# Chapter 2
# Gravity Beyond General Relativity

*Is this quintessence o[r] dust?*

Hamlet, *Hamlet*, 2.2

At the core of this thesis is the question of modifying general relativity. In the previous chapter, we introduced general relativity and its cosmological solutions, culminating in a discussion of two key aspects of the cosmological standard model: $\Lambda$CDM at late times and inflation at early times. In this chapter, we extend that discussion to theories of gravity beyond general relativity, and in particular the theories which will receive our attention in this thesis: massive gravity, massive bigravity, and Einstein-aether theory.

General relativity is the unique Lorentz-invariant theory of a massless spin-2 field [1–5]. To move beyond this theory, we must therefore modify its degrees of freedom. In massive gravity, this is done by endowing the graviton with a small but nonzero mass. Bigravity extends this by giving dynamics to a second tensor field which necessarily appears in the action for massive gravity; its dynamical degrees of freedom are two spin-2 fields, one massive and one massless. Finally, in Einstein-aether theory the massless graviton is supplemented by a vector field. This vector is constrained to always have a timelike vacuum expectation value (vev), and so spontaneously breaks Lorentz invariance by picking out a preferred time direction. It is thus a useful model for low-energy gravitational Lorentz violation.

## 2.1 Massive Gravity and Bigravity

The history of massive gravity is an old one, dating back to 1939 when the linear theory of Fierz and Pauli was published [6]. Studies of interacting spin-2 field theories also have a long history [7]. However, there had long been an obstacle to



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the construction of a fully nonlinear theory of massive gravity in the form of the notorious Boulware-Deser ghost [8], a pathological mode that propagates in massive theories at nonlinear order. This ghost mode was thought to be fatal to massive and interacting bimetric gravity until only a few years ago, when a way to avoid the ghost was discovered by utilising a very specific set of symmetric potential terms [9–16]. In this section we review that history, before moving onto the modern formulations of ghost-free massive gravity and bigravity, and elucidating their cosmological solutions. We refer the reader to [17, 18] for in-depth reviews on massive gravity and its history.

### *2.1.1 Building the Massive Graviton*

#### The Fierz–Pauli Mass Term

Let us begin by considering linearised gravity described by a spin-2 field, $h_{\mu\nu}$. We will routinely refer to this field as the graviton. The theory of a massless graviton is given by linearising the Einstein–Hilbert action around Minkowski space, i.e., by splitting the metric up as

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\rm Pl}} h_{\mu\nu}, \tag{2.1}$$

where $h_{\mu\nu}/M_{\rm Pl} \ll 1$, and keeping in the action only terms quadratic in $h_{\mu\nu}$. Doing this, we obtain the Lagrangian of linearised general relativity,

$$\mathscr{L}_{\rm GR,linear} = -\frac{1}{4} h^{\mu\nu} \hat{\mathscr{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta}, \tag{2.2}$$

where indices are raised and lowered with the Minkowski metric, $\eta_{\mu\nu}$, and we have defined the *Lichnerowicz operator*, $\hat{\mathscr{E}}$, by

$$\hat{\mathscr{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} \equiv -\frac{1}{2}\left(\Box h_{\mu\nu} - 2\partial_{(\mu}\partial_\alpha h^\alpha_{\nu)} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta})\right), \tag{2.3}$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$ is the trace of $h_{\mu\nu}$. No other kinetic terms are consistent with locality, Lorentz invariance, and gauge invariance under linearised diffeomorphisms,

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}. \tag{2.4}$$

Indeed, this uniqueness is a necessary (though not sufficient) part of the aforementioned uniqueness of general relativity as the nonlinear theory of a massless spin-2 field.

The role of gauge invariance is to ensure that there are no *ghosts*, i.e., no degrees of freedom with higher derivatives or wrong-sign kinetic terms. Ostrogradsky's theorem tells us that, up to a technical condition,[1] a Lagrangian with higher than second derivatives will lead to a Hamiltonian which is unbounded from below, and thus states with arbitrarily negative energy are allowed (for a thorough, modern derivation, see [19]). If we had included in Eq. (2.2) other terms that can be constructed out of $h_{\mu\nu}$ and its first and second derivatives, then the action would no longer be invariant under Eq. (2.4). In that case, we could split $h_{\mu\nu}$ into a transverse piece, $h^T_{\mu\nu}$, and a vector field, $\chi_\mu$, as

$$h_{\mu\nu} = h^T_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}, \tag{2.5}$$

and any terms not included in the action (2.2) would contain pieces with higher derivatives of $\chi_\mu$. Therefore we can see the linearised Einstein–Hilbert term as the kinetic term uniquely set by three requirements: locality, Lorentz invariance, and the absence of a ghost.

We would like to give $h_{\mu\nu}$ a mass—i.e., add a nonderivative interaction term—while maintaining those three requirements. Unfortunately, it is impossible to construct a local interaction term which is consistent with diffeomorphism invariance (2.4). Since this was useful in exorcising ghost modes, we will need to take care to ensure that no ghost is introduced by the mass term. At second order, this is not especially difficult as there are only two possible terms we can consider: $h^{\mu\nu}h_{\mu\nu}$ and $h^2$. We can then consider a general quadratic mass term,

$$\mathscr{L}_{\text{mass}} = -\frac{1}{8}m^2\left(h_{\mu\nu}h^{\mu\nu} - (1-a)h^2\right). \tag{2.6}$$

This leads to a ghostlike, scalar degree of freedom with mass $m_g^2 = \frac{3-4a}{2a}m^2$. The only way to remove the ghost from this theory, besides setting $m = 0$, is to set $a = 0$. The ghost then has infinite mass and is rendered nondynamical. We see the unique ghost-free action for a massive graviton at quadratic order is the *Fierz–Pauli action*,

$$\mathscr{L}_{\text{FP}} = -\frac{1}{4}h^{\mu\nu}\hat{\mathscr{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{8}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right). \tag{2.7}$$

**The Stückelberg "Trick"**

Before moving on to higher orders in $h_{\mu\nu}$, let us take a moment to count and classify the degrees of freedom it contains at linear order. Recall that a massless graviton contains two polarisations. Because we lose diffeomorphism invariance when we give the graviton a mass, it will contain more degrees of freedom. In fact, there are five in total. In principle, a sixth mode can arise, but it is always ghost-like and must therefore be removed from any healthy theory of massive gravity. To separate the degrees of freedom contained in the massive graviton, we use the Stückelberg "trick."

Stückelberg's idea is based on the observation that a gauge freedom such as diffeomorphism is not a physical property of a theory so much as a redundancy in

[1]Namely that the Lagrangian be nondegenerate, i.e., that $\partial L/\partial\dot{q}$ depend on $\dot{q}$.

description, and that redundancy can always be introduced by bringing in redundant variables. Let us consider splitting up $h_{\mu\nu}$ as

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{2}{m}\partial_{(\mu}A_{\nu)} + \frac{2}{m^2}\partial_\mu\partial_\nu\phi. \tag{2.8}$$

Defining the field strength tensor for $A_\mu$ analogously to electromagnetism, $F_{\mu\nu} \equiv \frac{1}{2}\partial_{[\mu}A_{\nu]}$, as well as $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\phi$ and the trace notation $[A] \equiv \eta^{\mu\nu}A_{\mu\nu}$, the Fierz–Pauli action (2.7) becomes

$$\begin{aligned}\mathscr{L}_{\rm FP} = &-\frac{1}{4}h^{\mu\nu}\hat{\mathscr{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{2}h^{\mu\nu}\left(\Pi_{\mu\nu} - [\Pi]\eta_{\mu\nu}\right) - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} \\ &- \frac{1}{8}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) - \frac{1}{2}m\left(h^{\mu\nu} - h\eta^{\mu\nu}\right)\partial_{(\mu}A_{\nu)}.\end{aligned} \tag{2.9}$$

This action is invariant under the simultaneous gauge transformations

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}, \qquad A_\mu \to A_\mu - \frac{m}{2}\xi_\mu \tag{2.10}$$

for $h_{\mu\nu}$ and

$$A_\mu \to A_\mu + \partial_\mu\lambda, \qquad \phi \to \phi - m\lambda \tag{2.11}$$

for $A_\mu$. With these gauge invariances restored, one can find that $h_{\mu\nu}$ contains the usual two independent components of a spin-2 degree of freedom, $A_\mu$ similarly contains the standard two independent components, and $\phi$ contains one, leading to a total of $2 + 2 + 1 = 5$ degrees of freedom for a Fierz–Pauli massive graviton.

Before moving on, let us briefly consider the limit $m \to 0$. Intuitively we would expect this to reduce to general relativity. In this limit, the vector completely decouples from the other two fields, while the scalar remains mixed with the tensor. They can be unmixed by transforming $h_{\mu\nu} \to h_{\mu\nu} + \phi\eta_{\mu\nu}$. However, this transformation introduces a coupling between $\phi$ and the stress-energy tensor for matter (which, for simplicity, we have neglected so far) which does not vanish in the massless limit. This is the origin of the van Dam–Veltman–Zakharov (vDVZ) discontinuity [20, 21]. While linearised Fierz–Pauli theory with matter indeed does not reduce to general relativity in the limit $m \to 0$, nonlinear effects cure this discontinuity: this is the celebrated Vainshtein mechanism which restores general relativity in environments where $\phi$ is nonlinear and allows theories like massive gravity to agree with solar system tests of gravity [22]. For a modern introduction to the Vainshtein mechanism, see Ref. [23].

