Definition 2.1 (*Laplace Transform*) Let $f(t)$ be a function of time t . The **Laplace transform** of $f(t)$ is $F(s)$, which is defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Table 2.1 The Laplace transform of some common functions

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	$\sinh bt$	$\frac{b}{s^2 - b^2}$
t	$\frac{1}{s^2}$	$\cosh bt$	$\frac{s}{s^2 - b^2}$
t^2	$\frac{2}{s^3}$	$e^{-at} \sinh bt$	$\frac{b}{(s+a)^2 - b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$e^{-at} \cosh bt$	$\frac{s+a}{(s+a)^2 - b^2}$
e^{at}	$\frac{1}{s-a}$	$t \sin bt$	$\frac{2bs}{s^2 + b^2}$
e^{-at}	$\frac{1}{s+a}$	$t \cos bt$	$\frac{s^2 - b^2}{s^2 + b^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$1(t)$	$\frac{1}{s}$
$\sin bt$	$\frac{b}{s^2 + b^2}$	$1(t-d)$	$\frac{e^{-sd}}{s}$
$\cos bt$	$\frac{s}{s^2 + b^2}$	$\delta(t)$	1
$e^{-at} \sin bt$	$\frac{b}{(s+a)^2 + b^2}$	$\delta(t)$	e^{-sd}
$e^{-at} \cos bt$	$\frac{s+a}{(s+a)^2 + b^2}$		

To find the Laplace transform of a function $f(t)$, we multiply it by e^{-st} and integrate between the limits 0 and ∞ . Determining the Laplace transform of a given functions $f(t)$ is essentially an exercise in integration. In Table 2.1, we have listed some common functions and their corresponding Laplace transform.

There are some useful properties of the Laplace transform that can be exploited, which are namely listed as follows.

1. Linearity:

Let $f(t)$ and $g(t)$ be two functions of time t , and k be a constant which may be negative, and then

$$\begin{aligned}\mathcal{L}\{f(t) + g(t)\} &= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, \\ \mathcal{L}\{kf(t)\} &= k\mathcal{L}\{f(t)\}.\end{aligned}$$

2. Shift theorems:

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{-at} f(t)\} = F(s + a),$$

where a is a constant.

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{u(t-d)f(t-d)\} = e^{-sd}F(s), \quad d > 0.$$

3. Final value theorem:

$$\lim_{s \rightarrow 0} sF(s) = \lim_{t \rightarrow \infty} f(t).$$

It is possible to obtain a mathematical model of an engineering system that consists of one or more differential equations. We have already seen that the solution of differential equations can be found using the Laplace transform, which naturally leads to the concept of a TF.

Consider a first-order differential equation

$$\frac{dx(t)}{dt} + x(t) = f(t), \quad x(0) = x_0, \quad (2.1)$$

where $f(t)$ represents the control input of system (2.1), and $x(t)$ is the output or the response of the system. Taking the Laplace transform of (2.1), we obtain

$$sX(s) - x_0 + X(s) = F(s). \quad (2.2)$$

Assuming $x_0 = 0$, the equivalent form of (2.2) can be formulated by

$$\frac{X(s)}{F(s)} = \frac{1}{1 + s}. \quad (2.3)$$

The function (2.3) is also referred to as a TF, which is the ratio of the Laplace transform of the output to the Laplace transform of the input for the single-input and single-output (SISO) system. Therefore, the TF for the system (2.1) from $f(t)$ to the output $x(t)$ is

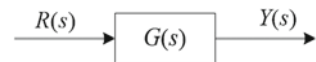
$$G(s) = \frac{1}{1 + s}.$$

The concept of a TF is very useful in engineering applications, which provides a simple algebraic relationship between the input and the output. In other words, it allows the analysis of dynamic system based on the differential equation to proceed in a relatively straightforward manner. Earlier we noted that it was necessary to assume zero initial conditions in order to form the TF. Without the assumption, the relationship between the input and the output would have been more complicated, and the relationship would vary depending on how much energy is stored in the system at $t = 0$. Assuming the zero initial conditions, the TF depends purely on the system characteristics.

Let $R(s) = \mathcal{L}\{r(t)\}$ be the Laplace transform of the input signal, and $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of the output signal. It can be seen from Fig. 2.1 that

$$Y(s) = G(s)R(s).$$

Fig. 2.1 Block diagram



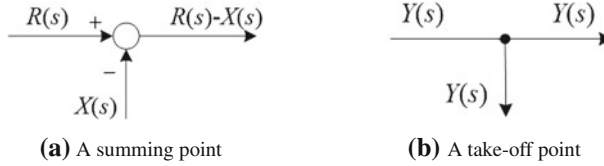


Fig. 2.2 Two components of block diagram

Block diagram consists of two basic components in Fig. 2.2. A summing point adds together the incoming signals to the summing point and produces an outgoing signal. A take-off point is the place where a signal is tapped. The process of tapping the signal has no effects on the signal value.

