

Chapter 2

Elements of General Topology

'Would you tell me, please, which way I ought to go from here?'
'That depends a good deal on where you want to get to.'
'I don't much care where -'
'Then it doesn't matter which way you go.'

Lewis Carroll, Alice in Wonderland

Let me start this chapter with a simple why-question: Why general topology? What is the main problem it wishes to solve? The answer is deceptively simple: general topology aims at analyzing and describing topological spaces. I will start this chapter by introducing the basic concepts of this field of research. I define notions like the axiomatic topology of a space, finite topological spaces, discrete spaces, indiscrete spaces, open, closed and clopen sets as well as some basic notions about limits and how various objects easily defined in calculus have to generalize in order to make sense in a general topological context. I mainly follow here reference [1] for a basic but very enlightening introduction. Topology, like most of the other branches of mathematics, can be described axiomatically [2]. In this sense, a topology can be defined as follows:

Definition 2.1 Let X be a non-empty set. A collection τ of subsets of X is said to be a topology on X if

- X and the empty set belong to τ
- the union of any (finite or infinite) number of sets in τ belongs to τ and
- the intersection of any two sets in τ belongs to τ

The pair (X, τ) is called a topological space.

If X is a non-empty set and τ is the collection of all subsets of X then τ is called the discrete topology on the set X . The topological space (X, τ) is called a discrete space. The indiscrete topology on the other side is given by $\tau = \{X, \emptyset\}$ and

then (X, τ) is called the indiscrete space. In both these cases each type of topology satisfies the conditions in the general definition of the topology.

At this point I can remind the reader why an axiomatic definition of a notion is useful [3]. Axioms are a method of restraining the means used to define an object such that the validity of the object defined using them is as general as possible. By being able to axiomatize a definition we become capable of observing the appearance of the defining axioms even in some unexpected situations [4]. For example, in this case we can already see that a discrete space connects all the elements of a space to each other by defining an open set for each and every subset of the original space. The set of single elements-subsets will also be part of this discrete topology, hence the name “discrete”. This last property allows therefore the possibility to distinctly specify each point or subset of points in the space offering the possibility of fine-graining the space. On the other side, we may think in terms of a coarse topology having only the empty set and the original set itself in it. This is an “indiscrete topology”. It also connects all the points in the space but doesn’t allow us to speak of them distinctly or to specify only certain collection of points in that space.

Instead of referring to “members of τ ” we may give to these sets more appropriate names. Let us call them open sets. The complements of the open sets with respect to the space X are called “closed sets”. This way of speaking leads to what is known as “open intervals” and “closed intervals” on the real number line. If (X, τ) is any topological space then \emptyset and X are closed sets. Also, the intersection of any finite or infinite number of closed sets is a closed set and the union of any finite number of closed sets is a closed set. The same is valid for open sets. Therefore, one observes that while any finite or infinite union of open sets is open, only finite intersections of open sets are open. Infinite intersections of open sets are not always open. I will show this in the next example:

Example 2.2 Let \mathbb{N} be the set of all positive integers and let τ consist of \emptyset and each subset S of \mathbb{N} such that the complement of S in \mathbb{N} , $\mathbb{N} - S$, is a finite set. It can be verified that τ is a topology on \mathbb{N} . It is called the finite-closed topology. For each natural number n , define the set S_n as

$$S_n = \{1\} \cup \{n+1\} \cup \{n+2\} \cup \{n+3\} \cup \dots = \{1\} \cup \bigcup_{m=n+1}^{\infty} \{m\} \quad (2.1)$$

Clearly each S_n is an open set in the topology τ , since its complement is a finite set. However,

$$\bigcap_{n=1}^{\infty} S_n = \{1\} \quad (2.2)$$

As the complement of $\{1\}$ is neither \mathbb{N} nor a finite set, $\{1\}$ is not open. So this shows that the intersection of the open sets S_n is not open.

It is important to observe that both union and intersection must be verified in order to prove that a subset is open. Now that the open and closed sets are defined, one

needs to notice that some open sets can also be closed at the same time. For example in a discrete space every set is both open and closed while in an indiscrete space (X, τ) all subsets of X except X and \emptyset are neither open nor closed. Hence there is the

Definition 2.3 A subset S of a topological space (X, τ) is said to be clopen if it is both open and closed in (X, τ) .