**The Boulware-Deser Ghost**

Upon moving beyond linear order, disaster strikes. While the Fierz–Pauli tuning (expressed above as $a = 0$) removes a sixth, ghostlike degree of freedom from the linear theory, Boulware and Deser found that this mode generically reappears at higher orders [8]. This is the notorious *Boulware-Deser ghost*. It is infinitely heavy on flat backgrounds, as evidenced by its infinite mass at the purely linear level, but can become light around nontrivial solutions [9], including cosmological backgrounds [24] and weak-field solutions around static matter [25–27].

A full history of this ghost mode is beyond the scope of this thesis; we will simply, following the review [17], introduce the simplest nonlinear extension of the Fierz–Pauli mass term and demonstrate the existence of a ghost mode, as an illustration of the fact that nontrivial interaction terms are required beyond the linear order in order to obtain a ghost-free theory of massive gravity. Let us define the matrix

$$\mathbb{M}^\mu{}_\nu \equiv g^{\mu\alpha}\eta_{\nu\alpha}. \tag{2.12}$$

Linearising this matrix around $\eta_{\mu\nu}$ as above, we find

$$\frac{1}{M_{\rm Pl}}h^\mu{}_\nu \approx \delta^\mu{}_\nu - \mathbb{M}^\mu{}_\nu. \tag{2.13}$$

The Fierz–Pauli term (2.7) can be obtained by linearising ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{\rm Pl}$) the nonlinear action [25]

$$\mathscr{L}_{\rm FP,nonlinear} = -m^2M_{\rm Pl}^2\sqrt{-g}\left\{[(\mathbb{I}-\mathbb{M})^2] - [\mathbb{I}-\mathbb{M}]^2\right\}, \tag{2.14}$$

where $\mathbb{I}$ is the identity matrix in four dimensions.

With a candidate nonlinear completion of massive gravity in hand, let us examine the behaviour of the helicity-0 mode, $\phi$, ignoring the helicity-1 and helicity-2 modes for simplicity. The matrix $\mathbb{M}$ becomes

$$\mathbb{M}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{2}{M_{\rm Pl}m^2}\Pi^\mu{}_\nu + \frac{1}{M_{\rm Pl}^2m^4}\Pi^\mu{}_\alpha\Pi^\alpha{}_\nu. \tag{2.15}$$

Plugging this into our nonlinear action, we find

$$\mathscr{L}_{\rm FP,nonlinear} = -\frac{4}{m^2}\left([\Pi^2]-[\Pi]^2\right) + \frac{4}{M_{\rm Pl}m^2}\left([\Pi^3]-[\Pi][\Pi^2]\right) + \frac{1}{M_{\rm Pl}^2m^6}\left([\Pi^4]-[\Pi^2]^2\right). \tag{2.16}$$

This clearly contains higher time derivatives for $\phi$, and will therefore lead to an Ostrogradsky instability and hence to ghosts. In fact, even the first, quadratic term—the most obvious ancestor of the Fierz–Pauli term—has higher derivatives. However, they turn out to be harmless for the reason that this quadratic term is, after integration by parts, a total derivative and therefore does not contribute to the dynamics. This miracle does not extend to any of the higher terms, which lead to a genuine sixth degree of freedom.[2]

[2] While we have seemingly split the metric up into five degrees of freedom—two tensor, two vector, and one scalar—and focused on the scalar, when $\phi$ has higher time derivatives it in fact contains two degrees of freedom, one of which is generically a ghost.

The Boulware-Deser ghost is not specific to this particular, simple nonlinear completion of the Fierz–Pauli term. It is a very generic problem, to the point that as of a decade ago it was thought to plague *all* nonlinear massive gravity theories [26]. The ghost can, however, be removed: one can consider general higher-order extensions and, at each order, tune the coefficients to eliminate higher time derivatives by packaging them into total derivative terms [9]. This led to a ghost-free, fully nonlinear theory of massive gravity in [10], nearly four decades after the discovery of the Boulware-Deser ghost.

**To the Nonlinear Theory**

Taken on its own, linearised gravity in the form (2.2) is a perfectly acceptable theory without being seen as a truncation of a nonlinear theory such as general relativity. Indeed, it is even a perfectly fine *gauge theory*, as its linearised diffeomorphism invariance is exact (as long as we include the Stückelberg fields, if we take the graviton to be massive). The wrench in the works comes when we add matter in. Unfortunately, the coupling to matter is necessary, as we prefer our theories to communicate with the rest of the Universe and hence be subject to experiment.

The coupling to matter at the linear level is of the form

$$\mathscr{L}_{\text{matter,linear}} = \frac{1}{2M_{\text{Pl}}} h_{\mu\nu} T_0^{\mu\nu}, \tag{2.17}$$

where $T_0^{\mu\nu}$ is the stress-energy tensor of our matter source. Diffeomorphism invariance is preserved in the matter sector if stress-energy is conserved, i.e., if $\partial_\mu T_0^{\mu\nu} = 0$. However, the coupling to $h_{\mu\nu}$ necessarily induces a violation of this conservation. For a simple example of this using a scalar field, see Ref. [17]. This problem is fixed by adding nonlinear corrections both to the matter coupling,

$$\mathscr{L}_{\text{matter,nonlinear}} = \frac{1}{2M_{\text{Pl}}} h_{\mu\nu} T_0^{\mu\nu} + \frac{1}{2M_{\text{Pl}}^2} h_{\mu\nu} h_{\alpha\beta} T_1^{\mu\nu\alpha\beta}, \tag{2.18}$$

for some tensor $T_1^{\mu\nu\alpha\beta}$, and to the gauge symmetry, symbolically written as

$$h \to h + \partial\xi + \frac{1}{M_{\text{Pl}}} \partial(h\xi). \tag{2.19}$$

While this ensures the conservation of the stress-energy tensor at the linear level, it is broken at the next order, and so we must continue this procedure order by order, *ad infinitum*.

For a massless spin-2 field, the end result of this procedure is well-known: it is general relativity. The fully nonlinear gauge symmetry is diffeomorphism invariance, and as long as the matter action is invariant under this symmetry, the stress-energy tensor is covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$. The linear action (2.2) must be promoted to something which is also consistent with this symmetry, and there is one answer: the Einstein–Hilbert action (1.4).

If we wish to extend this procedure to a massive graviton, i.e., nonlinearly complete the Fierz–Pauli mass term, then, as discussed in Sect. 2.1.1, the Boulware-Deser ghost looms as a pitfall. Demanding that this ghost be absent will severely restrict the allowed potentials to a very specific and special set of functions of $\mathbb{M}$. However, the gauge invariance can still be restored quite easily by a nonlinear version of the Stückelberg trick. Because this elucidates several of the properties of massive gravity, we will review it here.

When constructing a nonlinear candidate theory of massive gravity in Sect. 2.1.1, we employed the matrix $\mathbb{M} = g^{-1}\eta$, where $g^{-1}$ is the inverse of the dynamical metric and $\eta$ is the Minkowski metric. The appearance of a second, fixed metric in addition to the dynamical one is new to massive gravity. Indeed, it is necessary to have such a second metric, or *reference metric*, in order to give the graviton a mass. A mass term is a nonderivative interaction,[3] and the only nonderivative scalars or scalar densities we can construct out of $g_{\mu\nu}$ alone are $\mathrm{tr}\, g = 4$ and $\det g$. The first possibility is trivial, while it was shown in Ref. [8] that functions of the metric determinant can only consistently lead to a cosmological constant as well. Therefore we need a reference metric in order to construct a massive graviton.[4]

Note that, while we have so far taken the reference metric to be Minkowski, in principle we could extend the theory to a more general reference metric, $f_{\mu\nu}$. Consequently, even once we have specified the interaction potential there are many different massive gravity theories, one for each reference metric. Alternatively, $f_{\mu\nu}$ can be viewed as a "constant tensor" which must be specified by hand. Physically, the reference metric corresponds to the background around which linear fluctuations acquire the Fierz–Pauli form [15, 34]. This is why we have naturally discussed theories with a Minkowski reference metric: we began by considering fluctuations around that metric, and so it remains when extending to the nonlinear theory. Note that $f_{\mu\nu} = \eta_{\mu\nu}$ is a natural choice, as the theory then possesses a Poincaré-invariant preferred metric, allowing us to define mass and spin regardless of the solutions of the theory.[5]

The nonlinear Stückelberg trick is simply to introduce into the reference metric four Stückelberg fields, $\phi^a$, as

$$f_{\mu\nu} \to \tilde{f}_{\mu\nu} \equiv f_{ab}\partial_\mu\phi^a\partial_\nu\phi^b. \tag{2.20}$$

Note that here Latin indices are in field space, not spacetime; in particular, each of the $\phi^a$ fields transforms as a spacetime scalar. Consequently, $\tilde{f}_{\mu\nu}$ transforms as a tensor under general coordinate transformations, as well as $\mathbb{M}^\mu{}_\nu$ and all of the nonlinear completions of the Fierz–Pauli term constructed out of it, as long as we

[3] We could not have kinetic interactions anyway; in four dimensions, the Einstein–Hilbert term is unique [28].

[4] We have assumed locality in this discussion. A nonlocal theory can give the graviton a mass without the need for a reference metric [29–31]; see Refs. [32, 33] for studies of two interesting realisations of this idea.

[5] We thank Claudia de Rham for emphasising this point.

replace $f_{\mu\nu}$ with $\tilde{f}_{\mu\nu}$. If we choose the coordinate axes to align with the Stückelberg fields, $x^a = \phi^a$, then we have $\tilde{f}_{\mu\nu} = f_{\mu\nu}$ and recover the previous description. This is called *unitary gauge*. For simplicity we will usually work in unitary gauge when dealing with massive gravity.