There are several rules governing the manipulation of block diagrams as follows:

1. Cascade (Series) connection. The TF equivalent to a series connection of 2 blocks with TFs $G_1(s)$ and $G_2(s)$, represented in Fig. 2.3, is given by

$$G_c(s) = G_1(s)G_2(s).$$

2. Parallel connection. The equivalent TF for such a connection representing a summation of signals, given in Fig. 2.4, is obtained as

$$G_c(s) = G_1(s) + G_2(s).$$

3. Feedback connection. The simplest form of a feedback control system is given in Fig. 2.5. For such a system connection the TF is given by

$$G_c(s) = \frac{G(s)}{1 + KG(s)}.$$

Fig. 2.3 Series connection



Fig. 2.4 Parallel connection

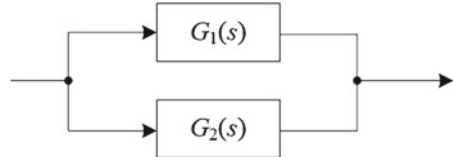
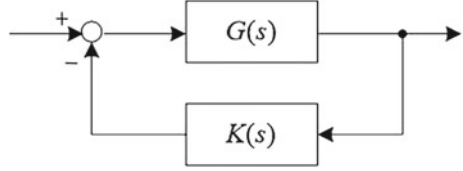


Fig. 2.5 Feedback connection



2.2 Frequency Division Strategies

2.2.1 Weighting Functions

Up till now, much progress has been made in terms of synthesizing H_∞ controllers. However, the selection method of appropriate weighting function is still very much an art. In this subsection, we consider how to formulate some performance objectives in finite frequency ranges using the weighting functions. For example, the finite frequency performance represented in the form of a TF $G(s)$ can be specified as

$$\begin{cases} \|G(j\omega)\|_\infty \leq \alpha < 1, & \forall \omega \leq \omega_0, \\ \|G(j\omega)\|_\infty \leq \beta > 1, & \forall \omega > \omega_0. \end{cases} \quad (2.4)$$

where ω_0 is the cut-frequency of high-low frequency ranges. We generally reflect the system performance objectives via the appropriate selection of weighting functions. On this basis, the finite frequency control specifications, such as the H_∞ performance index (2.4) can be obtained the following weighted form,

$$\|W_s(j\omega)G(j\omega)\| \leq 1,$$

with

$$\|W_s(j\omega)G(j\omega)\|_\infty = \begin{cases} \alpha^{-1}, & \forall \omega \leq \omega_0, \\ \beta^{-1}, & \forall \omega > \omega_0. \end{cases}$$

The meaning of weighted performance specifications can be shown as follows:

1. Some particular frequency components of a signal usually play very important roles in the control design.
2. Each of the signal component may not be measured in the same metric.

Note that weighting functions are essential to identify particular frequency components. On this basis, control design may be regarded as a process of choosing a controller such that certain weighted control specification are satisfied in some sense. In addition, we can see that low-frequency and high-frequency components of a plant can be extracted by band-pass filters. The TF of a band-pass filter can be used as a weighting function to form a weighted control specification.

Consider a weighting function $W(s)$ after normalization in the form of

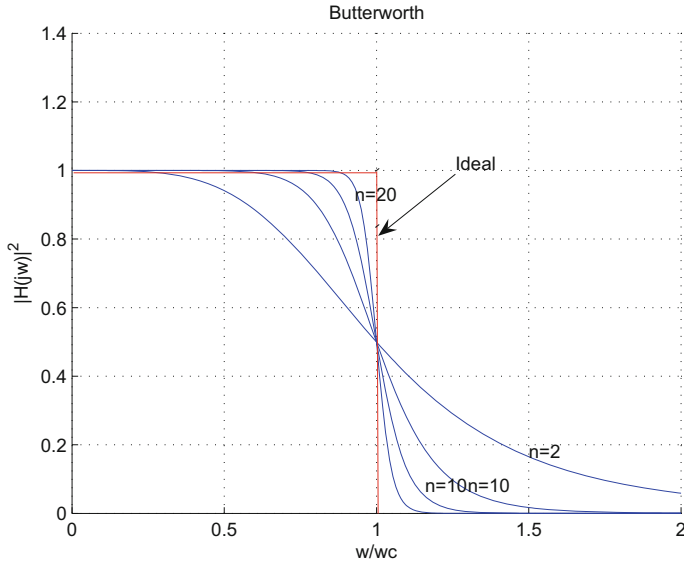


Fig. 2.6 Amplitude-frequency characteristic of butterworth low-pass filter

$$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0}.$$

If $n > m$, that is $W(s)$ is rational fraction, $W(s)$ has the low-pass property. If $n \leq m$, then $W(s)$ achieves the high-pass property. One of the most commonly used filter is the Butterworth LC filter. There are two classes Butterworth filter as follows:

- Butterworth low-pass filter

$$H_l(s) = \frac{b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

where $b_0 = \omega_c^n$. Setting $\omega_c = 1$ rad/s, the normalized results of the butterworth low-pass filter are obtained. The amplitude response, denoted by $|H_l(j\omega)|$, is

$$|H_l(j\omega)| = \frac{1}{\sqrt{1 + (\frac{\omega}{\omega_c})^{2n}}}.$$

Its amplitude-frequency characteristics are in Fig. 2.6.

