Chapter 2
Stress

2.1 Cauchy Stress

The concept of stress was conceptualized best by Cauchy (1789) and put in final form by Saint-Venant (1797). Consider an infinitesimal cubical element (Fig. 2.1) that is in a state of static equilibrium. The forces per unit area on each face of the element are the *stresses* on that face. There are two types of stresses; normal stresses are perpendicular to the face of the element and shear stresses are parallel to the element face. Different notations have been used to denote stresses. In this treatise, σ will be used to denote normal stresses, and either σ or τ will be used to denote shear stress. The notation is further augmented with subscripts indicating the face upon which the stress is acting and the direction of the stress. Thus, σ_{xx} is the normal stress in the x-direction acting upon a face whose normal is in the x-direction. The shear stresses on the x-face are denoted τ_{xy} and τ_{xz} for shear stresses acting in the y- and z-directions, respectively.

Normal stresses are considered positive when tensile, and shear stresses are considered positive when the force vector is in the positive direction of the corresponding axis. All stresses shown in Fig. 2.1 are positive. It is noted that the stresses on a face correspond to the three components of an arbitrary force vector (per unit area) acting on the face. Further, the normal and shear stresses on opposite faces (not shown in Fig. 2.1) must be in opposite directions in order to maintain force and moment equilibrium. There are nine unique components of stress at a point.

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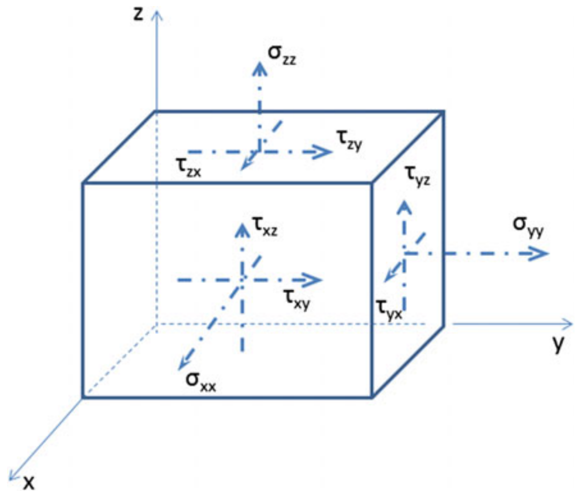


Fig. 2.1 Stress components

2.2 Plane Stress

A complete representation of the two-dimensional state of stress (plane stress with all z-components of stress = 0) at a point is depicted in Fig. 2.2. Note that the shear stresses on the three sets of opposite faces must be equivalent to couples that are in moment equilibrium. This establishes the condition that the shear stresses at a point are of equal magnitude, namely:

$$\tau_{xy} = \tau_{yx} \tag{2.1}$$

For the conditions of plane stress to exist in an x–y plane, it is understood that the z-components of stress are zero, i.e., $\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$.

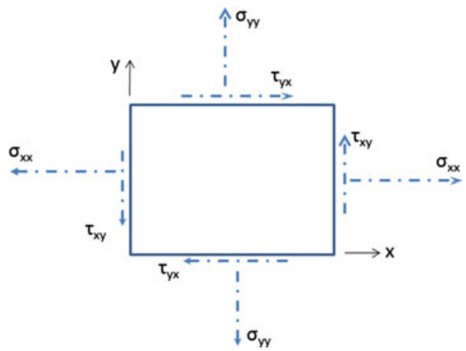


Fig. 2.2 Plane stress at a point

2.3 Stress Transformation

The plane state of stress at any point is a function of the plane of interest through that point. Ideally, it is desired to express the state of stress on any arbitrary plane passing through the point. This can be accomplished through consideration of the equilibrium equations for an arbitrary section passing through the element in Fig. 2.2. The resulting equations are called *stress transformation equations*.

Consider the free body diagram of a section of an element as shown in Fig. 2.3. Force equilibrium in the x′- and y′-directions results in the following plane stress transformation equations.

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \tag{2.2}$$

$$\tau_{x'y'} = -\left(\sigma_{xx} - \sigma_{yy}\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) \tag{2.3}$$

Following along similar lines for equilibrium, the normal stress $\sigma_{y'y'}$ is:

$$\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \tag{2.4}$$

These stress transformation equations can be expressed in a more convenient form through the use of trigonometric identities (see Appendix) with the results:

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{2.5}$$

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \tag{2.6}$$

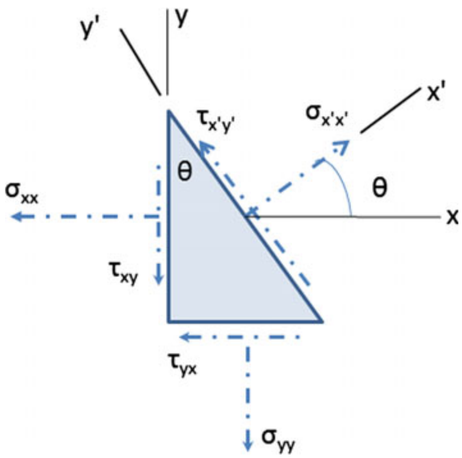


Fig. 2.3 Arbitrary section of plane stress element

$$\tau_{x'y'} = -\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \tag{2.7}$$