In general in every topological space (X, τ) both X and \emptyset are clopen, in a discrete space all subsets of X are clopen and in an indiscrete space the only clopen subsets are X and \emptyset .

In what follows I will discuss the notions that can be defined generally on a topological space. The analogy with the real line has its limits. First, on the real line we have a notion of “closeness”. For example, if we have a sequence of the form

$$0.1, 0.01, 0.001, \dots, \quad (2.3)$$

every element of this sequence is closer to zero than the previous. This means one can say that 0 is the limit point of this sequence. However, the interval $(0, 1]$ is not closed as it does not contain the limit of any sequence in it, in particular it does not contain the element 0.

A topological space is in some sense a general notion. For example we do not need to have notions like a metric over a topological space and the distance is therefore not always well defined. If we do not have a distance we must define the limit point differently, without considering the distance between two points as has been done in standard calculus.

Also, the topological spaces are defined by employing the concept of connectedness. This will also be defined in what follows. Let me start with a topological space (X, τ) . The elements of this space are referred to as points. Let A be a subset of a topological space (X, τ) . A point $x \in X$ is said to be a limit point (or accumulation point or cluster point) of A if every open set, U , containing x contains a point of A different from x .

In general a test whether a set is closed or not is the following

Proposition 2.4 Let A be a subset of a topological space (X, τ) . Then A is closed in (X, τ) if and only if A contains all of its limit points.

Proposition 2.5 Let A be a subset of a topological space (X, τ) and A' the set of all limit points of A . Then $A \cup A'$ is a closed set.

Definition 2.6 Let A be a subset of a topological space (X, τ) . Then the set $A \cup A'$ consisting of A and all its limit points is called the closure of A and is denoted \bar{A} .

Definition 2.7 Let A be a subset of a topological space (X, τ) . Then A is said to be dense in X or everywhere dense in X if $\bar{A} = X$. As an example \mathbb{Q} is a dense subset of \mathbb{R} .

As an example consider again the discrete topological space (X, τ) . Then, every subset of X is closed (since its complement is open). Therefore the only dense subset of X is X itself, since each subset of X is its own closure.

Proposition 2.8 *Let A be a subset of a topological space (X, τ) . Then A is dense in X if and only if every non-empty open subset of X intersects A non-trivially (that is, if $U \in \tau$ and $U \neq \emptyset$ then $A \cap U \neq \emptyset$).*

In what follows we need the concept of neighborhood. Again, for topological spaces where a metric is not defined and there is no notion of distance, this concept will prove to be not only important for what follows, but also interesting from a logical point of view.

Definition 2.9 Let (X, τ) be a topological space, N a subset of X and p a point in N . Then N is said to be a neighborhood of the point p if there exists an open set U such that $p \in U \subseteq N$.

As an example, the closed interval $[0, 1] \in \mathbb{R}$ is a neighborhood of the point $\frac{1}{2}$ since $\frac{1}{2} \in (\frac{1}{4}, \frac{3}{4}) \subseteq [0, 1]$.

Proposition 2.10 *Let A be a subset of a topological space (X, τ) . A point $x \in X$ is a limit point of A if and only if every neighborhood of x contains a point of A different from x .*

As a set is closed if and only if it contains all its limit points we deduce the following

Corollary 2.11 *Let A be a subset of a topological space (X, τ) . Then the set A is closed if and only if for each $x \in X - A$ there is a neighborhood N of x such that $N \subseteq X - A$.*

Corollary 2.12 *Let U be a subset of a topological space (X, τ) . Then $U \in \tau$ if and only if for each $x \in U$ there exists a neighborhood N of x such that $N \subseteq U$.*

Corollary 2.13 *Let U be a subset of a topological space (X, τ) . Then $U \in \tau$ if and only if for each $x \in U$ there exists a $V \in \tau$ such that $x \in V \subseteq U$.*

The last corollary provides a practical test of whether a set is open or not. A set is open if and only if it contains an open set about each of its points. In what follows, a brief discussion about connectedness [5] will be given. Some simple definitions and facts are given in an informal way, mainly following reference [1] which is a source of inspiration for the major part of this section. Let therefore S be a set of real numbers. If there is an element $b \in S$ such that $x \leq b$, for all $x \in S$ then b is said to be the greatest element of S . Similarly if S contains an element a such that $a \leq x$ for all $x \in S$ then a is called the least element of S . A set S of real numbers is said to be bounded above if there exists a real number c such that $x \leq c$ for all $x \in S$, and c is called an upper bound for S . Similarly, the terms “bounded below” and “lower bound” are defined. A set which is bounded above and bounded below is said to be bounded [6].