This type of filter has some specific characteristics.

1. For all n , $|H_l(j0)|^2 = 1$, if $\omega = 0$.
2. For all n , $|H_l(j\omega_c)|^2 = 1/2$, if $\omega = \omega_c$, such that there exists 3 dB amplitude attenuation at $\omega = \omega_c$.

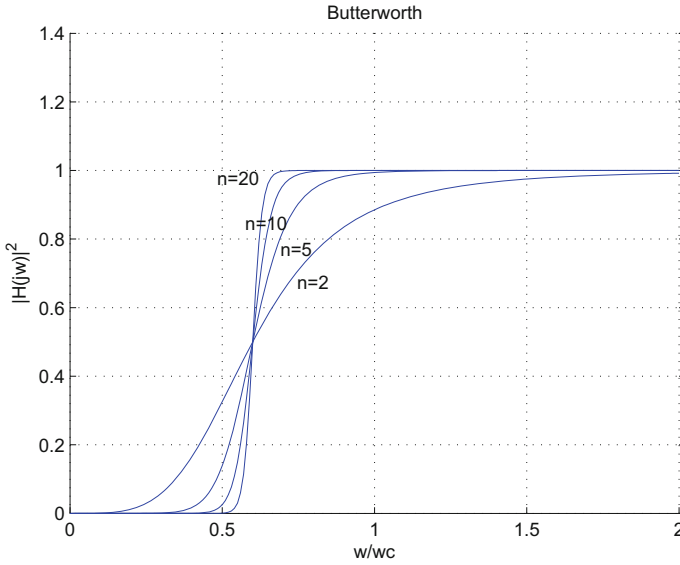


Fig. 2.7 Amplitude-frequency characteristic of butterworth high-pass filter

3. $|H_l(j\omega)|^2$ is a monotonically decreasing and continuous function about ω .
 4. When $n \rightarrow +\infty$, the butterworth low-pass filter can be viewed as a perfect low-pass filter.
 5. When $\omega = 0$, all levels of the derivative of $|H_l(j\omega)|^2$ are zero so that $|H_l(j\omega)|^2$ gets the maximum value and achieves the largest plain features at this point.
- Butterworth high-pass filter

$$H_h(s) = \frac{s^n}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}$$

The amplitude response, denoted by $|H_h(j\omega)|$, is

$$|H_h(j\omega)| = \frac{1}{\sqrt{1 + (\frac{\omega_c}{\omega})^n}}.$$

Its amplitude-frequency characteristics are in Fig. 2.7.

This type of filter has some specific characteristics.

1. For all n , $|H_h(j0)|^2 = 0$, if $\omega = 0$.
2. For all n , $|H_h(j\omega_c)|^2 = 1/2$, if $\omega = \omega_c$, such that there exists 3dB amplitude attenuation at $\omega = \omega_c$.
3. $|H_h(j\omega)|^2$ is a monotonically increasing and continuous function about ω .

4. When $n \rightarrow +\infty$, the butterworth high-pass filter can be viewed as an ideal high-pass filter.
5. When $\omega = +\infty$, all levels of the derivative of $|H_h(j\omega)|^2$ are zero so that $|H_h(j\omega)|^2$ gets the maximum value and achieves the largest plain features at this point.

Remark 2.1 We would emphasize that the weighing function method is particularly suitable for SISO systems. It is difficult for control engineers to construct satisfied weighing function matrices for multiple-input and multiple-output (MIMO) systems, which leads to the wide application of the general Kalman-Yakubovich-Popov (GKYP) lemma.

2.2.2 General Kalman-Yakubovich-Popov Lemma

Up till now, a wide range of state-space approaches for controller design problems adopt the Kalman-Yakubovich-Popov (KYP) lemma that transforms a frequency domain inequality (FDI) into a numerically tractable linear matrix inequality (LMI) [4, 6, 8, 9]. Although that the standard KYP lemma specifies FDIs in the entire frequency range, practical requirements are usually described by multiple FDIs in finite frequency ranges. For example, usually, small sensitivity in a low-frequency range and control roll-off in a high-frequency range are utilized to ensure desired tracking performance and disturbance attenuation capability. As a result, some sort of adaptors, such as the weighting functions, have been adopted to the requirements into the KYP framework. However, the design costs to search for the suitable weighting functions would be tedious and time-consuming, and the controller complexity tends to increase with respect to the complexity of the weighting functions, which leads to the development of GKYP lemma.