### *2.1.2 Ghost-Free Massive Gravity*

**de Rham-Gabadadze-Tolley Massive Gravity**

Recent years have seen a breakthrough in massive gravity, stemming from the development of a fully ghost-free and nonlinear theory by de Rham, Gabadadze, and Tolley (dRGT) [9, 10], following the important preliminary steps taken in [35, 36] using auxiliary extra dimensions. The full proof that the dRGT theory is ghost-free, including for a general reference metric, was completed in [14–16]. There are indications that this theory is the unique ghost-free massive gravity; in particular, new kinetic interactions do not appear to be consistent [37, 38]. We will use the formulation of the dRGT interaction potential developed in Ref. [13].

The action for dRGT massive gravity around a general reference metric, $f_{\mu\nu}$,[6] is

$$S_{\mathrm{dRGT}} = -\frac{M_{\mathrm{Pl}}^2}{2}\int d^4x\sqrt{-g}R + m^2 M_{\mathrm{Pl}}^2\int d^4x\sqrt{-g}\sum_{n=0}^{4}\alpha_n e_n(\mathbb{K}) + \int d^4x\sqrt{-g}\mathscr{L}_m\left(g,\Phi_i\right), \tag{2.21}$$

or, equivalently,

$$S_{\mathrm{dRGT}} = -\frac{M_{\mathrm{Pl}}^2}{2}\int d^4x\sqrt{-g}R + m^2 M_{\mathrm{Pl}}^2\int d^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n(\mathbb{X}) + \int d^4x\sqrt{-g}\mathscr{L}_m\left(g,\Phi_i\right), \tag{2.22}$$

where we have defined the square-root matrix $\mathbb{X}$ as

$$\mathbb{X} \equiv \sqrt{g^{-1}f}, \tag{2.23}$$

i.e., defined $\mathbb{X}$ so that (reintroducing explicit spacetime indices) $\mathbb{X}^\mu{}_\alpha \mathbb{X}^\alpha{}_\nu = g^{\mu\alpha} f_{\alpha\nu}$, and the related matrix $\mathbb{K}$ by

$$\mathbb{K} \equiv \mathbb{I} - \mathbb{X}. \tag{2.24}$$

Here, $\alpha_n$ and $\beta_n$ are dimensionless coupling constants, generally taken to be free parameters, and $e_n$ are the *elementary symmetric polynomials* of the eigenvalues $\lambda_i$ of the matrix argument. In terms of the eigenvalues, assuming $i$ runs from 1 to 4, these are (taking as the argument a general matrix, $\mathbb{A}$, for concreteness)

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[6] As discussed above, we are, strictly speaking, writing the action for massive gravity in unitary gauge. Extending to a more general gauge by including Stückleberg fields by promoting $f_{\mu\nu}$ to $f_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$ is trivial.

$$
\begin{aligned}
e_0(\mathbb{A}) &= 1, \\
e_1(\mathbb{A}) &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \\
e_2(\mathbb{A}) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \\
e_3(\mathbb{A}) &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \\
e_4(\mathbb{A}) &= \lambda_1\lambda_2\lambda_3\lambda_4 = \det \mathbb{A}.
\end{aligned} \tag{2.25}
$$

It is often more useful to write these polynomials directly in terms of the matrix,

$$
\begin{aligned}
e_0(\mathbb{A}) &\equiv 1, \\
e_1(\mathbb{A}) &\equiv [\mathbb{A}], \\
e_2(\mathbb{A}) &\equiv \frac{1}{2}\left([\mathbb{A}]^2 - [\mathbb{A}^2]\right), \\
e_3(\mathbb{A}) &\equiv \frac{1}{6}\left([\mathbb{A}]^3 - 3[\mathbb{A}][\mathbb{A}^2] + 2[\mathbb{A}^3]\right), \\
e_4(\mathbb{A}) &\equiv \det(\mathbb{A}).
\end{aligned} \tag{2.26}
$$

The formulations (2.21) and (2.22) in terms of $\mathbb{K} = \mathbb{I} - \sqrt{g^{-1}f}$ and $\mathbb{X} = \sqrt{g^{-1}f}$ are both very common in the literature. For reasons partly physical and partly historical, the $\mathbb{K}$ formulation is more common in massive gravity, while $\mathbb{X}$ is predominant in bigravity. The free parameters in the two formulations, $\alpha_n$ and $\beta_n$, are related by [13]

$$
\beta_n = (4-n)! \sum_{i=n}^{4} \frac{(-1)^{i+n}}{(4-i)!(i-n)!}\alpha_i. \tag{2.27}
$$

Throughout this thesis we will use the $\mathbb{X}$ basis and $\beta_n$ parametrisation except when stated otherwise.

Notice that although these potential terms are very complicated, they have a significant amount of structure. This can be better understood by decomposing the metrics into their *vielbeins*, defined by

$$
g_{\mu\nu} = \eta_{ab} E^a{}_\mu E^b{}_\nu, \qquad f_{\mu\nu} = \eta_{ab} L^a{}_\mu L^b{}_\nu. \tag{2.28}
$$

Since vielbeins are, in a sense, the "square roots" of the metrics, and the dRGT potential terms are built out of a square root matrix, this is a natural language in which to formulate massive gravity. The interaction terms are, up to numerical constants, given by [39]

$$
\begin{aligned}
e_0(\mathbb{X}) &\propto \tilde{\epsilon}_{abcd}\tilde{\epsilon}^{\mu\nu\alpha\beta} E^a{}_\mu E^b{}_\nu E^c{}_\alpha E^d{}_\beta, \\
e_1(\mathbb{X}) &\propto \tilde{\epsilon}_{abcd}\tilde{\epsilon}^{\mu\nu\alpha\beta} E^a{}_\mu E^b{}_\nu E^c{}_\alpha L^d{}_\beta, \\
e_2(\mathbb{X}) &\propto \tilde{\epsilon}_{abcd}\tilde{\epsilon}^{\mu\nu\alpha\beta} E^a{}_\mu E^b{}_\nu L^c{}_\alpha L^d{}_\beta,
\end{aligned}
$$

$$\begin{aligned} e_3(\mathbb{X}) &\propto \tilde{\epsilon}_{abcd}\tilde{\epsilon}^{\mu\nu\alpha\beta}E^a{}_\mu L^b{}_\nu L^c{}_\alpha L^d{}_\beta, \\ e_0(\mathbb{X}) &\propto \tilde{\epsilon}_{abcd}\tilde{\epsilon}^{\mu\nu\alpha\beta}L^a{}_\mu L^b{}_\nu L^c{}_\alpha L^d{}_\beta. \end{aligned} \tag{2.29}$$

We can now understand the simplicity of the dRGT interaction potential: it is a linear combination of the only possible wedge products one can construct from $\mathbf{E}^a = E^a{}_\mu dx^\mu$ and $\mathbf{L}^a = L^a{}_\mu dx^\mu$.

Finally, notice that we have only coupled matter minimally to the dynamical metric, $g_{\mu\nu}$. More general matter couplings are certainly possible, but the vast majority of attempts to couple the same matter sector to both metrics reintroduce the Boulware-Deser ghost [40–42]. The question of formulating and studying more general matter couplings will form a significant part of this thesis, lying at the heart of Chaps. 5–7.

**Hassan–Rosen Bigravity**

In dRGT massive gravity, the reference metric, $f_{\mu\nu}$, is fixed and must be put into the theory by hand. This leads to a multiplicity of theories of massive gravity: massive gravity around Minkowski, around de Sitter, around anti-de Sitter, and so on. As shown by Hassan and Rosen in Ref. [43], the reference metric can be freely given dynamics without spoiling the ghost-free nature of the theory, as long as its kinetic term is also of the Einstein–Hilbert form. This leads to *Hassan–Rosen bigravity* or *massive bigravity*,

$$\begin{aligned} S_{\rm HR} = &-\frac{M_g^2}{2}\int d^4x\sqrt{-g}R(g) - \frac{M_f^2}{2}\int d^4x\sqrt{-f}R(f) \\ &+ m^2M_g^2\int d^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n(\mathbb{X}) + \int d^4x\sqrt{-g}\mathscr{L}_m\left(g,\Phi_i\right), \end{aligned} \tag{2.30}$$

where $R(g)$ and $R(f)$ are the Ricci scalars corresponding to each of the metrics. We have allowed for the two metrics to have different Planck masses, $M_g$ and $M_f$, although as long as both are finite, they can be set equal to each other by performing the constant rescalings [44]

$$f_{\mu\nu} \to M_\star^{-2} f_{\mu\nu}, \qquad \beta_n \to M_\star^n \beta_n, \tag{2.31}$$

where $M_\star \equiv M_f/M_g$. Therefore the $f$-metric Planck mass is a redundant parameter. We will generally perform this rescaling implicitly in later chapters, although for now we will leave both Planck masses in to help to elucidate some of the physical features of the theory.

In terms of free parameters bigravity is simpler than massive gravity: we have traded a constant matrix ($f_{\mu\nu}$) for a constant number ($M_f$) which is not even physically relevant. We thus need to specify fewer theory ingredients to test its solutions.[7]

[7]Note however that the $f$-metric cosmological constant, $\beta_4$, is physically relevant in bigravity but not in massive gravity, as it is independent of $g_{\mu\nu}$ and hence only contributes to the equation of motion for $f_{\mu\nu}$.