Least Upper Bound Axiom 2.14 Let S be a non-empty set of real numbers. If S is bounded above, then it has a least upper bound.

The upper bound also called the supremum of S , denoted $\sup(S)$, may or may not belong to the set S . Indeed the supremum of S is an element of S if and only if S has a greatest element. Any set S of real numbers which is bounded below has a greatest lower bound which is also called the infimum and is denoted by $\inf(S)$.

Lemma 2.15 Let S be a subset of \mathbb{R} which is bounded above and let p be the supremum of S . If S is a closed subset of \mathbb{R} , then $p \in S$.

Proof See appendix.

Proposition 2.16 Let T be a clopen subset of \mathbb{R} . Then either $T = \mathbb{R}$ or $T = \emptyset$.

Proof See appendix.

Definition 2.17 Let (X, τ) be a topological space. Then it is said to be connected if the only clopen subsets of X are X and \emptyset . As an example, the topological space \mathbb{R} is connected.

From the definition follows that a topological space (X, τ) is not connected (i.e. disconnected) if and only if there are non-empty open sets A and B such that $A \cap B = \emptyset$ and $A \cup B = X$. This fact is important because it constitutes the basis for the future generalizations to connected manifolds, groups, etc.

In what follows I will briefly discuss what means when we say that two structures are equivalent [7]. The distinction between objects implies two items: the objects themselves and the criteria by which the notion of “distinctiveness” is defined. In set theory, two sets are said to be equivalent from the perspective of set theory if there exists a bijective function which maps one set onto another. Two groups are equivalent, also said to be isomorphic, if there exists a homomorphism of one to the other which is one-to-one and onto. Two topological spaces are equivalent, also said to be homeomorphic if there exists a homeomorphism of one onto the other. Hence, first we need a definition for the objects we want to compare. Then we need to explain what means “equivalent” in our theory. I will start by defining the objects that are important in this context, and these objects are the topological spaces. Hence, we will want to compare subspaces of a given space.

Definition 2.18 Let Y be a non-empty subset of a topological space (X, τ) . The collection $\tau_Y = \{O \cup Y : O \in \tau\}$ of subsets of Y is a topology on Y called the subspace topology (or relative topology, or induced topology on Y by τ). The topological space (Y, τ_Y) is said to be a subspace of (X, τ) .

One can check that τ_Y is indeed a topology on Y . Now we turn to the notion of equivalence defined for the topological spaces. We may start with an example

$$X = \{a, b, c, d, e\}, Y = \{g, h, i, j, k\} \quad (2.4)$$

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\} \quad (2.5)$$

and

$$\tau_1 = \{Y, \emptyset, \{g\}, \{i, j\}, \{g, i, j\}, \{h, i, j, k\}\} \quad (2.6)$$

It is intuitively clear that (X, τ) is equivalent to (Y, τ_1) . The function $f : X \rightarrow Y$ defined by $f(a) = g$, $f(b) = h$, $f(c) = i$, $f(d) = j$ and $f(e) = k$, provides the equivalence.

Definition 2.19 Let (X, τ) and (Y, τ_1) be topological spaces. Then we say they are homeomorphic if there exists a function $f : X \rightarrow Y$ which has the following properties:

- f is one-to-one (that is $f(x_1) = f(x_2)$ implies $x_1 = x_2$).
- f is onto (that is, for any $y \in Y$ there exists an $x \in X$ such that $f(x) = y$)
- for each $U \in \tau_1$, $f^{-1}(U) \in \tau$ and
- for each $V \in \tau$, $f(V) \in \tau_1$

Further, the map f is said to be a homeomorphism between (X, τ) and (Y, τ_1) . We write $(X, \tau) \cong (Y, \tau_1)$.

It can be shown that \cong is an equivalence relation and that all open intervals (a, b) are homeomorphic to each other. Length is not a topological property [8]. In particular, an open interval of finite length such as $(0, 1)$ is homeomorphic to one of infinite length such as $(-\infty, 1)$. In fact, all open intervals are homeomorphic with \mathbb{R} . There is an important aspect related to the methods of proof. In order to prove that two topological spaces are homeomorphic we have to find a homeomorphism between them. However, to prove that two topological spaces are not homeomorphic is often much harder as we have to show that no homeomorphism exists. In order to show this difficulty the next example is important.