The purpose of this section is to develop the state-space design theory that is capable of directly treating multiple FDI specifications in various frequency ranges. The GKYP lemma has been proven to be effective tools to handle inequalities on curves in the complex plane defined as [9, 18],

$$\Lambda(\Phi, \Psi) = \{\lambda \in \mathbf{C} \mid \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Phi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0, \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Psi \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \geq 0\}, \quad (2.5)$$

with $\Phi, \Psi \in \mathbf{H}_2$, where $\mathbf{H}_n$ stands for $n \times n$ Hermitian matrices set and $[\cdot]^*$ denotes the conjugate transpose. The set $\Lambda(\Phi, \Psi)$ represents a curve in the complex plane if and only if [16]

1. $\det(\Phi) < 0$, so that $\Lambda(\Phi, 0)$ corresponds to a circle or a straight line.
2. Φ and Ψ satisfy an additional condition that excludes empty or singleton sets.

The latter condition is most easily expressed with the help of a congruence transformation introduced. It can be demonstrated in [9] that if $\det(\Phi) < 0$, there exists a nonsingular $T \in \mathbf{C}^{2 \times 2}$ such that

$$\Phi = T^* \Phi_0 T, \quad \Psi = T^* \Psi_0 T,$$

where $\Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\Psi_0 = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ and $\alpha, \beta, \gamma \in \mathbf{R}$ with $\alpha \leq \gamma$. In the former case, $\Lambda(\Phi, \Psi)$ coincides with the circle or straight line given by $\Lambda(\Phi, 0)$, while in the latter case it is a segment of this circle or line. The imaginary axis and the unit circle are special cases of (2.5) with $\Psi = 0$, and Φ equal to

$$\Phi_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Phi_d = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for the imaginary axis and the unit circle, respectively.

Next, we present a dual version of the GKYP lemma which is more suitable than the original GKYP lemma for feedback synthesis. A multiplier method is then developed to render the synthesis conditions convex through a simple substitution of variable, in the static gain feedback setting.

Consider a TF for a given system with state-space matrix (A, B, C, D) ,

$$G(\lambda) = C(\lambda I - A)^{-1}B + D,$$

where λ is the frequency variable.

In order to more clearly describe GKYP lemma, we give the following function definition. For $G \in \mathbf{C}^{s \times r}$ and $\Pi \in \mathbf{H}_{s+r}$, a function $\sigma : \mathbf{C}^{s \times r} \times \mathbf{H}_{s+r} \rightarrow \mathbf{H}_r$ is defined by

$$\sigma(G(\lambda), \Pi) = \begin{bmatrix} G(\lambda) \\ I_r \end{bmatrix}^* \Pi \begin{bmatrix} G(\lambda) \\ I_r \end{bmatrix} < 0,$$

for all $\lambda \in \Lambda(\Phi, \Psi)$. The formulation of the performance matrix Π will be introduced detailedly in Sect. 2.3.3.

In the extended definition, the set $\Lambda(\Phi, \Psi)$ can be interpreted as the integration of elements (u, v) ,

$$\Sigma(\Phi, \Psi) = \{(u, v) \in \mathbf{C} \times \mathbf{C} | (u, v) \neq 0, \begin{bmatrix} u \\ v \end{bmatrix}^* \Phi \begin{bmatrix} u \\ v \end{bmatrix} = 0, \begin{bmatrix} u \\ v \end{bmatrix}^* \Psi \begin{bmatrix} u \\ v \end{bmatrix} \geq 0\}.$$

If $v \neq 0$, then $\lambda \neq u/v$ is a finite point in $\Lambda(\Phi, \Psi)$ and if $v = 0$, then $\Lambda(\Phi, \Psi)$ extends $\lambda = \infty$.

Lemma 2.1 (The GKYP Lemma) *Let $A \in \mathbf{C}^{n \times n}$, $B \in \mathbf{C}^{n \times r}$, and $\Theta \in \mathbf{H}_{n+r}$. Suppose $\Phi, \Psi \in \mathbf{H}_2$. Define a curve $\Lambda(\Phi, \Psi)$ in the complex plane, the following two statements are equivalent.*

(1) *If $\lambda \in \Lambda(\Phi, \Psi)$,*

$$\begin{bmatrix} u \\ v \end{bmatrix}^* \Theta \begin{bmatrix} u \\ v \end{bmatrix} < 0,$$

for all nonzero $(u, v) \in \Sigma(\Phi, \Psi)$.

(2) *There exist $P, Q \in \mathbf{H}_n$, that satisfy*

$$Q > 0, \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} + \Theta < 0.$$

If A has no eigenvalues in $\Lambda(\Phi, \Psi)$, the first statement reduces to the FDI

$$\begin{bmatrix} (\lambda I_n - A)^{-1} B \\ I_r \end{bmatrix}^* \Theta \begin{bmatrix} (\lambda I_n - A)^{-1} B \\ I_r \end{bmatrix} < 0, \forall \lambda \in \Lambda(\Phi, \Psi).$$

For the sake of brevity, we point out that only strict FDIs are considered in this monograph. The GKYP lemma readily extends to non-strict inequalities if a regularity condition is imposed.

Two critical lemmas in [8] are presented, which are treated as efficient mathematical tools throughout the following discussion.