Defining $m = \cos\theta$ and $n = \sin\theta$, the transformation equations (2.2–2.4) can be written in matrix form as:

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \tag{2.8}$$

2.4 Principal Stresses

The normal stresses at a point vary with the orientation of the plane passing through the point as determined from equilibrium. The maximum and minimum values of the normal stresses are called the *principal stresses*. Taking the derivative of the normal stress equilibrium equation (2.4) with respect to the angle θ and equating to zero gives the angles, θ_P, corresponding to the principal planes. The resulting equation is:

$$\tan(2\theta_P) = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{2.9}$$

The two angles θ_P from this equation differ by 90°. The planes defined by these angles are called the *principal planes*. One of the planes corresponds to the maximum normal stress and the other corresponds to the minimum normal stress. Combining these results (2.4 and 2.6) gives the expression for the principal stresses:

$$\sigma^P_{11,22} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{2.10}$$

Using the (+) sign in this equation gives the maximum normal stress, $\sigma_{11}^{\max}$, and the (–) sign gives the minimum normal stress, $\sigma_{22}^{\min}$.

The maximum shear stress at a point may also be of interest. Setting the derivative with respect to θ of the shear stress equation (2.5) equal to zero gives

$$\tan(2\theta_s) = -\left(\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}}\right) \tag{2.11}$$

The two solutions for θ_S correspond to the planes of maximum and minimum shear stress. They are equal in magnitude with the maximum shear stress being positive and the minimum shear stress being negative. The magnitude of the maximum shear stress is determined by combining the above equations with the result:

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{2.12}$$

It can also be shown that the maximum shear stress can be expressed in terms of the principal stresses as:

$$\tau_{\max} = \frac{\sigma_{11} - \sigma_{22}}{2} \tag{2.13}$$

2.5 Pure Shear Stress

The condition of pure shear is a most interesting state of stress. As shown in Fig. 2.4, pure shear in an x–y plane is equivalent to positive and negative, pure, normal stresses, of the same magnitude as the shear stress, on the planes at angles 45° to the x–y axes. This explains why a brittle material such as a piece of chalk fails along a 45° plane when subjected to pure torsion (Chap. 7). The brittle material is relatively weak in tension and thus it fails along the plane that has the highest level of tensile, normal stresses.

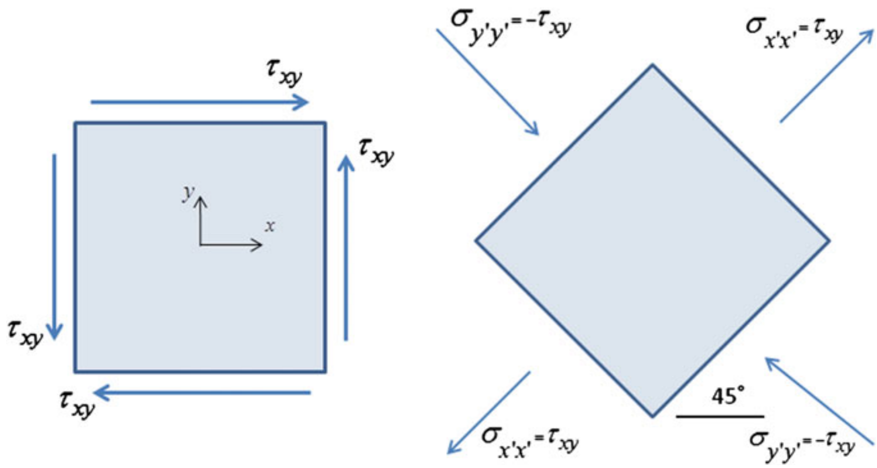


Fig. 2.4 Pure shear stress

2.6 Stress Tensor

The nine components of stress can be represented as a tensor quantity that obeys tensor transformation laws as described in the Appendix. The standard notation for the stress tensor is:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.14}$$

In (2.14), the indices 1, 2, 3 represent x, y, z in the rectangular Cartesian coordinate system. Further, we note that equilibrium requires that the stress tensor is symmetric with:

$$\sigma_{ij} = \sigma_{ji} \tag{2.15}$$

2.7 Units of Stress

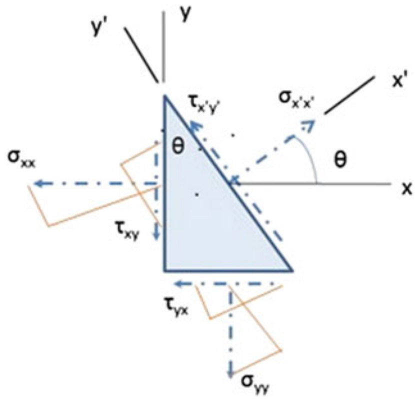
Stress is the force per unit area. Thus, stress can be expressed as pounds per square inch (psi) or Pascals [Pa, which is defined as newtons per square meter in the International (SI) System of units]. The units on force are derived from Newton's second law of motion which states that F = m • a. One newton (N) is the force required to accelerate one kilogram (kg) of mass (m) at the rate (a) of one meter per second squared (m/s^2). Thus 1 N = 1 kg • 1 m/s^2. The unit of mass in the Imperial and United States customary systems is called a slug. It is the mass that accelerates at one ft/s^2 when a force of one pound (lb) is exerted on it. Thus, a slug has the units $1\ \text{slug} = 1\,\frac{\text{lb}\cdot\text{s}^2}{\text{ft}}$.