Example 2.20 We want to prove that the interval $[0, 2]$ is not homeomorphic to the subspace $[0, 1] \cup [2, 3]$ or \mathbb{R} . Let for this $(X, \tau) = [0, 2]$ and $(Y, \tau_1) = [0, 1] \cup [2, 3]$. Then $[0, 1] = [0, 1] \cap Y \Rightarrow [0, 1]$ is closed in (Y, τ_1) and $[0, 1] = (-1, 1\frac{1}{2}) \cap Y \Rightarrow [0, 1]$ is open in (Y, τ_1) . Thus Y is not connected as it has $[0, 1]$ as a proper non-empty clopen subset.

Suppose that $(X, \tau) \cong (Y, \tau_1)$. Then there exists a homeomorphism $f : (X, \tau) \rightarrow (Y, \tau_1)$. So, $f^{-1}([0, 1])$ is a clopen subset of X , and hence X is not connected. This is false as $[0, 2] = X$ is connected. So we have a contradiction and thus the two topological spaces are not homeomorphic. Hence, we can observe the following

Proposition 2.21 *Any topological space homeomorphic to a connected space is connected.*

This observation is extremely important in simplifying the proofs that objects (hence also topological spaces) are not homeomorphic with each other [9]. Instead of actually

searching every possible homeomorphism and eliminating each of them, it is far easier to find one single property preserved by homeomorphisms which can be proven that one space has and the other does not. In this way, the “checking” of all possible homeomorphisms is avoided leading to a major simplification. There are several such properties preserved by homeomorphisms that can be used. However, when faced with a specific problem we may not be able to find the best property we would like to use. The art is to decide when it is easier to check all homeomorphisms and when it is easier to check all preserved properties [10]. One can however make statements about the real line for which we have the following

Definition 2.22 A subset S of \mathbb{R} is said to be an interval if it has the following property: if $x \in S$, $z \in S$ and $y \in \mathbb{R}$ are such that $x < y < z$ then $y \in S$.

Connectedness for the real line is easily prescribed by the following

Proposition 2.23 A subspace S of \mathbb{R} is connected if and only if it is an interval.

Up to now we discussed the objects and the equivalence relations. The next structure, specific to category theory is called the set of arrows [11]. They represent different things when analyzed in different branches of mathematics. In linear algebra we have as objects the vector spaces and as arrows the linear transformations. In group theory the objects are the groups while the arrows are the homomorphisms, while in set theory the objects are sets and the arrows are functions. In topology the objects are the topological spaces and the arrows are the continuous mappings. However, how can we define a notion such as “continuity” in a general topological space? Of course for functions from \mathbb{R} to \mathbb{R} this is simple: a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be continuous if for each $a \in \mathbb{R}$ and each positive real number ϵ , there exists a positive real number δ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. This construction however is very dependent on the definition of absolute value, subtraction and in general distance [12]. All these notions do not need to exist (although can certainly be defined for some cases) in general topological spaces [13]. Hence we need a different definition of continuity, more suitable for generalizations. We can see that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous iff for each $a \in \mathbb{R}$ and each interval $(f(a) - \epsilon, f(a) + \epsilon)$, for $\epsilon > 0$ there exists a $\delta > 0$ such that $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$ for all $x \in (a - \delta, a + \delta)$. This definition does not involve the notion of distance or of absolute value but it still involves the notion of subtraction which may not make sense in general i.e. the inversion of addition may not be defined [14]. In order to avoid subtraction completely we can introduce the following

Lemma 2.24 Let f be a function mapping \mathbb{R} into itself. Then f is continuous if and only if for each $a \in \mathbb{R}$ and each open set U containing $f(a)$, there exists an open set V containing a such that $f(V) \subseteq U$.

Proof See appendix.

One could use the property described in the above lemma to define continuity but the following lemma makes the definition more elegant.

Lemma 2.25 *Let f be a mapping of a topological space (X, τ) into a topological space (Y, τ') . Then the following two conditions are equivalent:*

- *for each $U \in \tau'$, $f^{-1}(U) \in \tau$*
- *for each $a \in X$ and each $U \in \tau'$ with $f(a) \in U$, there exists a $V \in \tau$ such that $a \in V$ and $f(V) \subseteq U$.*

Proof See appendix.