On the other hand, it is less simple from the more theoretical point of view that it contains more degrees of freedom. The mass spectrum of bigravity contains two spin-2 fields, one massive and one massless. We can see this at the linear level by expanding each metric around the same background, $\bar{g}_{\mu\nu}$, as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu}, \qquad f_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_f} l_{\mu\nu}. \tag{2.32}$$

For simplicity we assume the "minimal model" introduced in Ref. [13], given by

$$\beta_0 = 3, \qquad \beta_1 = -1, \qquad \beta_2 = 0, \qquad \beta_3 = 0, \qquad \beta_4 = 1. \tag{2.33}$$

The quadratic Lagrangian is given by [43]

$$\begin{aligned}\mathscr{L}_{\text{HR,linear}} = &-\frac{1}{4} h^{\mu\nu} \hat{\mathscr{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} - \frac{1}{4} l^{\mu\nu} \hat{\mathscr{E}}^{\alpha\beta}_{\mu\nu} l_{\alpha\beta} \\ &- \frac{1}{8} m^2 M_{\text{eff}}^2 \left[ \left( \frac{h^\mu{}_\nu}{M_g} - \frac{l^\mu{}_\nu}{M_f} \right)^2 - \left( \frac{h}{M_g} - \frac{l}{M_f} \right)^2 \right],\end{aligned} \tag{2.34}$$

where we have defined $M_{\text{eff}}^{-2} \equiv M_g^{-2} + M_f^{-2}$. Indices are raised and lowered with the background metric. We notice the usual Einstein–Hilbert terms for each of the two metric perturbations, as well as two Fierz–Pauli terms with some additional mixing between $h_{\mu\nu}$ and $l_{\mu\nu}$. This can be easily diagonalised by performing the change of variables

$$\frac{1}{M_{\text{eff}}} u_{\mu\nu} \equiv \frac{1}{M_f} h_{\mu\nu} + \frac{1}{M_g} l_{\mu\nu}, \qquad \frac{1}{M_{\text{eff}}} v_{\mu\nu} \equiv \frac{1}{M_f} h_{\mu\nu} - \frac{1}{M_g} l_{\mu\nu}. \tag{2.35}$$

The resultant unmixed Lagrangian,

$$\mathscr{L}_{\text{HR,linear}} = -\frac{1}{4} u^{\mu\nu} \hat{\mathscr{E}}^{\alpha\beta}_{\mu\nu} u_{\alpha\beta} - \frac{1}{4} v^{\mu\nu} \hat{\mathscr{E}}^{\alpha\beta}_{\mu\nu} v_{\alpha\beta} - \frac{1}{8} m^2 \left( v_{\mu\nu} v^{\mu\nu} - v^2 \right), \tag{2.36}$$

contains a Fierz–Pauli term for $v_{\mu\nu}$ and no interaction term for $u_{\mu\nu}$. Therefore in the linearised theory $u_{\mu\nu}$ corresponds to a massless graviton and $v_{\mu\nu}$ to a ghost-free massive one with mass $m$. We can see that in the limit where one Planck mass is much larger than the other, the metric with the larger Planck mass corresponds mostly to the massless graviton: if $M_f \gg M_g$, then $(u_{\mu\nu}, v_{\mu\nu}) \to (l_{\mu\nu}, h_{\mu\nu})$, and similarly if $M_g \gg M_f$, then $(u_{\mu\nu}, v_{\mu\nu}) \to (h_{\mu\nu}, -l_{\mu\nu})$. This formalises the notion, which is intuitive from Eq. (2.30), that we can recover dRGT massive gravity by taking one of the Planck masses to infinity. In that case, the massless mode corresponds entirely to the metric with the infinite Planck mass, its dynamics freeze out so that it becomes fixed, and the massless mode decouples from the theory, leaving us with what we expect for massive gravity.[8] Note, finally, that the notion of mass is only really well-defined around Minkowski space, as it follows from Poincaré invariance. More generally we can identify modes with a Fierz–Pauli term as massive, by analogy to the Minkowski case. As shown above, one can identify massive and massless linear fluctuations in this way around equal backgrounds for a special parameter choice, and indeed this can be done for general parameters as long as $g_{\mu\nu}$ and $f_{\mu\nu}$ are conformally related [40], but for general backgrounds there is no unambiguous splitting of the massive and massless modes in bigravity.

An interesting and useful property of Hassan–Rosen bigravity is that while $g_{\mu\nu}$ and $f_{\mu\nu}$ do not appear symmetrically in the action (2.30), it nevertheless does treat does metrics on equal footing, ignoring the matter coupling. In particular, the mass term has the property

$$\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right)=\sqrt{-f}\sum_{n=0}^{4}\beta_{4-n}e_n\left(\sqrt{f^{-1}g}\right), \tag{2.37}$$

which follows from the identity $\sqrt{-g}e_n\left(\sqrt{g^{-1}f}\right) = \sqrt{-f}e_{4-n}\left(\sqrt{f^{-1}g}\right)$ [43]. This can easily be seen by formulating the $e_n$ polynomials in terms of the eigenvalues of $\mathbb{X}$ as in Eq. (2.26). The result follows from using basic properties of the determinant and the fact that, because $\sqrt{g^{-1}f}$ and $\sqrt{f^{-1}g}$ are inverses of each other, their eigenvalues are inverses as well. As a consequence, the entire Hassan–Rosen action in vacuum is invariant under the exchange of the two metrics up to parameter redefinitions,

$$g_{\mu\nu}\leftrightarrow f_{\mu\nu}, \qquad M_g\leftrightarrow M_f, \qquad \beta_n\to\beta_{4-n}. \tag{2.38}$$

The fact that the matter coupling breaks this duality by coupling matter to only one metric will motivate the search for "double couplings" in later chapters.

By varying the action (2.30) with respect to the $g$ and $f$ metrics we obtain the generalised Einstein equations for massive bigravity [13],

$$G_{\mu\nu}(g)+m^2\sum_{n=0}^{3}(-1)^n\,\beta_n g_{\mu\alpha}Y^{\alpha}_{(n)\nu}\left(\sqrt{g^{-1}f}\right)=\frac{1}{M_g^2}T_{\mu\nu}, \tag{2.39}$$

$$G_{\mu\nu}(f)+\frac{m^2}{M_\star^2}\sum_{n=0}^{3}(-1)^n\,\beta_{4-n}f_{\mu\alpha}Y^{\alpha}_{(n)\nu}\left(\sqrt{f^{-1}g}\right)=0, \tag{2.40}$$

where $G_{\mu\nu}$ is the Einstein tensor computed for a given metric. The interaction matrices $Y_{(n)}(\mathbb{X})$ are defined as

[8] See, however, [45] for some caveats on taking the massive-gravity limit of bigravity.

$$
\begin{aligned}
Y_{(0)}(\mathbb{X}) &\equiv \mathbb{I}, \\
Y_{(1)}(\mathbb{X}) &\equiv \mathbb{X} - \mathbb{I}[\mathbb{X}], \\
Y_{(2)}(\mathbb{X}) &\equiv \mathbb{X}^2 - \mathbb{X}[\mathbb{X}] + \frac{1}{2}\mathbb{I}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right), \\
Y_{(3)}(\mathbb{X}) &\equiv \mathbb{X}^3 - \mathbb{X}^2[\mathbb{X}] + \frac{1}{2}\mathbb{X}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right) \\
&\quad - \frac{1}{6}\mathbb{I}\left([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]\right).
\end{aligned} \tag{2.41}
$$

Notice that they satisfy the relation [40]

$$
Y_{(n)}(\mathbb{X}) = \sum_{i=0}^{n} (-1)^i \mathbb{X}^{n-i} e_i(\mathbb{X}). \tag{2.42}
$$

The tensors $g_{\mu\lambda} Y^{\lambda}_{(n)\nu}$ are symmetric and so do not need to be explicitly symmetrised [40], although this fact has gone unnoticed in much of the literature. Finally, $T_{\mu\nu}$ is the stress-energy tensor defined with respect to the matter metric, $g$,

$$
T_{\mu\nu} \equiv -\frac{2}{\sqrt{-\det g}} \frac{\delta(\sqrt{-\det g}\,\mathscr{L}_m^g)}{\delta g^{\mu\nu}}. \tag{2.43}
$$

It is not difficult to check that when $T_{\mu\nu} = 0$, the Einstein equations are symmetric under the interchanges (2.38).

General covariance of the matter sector implies conservation of the stress-energy tensor as in general relativity,

$$
\nabla_g^{\mu} T_{\mu\nu} = 0. \tag{2.44}
$$

Furthermore, by combining the Bianchi identities for the $g$ and $f$ metrics with the field Eqs. (2.39) and (2.40), we obtain the following two Bianchi constraints on the mass terms:

$$
\nabla_g^{\mu} \frac{m^2}{2} \sum_{n=0}^{3} (-1)^n \beta_n g_{\mu\alpha} Y^{\alpha}_{(n)\nu}\left(\sqrt{g^{-1}f}\right) = 0, \tag{2.45}
$$

$$
\nabla_f^{\mu} \frac{m^2}{2M_\star^2} \sum_{n=0}^{3} (-1)^n \beta_{4-n} f_{\mu\alpha} Y^{\alpha}_{(n)\nu}\left(\sqrt{f^{-1}g}\right) = 0, \tag{2.46}
$$

after using Eq. (2.44). Only one of Eqs. (2.45) and (2.46) is independent: a linear combination of the two divergences can be formed which vanishes as an *identity*, i.e., regardless of whether $g_{\mu\nu}$ and $f_{\mu\nu}$ satisfy the correct equations of motion [46], so either of the Bianchi constraints implies the other.

