Chapter 2
On the Linearization Band

In this chapter we will review the linearization process, following the general approach of T. Krarup, as presented in his famous three letters on Molodensky's Problem (see [1]), but adapted to the case of the Scalar Molodensky Problem, as introduced in Chap. 1 and discussed also in [2], Sect. 2.3.

There are several reasons to do that:

(i) First of all, from a general theoretical point of view, we aim at clarifying that in the linearization procedure, although the normal potential U and "some telluroid", $\{h = \widetilde{h}(\sigma)\}$, are introduced as approximations respectively of the gravity potential W and of the Earth surface $\{h = h(\sigma)\}$, so that the increments $T = W - U$ and $\zeta = h - \widetilde{h}$ can be considered as "first-order" infinitesimals, such a hypothesis cannot be considered as acceptable, unless a suitable compatibility condition is introduced relating the orders of magnitude of the two quantities.

(ii) Based on the above remark, once the order of magnitude of T and ζ are assessed, the Molodensky BVP can be expanded up to second-order terms with the purpose of verifying that they can be neglected for the level of accuracy we aim at. This defines the linearization band.

(iii) Finally, we shall establish the principle of equivalence stating that all the problems formulated by linearization of the same scalar non-linear Molodensky problem, with approximate reference potential and telluroid chosen in the linearization band, are essentially equivalent, up to second order of magnitude errors.

A similar problem has already been analyzed in literature (see [3]) with a strong numerical apparatus, taking into account also the spatial behaviour of the gravity field. However, in the quoted paper the purpose was more to compare different formulations of the GBVP, arriving at the conclusion that the scalar non-linear Molodensky problem was the most natural and useful formulation for geodetic purposes. So we

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build here in a sense on this conclusion. Our approach however is more elementary, though sufficient to achieve the most important result of the chapter, namely the definition of a "linearization band".

Before we get started, let us comment on the meaning of what we will use as "order of magnitude" for the different quantities, q, usually defined on the Earth sphere, namely the projection of the Earth surface on the unit sphere.

One first rigorous definition could be the mean square value of the distribution of $q(\sigma)$, namely

$$O(q) = \sigma(q) = \{\frac{1}{4\pi}\int q^2(\sigma)d\sigma\}^{1/2} \quad (\sigma = (\lambda, \varphi) \text{ spherical coordinates}) \, . \tag{2.1}$$

In (2.1) σ is used with two different meanings: to represent spherical coordinates, but also to mean the r.m.s. of some quantity on the sphere. The two concepts however should be clear by the context. The disadvantage of using this measure though, is that the extreme value $\max_\sigma |q(\sigma)|$ is not easily related to $\sigma(q)$, in particular considering that every quantity has generally a different spectral signature when expressed in terms of spherical harmonics ([2], Sect. 3.8). For instance the value of $3\sigma(q)$ is not always a good guess of the maximum value of q. So, since in the present reasoning we want to be on the safe side in evaluating the error we should try to find an index more related to the maximum of q. For this purpose we shall use a value $O_M(q)$, which is a very high value of q, only seldom met on the Earth globe and even more rarely exceeded. For instance a 90 % quantile. Generally we shall agree on a value that at least satisfies the following relation

$$O_M(q) \leq \max |q(\sigma)| \leq 2O_M(q) \, . \tag{2.2}$$

To avoid ambiguity, in the rest of the Chapter we shall use the following table of orders of magnitudes:

Table 2.1 Orders of magnitude of various geodetic quantities

Quantity q	$O_M(q)$
a, b, R	6×10^6 m
e^2	$150^{-1} = 6.7\ 10^{-3}$
W	6×10^9 Gal $\times$ m
γ, g	10^3 Gal
$\frac{\partial\gamma}{\partial r}, \frac{\partial g}{\partial r}$	0.3 Gal km^{-1}
$\frac{\partial T}{\partial r}, \delta g, \Delta g$	0.1 Gal
$\frac{\partial^2 T}{\partial r^2}$	6×10^{-4} Gal km^{-1}
δ	3×10^{-4} rad
H	6×10^3 m

where a, b are the semi-axes of the ellipsoid, R the mean Earth radius, e^2 the square of the first eccentricity, W, g are potential and gravity on the Earth surface, T is the anomalous potential, δg, Δg are gravity disturbance and anomaly, δ is the deflection of the vertical, H is the topographic height.