Hence the notion of continuity for a function between two topological spaces becomes

Definition 2.26 Let (X, τ) and (Y, τ_1) be topological spaces and f a function from X into Y . Then $f : (X, \tau) \rightarrow (Y, \tau_1)$ is said to be a continuous mapping if for each $U \in \tau_1$, $f^{-1}(U) \in \tau$.

Now we can write the following

Proposition 2.27 *Let f be a mapping of a topological space (X, τ) into a space (Y, τ') . Then f is continuous if and only if for each $x \in X$ and each $U \in \tau'$ with $f(x) \in U$, there exists a $V \in \tau$ such that $x \in V$ and $f(V) \subseteq U$.*

Proposition 2.28 *Let (X, τ) , (Y, τ_1) and (Z, τ_2) be topological spaces. If $f : (X, \tau) \rightarrow (Y, \tau_1)$ and $g : (Y, \tau_1) \rightarrow (Z, \tau_2)$ are continuous mappings, then the composite function $g \circ f : (X, \tau) \rightarrow (Z, \tau_2)$ is continuous.*

Of course, the next result shows that we can interchange closed sets with open sets in the definition of continuity

Proposition 2.29 *Let (X, τ) and (Y, τ_1) be topological spaces. Then $f : (X, \tau) \rightarrow (Y, \tau_1)$ is continuous if and only if for every closed subset S of Y , $f^{-1}(S)$ is a closed subset of X .*

Proof See appendix.

There is a connection between continuous maps and homeomorphisms. If $f : (X, \tau) \rightarrow (Y, \tau_1)$ is a homeomorphism then it is a continuous map. Obviously not every continuous map is a homeomorphism.

Proposition 2.30 *Let (X, τ) and (Y, τ') be topological spaces and f a function from X to Y then f is a homeomorphism iff*

- *f is continuous*
- *f has an inverse*
- *f^{-1} is continuous*

Proposition 2.31 *Let (X, τ) and (Y, τ_1) be topological spaces, $f : (X, \tau) \rightarrow (Y, \tau_1)$ a continuous mapping, A a subset of X and τ_2 the induced topology on A . Further, let $g : (A, \tau_2) \rightarrow (Y, \tau_1)$ be the restriction of f to A , that is $g(x) = f(x)$ for all $x \in A$. Then g is continuous.*

An important result is given by the following

Proposition 2.32 *Let (X, τ) and (Y, τ_1) be topological spaces and $f : (X, \tau) \rightarrow (Y, \tau_1)$ surjective and continuous. If (X, τ) is connected then (Y, τ_1) is connected.*

Proof See appendix.

Otherwise stated this proposition says that any continuous image of a connected set is connected. It also says that if (X, τ) is a connected space and (Y, τ') is not connected then there exists no mapping of (X, τ) onto (Y, τ') which is continuous. There exists a stronger definition of connectedness [15]:

Definition 2.33 A topological space (X, τ) is said to be path-connected if for each pair of distinct points a and b of X there exists a continuous mapping $f : [0, 1] \rightarrow (X, \tau)$ such that $f(0) = a$ and $f(1) = b$. The mapping f is said to be a path joining a to b .

Every path connected space is connected. At this point, I can introduce Weierstrass' Intermediate value Theorem [16], an application of topology to the theory of functions of a real variable. The topological concept important for this is that of connectedness.

Theorem 2.34 *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and let $f(a) \neq f(b)$. Then for every number p between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = p$.*

Proof See appendix.

Corollary 2.35 *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and such that $f(a) > 0$ and $f(b) < 0$ then there exists an $x \in [a, b]$ such that $f(x) = 0$.*

Corollary 2.36 (The fixed point theorem) *Let f be a continuous mapping of $[0, 1]$ into $[0, 1]$. Then there exists a $z \in [0, 1]$ such that $f(z) = z$. The point is called a fixed point.*

Proof See appendix.

This corollary is a special case for another theorem called the Brouwer fixed point theorem [17] which says that every continuous function from a convex compact subset \mathcal{K} of a Euclidean space to \mathcal{K} itself has a fixed point. Most proofs are of algebraic topological nature [18]. However, this theorem has many applications, from theoretical economics [19] to applied mathematics [20].