**Field Equations for Massive Gravity**

Note that we can also easily obtain the equations of motion for dRGT massive gravity from the bimetric equations: the Einstein equation is simply Eq. (2.39) with $f_{\mu\nu}$ fixed to the desired reference metric,

$$G_{\mu\nu} + m^2 \sum_{n=0}^{3} (-1)^n \beta_n g_{\mu\alpha} Y^{\alpha}_{(n)\nu} \left(\sqrt{g^{-1}f}\right) = \frac{1}{M_{\rm Pl}^2} T_{\mu\nu}, \tag{2.47}$$

and matter is conserved with respect to $g_{\mu\nu}$ as usual. This quick "derivation" should be taken purely as a heuristic—i.e., if we had started off with the dRGT action (2.21) and varied with respect to $g_{\mu\nu}$, we would clearly obtain Eq. (2.47) regardless of whether $f_{\mu\nu}$ is dynamical—and *not* as the outcome of a limiting procedure. Indeed, because $f_{\mu\nu}$ lacks dynamics there is no analogue of its Einstein equation (2.40), and including that equation would lead to extra constraints. It turns out that massive gravity can be obtained as a limit of bigravity, but the process is more subtle than simply taking $M_f \to \infty$ (which freezes out the massless mode and equates it with $f_{\mu\nu}$, as discussed above) [45, 47]. Alternatively, the dRGT action can be obtained from the bigravity action by taking $M_f \to 0$, but to obtain massive gravity we must throw away the $f_{\mu\nu}$ Einstein equation (2.40) by hand. If we leave it in then it is determined algebraically in terms of $g_{\mu\nu}$.[9] Plugging this into the mass term we simply obtain a cosmological constant; thus this is the general-relativity limit of massive gravity.[10] This agrees with the linear analysis above, where we found that in the limit $M_f \to 0$, the fluctuations of $g_{\mu\nu}$ become massless.

## *2.1.3 Cosmological Solutions in Massive Bigravity*

In this subsection we review the homogeneous and isotropic cosmology of massive bigravity. We will follow the framework derived in Refs. [48, 49], and use, with some generalisations, the notation and approach summarised in Ref. [50]. As discussed above, we will rescale $f_{\mu\nu}$ and $\beta_n$ so that the two Planck masses are equal, $M_f = M_g$.

**Cosmological Equations of Motion**

We assume that, at the background level, the Universe can be described by Friedmann–Lemaître–Robertson–Walker (FLRW) metrics for both $g_{\mu\nu}$ and $f_{\mu\nu}$. Specialising to spatially-flat metrics, we have

[9] This is because we are effectively taking the dRGT action and varying with respect to $f_{\mu\nu}$, treating it like a Lagrange multiplier. Hence $f_{\mu\nu}$ cannot be picked freely in this case but is rather constrained in terms of $g_{\mu\nu}$.

[10] We thank Fawad Hassan for helpful discussions on these points.

$$ds_g^2 = -N(t)^2 dt^2 + a(t)^2 d\vec{x}^2, \tag{2.48}$$

$$ds_f^2 = -X(t)^2 dt^2 + Y(t)^2 d\vec{x}^2, \tag{2.49}$$

where $a(t)$ and $Y(t)$ are the spatial scale factors for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, and $N(t)$ and $X(t)$ are their lapses. In the rest of this thesis we will leave the time dependences of these functions implicit. We will find it useful to define the ratios of the lapses and scale factors,

$$x \equiv \frac{X}{N}, \qquad y \equiv \frac{Y}{a}. \tag{2.50}$$

Notice that these quantities are *coordinate-independent*: while we can freely choose either lapse or rescale either scale factor, their ratio is fixed. This is because bigravity is still invariant under general coordinate transformations as long as the same transformation is acted on each metric.[11]

With these choices of metrics, the generalised Einstein equations (2.39) and (2.40), assuming a perfect-fluid source with density $\rho = -T^0{}_0$, give rise to two Friedmann equations,

$$3H^2 = \frac{N^2}{M_g^2}\rho + m^2 N^2\left(\beta_0 + 3\beta_1 y + 3\beta_2 y^2 + \beta_3 y^3\right), \tag{2.51}$$

$$3K^2 = m^2 X^2\left(\beta_1 y^{-3} + 3\beta_2 y^{-2} + 3\beta_3 y^{-1} + \beta_4\right), \tag{2.52}$$

where we have defined the Hubble rates as[12]

$$H \equiv \dot{a}/a, \qquad K \equiv \dot{Y}/Y, \tag{2.53}$$

and overdots denote time derivatives. We will specialise in this thesis to pressureless dust, which obeys

$$\dot{\rho} + 3H\rho = 0. \tag{2.54}$$

The Bianchi constraint—either Eq. (2.45) or Eq. (2.46)—yields

$$m^2 a^2 P\left(X\dot{a} - N\dot{Y}\right) = 0, \tag{2.55}$$

where we have defined

$$P \equiv \beta_1 + 2\beta_2 y + \beta_3 y^2. \tag{2.56}$$

---

[11] In group-theoretic terms, there are two diffeomorphism groups, one for each metric, and bigravity breaks the symmetry under each of them but maintains the symmetry under their diagonal subgroup. This is obvious from the fact that the mass term only depends on the metrics in the combination $g^{\mu\alpha} f_{\alpha\nu}$.

[12] In order to present the cosmological solutions for general lapses, we will define the $g$-metric Hubble rate differently here than in the rest of this thesis; in particular, $H$ is not necessarily the cosmic-time Hubble rate.

The Bianchi constraint has two branches of solutions:

$$\text{Algebraic branch:} \qquad P = 0,$$
$$\text{Dynamical branch:} \qquad x = \frac{\dot{Y}}{\dot{a}}.$$

The algebraic branch is satisfied if $\beta_1 + 2\beta_2 y + \beta_3 y^2 = 0$, which seems to be nongeneric as it requires tuned initial conditions. Because the solutions on this branch have $y = \text{const.}$, the mass term in Eq. (2.51) clearly reduces to a cosmological constant. Thus the algebraic branch, at the background level, is equivalent to $\Lambda$CDM [48, 51]. At the level of linear perturbations, evidence has been found for several modes being strongly coupled [52]. Consequently we will focus our attention on the dynamical branch. In this case, the Bianchi constraint implies that the ratio of the lapses, $x$, can be written in terms of other background quantities as

$$x = \frac{Ky}{H}. \tag{2.57}$$

The Friedmann equations, (2.51) and (2.52), and the Bianchi identity (2.57) can be combined to find a purely algebraic, quartic evolution equation for $y$,

$$\beta_3 y^4 + (3\beta_2 - \beta_4)\, y^3 + 3\,(\beta_1 - \beta_3)\, y^2 + \left(\frac{\rho}{M_g^2 m^2} + \beta_0 - 3\beta_2\right) y - \beta_1 = 0. \tag{2.58}$$

The $g$-metric Friedmann equation (2.51) and quartic equation (2.58) completely determine the expansion history of the Universe. As in standard cosmology, we see that the cosmic expansion is governed by a Friedmann equation. It is sourced by a mass term that depends on $y$, the evolution of which is in turn determined by the quartic equation.

It will be useful to simplify the dynamics by expressing all background quantities solely in terms of $y(t)$ and then solving for $y(a)$. We can rearrange Eq. (2.58) to solve for $\rho(y)$,

$$\frac{\rho}{m^2 M_g^2} = -\beta_3 y^3 + (\beta_4 - 3\beta_2) y^2 + 3(\beta_3 - \beta_1) y + 3\beta_2 - \beta_0 + \beta_1 y^{-1}. \tag{2.59}$$

We can then substitute this into the Friedmann equation to find $H(y)$,

$$3H^2 = m^2 N^2 \left(\beta_4 y^2 + 3\beta_3 y + 3\beta_2 + \beta_1 y^{-1}\right). \tag{2.60}$$

By taking a derivative of the quartic equation (2.58) and using the fluid conservation Eq. (2.54) and our solution (2.59) for $\rho(y)$ we can find an evolution equation for $y(a)$,

$$\frac{d\ln y}{d\ln a} = \frac{\dot{y}}{Hy} = -3\frac{\beta_3 y^4 + (3\beta_2 - \beta_4)y^3 + 3(\beta_1 - \beta_3)y^2 + (\beta_0 - 3\beta_2)y - \beta_1}{3\beta_3 y^4 + 2(3\beta_2 - \beta_4)y^3 + 3(\beta_1 - \beta_3)y^2 + \beta_1}. \tag{2.61}$$

Using the definition of $y$ to find $\dot{y}$, we can easily write $K(y)$,

$$K = H + \frac{\dot{y}}{y}. \tag{2.62}$$

We can now write any background quantity in terms of $y$ alone, except for $y$ and $a$ themselves, and further we have two avenues for determining $y(a)$: integrating Eq. (2.61), or using the quartic equation (2.58) with $\rho = \rho_0 a^{-3}$. These expressions will be crucial throughout this thesis since they reduce the problem of finding any parameter—background or perturbation—to solving for $y(z)$, where $z = 1/a - 1$ is the redshift.[13]

We note briefly that there has been some discussion in the literature over how to correctly take square roots in bigravity. There exist cosmological solutions in which $\det\sqrt{g^{-1}f}$ becomes zero at a finite point in time (and only at that time), and so it is important to determine whether to choose square roots to always be positive, per the usual mathematical definition, or to change sign on either side of the point where $\det\sqrt{g^{-1}f} = 0$. This was discussed in some detail in Ref. [53] (see also Ref. [54]), where continuity of the vielbein corresponding to $\sqrt{g^{-1}f}$ demanded that the square root *not* be positive definite. We will take a similar stance here, and make the only choice that renders the action differentiable at all times, i.e., such that the derivative of $\sqrt{g^{-1}f}$ with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$ is continuous everywhere. In particular, for the FLRW backgrounds we are dealing with in this section, this choice implies that $\sqrt{-\det f} = XY^3$. This is important because, as we will see in Chap. 3, it turns out that in the only cosmology with linearly-stable perturbations, the $f$ metric bounces, so $X = KY/H$ changes sign during cosmic evolution. With our square-root convention, the square roots will change sign as well, rather than develop cusps. Note that sufficiently small perturbations around the background will not lead to a different sign of this square root.