We shall use the symbol $\sim$ to express that $O_M(q)$ attains a certain numerical value, for instance $\gamma \sim 10^3$ Gal. Noteworthy, with the figures of Table 2.1, the following relations hold

$$T \sim 1.6 \times 10^{-5} W \tag{2.3}$$

$$\Delta g \sim 1 \times 10^{-4} \gamma \tag{2.4}$$

$$\frac{T}{\gamma} \sim 1.6 \times 10^{-5} \frac{W}{\gamma} \sim 10^2 \text{ m} \tag{2.5}$$

After these remarks let us go back to the linearization of the scalar Molodensky problem. We introduce the approximate potential U and some telluroid $\widetilde{S} = \{h = \widetilde{h}(\sigma)\}$, with $\sigma = (\lambda, \varphi)$ ellipsoidal coordinates, such that

$$W - U|_{\widetilde{S}} = T|_{\widetilde{S}} \tag{2.6}$$

$$\zeta(\sigma) = h(\sigma) - \widetilde{h}(\sigma) \tag{2.7}$$

should be considered as first-order infinitesimals. Note that by taking U, the normal potential, as an approximate solution for W, we will define a certain linearization band, that however could change with a different approximate potential, typically becoming narrower. Note also that with (2.6), $O_M(T)$ is *fixed* by Table 2.1 and (2.3).

The free-boundary relations to be linearized are

$$W_0(\sigma) = U(\sigma, h(\sigma)) + T(\sigma, h(\sigma)) \tag{2.8}$$

$$\begin{aligned} g_0(\sigma) &= |\nabla U(\sigma, h(\sigma)) + \nabla T(\sigma, h(\sigma))| = \\ &= |\boldsymbol{\gamma}(\sigma, h(\sigma)) + \nabla T(\sigma, h(\sigma))| \,. \end{aligned} \tag{2.9}$$

In order to appreciate the order of magnitude of the errors committed by substituting (2.8) and (2.9) with the linearized relations, we will push the Taylor development to the second order. For the sake of conciseness we shall use the symbol q' to express the vertical or radial derivative of q, according to the context.

Linearization of (2.8): we have

$$\begin{aligned} W_0 &= U(\widetilde{h} + \zeta) + T(\widetilde{h} + \zeta) = \\ &= U(\widetilde{h}) + U'(\widetilde{h})\zeta + \frac{1}{2}U''(\widetilde{h})\zeta^2 + \\ &+ T(\widetilde{h}) + T'(\widetilde{h})\zeta + O_3 \,. \end{aligned} \tag{2.10}$$

We call *geodetic anomaly of the potential* W the quantity

$$DW = W_0 - U(\widetilde{h}) \,. \tag{2.11}$$

We also note that

$$U'(\widetilde{h}) \sim -\gamma(\widetilde{h})$$
$$U''(\widetilde{h}) \sim -\gamma'(\widetilde{h}) \,,$$

so that (2.10) can be reorganized as

$$\begin{aligned} DW = T(\widetilde{h}) - \gamma(\widetilde{h})\zeta - \frac{1}{2}\gamma'(\widetilde{h})\zeta^2 \\ +T'(\widetilde{h})\zeta + O_3 \end{aligned} \tag{2.12}$$

Now consider that in (2.12) we expect $T, \gamma\zeta$ to be the first-order terms, while $\frac{1}{2}\gamma'\zeta^2, T'\zeta$ should be the second-order terms, candidate to be neglected.