As I mentioned several times until now, the discussion in this first part was intentionally as general as possible. This implied the definition of notions like continuity such that they do not depend on notions related to metric spaces like distances, absolute values, etc. In what follows I will particularize the discussion a bit, making however as clear as possible that most of the interesting applications appear when the notions of metric and distance are not readily available. One may ask if there are situations when we do not wish to measure distances or distances are not well defined. Indeed, the basic question in topology is to describe structures that do not depend on

continuous deformations, and obviously, distance is one concept that changes in continuous deformations. Therefore, topological notions are in the most general sense not dependent on structures like distance. Additional structure must be added to the topological structure so that we are capable of discussing about distances. However, there do exist situations where distance is not necessary, for example quantum entanglement is a correlation which does not, a-priori, depend on distance. Once topology itself becomes uncertain, the notion of distance will become even more ambiguous. Most of the applications of topology to analysis are via metric spaces [21]. Because of this I will start with a definition

Definition 2.37 Let X be a non-empty set and d a real valued function defined on $X \times X$ such that for $a, b \in X$:

- $d(a, b) \geq 0$ and $d(a, b) = 0$ if and only if $a = b$
- $d(a, b) = d(b, a)$
- $d(a, c) \leq d(a, b) + d(b, c)$ for all $a, b, c \in X$

Then d is said to be a metric on X , (X, d) is a metric space and $d(a, b)$ is the distance between a and b .

Having a metric space (X, d) and r a positive real number we can define the open ball about $a \in X$ of radius r as the set

$$B_r = \{x : x \in X; d(a, x) < r\} \quad (2.7)$$

In what follows I wish to connect the metric spaces to the topological spaces. For this I will need the following

Lemma 2.38 Let (X, d) be a metric space and a and b points of X . Further, let δ_1 and δ_2 be positive real numbers. If $c \in B_{\delta_1}(a) \cap B_{\delta_2}(b)$ then there exists a $\delta > 0$ such that $B_\delta(c) \subseteq B_{\delta_1}(a) \cap B_{\delta_2}(b)$.

Corollary 2.39 Let (X, d) be a metric space and B_1 and B_2 open balls in (X, d) . Then $B_1 \cap B_2$ is a union of open balls in (X, d) .

Proposition 2.40 Let (X, d) be a metric space. Then the collection of open balls in (X, d) is a basis for a topology τ on X . This is the topology induced by the metric d and (X, τ) is called the induced topological space [22] or the corresponding topological space.

As an example consider d the euclidean metric on \mathbb{R} . Then a basis for the topology τ induced by the metric d is the set of all open balls. But $B_\delta(a) = (a - \delta, a + \delta)$. From this it is easy to see that τ is the euclidean topology on \mathbb{R} . Hence the euclidean metric on \mathbb{R} induces the euclidean topology on \mathbb{R} .

From the perspective of how a set of numbers can be completed, there exist other types of metrics. Among non-euclidean metrics one can cite the non-Archimedean

metric which gives rise to the so called p-adic numbers. This is one of the three possible completions of the rationals, the other two being the real numbers and the complex numbers. The p-adic numbers do not obey the Archimedean axiom, one of the axioms introduced by Hilbert in his general approach to geometry. The basic formulation of Archimedes' axiom is that given two magnitudes having a ratio, one can find a multiple of either which will exceed the other. This multiple must be finite. By this one excludes the existence of differential objects. Just as the real numbers are a completion of the rationals with respect to the usual norm, the p-adic numbers are the completion of the rationals with respect to the p-adic norm.

Let me now consider d the discrete metric on a set X . Then for each $x \in X$, $B_{\frac{1}{2}}(x) = \{x\}$. So, all the singleton sets are open in the topology τ induced on X by d . As a consequence, τ is the discrete topology.

Definition 2.41 Metrics on a set X are equivalent if they induce the same topology on X .

Proposition 2.42 *Let (X, d) be a metric space and τ the topology induced on X by the metric d . Then a subset U of X is open in (X, τ) if and only if for each $a \in U$ there exists an $\epsilon > 0$ such that the open ball $B_\epsilon(a) \subseteq U$.*

Proof See appendix.

It was noticed that every metric on a set X induces a topology on the set X . However, the reverse is not always true i.e. not every topology on a set is induced by a metric.