Lemma 2.2 *Let $\Phi, \Psi \in \mathbf{H}_2$, $\Pi \in \mathbf{H}_{s+r}$, and the system*

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}, \quad (2.6)$$

where $x(t) \in \mathbf{R}^n$ is state vector, $w(t) \in \mathbf{R}^r$ is the external disturbance, $u(t) \in \mathbf{R}^p$ is the control input, $y(t) \in \mathbf{R}^q$ is the measurement output, and $z(t) \in \mathbf{R}^s$ is the controlled output. A, B_i, C_i and D_{ij} ($i, j = 1, 2$) are all appropriate dimensions matrices. The system (2.6) can be given with TFM from $w(t)$ to $z(t)$ formulated by

$$G(s) = C_1(sI - A)^{-1}B_1 + D_{11}.$$

Consider $\Lambda(\Phi, \Psi)$ defined by (2.5). Suppose $\Lambda(\Phi, \Psi)$ represents curves on the complex plane and A has no eigenvalues in $\Lambda(\Phi, \Psi)$. The following statements are equivalent:

- (1) $\sigma(G(\lambda)^*, \Pi) < 0$ holds for all $\lambda \in \overline{\Lambda}(\Phi^T, \Psi^T)$, where $\overline{\Lambda} = \Lambda$ if Λ is bounded and $\overline{\Lambda} = \Lambda \cup \{\infty\}$ if unbounded.
- (2) There exist $P = P^*$ and $Q = Q^* > 0$ such that

$$N \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} N^* < 0,$$

where $N = \begin{bmatrix} M & I_{n+s} \end{bmatrix} T$, $M = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}$, and T is the permutation matrix such that for arbitrary matrices M_1, M_2, M_3 and M_4

$$\begin{bmatrix} M_1 & M_2 & M_3 & M_4 \end{bmatrix} T = \begin{bmatrix} M_1 & M_3 & M_2 & M_4 \end{bmatrix},$$

where matrices M_1, M_2, M_3 and M_4 have column dimensions n, r, n and s , respectively.

Define $G(\lambda) * K$ as the closed-loop TF from $w(t)$ to $z(t)$. A synthesis problem may be formulated as the search for the parameters $Q > 0$, P and K with $\mathcal{M}$ defined to be the state-space matrices of $G(\lambda) * K$ as

$$\mathcal{M} = \mathcal{A} + \mathcal{B}K\mathcal{C} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} K \begin{bmatrix} C_2 & D_{21} \end{bmatrix}.$$

The resulting condition is not convex due to the existence of the product term. Lemma 2.2 in [8] is developed as a multiplier method to re-parameterize the condition so that the problem becomes convex which can be further transformed into LMIs.

Lemma 2.3 *Let $R \in \mathbf{C}^{q \times (2n+r+s)}$, $\Phi, \Psi \in \mathbf{H}_2$, $\Pi \in \mathbf{H}_{r+s}$, $P, Q \in \mathbf{H}_n$ and the system (2.6) be given. The following statements are equivalent.*

- (1) *There exist a feedback gain K and a real scalar $\mu > 0$ such that*

$$NXN^* < 0, \quad S(TXT^* - \mu R^*R)S^* < 0,$$

where $S = \begin{bmatrix} \mathcal{M} & I_{n+s} \\ \mathcal{C} & 0 \end{bmatrix}$, $X = \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix}$, and $\mathcal{M}$ is defined in Lemma 2.2.

- (2) *There exist matrices $\chi \in W(\mathcal{C}, R)$ and κ such that*

$$T \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} T^* < \text{He} \begin{bmatrix} -\chi & \\ \mathcal{A}\chi + \mathcal{B}\kappa R & \end{bmatrix}.$$

If (2) holds, the gain in (1) can be given by $K = \kappa W^{-1}$.

To make the problem trackable, the multiplier χ can be specified by

$$W(\mathcal{C}, R) = \{\mathcal{C}^\dagger WR + (I - \mathcal{C}^\dagger \mathcal{C})V \mid W \in \mathbf{C}^{q \times q}, \det(W) \neq 0, V \in \mathbf{C}^{(n+r) \times (2n+r+s)}\},$$

where W and V are matrices to be solved, and superscript $\dagger$ denotes the Moore-Penrose inverse of matrix.

Remark 2.2 Here, for notational convenience throughout the monograph, the Hermitian part of a square matrix M is denoted by $\text{He}(M) = M + M^*$.

The condition in statement (2) of Lemma 2.3 has been characterized in terms of LMIs, which can thereby be numerically solved by semi-definite programming. Up till now, many commercial software packages are now available for solving this task. We strongly believe that it is beneficial for most engineers to have one of these computer programs. Software to synthesize H_∞ controllers has been available sometimes, such as the Robust Control Toolbox in MATLAB[®]. Recently, a LMI Toolbox for MATLAB has been widely used, which includes LMI solving program, alternative

H_∞ software, and μ -analysis and synthesis. All computation presented in this monograph has been done employing such toolbox.