**Properties of Bimetric Cosmologies**

We can understand the qualitative behaviour of bimetric cosmologies by taking the early- and late-time limits, $\rho \to \infty$ and $\rho \to 0$, respectively. We will use heuristic arguments to motivate results which were determined more rigorously in Ref. [55] and can also be seen from a statistical comparison to observations of the expansion history [49]. At early times, the quartic equation (2.58) is solved either by $y \to 0$ or $y \to \infty$. The former solution is quite easy to see: the quartic equation is of the form $\ldots + \rho y = 0$, where $\ldots$ contains only positive powers of $y$, so $y \to 0$ will clearly be a solution. These are called *finite-branch* solutions. The solutions with $y \to \infty$

[13] Note that while we have expressed all background quantities in terms of $y$ only, perturbations will in general depend on both $y$ and $a$.

at early times, or *infinite-branch* solutions, occur when one of the higher powers of $y$ in the quartic equation scales at just the right rate to cancel out the $\rho y$ term. These solutions are rather less common; in order to enforce $\Omega_{\mathrm{m}} \to 1$ and $H^2 > 0$ at early times, viable infinite-branch solutions require $\beta_2 = \beta_3 = 0$ and $\beta_4 > 0$ [55]. To see this, notice that $\Omega_{\mathrm{m}} = N^2\rho/(3M_g^2H^2)$ can be written in the limit $y \to \infty$, using Eqs. (2.59) and (2.60), as

$$\Omega_{\mathrm{m}} = 1 - \frac{\beta_3}{\beta_4}y + 3\frac{\beta_3^2}{\beta_4^2} - 3\frac{\beta_2}{\beta_4}. \tag{2.63}$$

The condition $\beta_4 > 0$ (rather than just $\beta_4 \neq 0$) arises from demanding, per Eq. (2.60), that $H^2$ be positive at all times.

On either branch, at late times $y$ will always flow to a constant, $y_c$, given by the quartic equation with $\rho = 0$,

$$\beta_3 y_c^4 + (3\beta_2 - \beta_4)\, y_c^3 + 3\,(\beta_1 - \beta_3)\, y_c^2 + (\beta_0 - 3\beta_2)\, y_c - \beta_1 = 0. \tag{2.64}$$

Moreover, by taking a derivative of the quartic equation we see that $\dot{y} \neq 0$ unless either $\dot{\rho} = 0$ or $y = 0$. Therefore $y$ evolves monotonically throughout cosmic history, flowing either from 0 up to $y_c$ or from $\infty$ down to $y_c$.[14]

As long as $y_c > 0$, the mass term asymptotes in the future to a cosmological constant. Hence bimetric cosmologies generally possess late-time acceleration with $H^2 \sim \mathcal{O}(m^2\beta_i)$. This is the case even if the $g$-metric cosmological constant, $\beta_0$, is turned off, hence these theories *self-accelerate*. Because $y$ cannot be constant in these models,[15] the effective dark energy is dynamical. In particular, we do not have $w = -1$ except at the asymptotic future. Crucially, the parameters and the potential structure leading to the accelerated expansion are thought to be stable under quantum corrections [56], in stark contrast to a cosmological constant, which would need to be fine-tuned against the energy of the vacuum [57–59].[16] Thus we find that bigravity is an excellent candidate for technically-natural self-acceleration. Comparisons to background data—specifically the cosmic microwave background, baryon acoustic oscillations, and type Ia supernovae—show that these cosmological models can agree well with observations [48, 49, 55].

Before ending this subsection, let us consider a worked example: the model with only $\beta_1$ nonzero. Because theoretical viability conditions require this term to be nonzero (ignoring the exact $\Lambda$CDM case with $\beta_1 = \beta_3 = 0$), it is the simplest nontrivial one-parameter model which will lead to sensible cosmologies [55]. It has

[14] Note that these are not necessarily the same $y_c$, as Eq. (2.64) can have multiple roots.

[15] Unless it is either 0 at all times, which is trivial, or the special case $\beta_1 = \beta_3 = 0$, in which case the Friedmann equation can be rewritten, with the help of the quartic equation, as a $\Lambda$CDM Friedmann equation with a rescaled gravitational constant [48].

[16] If matter couples to $g_{\mu\nu}$ then matter loops will still contribute to $m^2\beta_0$ as usual. It is the rest of the dRGT potential which is stable under quantum corrections. Consequently we focus on self-accelerating models and assume—as is common in the literature—that some unknown mechanism removes the dangerous cosmological constant.

been shown in Refs. [49, 55] that this model provides a self-accelerating evolution which agrees with background cosmological observations and, as it possesses the same number of free parameters as the standard ΛCDM model, is a viable alternative to it. Indeed, it may be more viable than ΛCDM if the graviton mass turns out to be stable to quantum corrections, as mentioned above. The graviton mass in this case is given by $\sqrt{\beta_1}m$. Note that in order to give rise to acceleration at the present era, the graviton mass typically should be comparable to the present-day Hubble rate, $\beta_1 m^2 \sim H_0^2$.

In this simple case, the evolution equation (2.61) for $y(a)$,

$$\frac{d\ln y}{d\ln a} = -3\left(1 - \frac{2}{1+3y^2}\right), \tag{2.65}$$

can be integrated exactly to find

$$y(a) = \frac{1}{6}a^{-3}\left(-C \pm \sqrt{12a^6 + C^2}\right). \tag{2.66}$$

Assuming $y > 0$ forces us to select the positive branch. Using the Friedmann and quartic equations, we can set the value of $C$ using initial conditions,

$$C = -\frac{m^2\beta_1}{H_0^2} + 3\frac{H_0^2}{\beta_1 m^2}, \tag{2.67}$$

where $H_0$ is the cosmic-time Hubble rate today. Equivalently, we can use the quartic equation to solve for $y$ and express the Friedmann equation as a modified expression for $H(\rho)$,

$$H^2 = \frac{N^2}{6M_g^2}\left(\rho + \sqrt{12m^4 M_g^4 \beta_1^2 + \rho^2}\right). \tag{2.68}$$

In either formulation, the late-time approach to a Λ-like behaviour is evident.

The minimal $\beta_1$-only model is also distinctive for having a phantom equation of state, $w(z) \approx -1.22^{+0.02}_{-0.02} - 0.64^{+0.05}_{-0.04} z/(1+z)$ at small redshifts. Moreover, $w$ is related in a simple way to the matter density parameter [55]. This provides a concrete and testable prediction of the model that can be verified by future large-scale structure experiments, such as Euclid [60, 61], intensity mappings of neutral hydrogen [62, 63], and combinations of structure and cosmic microwave background measurements [64]. The model has also been proven in [65] to satisfy an important stability bound at all times, avoiding the *Higuchi ghost* which plagues theories of a massive graviton on expanding backgrounds [66]. It is, however, worth noting that its linear cosmological perturbations are unstable until $z \sim 0.5$, as shown in Chap. 3 of this thesis. We emphasise that this instability does not rule out the $\beta_1$-only model, but rather impedes our ability to use linear perturbation theory to describe perturbations at all times. This raises the interesting question of how to make predictions for structure formation during the unstable period, a question which is beyond the scope of this thesis.

The aforementioned studies have largely been restricted to the background cosmology of the theory. As the natural next step, in Chaps. 3 and 4 we will extend the predictions of massive bigravity to the perturbative level, and study how consistent the models are with the observed growth of structures in the Universe.

**A No-Go Theorem for Massive Cosmology**

As discussed in the previous section, we can easily obtain the equations of motion for massive gravity from those of bigravity; the $g$-metric equation is the same, and we lose the $f$-metric equation. Note that we also still have the Bianchi constraint (2.55). This fact will turn out to be crucial.

Let us assume that the reference metric is Minkowski, $f_{\mu\nu} = \eta_{\mu\nu}$. Under the assumption of homogeneity and isotropy for $g_{\mu\nu}$ in unitary gauge, the Friedmann equation is simply equation (2.51) with $y = a^{-1}$. This alone would define perfectly acceptable cosmologies. However, the Bianchi constraint (2.55) causes trouble. In the bigravity language we now have $X = Y = 1$, so the constraint becomes

$$m^2 a^2 P \dot{a} = 0. \tag{2.69}$$

Because $\dot{Y} = 0$, we no longer have an interesting dynamical branch: it merely suggests $\dot{a} = 0$. Unfortunately, the algebraic branch, $P = 0$, also only has $a = \text{const.}$ as a solution; in bigravity it would fix $y = Y/a$ while allowing $a$ and $Y$ to change, but in massive gravity, it is $a$ which is fixed. Therefore $a$ is generally fixed to be constant, and this system has no dynamical solutions. This is the famous no-go theorem on cosmological solutions in dRGT massive gravity, and it is present for both flat and closed universes [67]. If we instead consider open universes or different reference metrics, such as FLRW or de Sitter, then FLRW solutions do exist, but they are unstable to the aforementioned Higuchi ghost and other linear and nonlinear instabilities [68–73].