Definition 2.43 A topological space (X, τ) is said to be a Hausdorff space (or a T_2 -space) if for each pair of distinct points a and b in X , there exist open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

It can be seen that \mathbb{R} , \mathbb{R}^2 and all discrete spaces are Hausdorff [23]. However, any set with at least 2 elements which has the indiscrete topology is not a Hausdorff space. It may be relevant to note that \mathbb{Z} with finite-closed topology is also not a Hausdorff space.

Proposition 2.44 *Let (X, d) be any metric space and τ the topology induced on the X by d . Then (X, τ) is Hausdorff.*

Proof See appendix.

We can see out of this proposition that an indiscrete space with at least two points has a topology which is not induced by any metric. Also, \mathbb{Z} with the finite-closed topology τ is such that τ is not induced by any metric on \mathbb{Z} .

Proposition 2.45 *A space (τ, X) is said to be metrizable if there exists a metric d on the set X with the property that τ is the topology induced by d .*

For example the set \mathbb{Z} with the finite-closed topology is not a metrizable space. One should not believe that any Hausdorff space is metrizable. In fact there exist Hausdorff spaces which are not metrizable [24].

In what follows, I will review briefly the notions surrounding the convergence of sequences. It is clear what a convergent sequence of real numbers is. In order to remind the reader, the definition is as follows. The sequence $x_1, x_2, \dots, x_n, \dots$ of the real numbers is said to converge to the real number x if given any $\epsilon > 0$, there exists an integer n_0 such that for all $n \geq n_0$, $|x_n - x| < \epsilon$. The generalization of this definition from \mathbb{R} to any metric space is obvious

Definition 2.46 Let (X, d) be a metric space and x_1, \dots, x_n, \dots a sequence of points in X . Then the sequence is said to converge to $x \in X$ if given any $\epsilon > 0$ there exists an integer n_0 such that for all $n \geq n_0$, $d(x, x_n) < \epsilon$. This is denoted by $x_n \rightarrow x$. The sequence $y_1, y_2, \dots, y_n, \dots$ of points in (X, d) is said to be convergent if there exist a point $y \in X$ such that $y_n \rightarrow y$.

Proposition 2.47 Let $x_1, x_2, \dots, x_n, \dots$ be a sequence of points in a metric space (X, d) . Further, let x and y be points in (X, d) such that $x_n \rightarrow x$ and $x_n \rightarrow y$. Then $x = y$.

We say that a subset A of a metric space (X, d) is closed (resp. open) in the metric space (X, d) if it is closed (resp. open) in the topology τ induced on X by the metric d .

In fact, the topology of a metric space can be described entirely in terms of its convergent sequences.

Proposition 2.48 Let (X, d) be a metric space. A subset A of X is closed in (X, d) if and only if every convergent sequence of points in A converges to a point in A . This means that A is closed in (X, d) if and only if $a_n \rightarrow x$ where $x \in X$ and $a_n \in A$ for all n , implies $x \in A$.

Proof See appendix.

This finishes the introduction in general topology required for this work. Further information on the subject can be found in [25–31]. While the results seem trivial, they by themselves are only marginally the reason for this chapter. I introduced this chapter mainly because the method of thinking derived from it reflects back to algebraic topology and more advanced mathematical subjects. In fact, during my independent research I started precisely with these constructions the formal study of topology. This proved very useful mainly because I understood the distinction between mathematical proofs and physical proofs. In general physicists tend to perform robust and numerically intensive calculations and to regard those as proofs in a very specific sense. The mathematical proofs, no matter how rigorous are often regarded with skepticism. On the other side, mathematically oriented researchers tend to see physical proves as inelegant, dull and sometimes plain inefficient. However, in what follows I show that the two ways of thinking may fruitfully coexist.