Then, we will give particular choices of R that lead to LMI synthesis conditions which are nonconservative. Please see [8] for detailed explanation. Later, we will discuss some heuristic choices of R leading to sufficient conditions for synthesis:

1. For continuous-time, small gain condition and low-frequency case, R can be chosen as

$$R_l = \begin{bmatrix} 0 & 0 & I & (D_{11} B_1^\dagger)^* \end{bmatrix}.$$

2. For continuous-time, small gain condition and high-frequency case, R can be chosen as

$$R_h = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}.$$

2.3 Characterization of Control Performance Index

2.3.1 Characterization of Finite Frequency Ranges

Denote the frequency variable as (2.5) with its properties. The continuous-time and discrete-time variable frequency variables are, respectively, shown as

$$\Lambda_c = \{j\omega | \omega \in \Omega_c\}, \quad \Lambda_d = \{e^{j\theta} | \theta \in \Omega_d\},$$

where Ω_c and Ω_d are subsets of real numbers specified by an additional choice of Ψ in Tables 2.2 and 2.3, respectively, Φ in $\Lambda(\Phi, \Psi)$ is selected as

$$\Phi_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Phi_d = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

for the imaginary axis and the unit circle, respectively.

For convenience presentation of main results, we divide the frequency range in the continuous-time case into the following three parts. For low frequencies, the frequency set Λ is specified by

$$\Lambda_{cl} = \{\omega | |\omega| < \omega_l\},$$

and for middle frequencies, Λ can be represented by

$$\Lambda_{cm} = \{\omega | \omega_1 < \omega < \omega_2\},$$

in the similar way, the high frequencies can be characterized by

$$\Lambda_{ch} = \{\omega | |\omega| > \omega_h\}$$

Table 2.2 The choice of Ψ in continuous-time case

Cont.	Ω_c	Ψ
LF	Λ_{cl}	$\begin{bmatrix} -1 & 0 \\ 0 & \omega_l^2 \end{bmatrix}$
MF	Λ_{cm}	$\begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1\omega_2 \end{bmatrix}$
HF	Λ_{ch}	$\begin{bmatrix} 1 & 0 \\ 0 & -\omega_h^2 \end{bmatrix}$

Table 2.3 The choice of Ψ for the discrete-time case

Disc.	Ω_d	Ψ
LF	Λ_{dl}	$\begin{bmatrix} 0 & 1 \\ 1 & -2 \cos \theta_l \end{bmatrix}$
MF	Λ_{dm}	$\begin{bmatrix} 0 & e^{j\theta_c} \\ e^{-j\theta_c} & -2 \cos \theta_w \end{bmatrix}$
HF	Λ_{dh}	$\begin{bmatrix} 0 & -1 \\ -1 & 2 \cos \theta_h \end{bmatrix}$

and the related Ψ in frequency division is summarized in Table 2.2, where $\omega_c = (\omega_1 + \omega_2)/2$ and LF, HF, and MF stand for low, high, and middle-frequency ranges, respectively.

Similarly, for the discrete-time system, the selection method of Ψ is stated as follows. Consider the low-frequency condition with

$$\Lambda_{dl} = \{e^{j\theta} \mid \theta < \theta_l\}.$$

Noting that $|\theta| \leq \theta_l$ if and only if $z = e^{j\theta}$ satisfies

$$z + \bar{z} \geq 2 \cos \theta_l = \gamma,$$

we choose

$$\Psi = \begin{bmatrix} 0 & 1 \\ 1 & -2 \cos \theta_l \end{bmatrix}.$$

The middle-frequency condition is considered with

$$\Lambda_{dm} = \{e^{j\theta} \mid \theta_1 \leq \theta \leq \theta_2\}.$$

Note that the frequency interval condition can be written as

$$|\theta - \theta_c| \leq \theta_w$$

or

$$\cos(\theta - \theta_c) \geq \cos \theta_w,$$

where $0 \leq \theta_w \leq \pi$ and

$$\theta_c = \frac{\theta_2 + \theta_1}{2}, \quad \theta_w = \frac{\theta_2 - \theta_1}{2},$$

which can be rewritten as $\sigma(e^{j\theta}, \Psi) \geq 0$ with

$$\Psi = \begin{bmatrix} 0 & e^{j\theta_c} \\ e^{-j\theta_c} & -2 \cos \theta_w \end{bmatrix}.$$

Finally, the high-frequency condition with

$$\Lambda_{dh} = \{e^{j\theta} \mid \theta_h \leq |\theta| \leq \pi\}$$

can be treated similarly. In this case, we have

$$\Psi = \begin{bmatrix} 0 & -1 \\ -1 & 2 \cos \theta_h \end{bmatrix}.$$

The work above can be summarized in Table 2.3.

2.3.2 Window Norm

One natural extension is then to directly specify important frequency domain properties such as bandwidth and magnitude of resonance peaks on certain TFM. All the work related with the TFM of a system is established in the finite frequency range rather than the entire frequency so that the conservativeness of frequency domain constraints is much reduced. Such property is described by using window norm.