For many applications, stresses are often expressed as ksi [kips (10^3 lb) per square inch] or MPa (mega pascals, where mega = 10^6). These notations are employed simply to reduce the number of zeros (or digits) required to express the value of large numbers. Note that 1.0 MPa = 145.0 psi = 0.145 ksi.

2.8 Exercises

2.8.1 Confirm equation 2.2.
2.8.2 Confirm equation 2.5.
2.8.3 Confirm equation 2.7.
2.8.4 Confirm equation 2.9.
2.8.5 Plot the variation of normal stress on planes passing through a point if it is known that the state of stress is planar with $\sigma_{xx} = 40,\ \sigma_{yy} = -20,\ \tau_{xy} = 10.$

Appendix: Solutions



2.8.1 Confirm equation 2.2.

Solution

Equation (2.2) ⇒

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

Summation of forces in the $x' - x'$-direction gives:

$$\sum F_{x'x'} = 0$$
$$\sigma_{x'x'}A = \sigma_{xx}\cos\theta\, A\cos\theta + \sigma_{yy}\sin\theta\, A\sin\theta + \tau_{xy}\cos\theta\, A\sin\theta + \tau_{xy}A\cos\theta\sin\theta$$
$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\cos\theta\, A\sin\theta$$

2.8.2 Confirm equation 2.5.

Equation (2.5) ⇒

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Solution
Using the trigonometric identities in (2.2):

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\Rightarrow$$

$$\sigma_{x'x'} = \sigma_{xx}\frac{1+\cos 2\theta}{2} + \sigma_{yy}\frac{1-\cos 2\theta}{2} + \tau_{xy}\sin 2\theta$$
$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

2.8.3 Confirm equation 2.7.

Equation (2.7) $\Rightarrow$

$$\tau_{x'y'} = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Solution
Summation of forces in the $y' - y'$-direction and using trigonometric identities gives:

$$\sum F_{y'y'} = 0 \quad \Rightarrow$$
$$\tau_{x'y'}A = A\cos\theta\left(\sigma_{xx}\sin\theta - \tau_{xy}\cos\theta\right) + A\sin\theta\left(\tau_{xy}\sin\theta - \sigma_{yy}\cos\theta\right)$$

$$\tau_{x'y'} = (\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta + \tau_{xy}\left(\sin^2\theta - \cos^2\theta\right)$$
$$\tau_{x'y'} = (\sigma_{xx}-\sigma_{yy})\frac{\sin 2\theta}{2} + \tau_{xy}\cos 2\theta$$

2.8.4 Confirm equation 2.9.

Equation (2.9) $\Rightarrow$

$$\tan(2\theta_P) = \frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}$$

Solution

Normal stress on arbitrary plane is from (2.5) ⇒

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

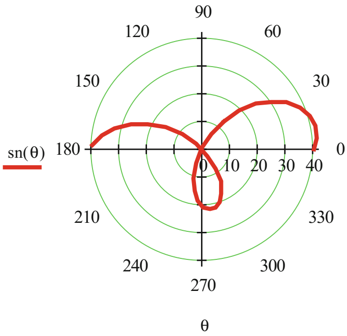
Setting derivative with respect to $\theta = 0$ for maximum and minimum ⇒

$$\begin{aligned}
\frac{d\sigma_{x'x'}}{d\theta} &= 0 = -2\sigma_{xx}\sin\theta\cos\theta + 2\sigma_{yy}\sin\theta\cos\theta + 2\tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)\\
0 &= \frac{-\sigma_{xx}\sin 2\theta}{2} + \frac{\sigma_{yy}\sin 2\theta}{2} + \tau_{xy}\left(\frac{1+\cos 2\theta}{2} - \frac{1-\cos 2\theta}{2}\right)\\
0 &= \frac{\sin 2\theta}{2}\left(\sigma_{yy} - \sigma_{xx}\right) + \tau_{xy}\cos 2\theta\\
\tan 2\theta &= \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}
\end{aligned}$$

2.8.5 Plot the variation of normal stress on planes passing through a point if it is known that the state of stress is planar with $\sigma_{xx} = 40,\ \sigma_{yy} = -20,\ \tau_{xy} = 10$

Solution

Plotting Eq. (2.2) ⇒



References

Cauchy (1789–1857) Cauchy, A. (1822). *Memoires de l'acdemie des sciences*, Paris, Vol. 7, (1827), pp. See, Bulletin de la societe philomathique, Paris (1823), p. 177.

Saint Venant (1797–1886) Saint-Venant, A. J. C. Barré de. (1845). Comptes Rendus, v20, p. 1765 and v21, p. 125.

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