References

1. S.A. Morris, *Topology Without Tears*, Online e-book (2007)
2. S.C. Kleene, *Introduction to metamathematics*, North-Holland (1959). ISBN-13: 978-0923891572
3. M. Hazewinkel, *Axiomatic method*, *Encyclopedia of Mathematics* (Springer, Berlin, 2001). ISBN 978-1-55608-010-4
4. R. Haag, D. Kastler, An algebraic approach to quantum field theory, *J. Math. Phys.* **5**, 848 (1964)
5. G.F. Simmons, *Introduction to Topology and Modern Analysis* (McGraw Hill Book Company, New York, 1968) p.144. ISBN 0-89874-551-9
6. A. Hatcher, Notes on Introductory Point-Set Topology, Cornell University lecture notes (2007)
7. E. Castellani, *Symmetry and Equivalence*, *Symmetries in Physics: Philosophical Reflections* (Cambridge University, Cambridge, 2003). ISBN 9-78052-152889-4
8. L.A. Steen, J.A. Seebach, *Counterexamples in Topology* (Springer, New York, 1978). ISBN-13: 978-0486687353
9. J.M. Moller, London Math Soc. Lecture Notes Series, 175, Cambridge University Press, p. 131 (1992)
10. M.F. Atiyah, Duality in Mathematics and Physics, lecture notes from the Institut de Matematica de la Universitat de Barcelona (IMUB) (2007)
11. S. Awodey, *Category Theory. Oxford Logic Guides* vol. 49 (Oxford University Press, Oxford, 2006). ISBN-13: 978-0199237180
12. F.W. Lawvere, Metric spaces, generalized logic, and closed categories, *Rend. Sem. Mat. Fis. Milano*, 43 (1973)
13. N. Bourbaki, *Topologie Generale* 5–10 (collective author) (1998). ISBN 3-54064-563-2
14. J.M. Howie, *Fundamentals of Semigroup Theory*, *London Mathematical Society Monographs*. New Series 12, (Oxford: Clarendon Press, Oxford, 1995). ISBN 0-19-851194-9
15. W. Browder, The homotopy type of differentiable manifolds, *Proceedings of the Aarhus Symposium*, p. 42 (1962)
16. R.G. Bartle, D.R. Sherbert, *Introduction to Real Analysis* (3rd edn.) (Wiley Publications, New York, 2001). ISBN-13: 978-0471433316
17. R.B. Kellogg, T.Y. Li, J.A. Yorke, A constructive proof of the Brouwer fixed point theorem and computational results, *SIAM J. Numer. Anal.* **13**(4), 473 (1976)
18. V.I. Istratescu, *Fixed Point Theory*, Reidel (1981). ISBN 90-277-1224-7
19. H.E. Scarf, Fixed-point theorems and economic analysis, Mathematical theorems can be used to predict probable effects of changes in economic policy, *Am. Sci.* **71**(3), 289 (1983)
20. S.L. Singh, S.N. Mishra, R. Chugh, R. Kamal, General common fixed point theorems and applications, *J. Appl. Math.* ID 902312, (2012)
21. W.A. Sutherland, *Introduction to Metric and Topological Spaces* (Oxford Science Publications, Oxford, 1975). ISBN 0-19-853161-3
22. L. Guilong, Topologies induced by equivalence relations, *Commun. Comput. Inf. Sci.* **163**, 204 (2011)
23. F. Hausdorff, Erweiterung einer Homeomorphie, *Fundamenta Mathematicae*, 16, p. 352 (1930)
24. K. Keremedis, E. Tachtsis, Countable Compact Hausdorff Spaces Need Not Be Metrizable in ZF, *Proc. Am. Math. Soc.* **135**(4), 1205 (2007)
25. S.S. Gabrielyan, Topologies on groups determined by sets of convergent sequences, *J. Pure Appl. Alg.* **217**(5), 786 (2013)
26. D.K. Burke, Cauchy Sequences in Semimetric Spaces, *Proc. Am. Math. Soc.* **33**(1), 161 (1972)
27. R. Prabir, Separability of metric spaces, *Trans. Am. Math. Soc.* **149**, 19 (1970)
28. G. Beer, A polish topology for the closed subsets of a polish space, *Proc. Am. Math. Soc.* **113**(4), 1123 (1991)
29. L. Nguyen Van, The structural Ramsey theory of metric spaces and topological dynamics of isometry groups, *Memoirs Am. Math. Soc.* **968**, 206 (2010). ISBN 978-0-8218-4711-4

30. S. Kasahara, On some generalizations of the Banach contraction theorem, Publ. RIMS, Kyoto Univ. **12**, 427 (1976)
31. S.H. Jones, Applications of the Baire category theorem, Real Anal. Exchange, **23**(2), 363 (1999)

The Universal Coefficient Theorem and Quantum Field
Theory

A Topological Guide for the Duality Seeker

Patrascu, A.-T.

2017, XVI, 270 p. 6 illus., 1 illus. in color., Hardcover

ISBN: 978-3-319-46142-7