But this is true only if both T and $\gamma\zeta$ are of the *same* order of magnitude; since $O_M(T)$ is fixed, we must introduce then a *compatibility condition* stating that

$$\gamma\zeta \sim T \,, \tag{2.13}$$

which, on account of (2.5), implies

$$\zeta \sim 10^2 \text{ m}. \tag{2.14}$$

Notice that (2.13) is not the Bruns relation (1.7), because in general $\widetilde{h}$ doesn't need to be the Marussi telluroid defined by (1.4), i.e., by the condition $DW = 0$, yet $O_M(\zeta)$ has to be 100 m, i.e., the telluroid has to be in a band of 100–200 m from the Earth surface at most, if we want the linearization procedure to work. A larger height anomaly might bring us to false conclusions. The fact that the Marussi telluroid satisfies the compatibility condition is a lucky empirical fact that is verified a posteriori, once the solution T has been found and not an a priori statement.

Given the above, we can pass to evaluate the second-order terms and decide whether they are negligible or not. Before we do that, we must fix the order of magnitude of the negligible errors, ε_w, in potential. We state the rule that ε_w is negligible if

$$O_M(\varepsilon_w) = 1 \text{ cm} \cdot \gamma = 10 \text{ Gal} \times \text{ m}. \tag{2.15}$$

In fact, by using the value in Table 2.1, and (2.14), we have

$$\frac{1}{\gamma}(\frac{1}{2}\gamma'\zeta^2) \sim 1.5 \text{ mm} \tag{2.16}$$

Moreover,

$$\frac{T'\zeta}{\gamma} \sim 1 \text{ cm}. \tag{2.17}$$

As we see, this term is still in our acceptable error range.

Linearization of (2.9): before starting our computation, we recall the differential formula, valid up to the second order,

$$|\mathbf{v} + d\mathbf{v}| = |\mathbf{v}| + \mathbf{e} \cdot d\mathbf{v} + \frac{1}{2|\mathbf{v}|} d\mathbf{v} \cdot (I - P_e) d\mathbf{v} \tag{2.18}$$

where

$$\mathbf{e} = \frac{\mathbf{v}}{|\mathbf{v}|} \ , \ P_e d\mathbf{v} = \mathbf{e}(\mathbf{e} \cdot d\mathbf{v}) \ .$$

By applying (2.18) to (2.9), we get

$$g_0 = \gamma(h) + \mathbf{e} \cdot \nabla T(h) + \frac{1}{2} \frac{1}{\gamma(h)} \nabla T \cdot (I - P_e) \nabla T + O_3 \tag{2.19}$$

where all quantities are still evaluated at h. Note that in (2.19) one can write, with a very good order of approximation,

$$\mathbf{e} \cong -\boldsymbol{\nu} \ , \ \mathbf{e} \cdot \nabla T \cong -T' \ ;$$

this is because the tangent to the normal field lines is equal to $\boldsymbol{\nu}$ on the ellipsoid and it varies very slowly with altitude, at least at topographic heights.

So we can write, developing to the second order,

$$\begin{aligned} g_0 = {} & \gamma(\widetilde{h}) + \gamma'(\widetilde{h})\zeta + \frac{1}{2}\gamma''(\widetilde{h})\zeta^2 + \\ & -T'(\widetilde{h}) - T''(\widetilde{h})\zeta + \\ & + \frac{1}{2} \frac{1}{\gamma(\widetilde{h})} \nabla T(\widetilde{h}) \cdot (I - P_e) \nabla T(\widetilde{h}) + O_3 \ . \end{aligned} \tag{2.20}$$

Again we define the geodetic anomaly of g as

$$Dg = g_0 - \gamma(\widetilde{h}) \ , \tag{2.21}$$

observing that Dg will coincide with the usual free air gravity anomaly Δg as soon as $\widetilde{h}$ is chosen as the height of the Marussi telluroid. We note as well that the last term, being already a second-order term, can be directly evaluated at $\widetilde{h}$. So, reorganizing (2.20), we get

$$\begin{aligned} Dg = g_0 - \gamma(\widetilde{h}) = {} & -T'(\widetilde{h}) + \gamma'(\widetilde{h})\zeta + \\ & + \frac{1}{2}\gamma''(\widetilde{h})\zeta^2 - T''\zeta + \frac{1}{2\gamma(\widetilde{h})} |\nabla_h T|^2 + O_3 \end{aligned} \tag{2.22}$$

where $\nabla_h T$ is just the horizontal gradient of T. It is immediate to verify that T' and $\gamma'\ \zeta$ are of the same order of magnitude, so we need to analyze second order terms.