The search for a viable cosmology with a massive graviton which avoids these conclusions has involved two routes. One is to extend dRGT by adding extra degrees of freedom. As discussed above, these problems are cured when the second metric is given dynamics. Other extensions of massive gravity, such as quasidilaton [74], varying-mass [67, 75], nonlocal [29, 76, 77], and Lorentz-violating [78, 79] massive gravity, also seem to possess improved cosmological behaviour. The other approach is to give up on homogeneity and isotropy entirely. While FLRW solutions are mathematically simple, the Universe could in principle have anisotropies which have such low amplitude, are so much larger than our horizon, or both, that we cannot easily observe them. Remarkably, these cosmologies are much better behaved in massive gravity than is the standard FLRW case [67]. The general scenario of an FLRW metric with inhomogeneous Stückelberg fields has been derived in Refs. [80, 81]. This includes, but is not limited to, the case in which the reference metric is still Minkowski space, but only has the canonical form $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ in coordinates where $g_{\mu\nu}$ is not of the FLRW form [82]. The inhomogeneous and anisotropic solutions are reviewed thoroughly in Ref. [17]. See Ref. [83] for a review of cosmology in massive gravity and some of its extensions.

## 2.2 Einstein-Aether Theory

In the final section of this chapter, we explore a different route to modifying gravity: allowing Lorentz symmetry to be violated. This is not a step taken lightly; Lorentz invariance is a cornerstone of modern physics. The two theories which have been separately successful at predicting nearly all experimental and observational data to date, general relativity to explain the structure of spacetime and gravity and the standard model of particle physics to describe particles and nongravitational forces in the language of quantum field theory, both contain Lorentz symmetry as a crucial underlying tenet.

What do we gain from exploring the breakdown of this fundamental symmetry? Given its foundational significance, the consequences of violating Lorentz invariance deserve to be fully explored and tested. Indeed, while experimental bounds strongly constrain possible Lorentz-violating extensions of the standard model [84], Lorentz violation confined to other areas of physics—such as the gravitational, dark, or inflationary sectors—is somewhat less constrained, provided that its effects are not communicated to the matter sector in a way that would violate the standard-model experimental bounds. Moreover, it is known that general relativity and the standard model should break down around the Planck scale and be replaced by a new, quantum theory of gravity. If Lorentz symmetry proves not to be fundamental at such high energies—for instance, because spacetime itself is discretised at very small scales—this may communicate Lorentz-violating effects to gravity at lower energies, which could potentially be testable. The study of possible consequences of its violation, and the extent to which they can be seen at energies probed by experiment and observation, may therefore help us to constrain theories with such behaviour at extremely high energies.

A pertinent recent example is Hořava–Lifschitz gravity, a potential ultraviolet completion of general relativity which achieves its remarkable results by explicitly treating space and time differently at higher energies [85]. The consistent nonprojectable extension [86–88] of Hořava–Lifschitz gravity is closely related to the model we will explore. Moreover, since we will be dealing with Lorentz violation in the gravitational sector, through a vector-tensor theory of gravity, the usual motivations for modifying gravity apply to this kind of Lorentz violation. Indeed, there are interesting models of cosmic acceleration, based on the low-energy limit of Hořava–Lifschitz gravity, in which the effective cosmological constant is technically natural [89, 90]. Generalised Lorentz-violating vector-tensor models have also been considered as candidates for both dark matter and dark energy [91, 92].

Lorentz violation need not have such dramatic, high-energy origins. Indeed, many theories with fundamental Lorentz violation may face fine-tuning problems in order to avoid low-energy Lorentz-violating effects that are several orders of magnitude greater than existing experimental constraints [93]. However, even a theory which possesses Lorentz invariance at high energies could spontaneously break it at low energies, and with safer experimental consequences.

Spontaneous violation of Lorentz invariance will generally result when a field that transforms nontrivially under the Lorentz group acquires a vacuum expectation value (VEV). A simple example is that of a vector field whose VEV is nonvanishing everywhere. As mentioned above, in order to avoid the experimental constraints such a vector field should not be coupled to the standard model fields, but in order to not be completely innocuous we will ask it to couple to gravity. Moreover, to model Lorentz violation in gravity without abandoning the successes of general relativity—in particular, without giving up general covariance—the (spontaneously) Lorentz-violating field must be a spacetime vector and must be dynamical.[17]

A particularly simple, yet quite general, example of a model with these features is Einstein-aether theory (æ-theory) [94, 95]. It adds to general relativity a dynamical, constant-length timelike vector field, called the aether and denoted by $u^a$, which spontaneously breaks Lorentz invariance by picking out a preferred frame at each point in spacetime while maintaining local rotational symmetry, thus breaking only the boost sector of Lorentz symmetry [95, 96]. The constant-length constraint plays two crucial roles. The first is phenomenological: it ensures that the aether picks a globally-nonzero VEV and so guarantees that Lorentz symmetry is in fact broken. The other role is to ensure that the theory is not sick: if the length is not fixed and the kinetic term is not gauge-invariant[18] then the length-stretching mode has a wrong-sign kinetic term and hence is ghostlike [97]. Note that æ-theory is the most general effective field theory in which the rotation group is unbroken [98], and hence it can be seen as the low-energy limit of any theory which violates boosts but maintains rotational symmetry.

### 2.2.1 Pure Aether Theory

Einstein-aether theory (which we will often refer to as "pure" Einstein-aether theory or æ-theory) is a theory of the spacetime metric $g_{\mu\nu}$ and a vector field (the "aether") $u^\mu$. The action is [95, 99]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - K^{\mu\nu}{}_{\rho\sigma}\nabla_\mu u^\rho \nabla_\nu u^\sigma + \lambda\left(u^\mu u_\mu + m^2\right)\right], \qquad (2.70)$$

where we have defined

$$K^{\mu\nu}{}_{\rho\sigma} \equiv c_1 g^{\mu\nu}g_{\rho\sigma} + c_2\delta^\mu_\rho\delta^\nu_\sigma + c_3\delta^\mu_\sigma\delta^\nu_\rho + c_4 u^\mu u^\nu g_{\rho\sigma}. \qquad (2.71)$$

[17] The requirement that the field be dynamical stems from the fact that there is no nontrivial (i.e., nonzero) spacetime vector which is covariantly constant, i.e., if $\nabla_\mu u^\nu = 0$ everywhere then necessarily $u^\nu = 0$.

[18] Note that gauge invariance would uniquely pick out the Maxwell term, in which case we would simply have electromagnetism which clearly does not spontaneously break Lorentz symmetry.

The action (2.70) contains an Einstein–Hilbert term for the metric, a kinetic term for the aether with four dimensionless coefficients $c_i$ (coupling the aether to the metric through the covariant derivatives), and a nondynamical Lagrange multiplier $\lambda$. Varying this action with respect to $\lambda$ constrains the aether to be timelike with a constant norm, $u^\mu u_\mu = -m^2$. The aether has units of mass; its length, $m$, has the same dimensions and corresponds to the Lorentz symmetry breaking scale.

The action (2.70) is the most general diffeomorphism-invariant action containing the metric, aether, and up to second derivatives of each. Higher derivatives are excluded because they would generically lead to ghostlike degrees of freedom. Most terms one can write down involving the aether are eliminated by the fixed norm condition. One could consider a term $R_{\mu\nu}u^\mu u^\nu$, but this is equivalent under integration by parts to $(\nabla_\mu u^\mu)^2 + (1/2)F_{\mu\nu}F^{\mu\nu} - (\nabla_\mu u_\nu)(\nabla^\mu u^\nu)$, where $F_{\mu\nu} = 2\nabla_{[\mu}u_{\nu]}$ is the field strength tensor, and so is already included in the æ-theory action [94]. In what follows we will follow much of the literature on aether cosmology (e.g., Refs. [99, 100]) and ignore the quartic self-interaction term by setting $c_4 = 0$.

It is generally assumed that (standard-model) matter fields couple to the metric only. Any coupling to the aether would lead to Lorentz violation in the matter sector by inducing different maximum propagation speeds for different fields, an effect which is strongly constrained by experiments [84]. These problematic standard-model couplings may, however, be forbidden by a supersymmetric extension of æ-theory [101]. The work on æ-theory which we detail in Chap. 8 will be interested in exploring and constraining Lorentz violation in the gravitational sector and in a single non-standard-model scalar, hence we will not need to worry about such a coupling.

The gravitational constant $G$ that appears in Eq. (2.70) is to be distinguished from the gravitational constants which appear in the Newtonian limit and in the Friedmann equations, both of which are modified by the presence of the aether [99]. The Newtonian gravitational constant, $G_N$, and cosmological gravitational constant, $G_c$, are related to the bare constant $G$ by

$$G_N = \frac{G}{1 + 8\pi G\delta}, \tag{2.72}$$

$$G_c = \frac{G}{1 + 8\pi G\alpha}, \tag{2.73}$$

where

$$\delta \equiv -c_1 m^2, \tag{2.74}$$

$$\alpha \equiv (c_{13} + 3c_2)m^2. \tag{2.75}$$

We have introduced the notation $c_{13} \equiv c_1 + c_3$, etc., which we will use throughout.