A TF in terms of state-space model is denoted as

$$E(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = C(sI - A)^{-1}B + D.$$

This section reviews some mathematical preliminaries, in particular, the computation of H_2 and H_∞ norm of a TFM $E(s)$.

Definition 2.2 (*Finite Frequency H_∞ Norm*) For a stable TFM $E(s)$, the H_∞ **norm** is given by

$$\|E\|_\infty = \sup_{\omega} \sigma(E(j\omega)), \quad \omega \in \Lambda,$$

where $\sigma(\cdot)$ denotes the singular value of a matrix, and $\sup_{\omega} \sigma(E(j\omega))$ denotes the peak singular value in finite frequency set Λ .

The H_{∞} norm can be viewed as a direct generalization of frequency domain specifications used in classical control for systems. For instance, it can be used to minimize the weighted sensitivity function, and select $E = \omega_p S$, where sensitivity function $S = (I + GC)^{-1}$. The following proposition provides an equivalent condition for the H_{∞} norm of a rational matrix.

Proposition 2.1 *Let $E(s)$ be a rational matrix, which does not have poles on the closed right half complex plane. Then, it can be obtained that*

$$\|E(s)\|_{\infty} = \sup_s \{\sigma_{\max}(E(s) | \operatorname{Re}(s) > 0), \operatorname{Im}(s) \in \Lambda\}.$$

Likewise, H_2 norm is also defined for rational matrix. The strict definition is as follows.

Definition 2.3 (*Finite Frequency H_2 Norm*) The H_2 **norm** of a TF $E(s)$, namely $\|E(s)\|_2$, which also corresponds to the impulse response energy $\|e(t)\|_2$, or L_2 norm of a real signal $e(t)$, can be formulated in either time or frequency domain as follows:

$$\|e(t)\|_2^2 = \int_0^{\infty} e(t)e(t)dt = \|E(s)\|_2^2 = \frac{1}{\pi} \int_0^{\infty} E(j\omega)E(-j\omega)d\omega.$$

The change from time domain to frequency domain is justified using Plancherels theorem (which discrete version is known as Parsevals theorem). H_2 norm can be used to characterize the input energy of signal $u(t)$.

Remark 2.3 If $U(s)$, the Laplace transform of input signal $u(t)$, is a vector function such that the integral $\int_{-\infty}^{\infty} U^*(j\omega)U(j\omega)d\omega$ exists, then, obviously the H_2 norm of $U(s)$ reduces to

$$\|U(s)\|_2 = (\operatorname{trace}(\frac{1}{2\pi} \int_{-\infty}^{\infty} U^*(j\omega)U(j\omega)d\omega))^{\frac{1}{2}}.$$

To investigate H_-/H_{∞} performance of continuous-time control systems in local frequency range, a new conception of window H_-/H_{∞} norm is presented based on traditional H_-/H_{∞} norm, and it is stated that traditional H_-/H_{∞} norm is a special case of window H_-/H_{∞} norm.

Definition 2.4 (*Finite Frequency H_-/H_{∞} Norm*) The H_- index of a TFM $E(s)$ over the finite frequency range $[\omega_1, \omega_2]$ is defined as

$$\|E(s)\|_{-}^{[\omega_1, \omega_2]} = \inf_{\omega \in [\omega_1, \omega_2]} \underline{\sigma}[E(j\omega)],$$

where $\underline{\sigma}$ denotes the minimum singular value of the matrix $E(j\omega)$.

The H_∞ norm of a TFM $E(s)$ over the finite frequency range $[0, \omega_l] \cup [\omega_h, \infty)$ is defined as

$$\|E(s)\|_\infty^{[0, \omega_l] \cup [\omega_h, \infty)} = \sup_{\omega \in [0, \omega_l] \cup [\omega_h, \infty)} \bar{\sigma}[E(j\omega)],$$

where $\bar{\sigma}$ denotes the maximum singular value of the matrix $E(j\omega)$.

To indicate the dependency on the finite frequency range $[\omega_1, \omega_2]$, we define the window H_- norm as $\|E(s)\|_-^{[\omega_1, \omega_2]}$, which is simplified into $\|E(s)\|_-$ when the frequency range is made with certainty. H_- norm is used as the worst-case fault sensitivity measure.

2.3.3 Frequency Domain Inequalities

Much of the recent work on robustness has been focused on frequency domain characterizations. The significant results in system and control literature, such as design specifications for practical control synthesis, depend on the characterizations of system in terms of FDIs. Classical FDIs are specified in the full frequency range. However, recent developments have demonstrated that finite frequency FDIs can increase flexibility in system analysis and synthesis because that system usually works in the specific frequency range due to the limits of external conditions or nature of pumping signals.