To verify whether the second-order terms are negligible we need to fix the order of a negligible error in g. We fix such error ε_g at the level

$$\varepsilon_g \sim 30\,\mu\text{Gal} \tag{2.23}$$

on the basis of the following simplistic reasoning. Since Δg, with an order of magnitude of 100 mGal, gives rise to a ζ of the order of 100 m, we could expect that, if we had an error with the same spectral shape as the signal and a mean square value of $10\ \mu\text{Gal} = 10^{-4} \times 100\,\text{mGal}$, the corresponding error in T/γ would be of the order of $10^{-4} 100\ \text{m} = 1\ \text{cm}$, which is compatible with our previous choice. This however, as we shall see soon, is a severe restriction that we decide to relax at least by a factor of 3. The justification of this choice is that we expect the errors we are going to study (particularly the error related to T) to have more energy in the higher degrees, and since the operator that brings from Δg to T is a smoother, we would expect a more favorable error propagation. Based on that and on the direct experience, we will accept the threshold (2.23).

We now examine the three second-order terms in (2.22).

We have, in simple spherical approximation, i.e., with $\gamma = \frac{GM}{r}$

$$O_M\left(\frac{1}{2}\gamma''\zeta^2\right) \cong O_M\left(3\gamma\left(\frac{\zeta}{r}\right)^2\right) \cong 8 \cdot 10^{-4}\,\text{mGal}\ ,$$

which is indeed totally irrelevant.

Let us consider then $O_M(T''\zeta)$. The value of $O_M(T'')$ in Table 2.1 is the 90 % quantile of T'' at zero level, according to a global model of the anomalous potential. With this value one has

$$O_M(T''\zeta) = 6 \times 10^{-5}\ \text{Gal} = 60\,\mu\text{Gal}.$$

With our definition (2.1) of O_M this is still compliant with (2.23), although it is clear that this term is mostly concerning us in the linearization procedure. As for the last term of (2.3), recalling that

$$\frac{|\nabla_h T|}{\gamma} \cong \delta\ ,$$

we have

$$O_M\left(\frac{1}{2}\frac{|\nabla_h T|^2}{\gamma}\right) = \frac{1}{2}O_M(\gamma\delta^2) = 4.5 \times 10^{-4}\ \text{Gal} = 45\,\mu\text{Gal}.$$

Also for this term we are close to the maximum admissible value.

All in all one has the impression that by keeping only linear terms in (2.12) and (2.20) it is difficult to guarantee that the overall committed error is 1 cm as a maximum. More probably a few centimeters could be a more realistic figure. However, in some cases our estimates are really pessimistic. In this sense we want to elaborate a little more on the term $T''\zeta$, not only because it is the one that seems to have the largest impact if neglected, but also because its introduction into the BVP would change its nature because of the second-order oblique derivative of T. To reconduct such a term to a more favourable figure we will use the two well-known relations, valid in spherical approximation,

$$T' = -\frac{2}{r}T - \Delta g \ , \tag{2.24}$$

$$\Delta g' = -\frac{2}{r}\Delta g \ . \tag{2.25}$$

The relation (2.25) in particular gives an approximate vertical derivative of Δg in free air ([2], Sect. 2.4), as it is correct in the present case because we do not take into account the effects of the masses between S and $\widetilde{S}$.

Combining the above relations, one finds

$$\begin{aligned} T'' &= \frac{2}{r^2}T - \frac{2}{r}T' - \Delta g' = \\ &= \frac{2}{r^2}T + \left(\frac{4}{r^2}T + \frac{2}{r}\Delta g\right) + \frac{2}{r}\Delta g = \\ &= \frac{6}{r^2}T + \frac{4}{r}\Delta g \ . \end{aligned}$$

Accordingly, one can write

$$\begin{aligned} O_M(T''\zeta) &= 6O_M\left(\frac{T}{r}\frac{\zeta}{r}\right) + 4O_M\left(\Delta g\frac{\zeta}{r}\right) = \\ &= 1.6 \times 10^{-6}\ \mathrm{Gal} + 6.6 \times 10^{-6}\ \mathrm{Gal} = 8.2\ \mu\mathrm{Gal} \ . \end{aligned}$$

As we see this estimate is almost one order of magnitude less than the one previously found.