### 2.2.2 Coupling to a Scalar Inflation

We now introduce to the theory a canonical scalar field $\phi$ which is allowed to couple kinetically to the aether through its *expansion*, $\theta \equiv \nabla_\mu u^\mu$ [102]. The full action reads

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - K^{\mu\nu}{}_{\rho\sigma}\nabla_\mu u^\rho \nabla_\nu u^\sigma + \lambda\left(u^\mu u_\mu + m^2\right) - \frac{1}{2}(\partial\phi)^2 - V(\theta,\phi)\right]. \tag{2.76}$$

Let us pause to motivate the generality of this model. Our aim in Chap. 8 will be to constrain couplings between a Lorentz-violating field and a scalar, in particular a canonical, slowly-rolling scalar inflaton, in as general a way as possible. As mentioned above, Einstein-aether theory is the unique Lorentz-violating effective field theory in which rotational invariance is maintained [98],[19] so any theory which spontaneously violates Lorentz symmetry without breaking rotational invariance will be described by the vector-tensor sector of our model at low energies. As for the scalar sector, the main restriction is that we have assumed a canonical kinetic term. While there are certainly coupling terms between the aether and the scalar which do not fall under the form $V(\theta,\phi)$, all of these terms have mass dimension greater than 4 and so are *irrelevant* operators. Such terms are nonrenormalisable. While this is not necessarily disastrous from an effective field theory perspective, these terms are also nevertheless mostly important at short distances, and so should not factor into the cosmological considerations at the heart of this thesis. To see that all terms with dimension 4 or less fall into the framework (2.76), notice that the aether, scalar, and derivative operators all have mass dimension 1, the aether norm is constant so $u_\mu u^\mu$ cannot be used in the coupling, and the aether and derivative operators carry spacetime indices which need to be contracted. Subject to these constraints, one can see that any terms which involve both $u^\mu$ and $\phi$ and have dimension 4 or less are either of the form $f(\theta,\phi)$ or can be recast into such a form under integration by parts. In particular, the only nontrivial interaction operators which are not irrelevant are $\phi\theta$ (dimension 3) and $\phi^2\theta$ (dimension 4).

This type of coupling was originally introduced with a more phenomenological motivation [102]. In a homogeneous and isotropic background, the aether aligns with the cosmic rest frame, so $\theta$ is essentially just the volume Hubble parameter, $\theta = 3mH$. Hence the introduction of the aether allows a scalar inflaton to couple directly to the expansion rate. This is impossible in GR where $H$ is not proportional to any Lorentz scalar.

The aether equation of motion, obtained by varying the action with respect to $u^\mu$, is

$$\lambda u^\nu = \nabla_\mu J^{\mu\nu} - \frac{1}{2}\nabla^\nu V_\theta \tag{2.77}$$

where the current tensor is defined by

[19]However, we note that there is an allowed term, the quartic self-interaction parametrised by $c_4$, which we have turned off.

$$J^{\mu}{}_{\sigma} \equiv -K^{\mu\nu}{}_{\sigma\rho}\nabla_{\nu}u^{\rho}, \tag{2.78}$$

and we are denoting partial derivatives of the potential by $V_\theta \equiv \partial V/\partial\theta$ and $V_\phi \equiv \partial V/\partial\phi$. Projecting this equation along $u^\mu$ allows us to obtain the Lagrange multiplier $\lambda$,

$$\lambda = -\frac{1}{m^2}u_\nu\nabla_\mu J^{\mu\nu} + \frac{1}{2m^2}u^\mu\nabla_\mu V_\theta. \tag{2.79}$$

We will account for the modification to gravity by leaving the Einstein equations in the standard form (1.6) and defining a stress-energy tensor for the combined aether-scalar system. Taking into account the contribution from the Lagrange multiplier term, this is given by

$$T_{\mu\nu} = 2\frac{\delta\mathscr{L}}{\delta g^{\mu\nu}} + u^\rho\frac{\delta\mathscr{L}}{\delta u^\rho}u_\mu u_\nu - \mathscr{L}g_{\mu\nu} \tag{2.80}$$

where $\mathscr{L}$ is the Lagrangian for the aether and scalar. Using this formula we find the stress-energy tensor,

$$\begin{aligned} T_{\mu\nu} = &\, 2c_1(\nabla_\mu u^\rho\nabla_\nu u_\rho - \nabla^\rho u_\mu\nabla_\rho u_\nu) \\ &- 2[\nabla_\rho(u_{(\mu}J^{\rho}{}_{\nu)}) + \nabla_\rho(u^\rho J_{(\mu\nu)}) - \nabla_\rho(u_{(\mu}J_{\nu)}{}^{\rho})] \\ &- 2m^{-2}u_\sigma\nabla_\rho J^{\sigma\rho}u_\mu u_\nu + g_{\mu\nu}\mathscr{L}_u \\ &+ \nabla_\mu\phi\nabla_\nu\phi - \left(\frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi + V - \theta V_\theta\right)g_{\mu\nu} \\ &+ (u^\rho\nabla_\rho V_\theta)(g_{\mu\nu} + m^{-2}u_\mu u_\nu), \end{aligned} \tag{2.81}$$

where $\mathscr{L}_u \equiv K^{\mu\nu}{}_{\rho\sigma}\nabla_\mu u^\rho\nabla_\nu u^\sigma$ is the Einstein-aether Lagrangian. Finally, the inflaton obeys the usual Klein–Gordon equation,

$$\Box\phi = V_\phi. \tag{2.82}$$

Notice that while this equation has the standard form, it couples the scalar to the aether since generally we will have $V_\phi = V_\phi(\theta, \phi)$.

Note that the equations of motion for the pure æ-theory follow simply by setting $\phi = 0$ and $V(\theta, \phi) = 0$.

### 2.2.3 *Einstein-Aether Cosmology*

In this section we examine the evolution of FLRW cosmological solutions in Einstein-aether theory. Consider a flat FLRW background geometry evolving in conformal time, $\tau$,

$$ds^2 = a^2(\tau)(-d\tau^2 + d\vec{x}^2). \tag{2.83}$$

In pure æ-theory, we take the 0–0 and trace Einstein equations to obtain the Friedmann equations,

$$\mathcal{H}^2 = \frac{8\pi G_c}{3} a^2 \rho_m, \tag{2.84}$$

$$\mathcal{H}' = -\frac{4\pi G_c}{3} a^2 \rho_m (1 + 3w), \tag{2.85}$$

where $\mathcal{H} \equiv a'/a = d \ln a/d\tau$ is the conformal time Hubble parameter. These are exactly the Friedmann equations of general relativity except that, as discussed above, the bare cosmological constant, $G$, is renormalised,

$$G_c = \frac{G}{1 + 8\pi G\alpha}, \tag{2.86}$$

with $\alpha = (c_1 + 3c_2 + c_3)m^2$. The aether does not change the cosmological dynamics at all, but just modifies the gravitational constant. This arises because in a homogeneous and isotropic background the Einstein-aether terms for the vector field only contribute stress-energy that tracks the dominant matter fluid, so the associated energy density is proportional to $H^2$ [99].[20]

The aether does contribute dynamical stress-energy once we couple it to a scalar. In the theory (2.76) the Friedmann equations are

$$\mathcal{H}^2 = \frac{8\pi G_c}{3} a^2 \left( V - \theta V_\theta + \rho_m + \frac{1}{2}\phi'^2 a^{-2} \right), \tag{2.87}$$

$$\mathcal{H}' = \frac{4\pi G_c}{3} a^2 \left[ -3\frac{m}{a}\left( 3\frac{m}{a} V_{\theta\theta}(\mathcal{H}' - \mathcal{H}^2) + V_{\theta\phi}\phi' \right) - \rho_m(1 + 3w) + 2(V - \theta V_\theta) - 2\phi'^2 a^{-2} \right], \tag{2.88}$$

For completeness we have included a matter component, but in the rest of this section and in Chap. 8 we will assume that $\phi$ is gravitationally dominant and ignore any $\rho_m$. The scalar field obeys the usual cosmological Klein–Gordon equation,

$$\phi'' + 2\mathcal{H}\phi' + a^2 V_\phi = 0. \tag{2.89}$$

As discussed above, the coupling to $\theta$ is contained in the function $V_\phi$. In the background, $\theta = 3mH$, with $H = \mathcal{H}/a$ the cosmic-time Hubble parameter, so this contributes either Hubble friction or a driving force [102].

We need not write down the aether field equations in the background. The vector field must be aligned with the cosmic rest frame due to homogeneity and isotropy, and its value,

$$u^\mu = \frac{m}{a}\delta^\mu{}_0, \tag{2.90}$$

[20] Note, however, that perturbations of the aether do carry some dynamics [100].

is determined completely by the normalisation condition, $u_\mu u^\mu = -m^2$. One can check that this solution satisfies the spatial component of the aether equation, while the temporal component only determines the Lagrange multiplier. In pure æ-theory this solution is stable perturbatively [100, 103–105] and that stability holds nonlinearly for most large perturbations [106]. This statement is subject to several constraints on the $c_i$ parameters which can be found in, e.g., [95, 100, 105], and we will assume throughout this thesis that these constraints hold. One of the important results in Chap. 8 is that the coupling between $u^\mu$ and $\phi$ can render cosmological solutions *unstable* for large regions of parameter space that are allowed by other experimental, observational, and theoretical constraints.

When the scalar potential is $V(\theta, \phi) = V(\phi)$, the background aether is irrelevant apart from rescaling Newton's constant, and many choices for the potential can lead to periods of slow-roll inflation [107]. Adding a coupling to the aether will change the dynamics but may still allow for inflation [102]. We will therefore aim to be as general about $V(\theta, \phi)$ as possible when discussing perturbation theory.

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