If the designs of the structure and the controller are integrated, control performance of mechanical structures can be significantly enhanced. In this subsection, we also present how to formulate the performance matrix Π . Generally, some characterizations of the system $G(s)$ are considered as follows:

1. Positive real property:

$$G(j\omega)^* + G(j\omega) > 0,$$

which is an important requirement for mechanical structure design to guarantee the existence of controllers that achieve high servo-bandwidth, where $G(j\omega)$ is the TF of the mechanical system to be designed. In this case, the performance matrix can be selected by

$$\Pi = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}.$$

2. Bounded real property:

$$G(j\omega)^* G(j\omega) < \gamma^2 I,$$

is an important requirement for mechanical structure design to guarantee the existence of controllers that achieves good disturbance attenuation ability with Π as

$$\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

3. Sensitivity shaping:

A typical control design with specifications on the closed-loop TF: a plant $G(s)$ and a controller $K(s)$. The sensitivity and the complementary sensitivity functions are defined by

$$\begin{aligned} S(j\omega) &= (1 - G(j\omega)K(j\omega))^{-1}, \\ T(j\omega) &= G(j\omega)(1 - G(j\omega)K(j\omega))^{-1}. \end{aligned}$$

The sensitivity-shaping problem typically consists of the following requirements:

$$\begin{aligned} \|S(j\omega)\|_\infty &< \alpha_1, \quad \omega \in \Lambda_l, \\ \|T(j\omega)\|_\infty &< \alpha_2, \quad \omega \in \Lambda_h, \end{aligned}$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$. These FDIs can be further represented in terms of open-loop TF, and the performance matrix Π can be formulated hereinafter.

Remark 2.4 There are close relationships between FDIs and TDIs. Most the significant results in systems and control literature generally characterize control system performances using FDIs and/or time domain inequalities. Certain system properties are characterized in terms of an inequality condition on the TF. Time domain interpretations of FDIs can provide more flexibility to capture various engineering requirements. For instance, we can see that the bounded-realness with gain bound γ is equivalent to the L_2 gain being less than or equal to $\sqrt{\gamma}$ in the time domain:

$$\int_0^\infty y^T(t)y(t)dt < \gamma \int_0^\infty u^T(t)u(t)dt, \quad u(t) \in L_2.$$

where the nonnegative number $\|u(t)\| = (\int_0^\infty u^T(t)u(t)dt)^{\frac{1}{2}}$ is the L_2 gain of $u(t)$, $y(t)$ is the response output with the input $u(t)$ for the system $G(s)$, and the space $L_2[0, \infty)$ is the space of all piecewise-continuous inputs defined on $[0, \infty)$ satisfying $\int_0^\infty \|u(t)\|^2 < \infty$. With this interpretation, the FDI for bounded-realness can encompass the worst-case disturbance attenuation (or tracking error, etc.) level in the L_2 sense, rather than the peak amplification factor with respect to a fixed sinusoidal input. Likewise, the positive-realness of $G(s)$ is equivalent to the passivity of the system in the time domain:

$$\int_0^\infty u^T(t)y(t)dt > 0, \quad u(t) \in L_2.$$

This interpretation allows for the concept of energy to be treated within the FDI framework for formalizing specifications.

In addition, some recent results [1, 3, 5, 7, 9, 14] have addressed this issue and generalized the standard KYP lemma [15, 19] to characterize FDIs within (semi)finite

frequency ranges in terms of LMIs. These results are based on the idea of the S-procedure, and are in connection with the literature on integral quadratic constraints [10, 12], indefinite linear quadratic control [19], and power distribution inequality [13]. These references demonstrated that structures of practical significance can be designed to achieve the specific property by solving LMI feasibility problem with the aid of an existing version of the finite frequency KYP lemma. To convert FDI to LMI, the following lemma is useful.

Lemma 2.4 (Schur Complement Lemma (SCL)) *The following statements are equivalent:*

- (1) $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0;$
- (2) $Q > 0, R - S^* Q^{-1} S > 0;$
- (3) $R > 0, Q - S R^{-1} S^* > 0.$

SCL can be extended into semi-definite program.

Lemma 2.5 (Semi-Definite Case of SCL) *The following statements are equivalent:*

- (1) $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0;$
- (2) $R \geq 0, Q - S R^{-1} S^* \geq 0.$

2.4 Conclusion

This chapter mainly introduces a few of mathematical analysis and useful lemmas, which will be used throughout the manuscript. In Sect. 2.1, the definition of the Laplace Transform has been presented, and some commonly used forms are given. In Sect. 2.2, we explain that both the weighting functions and the GKYP lemma are useful methods in the frequency division, where the former one is particularly suitable for SISO systems, and the latter one used in both SISO and MIMO systems enjoy wider application ranges. In Sect. 2.3, we present two methods to characterize system performances, namely window norm and FDIs. Concluding remarks are given in Sect. 2.4.

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Finite Frequency Analysis and Synthesis for Singularly
Perturbed Systems

Cai, C.; Wang, Z.; Xu, J.; Zou, Y.

2017, XIV, 227 p. 64 illus., 41 illus. in color., Hardcover

ISBN: 978-3-319-45404-7