With all the above discussions, we can finally say that, with an error of a few centimeters in geoid in the worst case, we can substitute the boundary relation of the non-linear Scalar Molodensky problem with the general linearized version

$$DW = T(\widetilde{h}) - \gamma(\widetilde{h})\zeta \tag{2.26}$$

$$Dg = -T'(\widetilde{h}) + \gamma'(\widetilde{h})\zeta \ ; \tag{2.27}$$

this estimate substantially agrees with the results of [3].

One has to recall that in the above boundary relations T' means $\frac{\partial T}{\partial h}$ and similarly γ'.

Solving (2.26) with respect to ζ, one gets the generalized Bruns relation

$$\zeta = \frac{T - DW}{\gamma} \tag{2.28}$$

and substituting into (2.27) one finds

$$-T' + \frac{\gamma'}{\gamma}T = Dg + \frac{\gamma'}{\gamma}DW , \tag{2.29}$$

which has to hold on the telluroid $\widetilde{S}$. All the above holds only if the compatibility condition (2.14) is verified.

After some reflections, the above discussion leads to conclude that the following *equivalence principle* holds:

two linearized formulations of the Molodensky problem

$$\begin{cases} \Delta T = O & \text{in } \widetilde{\Omega} \\ -T' + \frac{\gamma'}{\gamma}T = Dg + \frac{\gamma'}{\gamma}DW & \text{on } \widetilde{S} \\ T' = O\left(\frac{1}{r^3}\right) & r \to \infty \end{cases} \tag{2.30}$$

are equivalent if they can be transformed one into the other, with the respective boundary relations given on telluroids that are in the same linearization band, in particular the two telluroids should be different from one another by no more that 100–200 m.

Notice that any linear problem

$$Ax = y$$

can be transformed into an equivalent one

$$A\xi = \eta$$

with $\xi = x - x_0$ and $y = y - Ax_0$. So, what gives rise to the equivalence of two BVPS of the type (2.30) is in particular that the two telluroids are in the same linearization band. It is interesting to note that the idea of using a "gravimetric" telluroid, i.e., one for which $Dg = 0$, already considered by Krarup [1] and later on by Sansò [4] for more theoretical reasons, is in fact at the boundary of the equivalence to the classical Molodensky problem (1.9). In fact the condition $Dg = 0$ would lead to pseudo-Bruns relation for ζ_G (see (2.22))

$$\zeta_G = \frac{T'}{\gamma'} ,$$

such that

$$O(\zeta_G) \cong \frac{100\,\text{mGal}}{0.3\ \text{mGal}\ \text{m}^{-1}} \cong 300\ \text{m}.$$

This is three times the order of magnitude of the Marussi height anomaly

$$\zeta_M = \frac{T}{\gamma}\ .$$

Even more absurd is the conclusions that one would get by putting directly

$$g_0 - \gamma(\widetilde{h}) + \frac{\gamma'(\widetilde{h})}{\gamma'(\widetilde{h})}[W_0 - U(\widetilde{h})] = 0\ .$$

On the other hand the reason why such a relation cannot be used as a definition of the telluroid is precisely that it takes us out of the linearization band.

References

1. Krarup T (2006) Letters on molodensky's problem: III. A mathematical formulation of Molodensky's problem. In: Borre K (ed) Matematical foundation of geodesy. Springer, Berlin
2. Sansò F, Sideris M (2013) Geoid determination: theory and methods. Springer, Berlin
3. Heck B, Seitz K (1993) Effects of non-linearity in the geodetic boundary problems. D.G.K., A. N109. Verlag der Bayerische Academy der Wissenschaften, Münich
4. Sansò F (1977) The geodetic boundary value problem in gravity space. Memorie Acc Lincei 14(S8):n3

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